Optimization Models for Fairness

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May 2014

Modeling Fairness

- Why represent fairness in an optimization model?
  - In many applications, equitable distribution is an **objective**. How to formulate it mathematically?
  - Optimization models may provide **insight** into the consequences of ethical theories.

Modeling Equity

- Some applications□
  - Single-payer health system.
  - Facility location (e.g., emergency services).
  - Taxation (revenue vs. progressivity).
  - Relief operations.
  - Telecommunications (lexmax, Nash bargaining solution)

Outline

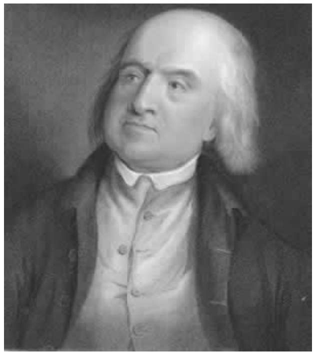
- Optimization models and their implications
  - Utilitarian
  - Rawlsian (lexmax)
- Axiomatics
  - Deriving utilitarian and Rawlsian criteria
- Measures of inequality
- An allocation problem
- Bargaining solutions
  - Nash
  - Raiffa-Kalai-Smorodinsky
- Combining utility and equity
  - Health care example

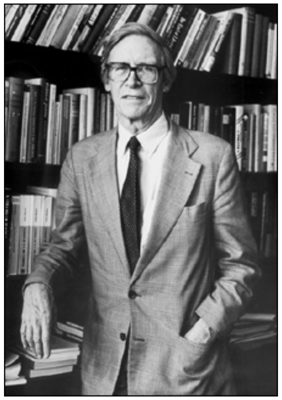
Optimization Models and Their Implications

- Utilitarianism
  - The optimization problem
  - Characteristics of utilitarian allocations
  - Arguments for utilitarianism
- Rawlsian difference principle
  - The social contract argument
  - The lexmax principle
  - The optimization problem
  - Characteristics of lexmax solutions

Efficiency vs. Equity

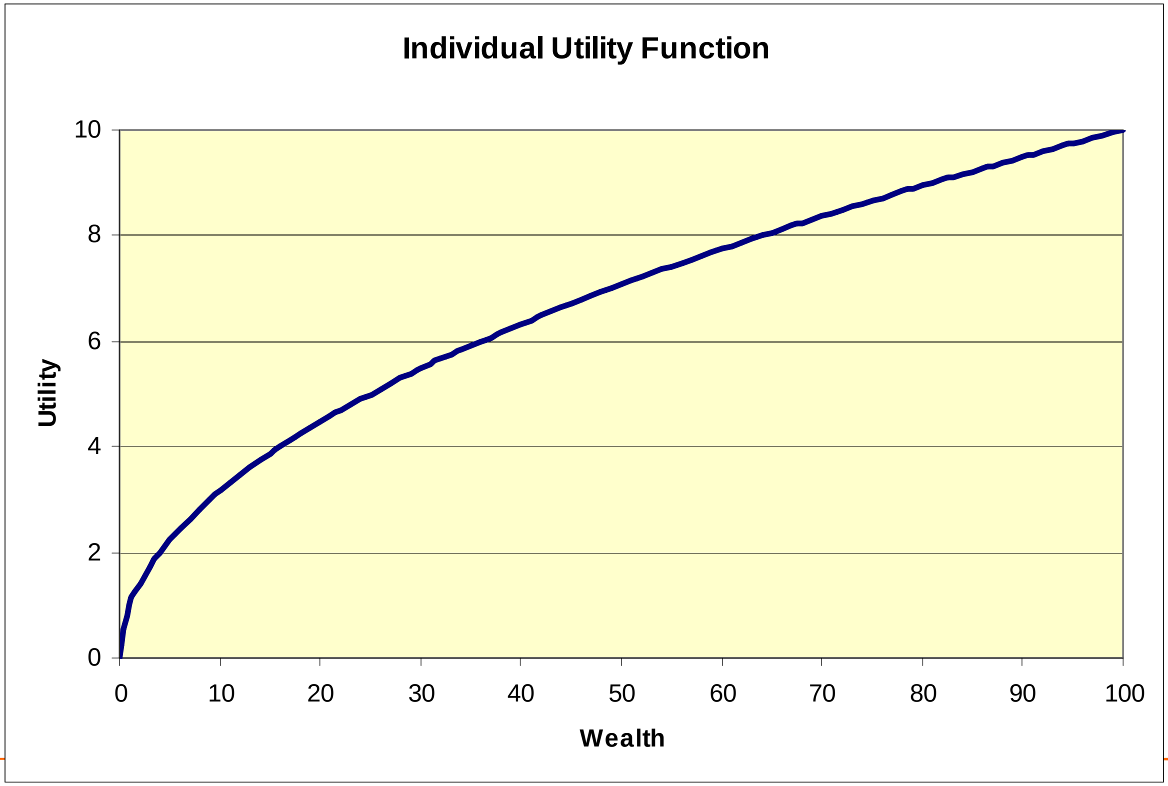
- Two classical criteria for distributive justice:
  - **Utilitarianism (efficiency)**
  - **Difference principle** of **John Rawls (equity)**
- These have the must studied philosophical underpinnings.





Utilitarian Principle

- We assume that every individual has a utility function *v(x)*, where *x* is the wealth allocation to the individual.



Utilitarian Principle

- A "just" distribution of wealth is one that maximizes total expected utility.

- Let x_i = wealth initially allocated to person *i*
  $u_i(x_i)$ = utility eventually produced by person *i*

Utilitarian Model

- The utility maximization problem:

$$\max \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

Total budget

$$x_i \geq 0, \quad \text{all } i$$

Utilitarian Model

- Elementary analysis yields the optimal solution:

$$u_1'(x_1) = \cdots = u_n'(x_n)$$

Marginal productivity

Distribute wealth so as to equalize marginal productivity.

Utilitarian Model

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Marginal productivity

Distribute wealth so as to equalize marginal productivity.

- If we index persons in order of marginal productivity, i.e.,

$$u_i'(\cdot) \le u_{i+1}'(\cdot), \text{ all } i$$

Then less productive individuals receive less wealth.

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Marginal productivity

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- If we index persons in order of marginal productivity, i.e.,

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