

# A spectral-based analysis of the separation between two-layer neural networks and linear methods

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**Editor:** Joan Bruna

## Abstract

We propose a spectral-based approach to analyze how two-layer neural networks separate from linear methods in terms of approximating high-dimensional functions. We show that quantifying this separation can be reduced to estimating the Kolmogorov width of two-layer neural networks, and the latter can be further characterized by using the spectrum of an associated kernel. Different from previous work, our approach allows obtaining upper bounds, lower bounds, and identifying explicit hard functions in a united manner. We provide a systematic study of how the choice of activation functions affects the separation, in particular the dependence on the input dimension. Specifically, for nonsmooth activation functions, we extend known results to more activation functions with sharper bounds. As concrete examples, we prove that any single neuron can instantiate the separation between neural networks and random feature models. For smooth activation functions, one surprising finding is that the separation is negligible unless the norms of inner-layer weights are polynomially large with respect to the input dimension. By contrast, the separation for nonsmooth activation functions is independent of the norms of inner-layer weights.

**Keywords:** Kolmogorov width, two-layer neural network, single neuron, spectral decay, curse of dimensionality

## 1. Introduction

Neural network-based machine learning has achieved remarkable performances in many applications, such as computer vision, natural language processing, and scientific computing. One of the key reasons behind these successes is that neural networks can efficiently approximate certain high-dimensional functions (Barron, 1993, 1994; Bach, 2017a; Ongie et al., 2019; E et al., 2019; Poggio and Liao, 2018; E et al., 2021; Gribonval et al., 2021; Suzuki, 2019; Klusowski and Barron, 2016), while traditional linear methods cannot (Barron, 1993, 1992; Kurková and Sanguinetti, 2002; Siegel and Xu, 2021). In this paper, we provide a comprehensive study of this phenomenon for two-layer neural networks.

Consider two-layer neural networks given by

$$f_m(x; \theta) = \sum_{j=1}^m a_j \sigma(w_j^T x + b_j), \quad (1)$$

where  $w_j \in \mathbb{R}^d, a_j \in \mathbb{R}, b_j \in \mathbb{R}$  and  $\sigma : \mathbb{R} \mapsto \mathbb{R}$  is the nonlinear activation function. We use  $\theta = \{a_j, w_j, b_j\}_{j=1}^m$  to denote all the parameters and  $m$  to denote the network width. Denote by  $\mathcal{N}_m$  the set of two-layer neural networks of width  $m$ :

$$\mathcal{N}_m = \left\{ f_m(\cdot; \theta) \mid \sum_{j=1}^m |a_j| \leq 1, \max_j (\|w_j\|_2 + |b_j|) \leq r \right\}. \quad (2)$$

To study the separation between neural networks and linear methods, we consider target functions in

$$\mathcal{N} = \overline{\cup_{m=1}^{\infty} \mathcal{N}_m}, \quad (3)$$

which is the closure (with respect to the  $L^\infty$  metric) of all finite-width neural networks with bounded norms defined above. Note that the function class  $\mathcal{N}$  depends on the activation function  $\sigma$  and the size of inner-layer weights  $r$ , which are omitted in the notation for simplicity. The function class  $\mathcal{N}$  has been widely used to analyze properties of two-layer neural networks including but not limited to the separation from linear methods. For more details, we refer to Barron (1993, 1992); Kurková and Sanguineti (2002); Bach (2017a); E et al. (2019, 2021) and the references therein. In this paper, we will focus on studying the separation in terms approximating functions in  $\mathcal{N}$ .

Denote by  $\rho$  the input distribution and the approximation error will be measured with respect to  $\rho$ . Mathematically speaking, quantifying the separation boils down to estimating the following two quantities:

- (1) The approximability by two-layer neural networks:

$$e_m(\mathcal{N}) := \sup_{f \in \mathcal{N}} \inf_{g \in \mathcal{N}_m} \|f - g\|_{L^2(\rho)}^2, \quad (4)$$

which describes the worst-case error of approximating functions in  $\mathcal{N}$  with two-layer neural networks.

- (2) The (in)approximability by linear methods:

$$\omega_m(\mathcal{N}) := \inf_{\phi_1, \dots, \phi_m \in L^2(\rho)} \sup_{f \in \mathcal{N}} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|f - \sum_{j=1}^m c_j \phi_j\|_{L^2(\rho)}^2, \quad (5)$$

which describes the worst-case error of approximating functions in  $\mathcal{N}$  with the best-possible fixed features. Note that  $\omega_m(\mathcal{N})$  is exactly the *Kolmogorov width* of  $\mathcal{N}$  (Kolmogorov, 1936; Lorentz, 1966).

When  $\omega_m(\mathcal{N})$  is significantly larger than  $e_m(\mathcal{N})$ , we say that there is a *separation* between two-layer neural networks and linear methods (with the best-possible fixed features). In other words, there exist functions in  $\mathcal{N}$  such that they can be efficiently approximated by neural networks but not by linear methods.

We are particularly interested in how the separation depends on the input dimension  $d$ . Does the separation become more significant in higher dimensions? To facilitate the statement, we use the concept of curse of dimensionality (CoD) (Bellman, 1957). A quantity  $q_m$  is said to be free of the CoD if  $q_m \leq \text{poly}(d)m^{-\beta}$  with  $\beta$  being a constant independent

of  $d$ . On the contrary, we say  $q_m$  exhibits or suffers from the CoD if  $q_m \geq \text{poly}(1/d)$  for all  $m \leq \exp(d)$ . A specific rate that achieves the CoD is  $q_m \geq \text{poly}(1/d)m^{-\beta/d}$  with  $\beta$  being a constant independent of  $d$ . In particular, when the Kolmogorov width  $\omega_m(\mathcal{N})$  exhibits the CoD, we must need exponentially many features for approximating functions in  $\mathcal{N}$ .

The seminal work (Barron, 1993) identified a subclass  $\mathcal{N}_F \subset \mathcal{N}$ , determined by the first-order moment of the Fourier transform, and showed  $e_m(\mathcal{N}_F) = O(1/m)$ . This bound means that approximating functions in  $\mathcal{N}_F$  with neural networks is free of the CoD; in other words, the approximation rate is dimension-independent. Later, Barron (1992) extended this result to the whole class  $\mathcal{N}$ . Along this line of work, (Breiman, 1993; Makovoz, 1998; Kurková et al., 1997; Kurková and Sanguineti, 2001, 2002; Bach, 2017a; E et al., 2019; Siegel and Xu, 2020a) improved and extended the upper bound, showing  $e_m(\mathcal{N}) = O(1/m)$  holds for very general activation functions. Given this upper bound of  $e_m(\mathcal{N})$ , what remains is estimating the Kolmogorov width  $\omega_m(\mathcal{N})$ .

Barron (1993, 1992) proved that  $\omega_m(\mathcal{N}) \geq Cd^{-1}m^{-1/d}$ , which implies that approximating functions  $\mathcal{N}$  with linear methods (even using the best-possible features) suffers from the CoD. Together with  $e_m(\mathcal{N}) = O(m^{-1})$ , Barron (1993, 1992) established the first CoD-type separation for neural networks and linear methods. However, the lower bound is limited to sigmoidal activation functions, such as the sigmoid and Heaviside step function. Barron (1993) obtained the lower bound by explicitly constructing exponentially many orthogonal functions in  $\mathcal{N}$  via the Fourier transform. Kurková and Sanguineti (2002) provided a systematic study of this orthogonal function argument for general function classes. But the application to neural networks is still limited to the case considered in Barron (1993), since the Fourier-based construction of orthogonal functions in Barron (1993) is specific to sigmoidal activations. It is unclear how to extend it to general cases. On the other hand, these works did not consider the upper bound of  $\omega_m(\mathcal{N})$ , which is critical for understanding when and how the separation from linear methods becomes negligible. For instance, if  $\omega_m(\mathcal{N}) = O(m^{-1})$ , then there is no separation between two-layer neural networks and linear methods for approximating functions in  $\mathcal{N}$  since  $e_m(\mathcal{N}) = O(m^{-1})$ .

In addition, the separation results mentioned above rely on worst-case analyses and only guarantee the existence of functions that are hard to approximate with linear methods. Another important but less-explored question is: Can we identify some specific hard functions that instantiate the separation? A recent progress was made by E et al. (2019) and Yehudai and Shamir (2019) for the separation from the specific random feature model (RFM) (Rahimi and Recht, 2007):  $h_m(x; a) = \sum_{j=1}^m a_j \sigma(w_j^T x)$ , where  $\{w_j\}$  are independently and uniformly drawn from the unit sphere. This model may look similar to two-layer neural networks. The difference is that the  $\{w_j\}$  in two-layer neural networks are adaptive to the target function, whereas  $\{w_j\}$  in the RFM are fixed.

Specifically, E et al. (2019) and Yehudai and Shamir (2019) showed that a single neuron:  $x \mapsto \sigma(v^T x + b)$  is enough to separate the two methods. E et al. (2019) numerically showed that approximating a single neuron with random features suffers from the CoD; Yehudai and Shamir (2019) provided a partial theoretical explanation. Yehudai and Shamir (2019) proved that there exist constants  $C_0, C_1 > 0$  such that, if  $m \leq e^{C_1 d}$  and the coefficients satisfy  $\max_{j \in [m]} |c_j| \leq e^{C_1 d}/m$ , then there exists a  $b^* \in \mathbb{R}$  such that for any  $v \in \mathbb{R}^d$  with  $\|v\| = d^3$ , the error of approximating  $x \mapsto \sigma(v^T x + b^*)$  with the random features is larger than  $C_0$ . However, this theoretical result is quite unsatisfying in two aspects: (1) The bias

term  $b^*$  still needs to be chosen in an adversary way, which means that this function is not completely explicit; (2) The coefficient magnitudes need to be bounded. Consequently, we cannot fully claim the separation for approximating this specific function. In addition, the analysis there is essentially also based on explicit constructions of nearly orthogonal functions (Malach and Shalev-Shwartz, 2020).

### 1.1 Overview of our contributions

In this paper, we present a spectral-based approach to analyze the separation, which reduces the problem to estimating the eigenvalues of an associated kernel. Specifically, consider a general parametric feature  $\varphi : X \times \Omega \mapsto \mathbb{R}$  and let  $\Phi = \{\varphi(\cdot; v) \mid v \in \Omega\}$  be the class of features. The key quantity in our analysis is the following average version of Kolmogorov width:

$$A_m^{(\pi)}(\Phi) = \inf_{\phi_1, \dots, \phi_m \in L^2(\rho)} \mathbb{E}_{v \sim \pi} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|\varphi(\cdot; v) - \sum_{j=1}^m c_j \phi_j\|_{L^2(\rho)}^2, \quad (6)$$

where  $\pi$  is a probability distribution on  $\Omega$ . We prove in Lemma 2 that  $A_m^{(\pi)}(\Phi) = \sum_{j=m+1}^{\infty} \lambda_j$ , where  $\{\lambda_j\}_{j \geq 1}$  are the eigenvalues in a non-increasing order of the kernel:  $k_{\pi}(x, x') = \langle \varphi(x; \cdot), \varphi(x'; \cdot) \rangle_{\pi}$ . Then we apply this general result to the two-layer neural network case where  $\varphi(x; v) = \sigma(w^T x + b)$ . The specific procedures go as follows.

- Let  $\mathcal{Q}^{\gamma, b} = \{\sigma_v(x) = \sigma(\gamma v^T x + b) \mid v \in \mathbb{S}^{d-1}\}$  be the class of single neuron and  $\tau_{d-1}$  be the uniform distribution over  $\mathbb{S}^{d-1}$ . We show in Theorem 10 that when  $\rho = \tau_{d-1}$ , there exists a  $C_d = \text{poly}(d)$  such that

$$\sup_{\gamma + |b| \leq r} A_m^{(\tau_{d-1})}(\mathcal{Q}^{\gamma, b}) \leq \omega_m(\mathcal{N}) \leq C_d \sup_{\gamma + |b| \leq r} A_m^{(\tau_{d-1})}(\mathcal{Q}^{\gamma, b}). \quad (7)$$

- Moreover, by Lemma 2,  $A_m^{(\tau_{d-1})}(\mathcal{Q}^{\gamma, b}) = \sum_{j=m+1}^{\infty} \lambda_j$ , where  $\{\lambda_j\}_{j \geq 1}$  denote the eigenvalues corresponding to the kernel:  $k^{(\gamma, b)}(x, x') = \mathbb{E}_{v \sim \tau_{d-1}} [\sigma(\gamma v^T x + b) \sigma(\gamma v^T x' + b)]$ .

We thus provide a tight characterization of  $\omega_m(\mathcal{N})$  using the spectrum of  $k^{(\gamma, b)}$ . For ReLU $^{\alpha}$  activations, we even have  $C_d = O(1)$ , where the characterization becomes exact. Here the ReLU $^{\alpha}$  activation function is defined by  $\sigma(z) = \max(0, z^{\alpha})$ . Note that the kernel  $k^{(\gamma, b)}$  is in a dot-product form, for which we can apply the harmonic analysis to obtain explicit estimates of the eigenvalues (Smola et al., 2001; Bach, 2017a).

The above approach does not need the explicit construction of orthogonal functions, thereby working for very general activation functions, including the commonly-used ReLU, Gaussian error linear unit (GELU) (Hendrycks and Gimpel, 2016), Swish/SiLU (Ramachandran et al., 2017; Elfving et al., 2018). Moreover, it allows obtaining upper bounds, lower bounds, and explicit hard functions simultaneously. Using this approach, we provide a comprehensive study of how the smoothness of  $\sigma$  affects the decay of  $\omega_m(\mathcal{N})$ , in particular the dependence on the input dimension. Our specific findings are summarized as follows.

- For the nonsmooth ReLU $^{\alpha}$  activation functions, we show in Proposition 4.1 that  $\omega_m(\mathcal{N}) \geq C(1/d, \alpha) m^{-\frac{2\alpha+1}{d-1}}$  with  $C(1/d, \alpha)$  depending on  $1/d$  polynomially. Combining with the known bounds  $e_m(\mathcal{N}) = O(1/m)$ , we establish the first CoD-type

separation for general  $\text{ReLU}^\alpha$  activations, which includes the result of Barron (1993) as a special case. In Theorem 6, we prove that the preceding lower bound holds for any single neuron, which removes all the restrictions of Yehudai and Shamir (2019), including the boundedness of coefficient magnitudes and the adversarial choice of bias term. Thus we can now truly claim the separation from the RFM for this specific function. Moreover, our results hold for any  $\text{ReLU}^\alpha$  activation, whereas Yehudai and Shamir (2019) only considers the ReLU case, i.e.,  $\alpha = 1$ .

- For smooth activation functions, we show that whether  $\omega_m(\mathcal{N})$  exhibits the CoD or not depends on the norms of inner-layer weights, i.e., the value of  $r$  in Eq. (2). Specifically, Theorem 11 shows that when  $r = 1$ ,  $\omega_m(\mathcal{N}) \leq d/m$  for activation functions that satisfy certain derivative boundedness condition. Proposition 12 further provides a fine-grained characterization of the dependence on  $r$  for the specific arctangent activation, showing  $\omega_m(\mathcal{N}) \leq d^4 r^2 m^{-\max(1/2, 1/r^2)}$ . These rates are dimension-independent and moreover, they can be achieved by using spherical harmonics, i.e., the homogeneous harmonic polynomials constrained on  $\mathbb{S}^{d-1}$ . Therefore, neural networks in these cases do not perform (significantly) better than polynomials. On the contrary, Theorems 14 and 15 show that the CoD-type separation can be recovered for sigmoid-like and ReLU-like smooth activations when  $r \geq d^C$  for some positive constant  $C$ . In particular,  $C > 1/2$  is enough for the specific arctangent activation function.

We notice that there is a concurrent work (Siegel and Xu, 2021), obtaining the same decay rates of  $\omega_m(\mathcal{N})$  for the  $\text{ReLU}^\alpha$  activations. However, the bounds there are not very useful in the high-dimensional regime since Siegel and Xu (2021) did not show their constants depending on  $d$  polynomially. On the other hand, the technique used in Siegel and Xu (2021) is still based on the orthogonal function argument developed in Barron (1993); Kurková and Sanguineti (2002), whereas ours is completely different.

Essentially, our spectral-based approach, in particular the bound (7), provides a tight characterization of the Kolmogorov width for two-layer neural networks, which is even almost exact for the  $\text{ReLU}^\alpha$  activation function. It is also applicable to analyze other properties that are related to Kolmogorov width. For instance, Siegel and Xu (2021) obtained a tight lower bound of the metric entropy (logarithm of the covering number) (Kolmogorov, 1958) of two-layer neural networks by using our results.

## 1.2 Other related work

**Approximation with random features** Our work is also related to Ghorbani et al. (2021), which shows that the random feature model effectively fits polynomials. Let  $\gamma = \mathcal{N}(0, 1)$  be the standard normal distribution. Specifically, for the single neuron target function, Ghorbani et al. (2021) showed that the approximating error with  $N$  random features is roughly controlled by  $\|\sigma_{>\ell}\|_{L^2(\gamma)}^2$  when  $d^{\ell+\delta} \leq N \leq d^{\ell+1-\delta}$  for small  $\delta > 0$ . Here,  $\sigma_{>\ell}(\cdot)$  is the projection of  $\sigma$  orthogonally to the subspace spanned by polynomials of minimum degree  $\ell$ . However, no explicit estimate is provided, and it is, therefore, unclear how the approximability depends on the input dimension. Moreover, the analysis of Ghorbani et al. (2021) is limited to the specific forms of features. By contrast, our analysis is applicable to very general features.

**Learning neural networks activated by smooth functions** Livni et al. (2014) proved that two-layer neural networks activated by the sigmoid function can be efficiently approximated by polynomials, if the norms of inner-layer weights are bounded by a constant independent of  $d$ . Then, they use this approximation result to show that these networks can be learned efficiently in polynomial time (see also Zhang et al. (2016)). In this paper, we extend this result to more general smooth activations, which include all the commonly-used ones, such as tanh, softplus, GELU. Moreover, we prove that when the norms of inner-layer weights are large than  $d^\beta$  with  $\beta$  being a small positive constant, there does not exist fixed features such that the linear method can avoid the CoD. These improvements over Livni et al. (2014) benefit from the spectral-based approach developed in our study, which is quite different from the techniques used in Livni et al. (2014).

**The spectrum of the associated kernel** It is well-known that the spectrum of the kernel  $k_\pi$  plays an important role in analyzing the corresponding RFM (Carratino et al., 2018; Bach, 2017b,a; Mei and Montanari, 2019). Recently, it is also extensively explored to understand neural networks in the kernel regime (Daniely, 2017; Xie et al., 2017; Jacot et al., 2018; Chen and Xu, 2021; Bietti and Mairal, 2019; Bietti and Bach, 2021; Scetbon and Harchaoui, 2021; Bach, 2017a). In contrast to these works, we reveal that the spectrum of that kernel also play a fundamental role in separating neural networks from linear methods, which is essentially a property beyond the kernel regime.

At a technical point, our work also bears similarity with Smola et al. (2001) and Bach (2017a) since we all rely on the harmonic analysis of functions on  $\mathbb{S}^{d-1}$ . Specifically, Smola et al. (2001) provided the integral representation of eigenvalues for general dot-product kernels; Bach (2017a) provided detailed calculations for the specific kernel  $k_\pi$ . In particular, Bach (2017a) obtained (complicated) analytic expressions of the eigenvalues for  $\text{ReLU}^\alpha$  activations, which are expressed in terms of Gamma functions. The reduced problem in our analysis is also estimating the eigenvalues of the kernel as explained above. We adopt the integral representation in Smola et al. (2001); Bach (2017a) and the analytic expressions in Bach (2017a). The specific differences from Bach (2017a) are listed below.

- For  $\text{ReLU}^\alpha$  activation functions, we obtain non-asymptotic estimates of the eigenvalues with the constants depending on  $d$  polynomially, whereas Bach (2017a) only considered the asymptotic regime without tracking the constants. This improvement is crucial for understanding the separation in high dimensions.
- We provide estimates of the eigenvalues for smooth activation functions, which are not covered in Bach (2017a). Moreover, for the specific arctangent activation, we further provide a fine-grained characterization by showing that the eigenvalue can be expressed analytically using Gaussian hypergeometric functions (Olver et al., 2010).
- Bach (2017a) used the spectrum of that kernel to compute the reproducing kernel Hilbert space (RKHS) (Aronszajn, 1950) norm induced by that RFM. However, our interest is analyzing how two-layer neural networks outperform linear methods.

## 2. Preliminaries

**Notation.** Let  $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ ,  $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  be the surface area of  $\mathbb{S}^{d-1}$ , and  $\tau_{d-1}$  be the uniform distribution over  $\mathbb{S}^{d-1}$ . For any  $\Omega$ , denote by  $\mathcal{P}(\Omega)$  the set of

probability measures over  $\Omega$ . For a probability measure  $\gamma$ , for simplicity we use  $\langle \cdot, \cdot \rangle_\gamma$  and  $\| \cdot \|_\gamma$  to denote the  $L^2(\gamma)$  inner product and norm, respectively. Given a subset  $B$ , denote by  $1_B$  the indicator function of  $B$ , which has value 1 at points of  $B$  and 0 otherwise. Let  $\Gamma(x)$  be the Gamma function and  $\binom{n}{m} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)}$  be the binomial coefficient. We shall use  $\text{poly}(\beta)$  to denote a quantity that depends on  $\beta$  polynomially. We use  $X \sim Y$ , if there exist absolute constants  $C_1, C_2 > 0$  such that  $C_1 Y \leq X \leq C_2 Y$ .  $X \lesssim Y$  means  $X \leq CY$  for an absolute constant  $C > 0$ , and  $X \gtrsim Y$  is defined analogously.

## 2.1 Legendre polynomials and spherical harmonics

We shall focus on the case of  $\rho = \tau_{d-1}$ , where the eigenvalues of the kernel of interest can be explicitly estimated. Hence, we need to prepare some basic techniques for analyzing functions on  $\mathbb{S}^{d-1}$ . Denote by  $P_k$  the associated Legendre polynomials of degree  $k$  in  $d$  dimensions (Atkinson and Han, 2012, Section 2.6), which satisfies the following recursive formula (Atkinson and Han, 2012, Equation 2.86):

$$\begin{aligned} P_0(t) &= 0, P_1(t) = t, \\ P_k(t) &= \frac{2k+d-4}{k+d-3} t P_{k-1}(t) - \frac{k-1}{k+d-3} P_{k-2}(t), \quad k \geq 2. \end{aligned} \quad (8)$$

Note that  $\{P_k\}_{k=0}^\infty$  are the orthogonal polynomials with respect to the distribution  $p_d(t) = (1-t^2)^{\frac{d-3}{2}} / B(\frac{1}{2}, \frac{d-1}{2})$ , i.e., the distribution of  $x_1$  for  $x \sim \tau_{d-1}$ . Specifically,

$$\int_{-1}^1 P_k(t) P_j(t) (1-t^2)^{(d-3)/2} dt = \delta_{jk} \frac{\omega_{d-1}}{\omega_{d-2}} \frac{1}{N(d, k)}, \quad (9)$$

where

$$N(d, k) = \frac{2k+d-2}{k} \binom{k+d-3}{d-2}$$

(Atkinson and Han, 2012, Equation 2.67 and 2.68). Notice that the above equation can be rewritten as follows:

$$\int_{\mathbb{S}^{d-1}} P_k(x^\top y) P_j(x^\top y) d\tau_{d-1}(y) = \frac{\delta_{jk}}{N(d, k)}, \quad \forall x \in \mathbb{S}^{d-1}.$$

The Rodrigues's formula gives a closed-form expression of  $P_k$ :

$$P_k(t) = \left(-\frac{1}{2}\right)^k \frac{\Gamma((d-1)/2)}{\Gamma(k+(d-1)/2)} (1-t^2)^{(3-d)/2} \left(\frac{d}{dt}\right)^k (1-t^2)^{k+(d-3)/2}. \quad (10)$$

The polynomial  $P_k$  is even (resp. odd) when  $k$  is even (resp. odd).

Let  $\mathcal{Y}_k^d$  be the space of all homogeneous harmonic polynomials of degree  $k$  in  $d$  dimensions restricted on  $\mathbb{S}^{d-1}$ ; the dimension of the space  $\mathcal{Y}_k^d$  is  $N(d, k)$ . Let  $\{Y_{k,j}\}_{1 \leq j \leq N(d,k)}$  be an orthonormal basis of  $\mathcal{Y}_k^d$  in  $L^2(\tau_{d-1})$ . Then  $Y_{k,j} : \mathbb{S}^{d-1} \mapsto \mathbb{R}$  denotes the  $j$ -th spherical harmonics of degree  $k$ . Then  $\{Y_{k,j}\}_{k \in \mathbb{N}, 1 \leq j \leq N(d,k)}$  forms an orthonormal basis of  $L^2(\tau_{d-1})$  (Atkinson and Han, 2012, Section 2.1.3, Corollary 2.15 and Theorem 2.38).

The spherical harmonics is related to the Legendre polynomials:

$$\sum_{j=1}^{N(d,k)} Y_{k,j}(x)Y_{k,j}(y) = N(d,k)P_k(x^\top y) \quad (11)$$

(Atkinson and Han, 2012, Theorem 2.24). For any  $f : [-1, 1] \mapsto \mathbb{R}$ ,  $x \in \mathbb{S}^{d-1}$  and  $Y_k \in \mathcal{Y}_k^d$ , the Hecke-Funk formula (Atkinson and Han, 2012, Theorem 2.22) is given by

$$\int_{\mathbb{S}^{d-1}} f(x^\top y)Y_k(y) d\tau_{d-1}(y) = \frac{\omega_{d-2}}{\omega_{d-1}}Y_k(x) \int_{-1}^1 f(t)P_k(t) (1-t^2)^{(d-3)/2} dt. \quad (12)$$

We refer to Schoenberg et al. (1942) and Atkinson and Han (2012) for more details about the harmonic analysis on  $\mathbb{S}^{d-1}$ .

### 3. A general result

Denote by  $\varphi : X \times \Omega \mapsto \mathbb{R}$  a general parametric feature. For any  $\pi \in \mathcal{P}(\Omega)$ , define

$$k_\pi : X \times X \mapsto \mathbb{R}, \quad k_\pi(x, x') = \langle \varphi(x; \cdot), \varphi(x'; \cdot) \rangle_\pi. \quad (13)$$

Assume that  $\varphi(\cdot; v)$  is continuous on  $X$  for any  $v \in \Omega$  and  $X$  is compact. By Mercer's theorem, we have the eigendecomposition:  $k_\pi(x, x') = \sum_{j=1}^{\infty} \lambda_j e_j(x) e_j(x')$ , where  $\{\lambda_j\}_{j \geq 1}$  are the eigenvalues in a non-increasing order and  $\{e_j\}_{j \geq 1}$  are the corresponding eigenfunctions that satisfy  $\mathbb{E}_{x' \sim \rho}[k_\pi(x, x') e_j(x')] = \lambda_j e_j(x)$ . The trace of  $k_\pi$  satisfies that  $\sum_{j=1}^{\infty} \lambda_j = \mathbb{E}_{x \sim \rho}[k_\pi(x, x)]$ . In particular, we are interested in the following quantity

$$\Lambda_\pi(m) = \sum_{j=m+1}^{\infty} \lambda_j. \quad (14)$$

Consider the model:  $g_m(x; \theta) = \sum_{j=1}^m a_j \varphi(x; v_j)$ , where  $\theta = \{(a_j, v_j)\}_{j=1}^m$  denote the learnable parameters. In the literature, this model is often called *variable-basis approximation* (Gnecco, 2012; Kurková and Sanguineti, 2001). Two-layer neural networks correspond to the special case where  $\varphi(x; v) = \sigma(w^\top x + b)$ . Other examples include the free-node splines ( DeVore and Lorentz, 1993, Chapter 11), trigonometric polynomials with free frequencies ( DeVore and Temlyakov, 1995), etc.

Recall that  $\Phi = \{\varphi(\cdot; v) \mid v \in \Omega\}$  is the class of parametric feature functions. Define

$$\mathcal{G} = \overline{\left\{ \sum_{j=1}^m a_j \varphi(\cdot; v_j) \mid \sum_{j=1}^m |a_j| \leq 1, m \in \mathbb{N}^+ \right\}}, \quad (15)$$

which is just the closure of the convex, symmetric hull of  $\Phi$ . When  $\varphi(x; v) = \sigma(w^\top x + b)$  and  $\Omega = \{(w^\top, b)^\top \in \mathbb{R}^{d+1} : \|w\|_2 + |b| \leq r\}$ , we have  $\mathcal{G} = \mathcal{N}$ .

The following theorem provides a lower bound of the Kolmogorov width of  $\mathcal{G}$ .

**Proposition 1 (Spectral-based lower bound)**  $\omega_m(\mathcal{G}) \geq \sup_{\pi \in \mathcal{P}(\Omega)} \Lambda_\pi(m)$ .



The proof of Proposition 1 needs the following characterization of the average Kolmogorov width of  $\Phi$ .

**Lemma 2**  $A_m^{(\pi)}(\Phi) = \mathbb{E}_{v \sim \pi} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|\varphi_v - \sum_{j=1}^m c_j e_j\|_\rho^2 = \Lambda_\pi(m)$ , where  $\varphi_v = \varphi(\cdot; v)$ .

It is implied that the best-possible features minimizing the average approximation error are the leading eigenfunctions.

**Proof** Without loss of generality (WLOG), assume  $\{\phi_j\}_{j=1}^m$  are orthonormal in  $L^2(\rho)$ , otherwise we can perform Gram-Schmidt orthonormalization. Then,

$$\begin{aligned} \mathbb{E}_{v \sim \pi} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|\varphi_v - \sum_{j=1}^m c_j \phi_j\|_\rho^2 &= \mathbb{E}_{v \sim \pi} \|\varphi_v - \sum_{j=1}^m \langle \varphi_v, \phi_j \rangle_\rho \phi_j\|_\rho^2 = \mathbb{E}_{v \sim \pi} [\|\varphi_v\|_\rho^2 - \sum_{j=1}^m \langle \varphi_v, \phi_j \rangle_\rho^2] \\ &= \mathbb{E}_{x \sim \rho} [k_\pi(x, x)] - \sum_{j=1}^m \mathbb{E}_{x, x' \sim \rho} [\phi_j(x) \phi_j(x') k_\pi(x, x')] \\ &\geq \sum_{j=1}^m \lambda_j - \sup_{\langle \psi_j, \psi_{j'} \rangle_\rho = \delta_{j,j'}} \sum_{j=1}^m \mathbb{E}_{x, x' \sim \rho} [\psi_j(x) \psi_j(x') k_\pi(x, x')]. \end{aligned} \quad (16)$$

The second term in (16) is a standard PCA problem for  $k_\pi$ , where the supremum is reached at  $\phi_j = e_j$  and  $\sum_{j=1}^m \mathbb{E}_{x, x' \sim \rho} [e_j(x) e_j(x') k_\pi(x, x')] = \sum_{j=1}^m \lambda_j$ . Plugging it back to (16) completes the proof.  $\blacksquare$

**Proof of Proposition 1** Lemma 2 provides a estimate of the average approximation error. The Kolmogorov width, which is the worst-case error, can be bounded as follows:

$$\omega_m(\Phi) = \inf_{\phi_1, \dots, \phi_m} \sup_{v \in \Omega} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|\varphi_v - \sum_{j=1}^m c_j \phi_j\|_\rho^2 \geq A_m^{(\pi)}(\Phi) = \Lambda_\pi(m), \quad \forall \pi \in \mathcal{P}(\Omega). \quad (17)$$

Hence,  $\omega_m(\mathcal{G}) \geq \omega_m(\Phi) \geq \sup_{\pi \in \mathcal{P}(\Omega)} \Lambda_\pi(m)$  because of  $\Phi \subset \mathcal{G}$ .  $\blacksquare$

The spectral-based lower bound of  $\omega_m(\mathcal{G})$  in Proposition 1 is quite general, holding for any parametric features. Even when explicit estimates of the eigenvalues of  $k_\pi$  are not tractable, we can still numerically compute ones, thereby obtaining a numerical lower bound of  $\omega_m(\mathcal{G})$ .

We notice that a similar lower bound of Kolmogorov width already appeared in Ismagilov (1968) for a completely different purpose. At first glance, the function class considered in Ismagilov (1968) may look different from  $\mathcal{G}$ , but they are in fact equivalent (see (Pinkus, 2012, Section 3.4)). However, to the best of our knowledge, our work is the first one exploiting this approach to study variable-basis approximations and two-layer neural networks. In contrast, the existing works on bounding  $\omega_m(\mathcal{G})$  (Kurková and Sanguineti, 2002; Gnecco, 2012; Siegel and Xu, 2021) all rely on the orthogonal function argument, whose applicability is limited to some special cases. Moreover, we obtain an upper bound that match the lower bound in Proposition 1 for two-layer neural networks.

In the remaining of this paper, we will apply the above general result to the case of two-layer neural networks and the input distribution  $\rho = \tau_{d-1}$ . In such case, we can have

explicit estimates of the eigenvalues, thereby the Kolmogorov widths. Moreover, taking  $\pi = \tau_{d-1}$  also yields an upper bound that matches the above lower bound. For simplicity, we will omit the subscript of  $\Lambda_\pi$  since  $\pi$  is always fixed to be  $\tau_{d-1}$  in the following analyses.

#### 4. Approximation of single neurons

We first consider the single neuron without bias:  $x \mapsto \sigma(v^T x)$ . The results established in this section will be utilized later to analyze two-layer neural networks.

Assume that  $\pi = \rho = \tau_{d-1}$ . By the rotational invariance,  $k_\pi$  can be written in a dot-product form:

$$k_\pi(x, x') = \int_{\mathbb{S}^{d-1}} \sigma(v^T x) \sigma(v^T x') d\tau_{d-1}(v) = \kappa(x^T x') \quad (18)$$

where  $\kappa : [-1, 1] \mapsto \mathbb{R}$ . Following Smola et al. (2001), the spectral decomposition of  $\kappa$  is given by:

$$\kappa(x^T x') = \sum_{k=0}^{\infty} \sum_{j=1}^{N(d,k)} \mu_k Y_{k,j}(x) Y_{k,j}(x'), \quad (19)$$

where  $\mu_k$  is the eigenvalue and the spherical harmonics  $Y_{k,j}$  is the corresponding eigenfunction that satisfies  $\mathbb{E}_{x' \sim \tau_{d-1}} [\kappa(x^T x') Y_{k,j}(x')] = \mu_k Y_{k,j}(x)$ . Note that  $\{\lambda_j\}$  are the eigenvalues counted with multiplicity, while  $\{\mu_k\}$  are the eigenvalues counted without multiplicity. We refer to (Schoenberg et al., 1942; Smola et al., 2001) for more details about the spectral decomposition of a dot-product kernel on  $\mathbb{S}^{d-1}$ .

Applying Lemma 2 to single neurons gives the following lower bound

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \geq \Lambda(m), \quad (20)$$

and the inequality can be achieved by using the spherical harmonics as the fixed features.

By exploiting the rotational invariance, one can show the following upper bound.

**Proposition 3 (Uniform approximability)** *Let  $\{\phi_j\}_{j=1}^m$  be the leading spherical harmonics. For any non-increasing function  $L : \mathbb{N}^+ \rightarrow \mathbb{R}^+$  that satisfies  $\Lambda(m) \leq L(m)$ , let  $q(d, L) = \sup_{k \geq 1} \frac{L(k)}{L((d+1)k)}$ . Then we have for any  $v \in \mathbb{S}^{d-1}$ ,*

$$\inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \lesssim q(d, L) L(m). \quad (21)$$

The proof is deferred to Appendix A.1. When  $L(m) \sim m^{-s}$ ,  $q(d, L) \lesssim d^s$ . Therefore,  $q(d, L)$  is at most polynomial in  $d$ . In fact, one can choose  $L(m) = \Lambda(m)$  to obtain the tightest bound. The introduction of  $L(m)$  is to facilitate the explicit calculation of constant  $q(d, L)$ , since we hardly have the exact rate of  $\Lambda(m)$ .

By applying the eigenvalue estimates given in the next subsections, we have the following specific results. For the nonsmooth activations considered in Section 4.1, taking  $L(m) \sim m^{-\frac{2\alpha+1}{d-1}}$  yields  $q(d, L) \lesssim d^{\frac{2\alpha+1}{d-1}} \leq C_\alpha$  with  $C_\alpha$  being a positive constant independent of  $d$ .

For smooth activation functions that satisfy Assumption 8, we can take  $L(m) = 1/m$ , for which  $q(d, L) \lesssim d$ .

What remains is to estimate the eigenvalues of  $k_\pi$  and the following integral representation allows both explicit estimations and numerical computations of the eigenvalues.

**Lemma 4** *For the dot-product kernel  $k_\pi$ ,  $\mu_k = \eta_k^2$  with*

$$\eta_k = \frac{\omega_{d-2}}{\omega_{d-1}} \int_{-1}^1 \sigma(t) P_k(t) (1-t^2)^{(d-3)/2} dt. \quad (22)$$

*In addition, assume that  $\sigma \in C^\infty(\mathbb{R})$ . Then,*

$$\eta_k = \frac{\Gamma(d/2)}{2^k \sqrt{\pi} \Gamma(k + (d-1)/2)} \int_{-1}^1 \sigma^{(k)}(t) (1-t^2)^{k+(d-3)/2} dt. \quad (23)$$

**Proof** By the Hecke-Funk formula (12),

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \kappa(x^T x') Y_{k,j}(x') d\tau_{d-1}(x') &= \int_{\mathbb{S}^{d-1}} \left( \int_{\mathbb{S}^{d-1}} \sigma(w^T x) \sigma(w^T x') d\tau_{d-1}(w) \right) Y_{k,j}(x') d\tau_{d-1}(x') \\ &= \int_{\mathbb{S}^{d-1}} \sigma(w^T x) \left( \int_{\mathbb{S}^{d-1}} \sigma(w^T x') Y_{k,j}(x') d\tau_{d-1}(x') \right) d\tau_{d-1}(w) \\ &= \int_{\mathbb{S}^{d-1}} \sigma(w^T x) \eta_k Y_{k,j}(w) d\tau_{d-1}(w) = \eta_k^2 Y_{k,j}(x). \end{aligned}$$

Substituting the Rodrigues formula (10) into (22) and applying the integration by parts give

$$\begin{aligned} \eta_k &= \left(-\frac{1}{2}\right)^k \frac{\omega_{d-2}}{\omega_{d-1}} \frac{\Gamma((d-1)/2)}{\Gamma(k + (d-1)/2)} \int_{-1}^1 \sigma(t) \left(\frac{d}{dt}\right)^k (1-t^2)^{k+(d-3)/2} dt \\ &= \frac{1}{2^k} \frac{\omega_{d-2}}{\omega_{d-1}} \frac{\Gamma((d-1)/2)}{\Gamma(k + (d-1)/2)} \int_{-1}^1 \sigma^{(k)}(t) (1-t^2)^{k+(d-3)/2} dt. \end{aligned}$$

Inserting  $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  completes the proof. ■

We remark that the integral representation (22) has been adopted in (Bach, 2017a, Appendix D) to calculate the eigenvalues of  $k_\pi$  for  $\text{ReLU}^\alpha$  activations. Eq. (23) also follows straightforwardly from Eq. (22). We provide the proof here since it is simple but very helpful for understanding what property of activation functions affect the eigenvalue. In particular, Eq. (23) shows that the smaller is the  $k$ -th order derivative, the smaller is the eigenvalue and this formula will be used later to estimate the eigenvalues for smooth activation functions.

**Numerical computation of  $\Lambda(m)$**  The following procedures provide a numerical way to compute  $\Lambda(m)$ , which works for any activation functions.

- First,  $\sum_{j=1}^\infty \lambda_j = \int_{\mathbb{S}^{d-1}} \kappa(x^T x) d\tau_{d-1}(x) = \kappa(1)$ . If  $\kappa(1)$  does not have an explicit expression, we can use Monte-Carlo integration to numerically compute it by Eq. (18).
- Second, the eigenvalues  $\{\lambda_j\}$  are computed through numerically integrating the right hand side of Eq. (22), where the Legendre polynomials can be efficiently computed using the recursive formula (8).
- The output is:  $\Lambda(m) = \kappa(1) - \sum_{j=1}^m \lambda_j$ .

#### 4.1 Nonsmooth activations

Consider the  $\text{ReLU}^\alpha$  activation function:  $\sigma(t) = \max(0, t)^\alpha$  with  $\alpha \in \mathbb{N}$ . The Heaviside step and ReLU function correspond to  $\alpha = 0$  and  $\alpha = 1$ , respectively. The case of  $\alpha > 1$  also has many applications in scientific computing (E and Yu, 2018; Siegel and Xu, 2020b; Li et al., 2019). In particular, for  $\alpha = 0, 1$ , Cho and Saul (2009) shows

$$\kappa(t) = \begin{cases} \frac{1}{2\pi}(\pi - \arccos(t)) & \text{if } \alpha = 0 \\ \frac{1}{2\pi d} \left( (\pi - \arccos(t))t + \sqrt{1-t^2} \right) & \text{if } \alpha = 1. \end{cases} \quad (24)$$

**Proposition 5** *Let  $\sigma(t) = \max(0, t)^\alpha$  with  $\alpha \in \mathbb{N}$ . There exists a constant  $C(1/d, \alpha)$  depending on  $1/d$  polynomially such that  $\Lambda(m) \geq C(1/d, \alpha)m^{-\frac{2\alpha+1}{d-1}}$ . In particular,  $C(1/d, 0) = 1/d$ .*

The proof is quite technical and deferred to Appendix A.2, where the analytic expression of the eigenvalue  $\mu_k$  obtained in (Bach, 2017a, Appendix D) is used. Figure 1 compares the above bounds of  $\Lambda(m)$  and numerical estimations for various  $d$ 's for the case of  $\alpha = 0$ . It is clear that the decay suffers from the CoD and the explicit rate  $m^{-(2\alpha+1)/(d-1)}$  given in Proposition 5 aligns very well with the ground truth for large  $m$ 's.

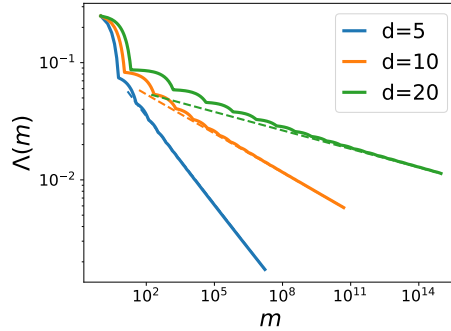


Figure 1: The decay of  $\Lambda(m)$  for various  $d$ 's and  $\alpha = 0$ . The solid curve corresponds to the numerical estimate, while the dashed curve corresponds the explicit estimate  $m^{-(2\alpha+1)/(d-1)}$  given in Proposition 5.

Then we have the following theorem, which shows that approximating single neurons activated by  $\text{ReLU}^\alpha$  with linear methods suffers from the CoD.

**Theorem 6** *Let  $\sigma_v(x) = \max(0, v^T x)^\alpha$  for  $\alpha \in \mathbb{N}$ . Then, there exists a constant  $C(1/d, \alpha)$  that depends on  $1/d$  polynomially such that the following statements hold.*

- For any fixed features  $\{\phi_j\}_{j=1}^m$ , we have

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m \in \mathbb{R}} \left\| \sigma_v - \sum_{j=1}^m c_j \phi_j \right\|_{\tau_{d-1}}^2 \geq C(1/d, \alpha) m^{-\frac{2\alpha+1}{d-1}}. \quad (25)$$

- Consider the random feature:  $f_j(\cdot; \cdot) : \mathbb{R}^d \times \mathbb{R}^{d \times q_j} \mapsto \mathbb{R}$ . We assume  $\{f_j\}_{j=1}^m$  are rotationally invariant, i.e., for any  $j \in [m]$ ,  $f_j(x; W_j) = f_j(Qx; QW_j)$  for any orthonormal matrix  $Q \in \mathbb{R}^{d \times d}$  and  $W_j$  is sampled from a rotation-invariant distribution  $\pi_j$ . Let  $\bar{W} = (W_1, \dots, W_m)$ . Then, for any  $v \in \mathbb{S}^{d-1}$ ,

$$\mathbb{E}_{\bar{W}} \inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j f_j(\cdot; W_j)\|_{\tau_{d-1}}^2 \geq C(1/d, \alpha) m^{-\frac{2\alpha+1}{d-1}}. \quad (26)$$

If  $m \leq 2^d$ , with probability larger than  $C(1/d, \alpha)$  over the sampling of  $\bar{W}$ , we have for any  $v \in \mathbb{S}^{d-1}$ ,

$$\inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j f_j(\cdot; W_j)\|_{\tau_{d-1}}^2 \geq C(1/d, \alpha). \quad (27)$$

Eq. (25) shows that the average error of approximating single neurons with any fixed features suffers from the CoD. This suggests that there exists a single neuron  $\sigma_v$  such that the approximation is difficult, whereas the specific choice of  $v$  is unknown and it depends on the features. Eq. (26) and (27) improve this by showing that the same conclusion holds for any  $v \in \mathbb{S}^{d-1}$  as long as the features are rotationally invariant. Specifically, Eq. (26) bounds the error in expectation with respect to the sampling of random features and Eq. (27) further provides a bound of the failure probability.

A typical form of rotation-invariant random features is  $f_j(x; W_j) = g_j(W_j^T x)$  with  $g_j : \mathbb{R}^{q_j} \mapsto \mathbb{R}^1$ , which include  $f_j(x; w) = \sigma(w^T x)$  (emerging in analyzing neural networks) and kernel predictors with dot-product kernels. Therefore, our results cover the setting considered in Yehudai and Shamir (2019) but successfully remove all the limitations of Yehudai and Shamir (2019). Specifically, we impose no restriction on the coefficient magnitudes and do not need to adversarially choose the bias term. Note that any single neuron can be approximated exactly by two-layer neural network of width  $m = 1$ , whereas the random feature approximation requires exponentially many features. They together provides a CoD-type separation between two methods for approximating this specific function.

**Proof of Theorem 6** Eq. (25) follows from a simple combination of Proposition 5 and Lemma 2. To prove Eq. (26), we need to exploit the rotational invariance of the random features. Let  $S_v(\bar{W}) = \inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j f_j(\cdot; W_j)\|_{\tau_{d-1}}^2$ . For any  $v \in \mathbb{S}^{d-1}$ , let  $Q_v \in \mathbb{R}^{d \times d}$  be an orthonormal matrix such that  $Q_v v = e_1$ . Then,

$$\begin{aligned} \mathbb{E}_{\bar{W}}[S_v(\bar{W})] &= \mathbb{E}_{\bar{W}} \inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j f_j(\cdot; W_j)\|_{\rho}^2 \stackrel{(i)}{=} \mathbb{E}_{\bar{W}} \inf_{c_1, \dots, c_m} \|\sigma_{e_1} - \sum_{j=1}^m c_j f_j(\cdot; Q_v W_j)\|_{\rho}^2 \\ &\stackrel{(ii)}{=} \mathbb{E}_{\bar{W}} \inf_{c_1, \dots, c_m} \|\sigma_{e_1} - \sum_{j=1}^m c_j f_j(\cdot; W_j)\|_{\rho}^2 = \mathbb{E}_{\bar{W}}[S_{e_1}(\bar{W})], \end{aligned}$$

where (i) and (ii) follow from the rotational invariance of  $\rho$  and  $\{\pi_j\}$ , respectively. Therefore,  $\mathbb{E}_{\bar{W}}[S_v(\bar{W})]$  is constant with respect to  $v$ . By Lemma 2, we have

$$\mathbb{E}_{\bar{W}}[S_v(\bar{W})] = \mathbb{E}_{v \sim \tau_{d-1}} \mathbb{E}_{\bar{W}}[S_v(\bar{W})] = \mathbb{E}_{\bar{W}} \mathbb{E}_{v \sim \tau_{d-1}}[S_v(\bar{W})] \geq \mathbb{E}_{\bar{W}}[\Lambda(m)] = \Lambda(m).$$

Then, applying Proposition 5 completes the proof of Eq. (26).

In addition, we have

$$S_v(\bar{W}) \leq \|\sigma_v\|_{\tau_{d-1}}^2 = \int_{\mathbb{S}^{d-1}} \sigma(v^T x)^2 d\tau_{d-1} = \int_0^1 t^{2\alpha} p_d(t) dt = \frac{\Gamma(d/2)}{\Gamma(d/2 + \alpha)} \lesssim \frac{1}{d^\alpha},$$

where  $p_d(t) = \frac{(1-t^2)^{\frac{d-3}{2}}}{B(\frac{1}{2}, \frac{d-1}{2})}$  is the density function of  $v^T x$ .

Then,

$$\Lambda(m) \leq \mathbb{E}[S_v(\bar{W})] \lesssim \mathbb{P}\{S_v(\bar{W}) \geq t\}d^{-\alpha} + \mathbb{P}\{S_v(\bar{W}) \leq t\}t \leq \mathbb{P}\{S_v(\bar{W}) \geq t\}d^{-\alpha} + t. \quad (28)$$

Taking  $t = 0.5\Lambda(m)$ , we have

$$\mathbb{P}\{S_v(\bar{W}) \geq 0.5\Lambda(m)\} \gtrsim \frac{d^\alpha}{2}\Lambda(m).$$

When  $m \leq 2^d$ ,  $\Lambda(m) \geq C(1/d, \alpha)2^{-2\alpha-1}$ . Hence, we complete the proof of Eq. (27). ■

## 4.2 Smooth activations

We now turn to smooth activation functions, such as sigmoid, softplus, arctan, GELU, and Swish/SiLU. We first have the following lemma, which bounds the eigenvalue  $\mu_k$  by using the  $k$ -th order derivative of  $\sigma$ .

**Lemma 7** *Assume that  $\sigma$  is smooth and let  $B_k = \sup_{t \in \mathbb{R}} |\sigma^{(k)}(t)|$ . Then, we have*

$$\mu_k \leq \frac{B_k^2}{2^{2k}} \frac{\Gamma(d/2)^2}{\Gamma(k + d/2)^2}.$$

**Proof** By the assumption,

$$\begin{aligned} \left| \int_{-1}^1 \sigma^{(k)}(t) (1-t^2)^{k+(d-3)/2} dt \right| &\leq B_k \int_{-1}^1 (1-t^2)^{k+(d-3)/2} dt \\ &= B_k \int_0^1 u^{-1/2} (1-u)^{k+(d-3)/2} du = B_k \frac{\Gamma(1/2)\Gamma(k+(d-1)/2)}{\Gamma(k+d/2)}, \end{aligned} \quad (29)$$

where the second equality follows from the change of variable  $u = t^2$ . Then, using Eq. (23) and  $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  gives rise

$$|\eta_k| \leq \frac{B_k \omega_{d-2}}{2^k \omega_{d-1}} \frac{\Gamma(1/2)\Gamma((d-1)/2)}{\Gamma(k+d/2)} = \frac{B_k}{2^k} \frac{\Gamma(d/2)}{\Gamma(k+d/2)}.$$

Then, applying  $\mu_k = \eta_k^2$  completes the proof. ■

**Assumption 8** *Assume that  $B_k := \max_{t \in \mathbb{R}} |\sigma^{(k)}(t)| \lesssim \Gamma(k+1)$ .*

All the popular smooth activation functions satisfy the above assumption as shown below.

- For  $\sigma(t) = \sin(t)$  and  $\sigma(t) = \cos(t)$ ,  $B_k = 1$ .
- Consider the sigmoid function:  $\sigma(z) = 1/(1 + e^{-z})$ , which can be viewed as a complex function  $\mathbb{C} \mapsto \mathbb{C}$ . The singular points of  $\sigma$  are  $\{z = (2k + 1)\pi i\}_{k \in \mathbb{Z}}$ . For any  $t \in \mathbb{R}$ , let  $C_t = \{z \in \mathbb{C} : |z - t| = 2\}$ . Then, all the singular points must be outside the curve  $C_t$ . Using Cauchy's integral formula, for any  $t \in \mathbb{R}$ , we have

$$\begin{aligned} |\sigma^{(k)}(t)| &= \left| \frac{\Gamma(k+1)}{2\pi i} \int_{C_t} \frac{\sigma(z)}{(z-t)^{k+1}} dz \right| \leq \frac{\Gamma(k+1)}{2\pi} \int_{C_t} \frac{|\sigma(z)|}{|z-t|^{k+1}} |dz| \\ &\leq \frac{\Gamma(k+1) \max_{z \in C_t} |\sigma(z)|}{2\pi 2^{k+1}} \int_{C_t} |dz| \leq \frac{\Gamma(k+1) \max_{z \in C_t} |\sigma(z)|}{2^k} \lesssim \frac{\Gamma(k+1)}{2^k}. \end{aligned} \quad (30)$$

- For all the other commonly-used smooth activation functions, we can obtain similar estimates of the  $k$ -th order derivatives by using Cauchy's integral formula.

**Proposition 9** *Under Assumption 8, we have  $\Lambda(m) \lesssim 1/m$ .*

The proof is deferred to Appendix A.3. We remark that the above estimate of  $\Lambda(m)$  is rather rough for most smooth activation functions, where  $B_k$  is much smaller than  $\Gamma(k+1)$  as demonstrated in Eq. (30). A simple combination of Proposition 9 and the proof of Lemma 2 gives

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \lesssim \frac{1}{m}, \quad (31)$$

where  $\{\phi_j\}_{j=1}^m$  are the leading spherical harmonics. Applying Proposition 3 to activation functions satisfying Assumption 8, we can obtain that

$$\sup_{v \in \mathbb{S}^{d-1}} \inf_{c_1, \dots, c_m} \|\sigma_v - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \lesssim \frac{d}{m}, \quad (32)$$

By comparing with Theorem 6, we see that for smooth activations, the approximation with fixed features does not suffer from the CoD. This is very different from the nonsmooth ones.

## 5. Kolmogorov widths of two-layer neural networks

We are now ready to estimate  $\omega_m(\mathcal{N})$ , which describes the (in)approximability of  $\mathcal{N}$  by linear methods. In this section, we use  $\mathcal{N}^r$  instead of  $\mathcal{N}$  for emphasizing the dependence on the norms of inner-layer widths. In addition, in order to deal with the bias term, we define  $\sigma^{(\gamma, b)}(t) = \sigma(\gamma t + b)$  for  $\gamma > 0, b \in \mathbb{R}$  and the associated kernel

$$k^{(\gamma, b)}(x, x') = \mathbb{E}_{v \sim \tau_{d-1}} [\sigma(\gamma v^T x + b) \sigma(\gamma v^T x' + b)]. \quad (33)$$

Let  $\Lambda^{(\gamma, b)}(\cdot)$  denote the trace decay of  $k^{(\gamma, b)}$  defined according to Eq. (14).

**Theorem 10** Let  $\Lambda_r(m) = \sup_{\gamma+|b|\leq r} \Lambda^{(\gamma,b)}(m)$  and  $q(d,r) = \sup_{k\geq 1} \frac{\Lambda_r(k)}{\Lambda_r((d+1)k)}$ . Then,

$$\Lambda_r(m) \leq \omega_m(\mathcal{N}^r) \lesssim q(d,r)\Lambda_r(m).$$

**Proof** Let  $\sigma_{w,b}(x) = \sigma(w^T x + b)$ . Then, for any  $\phi_1, \dots, \phi_m$ ,

$$\begin{aligned} \sup_{f \in \mathcal{N}^r} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|f - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 &\geq \sup_{\gamma+|b|\leq r} \mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m \in \mathbb{R}} \|\sigma_{\gamma v, b} - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \\ &\geq \sup_{\gamma+|b|\leq r} \Lambda^{(\gamma,b)}(m) = \Lambda_r(m), \end{aligned}$$

where the second inequality follows from Lemma 2. Hence, the lower bound is proved. Let  $(c_1(w,b), \dots, c_m(w,b)) = \operatorname{argmin}_{c_1, \dots, c_m} \|\sigma_{w,b} - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2$ . Then by Proposition 3 with taking  $L(m) = \Lambda_r(m)$ , we have

$$\|\sigma_{w,b} - \sum_{j=1}^m c_j(w,b) \phi_j\|_{\tau_{d-1}}^2 \lesssim q(d,r)\Lambda_r(m).$$

For any  $f \in \mathcal{N}^r$  and  $\varepsilon > 0$ , there exist  $\{(a_i, w_i, b_i)\}_i$  such that  $\sum_i |a_i| \leq 1, \max_i (\|w_i\|_2 + |b_i|) \leq r$  and

$$\|f - \sum_i a_i \sigma_{w_i, b_i}\|_{\tau_{d-1}} \leq \varepsilon.$$

Let  $\bar{c}_j = \sum_i a_i c_j(w_i, b_i)$ . Then,

$$\begin{aligned} \|f - \sum_{j=1}^m \bar{c}_j \phi_j\|_{\tau_{d-1}} &\leq \varepsilon + \left\| \sum_i a_i \sigma_{w_i, b_i} - \sum_{j=1}^m \sum_i a_i c_j(w_i, b_i) \phi_j \right\|_{\tau_{d-1}} \\ &\leq \varepsilon + \sum_i |a_i| \left\| \sigma_{w_i, b_i} - \sum_{j=1}^m c_j(w_i, b_i) \phi_j \right\|_{\tau_{d-1}} \\ &\leq \varepsilon + \sum_i |a_i| \sqrt{q(d,r)\Lambda_r(m)} \leq \varepsilon + \sqrt{q(d,r)\Lambda_r(m)}. \end{aligned}$$

Taking  $\varepsilon \rightarrow 0$ , we complete the proof.  $\blacksquare$

In the proof, the key ingredient is the uniform approximability of single neurons shown in Proposition 3. It is implied that  $\Lambda_r(m)$  provide a tight bounds of the Kolmogorov width  $\omega_m(\mathcal{N}^r)$ . When  $\Lambda_r(m) \sim m^{-s}$ ,  $q(d,r) \sim d^s$ . In particular, when  $\Lambda_r(m) = C_d m^{-\beta/d}$ , we have  $q(d,r) \leq d^{\beta/d} = O(1)$ . This means that when  $\Lambda_r(m)$  exhibits the CoD,  $\omega_m(\mathcal{N}^r) \sim \Lambda_r(m)$ , i.e., the spectral decay provides an exact description of the Kolmogorov width. Next, we study how the decay rate is affected by the norms of inner-layer widths and the smoothness of activation functions.



### 5.1 Influence of the norms of inner-Layer weights

For ReLU,  $\mathcal{N}^r$  (up to a rescaling) are obviously the same for different  $r$ 's because of the homogeneity of  $\sigma$ . In particular, Theorem 6 implies

$$\omega_m(\mathcal{N}^r) \geq \frac{C(1/d)r^2}{m^{3/(d-1)}}, \quad (34)$$

where  $C(1/d)$  depends on  $1/d$  polynomially. Hence,  $\omega_m(\mathcal{N}^r)$  exhibits the CoD for the ReLU activation function and the decay rate is independent of the value  $r$ . However, for general activation functions, restricting  $r$  may affect the decay rate.

**Theorem 11** *Suppose that  $\sigma \in C^\infty(\mathbb{R})$  satisfies Assumption 8 and  $r = 1$ . Then,  $\omega_m(\mathcal{N}^1) \lesssim d/m$ . The equality is reached by choosing the spherical harmonics as the fixed features.*

This theorem follows from a simple combination of Proposition 9 and Proposition 10. For the specific arctangent activation, we have a fine-grained characterization as follows.

**Proposition 12** *Assume  $\sigma(t) = \arctan(t)$ . We have*

$$\omega_m(\mathcal{N}^r) \lesssim \frac{d^4 r^2}{m^{\min(0.5, r^{-2})}}. \quad (35)$$

*The equality is reached by choosing the spherical harmonics as the fixed features.*

The key idea is to estimate the integral (23) in the Fourier domain using the Parseval's theorem. By using the explicit formula of the Fourier transform of  $\sigma^{(k)}$  for the arctangent function, we show that the eigenvalue can be expressed analytically using Gaussian hypergeometric functions. Then the integral representation of Gaussian hypergeometric functions is used for the estimation. The proof is quite technical and deferred to Appendix B.1.

Theorem 11 and Proposition 12 imply that two-layer neural networks have no clear separation from linear methods when the activation is smooth and the norms of inner-layer weights are bounded. Specifically, in this case, two-layer neural networks behave like polynomials in terms of approximation power. This is quite different from the ReLU case, where the separation of two type of methods is independent of the inner-layer weight norms.

Proposition 12 implies that the error rate decreases with  $r$  but independent of  $d$  if  $r = O(1)$ . We conjecture that similar results hold for general smooth activation functions and some numerical supports are provided in Figure 2. Specifically, we examine four activation functions including two sigmoid-like activations: Arctan and Sigmoid, and two ReLU-like activations: SiLU and softplus. According to Proposition 10,  $\Lambda_r$  is a good proxy of the Kolmogorov width. The eigenvalues are numerically computed using Eq. (22). In experiments, we find that  $\Lambda_r = \Lambda^{(r,0)}$  for all the activation functions examined. Figure 2 shows that for all the cases, the rate is independent of  $d$  for a fixed  $r$ , and decreases with  $r$  for a fixed  $d$ . This is consistent with Eq. (35), which is only proved for the arctangent activation function.

Note that a result similar to Proposition 12 has been provided in Livni et al. (2014) for the sigmoid activation function. Ours differs from Livni et al. (2014) in two aspects. First, we show that the same observation holds for more general smooth activation functions by

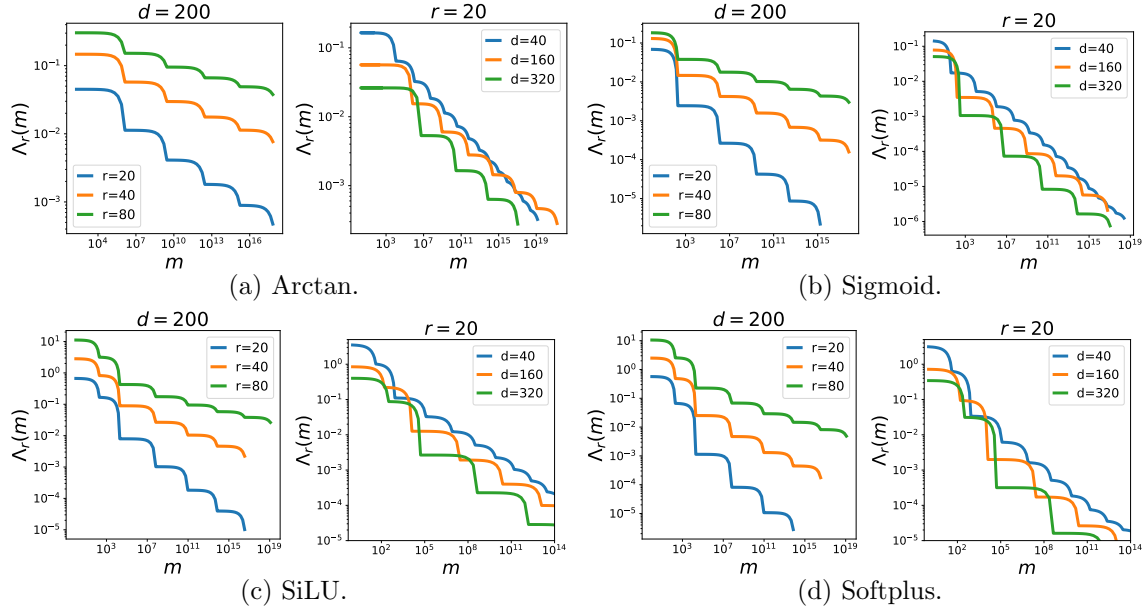


Figure 2: How the decay of  $\Lambda_r(m)$ , thereby the linear approximability, changes with  $r$  for fixed  $d$  (left), and changes with  $d$  for fixed  $r$  (right). Two sigmoid-like and two ReLU-like smooth activation functions are examined.

clear numerical evidences. In particular, Proposition 10 combined with Lemma 4 provides us an easy way to numerically compute upper bounds. Second, we can establish a hardness result given below. These improvements are benefiting from our spectral-based analysis.

Next we then show that when  $r$  is polynomially large with respect to  $d$ , even for smooth activation function,  $\omega_m(\mathcal{N}^r)$  exhibits the CoD.

**Assumption 13** *Suppose that the activation function  $\sigma$  satisfies either  $|\sigma(rt) - \text{step}(t)| \lesssim (1 + r|t|)^{-\beta}$  or  $|\sigma(rt) - r \text{ReLU}(t)| \leq (1 + r|t|)^{-\beta}$  for any  $t$  and some constant  $\beta > 0$ .*

This assumption is satisfied by all the commonly-used activation functions.

**Theorem 14** *Suppose that  $\sigma$  satisfies Assumption 13 and  $m \leq 2^d$ . Then, there exists constants  $C_1(\beta), C_2 > 0$  such that if  $r \geq d^{C_1(\beta)}$ , we have  $\omega_m(\mathcal{N}^r) \gtrsim d^{-C_2}$ .*

**Proof** We only present the proof for the sigmoid-like activation functions. The proof for ReLU-like ones is similar can be found in Appendix B.2. For any  $v \in \mathbb{S}^{d-1}$ ,

$$\|\sigma(rv^T \cdot) - \text{step}(v^T \cdot)\|_{\tau_{d-1}}^2 = \frac{1}{B(\frac{1}{2}, \frac{d-1}{2})} \int_{-1}^1 (\sigma(rt) - \text{step}(t))^2 (1-t^2)^{(d-3)/2} dt,$$

where we use the fact that the density function of  $v^T x$  is  $p(t) = \frac{(1-t^2)^{\frac{d-3}{2}}}{B(\frac{1}{2}, \frac{d-1}{2})}$ . Then, we have

$$\begin{aligned} \|\sigma(rv^T \cdot) - \text{step}(v^T \cdot)\|_{\tau_{d-1}}^2 &\leq \frac{2}{B(\frac{1}{2}, \frac{d-1}{2})} \left( \int_0^\delta (1-t^2)^{(d-3)/2} dt + \int_\delta^1 \frac{1}{(1+rt)^{2\beta}} (1-t^2)^{(d-3)/2} dt \right) \\ &\lesssim d^{1/2} \delta + \left( \frac{1}{r\delta} \right)^{2\beta}, \end{aligned}$$

Let  $h(t) = \sqrt{\frac{d}{r^2}}t + t^{-2\beta}$ . Then,  $h'(t) = \sqrt{\frac{d}{r^2}} - 2\beta t^{-2\beta-1} = 0$  leads to  $\bar{t} = (2\beta\sqrt{r^2/d})^{1/(1+2\beta)}$ . Hence,

$$\|\sigma(rv^T \cdot) - \text{step}(v^T \cdot)\|_{\tau_{d-1}}^2 \lesssim \inf_{\delta \in [0,1]} \left( d^{1/2} \delta + \left( \frac{1}{r\delta} \right)^{2\beta} \right) = C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}}.$$

By the triangle inequality,

$$\begin{aligned} \|\sigma(rv^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 &\gtrsim \|\text{step}(v^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 - \|\text{step}(v^T \cdot) - \sigma(rv^T \cdot)\|_{\tau_{d-1}}^2 \\ &\gtrsim \|\text{step}(v^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 - C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}}. \end{aligned}$$

Using Theorem 6, there exist a constant  $C_2 > 0$  such that

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m} \|\sigma(rv^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \gtrsim \frac{1}{d^{C_2} m^{1/(d-1)}} - C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}}.$$

Let  $C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}} \lesssim d^{-C_2}$ . This leads to  $r \geq C(\beta)^{\frac{(1+2\beta)}{\beta}} d^{\frac{1}{2} + \frac{C_2(1+2\beta)}{\beta}}$ . Hence, there exist  $C_1(\beta) > 0$  such that if  $m \leq 2^d$  and  $r \geq d^{C_1(\beta)}$ , we must have

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m} \|\sigma(rv^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \gtrsim \frac{1}{d^{C_2}}.$$

■

In particular, for the arctangent activation function, we have the following refined result, whose proof is deferred to Appendix B.3.

**Theorem 15** *Suppose that  $\sigma(t) = \arctan(t)$  and  $r = d^\alpha$ . If  $\alpha > \frac{1}{2}$ , there exists constants  $C_1, C_2(\alpha) > 0$  such that if  $m \leq d^{-C_1} 2^{C_2(\alpha) d^{2\alpha-1}}$ ,  $\omega_m(\mathcal{N}^r) \gtrsim d^{-3}$ .*

Let  $r = d^\alpha$ . Theorem 15 improves Theorem 14 by showing that  $\alpha > 1/2$  is sufficient to establish the CoD-type lower bound. We conjecture that the same results also hold for more general activation functions. We leave this to future work. These theorems imply that  $\omega_m(\mathcal{N}^r)$  also exhibits the CoD for sigmoid-like and ReLU-like smooth activations as long as the norms of inner-layer weights are polynomial in  $d$ .

This result is related to (Livni et al., 2014, Theorem 5), which shows that the time complexity of learning  $\mathcal{N}^{\text{poly}(d)}$  is exponential in  $d$ . However, (Livni et al., 2014, Theorem 5) relies on cryptographic assumptions. These assumptions mean that some standard hard problems cannot be learned in polynomial time; otherwise modern cryptosystems can be broken in polynomial time. The theoretical justification of these assumptions remains an open problem, although they are believed to be true. By contrast, our result is unconditional and does not rely on any hardness assumption. Note that two results are generally not comparable, since ours is for the approximation complexity whereas (Livni et al., 2014, Theorem 5) is for training complexity.

Lastly, we mention that some previous studies of two-layer neural networks constrain  $r$  to be finite (see, e.g., Chen et al. 2020). Our result suggests that one should be careful about the value of  $r$ , otherwise the result may be not able to distinguish neural networks from linear methods.

## 6. Conclusion

In this paper, we provide a systematic study of the separation between two-layer neural networks and linear methods in terms of approximation functions in high dimension. To this end, we develop a spectral-based approach, which reduces the problem to computing the eigenvalues of an associated kernel. Our approach allows obtaining upper bounds, lower bounds, and identifying explicit hard examples simultaneously. We extend and improve the previous separation results for the sigmoidal and  $\text{ReLU}^\alpha$  activation functions to general nonsmooth activation functions. We also find that for smooth activation functions, whether the separation exists or not crucially depends on the inner-layer weight norms.

Technically speaking, our spectral-based approach provides a way to accurately compute the Kolmogorov width of two-layer neural networks. This approach should be also applicable to analyze other properties that are related to the Kolmogorov width, e.g., the metric entropy. We leave this to future work.

## Appendix A. Missing proofs of Section 4

Here, we present the missing lengthy proofs of Section 4.

### A.1 Proof of Proposition 3

To prove Proposition 3, we first need the following lemma.

**Lemma 16** *For any activation function  $\sigma$  and any  $v \in \mathbb{S}^{d-1}$ ,*

$$\inf_{\{c_{i,j}\}_{1 \leq i \leq k, 1 \leq j \leq N(d,i)}} \left\| \sigma_v - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} c_{i,j} Y_{i,j} \right\|_{\tau_{d-1}}^2 = \sum_{i=k+1}^{\infty} N(d,i) \mu_i.$$

**Proof** Recalling that  $\{Y_{i,j}\}_{0 \leq i \leq k, 1 \leq j \leq N(d,i)}$  is orthonormal in  $L^2(\tau_{d-1})$ , we have

$$\inf_{\{c_{i,j}\}_{0 \leq i \leq k, 1 \leq j \leq N(d,i)}} \left\| \sigma_v - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} c_{i,j} Y_{i,j} \right\|_{\tau_{d-1}}^2 = \left\| \sigma_v \right\|_{\tau_{d-1}}^2 - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} \langle Y_{i,j}, \sigma_v \rangle_{\tau_{d-1}}^2.$$

Hence, using Eq. (11)

$$\begin{aligned}
 & \|\sigma_v\|_{\tau_{d-1}}^2 - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} \langle Y_{i,j}, \sigma_v \rangle_{\tau_{d-1}}^2 \\
 &= \int_{\mathbb{S}^{d-1}} |\sigma(v^\top x)|^2 d\tau_{d-1}(x) - \sum_{i=0}^k \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \sigma(v^\top x) \sigma(v^\top x') \sum_{j=1}^{N(d,i)} Y_{i,j}(x) Y_{i,j}(x') d\tau_{d-1}(x) d\tau_{d-1}(x') \\
 &= \int_{\mathbb{S}^{d-1}} |\sigma(v^\top x)|^2 d\tau_{d-1}(x) - \sum_{i=0}^k N(d,i) \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \sigma(v^\top x) \sigma(v^\top x') P_i(x^\top x') d\tau_{d-1}(x) d\tau_{d-1}(x').
 \end{aligned}$$

By the rotational invariance of  $\tau_{d-1}$ , we know that the above equation is constant for all  $v \in \mathbb{S}^{d-1}$ . Hence,

$$\begin{aligned}
 & \inf_{\{c_{i,j}\}_{0 \leq i \leq k, 1 \leq j \leq N(d,i)}} \left\| \sigma_v - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} c_{i,j} Y_{i,j} \right\|_{\tau_{d-1}}^2 \\
 &= \int_{\mathbb{S}^{d-1}} \left[ \int_{\mathbb{S}^{d-1}} |\sigma(v^\top x)|^2 d\tau_{d-1}(v) \right] d\tau_{d-1}(x) \\
 &\quad - \sum_{i=0}^k N(d,i) \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \left[ \int_{\mathbb{S}^{d-1}} \sigma(v^\top x) \sigma(v^\top x') d\tau_{d-1}(v) \right] P_i(x^\top x') d\tau_{d-1}(x) d\tau_{d-1}(x') \\
 &= \int_{\mathbb{S}^{d-1}} \kappa(x^\top x) d\tau_{d-1}(x) - \sum_{i=0}^k N(d,i) \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \kappa(x^\top x') P_i(x^\top x') d\tau_{d-1}(x) d\tau_{d-1}(x').
 \end{aligned}$$

Combining the last equation and Eq. (11) and Eq. (19), we obtain that

$$\begin{aligned}
 & \inf_{\{c_{i,j}\}_{0 \leq i \leq k, 1 \leq j \leq N(d,i)}} \left\| \sigma_v - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} c_{i,j} Y_{i,j} \right\|_{\tau_{d-1}}^2 \\
 &= \sum_{i=0}^{\infty} \sum_{j=1}^{N(d,i)} \mu_i - \sum_{i=0}^k \sum_{j=1}^{N(d,i)} \mu_i \left[ \int |Y_{i,j}(x)|^2 d\tau_{d-1}(x) \right]^2 = \sum_{i=k+1}^{\infty} N(d,i) \mu_i.
 \end{aligned}$$

■

**Proof of Proposition 3** Let  $m_k = \sum_{i=0}^k N(d,i)$ . By Lemma 16 and Proposition 9, we have

$$\inf_{c_1, \dots, c_{m_k}} \left\| \sigma_v - \sum_{j=1}^{m_k} c_j \phi_j \right\|_{\tau_{d-1}}^2 = \Lambda(m_k) \leq L(m_k).$$

For any  $m$ , assume  $m \in [m_{k-1} + 1, m_k]$ . Then,

$$\inf_{c_1, \dots, c_m} \left\| \sigma_v - \sum_{j=1}^m c_j \phi_j \right\|_{\tau_{d-1}}^2 \leq \inf_{c_1, \dots, c_{m_{k-1}}} \left\| \sigma_v - \sum_{j=1}^{m_{k-1}} c_j \phi_j \right\|_{\tau_{d-1}}^2 \leq L(m_{k-1}) \leq \frac{L(m_{k-1})}{L(m_k)} L(m) \tag{36}$$

By (A.2) and (37), we have

$$m_k = \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} + \frac{\Gamma(k+d-1)}{\Gamma(d)\Gamma(k)},$$

when  $k \geq 1$  and  $m_0 = 0$ . Then,

$$\begin{aligned} \frac{m_{k+1}}{m_k} &\leq \max\left\{\frac{\Gamma(k+d+1)}{\Gamma(d)\Gamma(k+2)} \Big/ \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)}, \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} \Big/ \frac{\Gamma(k+d-1)}{\Gamma(d)\Gamma(k)}\right\} \\ &= \max\left\{\frac{k+d}{k+1}, \frac{k+d-1}{k}\right\} \leq d+1, \end{aligned}$$

and

$$\frac{m_1}{m_0} = d+1.$$

Therefore,

$$\frac{m_k}{m_{k-1}} \leq d+1,$$

which means that

$$\frac{L(m_{k-1})}{L(m_k)} \leq \frac{L(m_{k-1})}{L((d+1)m_{k-1})} \leq q(d, L).$$

Plugging it into (36), we complete the proof.

## A.2 Proof of Proposition 5

We only give the proof when  $d \geq 3$ . The simple case  $d = 2$  can also be proven using similar argument. According to Appendix D.2 of Bach (2017a)<sup>1</sup>, we have

$$\eta_k = \begin{cases} C_1(\frac{1}{d}, \alpha) & \text{if } k \leq \alpha, \\ 0 & \text{if } k \geq \alpha + 1, \text{ and } k \equiv \alpha \pmod{2} \\ \frac{\Gamma(\alpha+1)}{\sqrt{2\pi} 2^k} \frac{\Gamma(d/2)\Gamma(k-\alpha)}{\Gamma(\frac{k-\alpha+1}{2})\Gamma(\frac{k+d+\alpha}{2})} & \text{otherwise,} \end{cases}$$

where  $C_1(\frac{1}{d}, \alpha)$  depends on  $\frac{1}{d}$  polynomially. By Stirling formula,  $\Gamma(t) \sim \sqrt{2\pi} t^{t-1/2} e^{-t+1}$  for any  $t \geq 1$ . Then, up to a constant only depending on  $\alpha$ , we have for  $k \geq \alpha + 1$  and  $k \equiv (\alpha + 1) \pmod{2}$ ,

$$\mu_k \sim d^{d-1} k^{k-\alpha-1} (k+d)^{-k-d-\alpha+1}.$$

Note that

$$N(d, k) = \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} - \frac{\Gamma(k+d-2)}{\Gamma(d)\Gamma(k-1)}$$

where up to a constant, we have

$$\frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} \sim (k+d)^{k+d-\frac{1}{2}} k^{-\frac{1}{2}-k} d^{\frac{1}{2}-d} \quad (37)$$

1. Note that the one provided in Bach (2017a) is not correct due to the miscalculation of  $\omega_{d-2}/\omega_{d-1}$ .

WLOG, assume that  $\alpha$  is odd. By definition,

$$\lambda_m = \mu_{2k}, \text{ for } m \in \left[ \sum_{i=0}^{\alpha} N(d, i) + 1 + \frac{\Gamma(2k+d-2)}{\Gamma(d)\Gamma(2k-1)} - \frac{\Gamma(\alpha+d-1)}{\Gamma(d)\Gamma(\alpha)}, \right. \\ \left. \sum_{i=0}^{\alpha} N(d, i) + \frac{\Gamma(2k+d)}{\Gamma(d)\Gamma(2k+1)} - \frac{\Gamma(\alpha+d-1)}{\Gamma(d)\Gamma(\alpha)} \right] \quad (38)$$

By (37), when

$$m \geq 1 + \sum_{i=0}^{\alpha} N(d, i) = C\left(\frac{1}{d}, \alpha\right), \quad (39)$$

and there exists  $k$  such that (38) is satisfied, we have

$$m \gtrsim (2k+d)^{2k+d-\frac{5}{2}} (2k)^{\frac{3}{2}-2k} d^{\frac{1}{2}-d},$$

and,

$$\lambda_m = \mu_{2k} \sim (2k+d)^{-2k-d-\alpha+1} (2k)^{2k-\alpha-1} d^d,$$

which means

$$\lambda_m m^{\frac{d+2\alpha}{d-1}} \geq C\left(\frac{1}{d}, \alpha\right) \left(\frac{2k+d}{2k}\right)^{\frac{2\alpha+1}{d-1}(2k+d-\frac{5}{2})-\alpha-\frac{3}{2}} \geq C\left(\frac{1}{d}, \alpha\right). \quad (40)$$

Noticing that when (39) is not satisfied, the above inequality still holds, we have

$$\Lambda(m) = \sum_{j=m+1}^{\infty} \lambda_j \geq C\left(\frac{1}{d}, \alpha\right) m^{-\frac{2\alpha+1}{d-1}}.$$

In particular, when  $\alpha = 0$ ,  $C(1/d, 0) = 1/d$ .

### A.3 Proof of Proposition 9

First, by (A.2) and (A.2),

$$\sum_{j=0}^k N(d, j) \lesssim \frac{2\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} \sim (k+d)^{k+d-1/2} k^{-1/2-k} d^{-d+1/2}.$$

Second, plugging  $B_k \lesssim k!$  into Lemma 7 and using the Stirling formula, we have

$$\mu_k \lesssim \frac{k^{2k+1} e^{-2k}}{2^{2k}} \left( \frac{(d/2)^{d/2-1/2} e^{-d/2+1}}{(k+d/2)^{k+d/2-1/2} e^{-k-d/2+1}} \right)^2 \sim k^{2k+1} d^{d-1} (2k+d)^{-2k-d+1}.$$

Let  $\bar{\lambda}_m = \mu_k$  if  $m \in [\sum_{j=0}^{k-1} N(d, j) + 1, \sum_{j=0}^k N(d, j)]$ , then

$$\Lambda(m) \leq \sum_{j=m+1}^{\infty} \bar{\lambda}_j$$

because  $\{\lambda_j\}_{j \geq 0}$  is the non-increasing rearrangement of  $\{\bar{\lambda}_j\}_{j \geq 0}$ .

By definition, if  $\bar{\lambda}_m = \mu_k$ , we must have  $m \leq \sum_{j=0}^k N(d, j)$ . Therefore,

$$\begin{aligned} \bar{\lambda}_m m^2 &\lesssim \mu_k \left( \sum_{j=0}^k N(d, j) \right)^2 \leq d^{-d} (k+d)^{2k+2d-1} (2k+d)^{-2k-d+1} \\ &= \left( \frac{k+d}{d} \right)^d \left( \frac{k+d}{2k+d} \right)^{2k+d-1} \leq \left( \frac{k+d}{d} \right)^d \left( \frac{k+d}{2k+d} \right)^{2k+d}. \end{aligned}$$

Let  $k = td$ , then we have

$$\bar{\lambda}_m m^2 \lesssim (t+1)^d \left( \frac{t+1}{2t+1} \right)^{(2t+1)d} =: h(t)^d, \quad (41)$$

where  $h(t) = (t+1)^{2t+2} / (2t+1)^{2t+1}$ . Hence,  $\log h(t) = (2t+2) \log(t+1) - (2t+1) \log(2t+1)$ .

$$\frac{d \log h(t)}{dt} = 2 \log(t+1) + 2 - 2 \log(2t+1) - 2 \leq 0, \quad \forall t \geq 0. \quad (42)$$

Combing with  $\log h(0) = 0$ , we have  $\log h(t) \leq 0$ , i.e.,  $h(t) \leq 1$ , for any  $t \geq 0$ . Plugging it into (41) leads to  $\bar{\lambda}_m m^2 \lesssim 1$ . In other words,  $\bar{\lambda}_m \lesssim 1/m^2$ . Therefore, we have

$$\Lambda(m) \leq \sum_{j=m+1}^{\infty} \bar{\lambda}_j \lesssim \frac{1}{m}.$$

■

## Appendix B. Missing proofs of Section 5

### B.1 Proof of Proposition 12

We show that for the arctangent activation function, the eigenvalues can be expressed analytically using the hypergeometric functions.

**Lemma 17** *Assume  $\sigma(z) = \arctan(rz + b)$ . Then,*

$$|\eta_k| \leq Q_{d,k} \int_0^1 \left( \frac{r^2}{1+r^2t} \right)^{\frac{k}{2}} t^{\frac{k-1}{2}} (1-t)^{\frac{k+d-3}{2}} dt, \quad (43)$$

where  $Q_{d,k} = \frac{\Gamma(k)\Gamma(\frac{d}{2})}{2^k \Gamma(\frac{k+1}{2}) \Gamma(\frac{k+d-1}{2})}$ . The equality is reached when  $b = 0$ .

**Proof** Recall (10),

$$\eta_k = \frac{1}{2^k} \frac{\omega_{d-2}}{\omega_{d-1}} \frac{\Gamma((d-1)/2)}{\Gamma(k+(d-1)/2)} \int_{-1}^1 \sigma^{(k)}(t) (1-t^2)^{k+(d-3)/2} dt.$$

By Parseval's theorem,

$$\eta_k = \frac{1}{2^k} \frac{\omega_{d-2}}{\omega_{d-1}} \frac{\Gamma((d-1)/2)}{\Gamma(k+(d-1)/2)} \frac{1}{2\pi} \int \mathcal{F}(\sigma^{(k)})(\xi) \mathcal{F}(\nu_{d,k})(\xi) d\xi, \quad (44)$$



where  $\nu_{k,d}(t) = (1 - t^2)^{k + \frac{d-3}{2}}$  and  $\mathcal{F}(\cdot)$  denote the Fourier transform, i.e.,  $\mathcal{F}(f)(\xi) = \int f(t)e^{-i\xi t} dt$ .

Notice that

$$\sigma'(z) = \frac{r}{1 + (rz + b)^2} \quad \mathcal{F}(\sigma')(\xi) = \pi e^{i\frac{b}{r}\xi - |\xi|/r} \quad \mathcal{F}(\sigma^{(k)})(\xi) = (i\xi)^{k-1} \pi e^{i\frac{b}{r}\xi - |\xi|/r}. \quad (45)$$

Also, using Eqn. (10.9.4) of Olver et al. (2010), we know that

$$\mathcal{F}(\nu_{k,d})(\xi) = \sqrt{2\pi} 2^{k + \frac{d-3}{2}} \Gamma\left(k + \frac{d-1}{2}\right) |\xi|^{-k - \frac{d-2}{2}} J_{k + \frac{d-2}{2}}(|\xi|), \quad (46)$$

where  $J_{k + \frac{d-2}{2}}$  is the Bessel function of the first kind with order  $k + \frac{d-2}{2}$  (see, e.g., (Olver et al., 2010, 10.2.2)). Plugging (45) and (46) into (44) leads to

$$\begin{aligned} \eta_k &= 2^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right) \frac{1}{2\pi} \int_{-\infty}^{\infty} |\xi|^{k-1} \pi e^{-|\xi|/r + i\frac{b}{r}\xi} |\xi|^{-k - \frac{d-2}{2}} J_{k + \frac{d-2}{2}}(|\xi|) d\xi \\ &= 2^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right) \int_0^{\infty} \xi^{-d/2} J_{k + \frac{d-2}{2}}(\xi) e^{-\xi/r} i^{k-1} [e^{i\frac{b}{r}\xi} + e^{-i\frac{b}{r}\xi}] d\xi. \end{aligned}$$

By Eq. (10.22.49) of Olver et al. (2010), the above integration can be expressed by using the Gaussian hypergeometric function  $\mathbf{F}$  (see (Olver et al., 2010, Chapter 15)):

$$\eta_k = \Gamma(k) \Gamma\left(\frac{d}{2}\right) \left(\frac{r}{2}\right)^k i^{k-1} \frac{1}{2} \left[ \mathbf{F}\left(\frac{k}{2}, \frac{k+1}{2}; k + \frac{d}{2}; -\frac{r^2}{(1+bi)^2}\right) + \mathbf{F}\left(\frac{k}{2}, \frac{k+1}{2}; k + \frac{d}{2}; -\frac{r^2}{(1-bi)^2}\right) \right], \quad (47)$$

Notice that  $\mathbf{F}(p, q; u; z)$  has the integral representation (Olver et al., 2010, Section 15.6) as follows:

$$\mathbf{F}(p, q; u; z) = \frac{1}{\Gamma(q)\Gamma(u-q)} \int_0^1 \frac{t^{q-1}(1-t)^{u-q-1}}{(1-zt)^p} dt.$$

Plugging it into (47) gives us

$$|\eta_k| \leq \frac{\Gamma(k)\Gamma(\frac{d}{2})}{2^k \Gamma(\frac{k+1}{2})\Gamma(\frac{k+d-1}{2})} \int_0^1 \left(\frac{r^2}{1+r^2t}\right)^{k/2} t^{\frac{k-1}{2}} (1-t)^{\frac{k+d-3}{2}} dt.$$

Thus, we complete the proof. ■

Let

$$I = \int_0^1 \left(\frac{r^2}{1+r^2t}\right)^{\frac{k}{2}} t^{\frac{k-1}{2}} (1-t)^{\frac{k+d-3}{2}} dt. \quad (48)$$

Now, our task is to estimate how  $I$  depends on  $r, k$ , and  $d$ . In order to achieve this, we consider two cases: (1)  $r^2 \geq \frac{2k+d-4}{k-1}$ ; (2)  $r^2 \leq \frac{2k+d-4}{k-1}$ , separately.

**Lemma 18** *Let  $h_{\alpha,\beta}(t) = t^\alpha(1-t)^\beta$ . Then,  $h_{\alpha,\beta}$  is increasing in  $[0, \frac{\alpha}{\alpha+\beta}]$  and decreasing in  $[\frac{\alpha}{\alpha+\beta}, 1]$ .*

**Proof** A simple calculation gives us

$$h'_{\alpha,\beta}(t) = \alpha t^{\alpha-1}(1-t)^\beta - \beta t^\alpha(1-t)^{\beta-1} = (\alpha - (\alpha + \beta)t)t^{\alpha-1}(1-t)^{\beta-1}.$$

Hence,  $h'_{\alpha,\beta}(t) \geq 0$  for  $t \in [0, \frac{\alpha}{\alpha+\beta}]$ , and  $h'_{\alpha,\beta}(t) \leq 0$  for  $t \in [\frac{\alpha}{\alpha+\beta}, 1]$  ■

**Proposition 19** Assume  $r^2 \geq \frac{2k+d-4}{k-1}$ . Then,  $I \lesssim r e^{-\frac{k+d}{2r^2}}$ .

**Proof** Consider the decomposition

$$\begin{aligned} I &= \left( \int_0^{\frac{1}{r^2}} + \int_{\frac{1}{r^2}}^1 \right) \left( \frac{r^2}{1+r^2t} \right)^{k/2} t^{\frac{k-1}{2}} (1-t)^{\frac{k+d-3}{2}} dt \\ &=: I_1 + I_2. \end{aligned}$$

Next, we estimate  $I_1, I_2$ , separately.

By Lemma 18,  $h_{\frac{k-1}{2}, \frac{k+d-3}{2}}(\cdot)$  is increasing for  $t \leq \bar{t} = \frac{k-1}{2k+d-4}$ . The assumption on  $r$  ensures that  $\frac{1}{r^2} \leq \bar{t}$ . Therefore, for  $I_1$ ,

$$I_1 \leq \int_0^{\frac{1}{r^2}} \frac{r^k}{(1+r^2t)^{k/2}} \left( \frac{1}{r^2} \right)^{\frac{k-1}{2}} \left( 1 - \frac{1}{r^2} \right)^{\frac{k+d-3}{2}} dt \lesssim \frac{1}{kr} \left( 1 - \frac{1}{r^2} \right)^{\frac{k+d-3}{2}} \lesssim \frac{1}{k} \left( 1 - \frac{1}{r^2} \right)^{\frac{k+d}{2}}, \quad (49)$$

where the last inequality uses the fact that  $r^2 \geq 2$ .

For  $I_2$ ,

$$\begin{aligned} I_2 &= \int_{\frac{1}{r^2}}^1 \left( \frac{r^2t}{1+r^2t} \right)^{\frac{k}{2}} t^{-\frac{1}{2}} (1-t)^{\frac{k+d-3}{2}} dt \leq \int_{\frac{1}{r^2}}^1 t^{-\frac{1}{2}} (1-t)^{\frac{k+d-3}{2}} dt \\ &\stackrel{(i)}{\leq} r \left( 1 - \frac{1}{r^2} \right)^{\frac{k+d-3}{2}} \lesssim r \left( 1 - \frac{1}{r^2} \right)^{\frac{k+d}{2}}, \end{aligned} \quad (50)$$

where (i) is due to that  $h_{-\frac{1}{2}, \frac{k+d-3}{2}}(\cdot)$  is decreasing in  $[0, 1]$ .

Combining (49) and (50), we have

$$I \lesssim r \left( 1 - \frac{1}{r^2} \right)^{\frac{k+d}{2}} \leq r e^{-\frac{k+d}{2r^2}}. \quad \blacksquare$$

**Lemma 20** For  $\alpha, \beta, \gamma > 0$ , let  $H_{\gamma,\alpha,\beta}(t) := \left( \frac{\gamma t}{1+\gamma t} \right)^\alpha (1-t)^\beta$  with  $t \in [0, 1]$ . Assume  $\beta \geq \alpha$  and  $\gamma\alpha/\beta \leq 2$ . Then,

$$H_{\gamma,\alpha,\beta}(t) \leq \left( \frac{\gamma\alpha}{\gamma\alpha + \beta} \right)^\alpha \left( 1 - \frac{\alpha}{2\beta} \right)^\beta.$$

**Proof** Taking the derivative gives us

$$\begin{aligned} H'_{\gamma,\alpha,\beta}(t) &= \alpha \left( \frac{\gamma t}{1+\gamma t} \right)^{\alpha-1} \frac{\gamma}{(1+\gamma t)^2} (1-t)^\beta - \beta \left( \frac{\gamma t}{1+\gamma t} \right)^\alpha (1-t)^{\beta-1} \\ &= \left( \alpha - \beta t(1+\gamma t) \right) \frac{\gamma}{(1+\gamma t)^2} \left( \frac{\gamma t}{1+\gamma t} \right)^{\alpha-1} (1-t)^{\beta-1} \end{aligned}$$

Let  $\delta = \alpha/\beta$ . Then, the maximal value of  $H_{\gamma,\alpha,\beta}(\cdot)$  is reached at

$$\bar{t} = \frac{1}{2\gamma} \left( \sqrt{4\gamma\delta + 1} - 1 \right).$$

Notice that  $1 + t/4 \leq \sqrt{1+t} \leq 1 + t/2$  for  $t \in [0, 8]$ . By the assumption,  $0 \leq 4\gamma\delta \leq 8$ , which implies that  $\frac{\delta}{2} \leq \bar{t} \leq \delta$ . Hence,

$$H_{\gamma,\alpha,\beta}(t) \leq H_{\gamma,\alpha,\beta}(\bar{t}) = \left( \frac{\gamma\bar{t}}{1+\gamma\bar{t}} \right)^\alpha (1-\bar{t})^\beta \leq \left( \frac{\gamma\delta}{1+r\delta} \right) \left( 1 - \frac{\delta}{2} \right)^\beta = \left( \frac{\gamma\alpha}{\gamma\alpha + \beta} \right)^\alpha \left( 1 - \frac{\alpha}{2\beta} \right)^\beta. \quad \blacksquare$$

**Lemma 21** *Assume that  $r^2 \leq \frac{2k+d-4}{k-1}$  and  $k, d \geq C$  for an absolute constant  $C$ . Then, we have*

$$I \lesssim d^{\frac{1}{2}} r \left( \frac{r^2 k}{(r^2 + 1)k + d} \right)^{\frac{k}{2}} e^{-k/4}.$$

Recalling (48),

$$I \leq r \int_0^1 \left( \frac{r^2 t}{1+r^2 t} \right)^{\frac{k-1}{2}} (1-t)^{\frac{k+d-3}{2}} dt = r \int_0^1 H_{r^2, \frac{k-1}{2}, \frac{k+d-3}{2}}(t) dt.$$

By the assumption,

$$r^2 \frac{(k-1)/2}{(k+d-3)/2} \leq \frac{2k+d-4}{k+d-3} = 2 - \frac{d-2}{k+d-3} \leq 2,$$

which means that the condition in Lemma 20 is satisfied. Therefore, we have

$$\begin{aligned} I &\leq r \left( \frac{r^2(k-1)}{r^2(k-1) + k + d - 3} \right)^{\frac{k-1}{2}} \left( 1 - \frac{k-1}{2(k+d-3)} \right)^{\frac{k+d-3}{2}} \\ &= r \left( \frac{r^2(k-1)}{(r^2+1)(k-1) + d-2} \right)^{\frac{k-1}{2}} \left( 1 - \frac{k-1}{2(k+d-3)} \right)^{\frac{k+d-3}{2}} \\ &\lesssim d^{\frac{1}{2}} r \left( \frac{r^2 k}{(r^2+1)k + d} \right)^{\frac{k}{2}} \left( 1 - \frac{k}{2(k+d)} \right)^{\frac{k+d}{2}}, \end{aligned} \quad (51)$$

where the last inequality follows from that  $d, k \geq C$  for some absolute constant  $C$ . Using the fact that  $(1 - 1/x)^x \leq e^{-1}$  for any  $x \geq 1$ , we complete the proof.

**Proof of Proposition 12** Following Theorem 10, what remains is to show that when  $\sigma(x) = \arctan(\gamma x + b)$  such that  $|\gamma| + |b| \leq r$ , we have

$$\Lambda(m) \lesssim d^2 r^4 m^{-\frac{1}{\max(2, r^2)}}.$$

According to Lemma 4 and Proposition 17,  $\mu_k = Q_{d,k}^2 I^2$  if  $k$  is odd, other wise  $\mu_k = 0$ . Assume that  $k$  is odd. Let

$$m_k = \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} \sim \frac{(k+d)^{k+d-\frac{1}{2}}}{k^{k+\frac{1}{2}}d^{d-\frac{1}{2}}} \lesssim \frac{(k+d)^{k+d}}{k^k d^d}. \quad (52)$$

Our task is to determine the smallest  $\beta$  such that

$$m_k^{1+1/\beta} \mu_k \leq m_k^{1+1/\beta} Q_{d,k}^2 I^2 = (m_k Q_{d,k}^2) \left( m_k^{1/\beta} I^2 \right) \leq C_{d,r},$$

where  $C_{d,r}$  depends on  $d, r$  polynomially.

First,

$$\begin{aligned} m_k Q_{d,k}^2 &= \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)} \left( \frac{\Gamma(k)\Gamma(\frac{d}{2})}{2^k \Gamma(\frac{k+1}{2})\Gamma(\frac{k+d-1}{2})} \right)^2 \\ &\sim \frac{(k+d)^{k+d-\frac{1}{2}}}{k^{\frac{1}{2}+k} d^{d-\frac{1}{2}}} \cdot \frac{k^{k-1} d^{d-1}}{(k+d)^{k+d-2}} \lesssim d. \end{aligned} \quad (53)$$

Next, we turn to estimate  $m_k^{1/\beta} I^2$ .

(1) We first consider the case where  $r^2 \leq \frac{2k+d-4}{k-1}$ . By Proposition 21 and (52), we have

$$\begin{aligned} m_k^{1/\beta} I^2 &\lesssim dr^2 \left( \frac{(k+d)^{k+d}}{k^k d^d} \right)^{1/\beta} \left( \frac{r^2 k}{(r^2+1)k+d} \right)^k e^{-k/2} \\ &\lesssim dr^2 \left( \frac{(k+d)(r^2 k)^\beta}{k((r^2+1)k+d)^\beta} \right)^{k/\beta} \left( \left( \frac{k+d}{d} \right)^{d/\beta} e^{-k/2} \right) \\ &=: dr^2 A^{k/\beta} B. \end{aligned} \quad (54)$$

Next, we estimate  $A$  and  $B$ , separately. To simplify the notation, we write  $k = sd, a = r^2$ . For  $A$ , if  $\beta \geq a$ ,

$$A = \frac{(k+d)(r^2 k)^\beta}{k((r^2+1)k+d)^\beta} = \frac{(s+1)(as)^\beta}{s((a+1)s+1)^\beta} \leq \frac{a^\beta s^{\beta+1} + a^\beta s^\beta}{(a+1)^\beta s^{\beta+1} + \beta(a+1)^{\beta-1} s^\beta} \leq 1. \quad (55)$$

For  $B$ , if  $\beta \geq 2$ ,

$$B = \left( 1 + \frac{k}{d} \right)^{d/\beta} e^{-k/2} \leq e^{k/\beta} \cdot e^{-k/2} \leq 1. \quad (56)$$

Combining (55) and (56),

$$m_k^{1/r^2} I^2 \leq dr^2.$$

(2) Now, we turn to the case where  $r^2 \geq \frac{2k+d-4}{k-1}$ . By Proposition (19) and (52),

$$\begin{aligned} m_k^{1/\beta} I^2 &\lesssim \left( \frac{(k+d)^{k+d}}{k^k d^d} \right)^{1/\beta} r^2 e^{-(k+d)/r^2} \\ &\leq r^2 \left( 1 + \frac{d}{k} \right)^{k/\beta} \left( 1 + \frac{k}{d} \right)^{d/\beta} e^{-(k+d)/r^2} \\ &\leq r^2 e^{d/\beta} e^{k/\beta} e^{-(k+d)/r^2}. \end{aligned}$$

Taking  $\beta = r^2$  gives rise to  $m_k^{1/r^2} I^2 \lesssim r^2$ .

Combining two cases, we arrive at

$$m^{1+\frac{1}{\max(2,r^2)}} \mu_k \lesssim d^2 r^2.$$

Therefore,

$$\Lambda(m) = \sum_{j=m+1}^{\infty} \bar{\lambda}_j \lesssim d^2 r^2 \sum_{j=m+1}^{\infty} j^{-1-\frac{1}{\max(2,r^2)}} \lesssim d^2 r^4 m^{-\frac{1}{\max(2,r^2)}}.$$

## B.2 The missing proof of Theorem 14

**Proof** Here, we provide the proof for the ReLU-like case, which is similar to the sigmoid-like case. For any  $v \in \mathbb{S}^{d-1}$ , we have

$$\begin{aligned} \|\sigma(rv^T \cdot) - r \operatorname{ReLU}(v^T \cdot)\|_{\tau_{d-1}}^2 &= \frac{1}{B(\frac{1}{2}, \frac{d-1}{2})} \int_{-1}^1 (\sigma(rt) - r \operatorname{ReLU}(t))^2 (1-t^2)^{\frac{d-3}{2}} dt \\ &\leq \frac{2}{B(\frac{1}{2}, \frac{d-1}{2})} \left( \int_0^\delta (1-t^2)^{\frac{d-3}{2}} dt + \int_\delta^1 \frac{1}{(1+rt)^{2\beta}} (1-t^2)^{\frac{d-3}{2}} dt \right) \\ &\lesssim d^{\frac{1}{2}} \delta + \frac{1}{(1+r\delta)^{2\beta}} \leq d^{\frac{1}{2}} \delta + \frac{1}{(r\delta)^{2\beta}}. \end{aligned}$$

Following the proof of Theorem 14, we have

$$\inf_{\delta \in [0,1]} \left( d^{\frac{1}{2}} \delta + \frac{1}{(r\delta)^{2\beta}} \right) \leq C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}}.$$

By the triangular inequality,

$$\begin{aligned} \|\sigma(rv^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 &\gtrsim \|r \operatorname{ReLU}(v^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 - \|\sigma(rv^T \cdot) - r \operatorname{ReLU}(v^T \cdot)\|_{\tau_{d-1}}^2 \\ &\gtrsim \|r \operatorname{ReLU}(v^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 - C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}}. \end{aligned}$$

Using Theorem 6, there exists a  $C_3 > 0$  such that

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m} \|\sigma(rv^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \gtrsim \frac{r^2}{d C_3 m^{3/(d-1)}} - C(\beta) \left( \frac{d}{r^2} \right)^{\frac{\beta}{2(1+2\beta)}}.$$

Obviously, there exists  $C'(\beta) > 0$  such that when  $m \leq 2^d$  and  $r \geq C'(\beta)d^{\max(1/2, C_3/2)}$ , we have

$$\mathbb{E}_{v \sim \tau_{d-1}} \inf_{c_1, \dots, c_m} \|\sigma(rv^T \cdot) - \sum_{j=1}^m c_j \phi_j\|_{\tau_{d-1}}^2 \gtrsim 1.$$

■

### B.3 Proof of Theorem 15

**Proof** Following the same approach with Proposition 17, we know that

$$|\eta_k| = \begin{cases} Q_{d,k} \int_0^1 \left( \frac{d^{2\alpha}}{1+d^{2\alpha}t} \right)^{\frac{k}{2}} t^{\frac{k-1}{2}} (1-t)^{\frac{k+d-3}{2}} dt & \text{when } k \text{ is odd} \\ 0 & \text{when } k \text{ is even.} \end{cases}$$

Therefore, when  $\frac{\Gamma(2k+d-3)}{\Gamma(d)\Gamma(2k-2)} + 1 \leq m \leq \frac{\Gamma(2k+d-1)}{\Gamma(d)\Gamma(2k)}$  or  $m = k = 1$ , we have

$$\mu_m^{\frac{1}{2}} = Q_{d,2k-1} \int_0^1 \left( \frac{d^{2\alpha}}{1+d^{2\alpha}t} \right)^{\frac{2k-1}{2}} t^{k-1} (1-t)^{\frac{2k+d-4}{2}} dt.$$

Noticing that  $\Lambda(m)$  is increasing with respect to  $\alpha$ , we only consider the case that  $\alpha \in (\frac{1}{2}, 1]$ .

First, noticing that

$$\begin{aligned} \int_0^1 \left( \frac{d^{2\alpha}}{1+d^{2\alpha}t} \right)^{\frac{2k-1}{2}} t^{k-1} (1-t)^{\frac{2k+d-4}{2}} dt &\geq \int_{\frac{1}{2d}}^{\frac{1}{d}} \left( \frac{d^{2\alpha}}{1+d^{2\alpha}t} \right)^{\frac{2k-1}{2}} t^{k-1} (1-t)^{\frac{2k+d-4}{2}} dt \\ &\gtrsim d^{-\frac{3}{2}} \left( \frac{d^{2\alpha-1}}{2+d^{2\alpha-1}} \right)^{\frac{2k-1}{2}} \left(1 - \frac{1}{d}\right)^{\frac{2k+d-4}{2}} \\ &\gtrsim d^{-\frac{3}{2}} \left( \frac{d^{2\alpha-1}}{2+d^{2\alpha-1}} \right)^{2k}. \end{aligned}$$

Also,

$$\begin{aligned} N(d, 2k-1) Q_{d,2k-1}^2 &= \frac{4k+d-4}{2k-1} \frac{\Gamma(2k+d-3)}{\Gamma(d-1)\Gamma(2k-1)} Q_{d,2k-1}^2 \\ &\gtrsim \frac{(2k+d)^{2k+d-\frac{7}{2}}}{(2k)^{2k-\frac{3}{2}} d^{d-\frac{3}{2}}} \frac{(2k)^{2k-2} d^{d-1}}{(2k+d)^{2k+d-3}} \\ &= \left( \frac{2kd}{(2k+d)} \right)^{\frac{1}{2}} \gtrsim 1. \end{aligned}$$

Therefore, for any  $k \geq 1$ ,

$$\begin{aligned} \Lambda \left( \frac{\Gamma(2k+d-3)}{\Gamma(d)\Gamma(2k-2)} \right) &= \sum_{j=k}^{+\infty} N(d, 2j-1) Q_{d,2j-1}^2 \left[ \int_0^1 \left( \frac{d^{2\alpha}}{1+d^{2\alpha}t} \right)^{\frac{2k-1}{2}} t^{k-1} (1-t)^{\frac{2k+d-4}{2}} dt \right]^2 \\ &\gtrsim d^{-3} \sum_{j=k}^{+\infty} \left( \frac{d^{2\alpha-1}}{2+d^{2\alpha-1}} \right)^{4j} \gtrsim d^{-3} \left( \frac{d^{2\alpha-1}}{2+d^{2\alpha-1}} \right)^{4k}. \end{aligned}$$

Therefore, when  $k \sim \frac{1}{2}d^{2\alpha-1}$ , we have

$$\Lambda \left( \frac{\Gamma(2k+d-3)}{\Gamma(d)\Gamma(2k-2)} \right) \gtrsim d^{-3}. \quad (57)$$

On the other hand, when  $k \sim \frac{1}{2}d^{2\alpha-1}$ , we have

$$m_{d,\alpha} := \frac{\Gamma(2k+d-3)}{\Gamma(d)\Gamma(2k-2)} \sim \frac{(2k+d)^{2k+d-\frac{7}{2}}}{(2k)^{2k-\frac{5}{2}}d^{d-\frac{1}{2}}} \quad (58)$$

Therefore, there exist  $C_1, C_2(\alpha) > 0$  such that

$$m_{d,\alpha} \gtrsim d^{(2-2\alpha)d^{2\alpha-1}-\frac{11}{2}+5\alpha} = d^{-\frac{11}{2}+5\alpha} 2^{C(\alpha)d^{2\alpha-1}\ln(d)} \geq \frac{1}{d^{C_1}} 2^{C_2(\alpha)d^{2\alpha-1}}. \quad (59)$$

■

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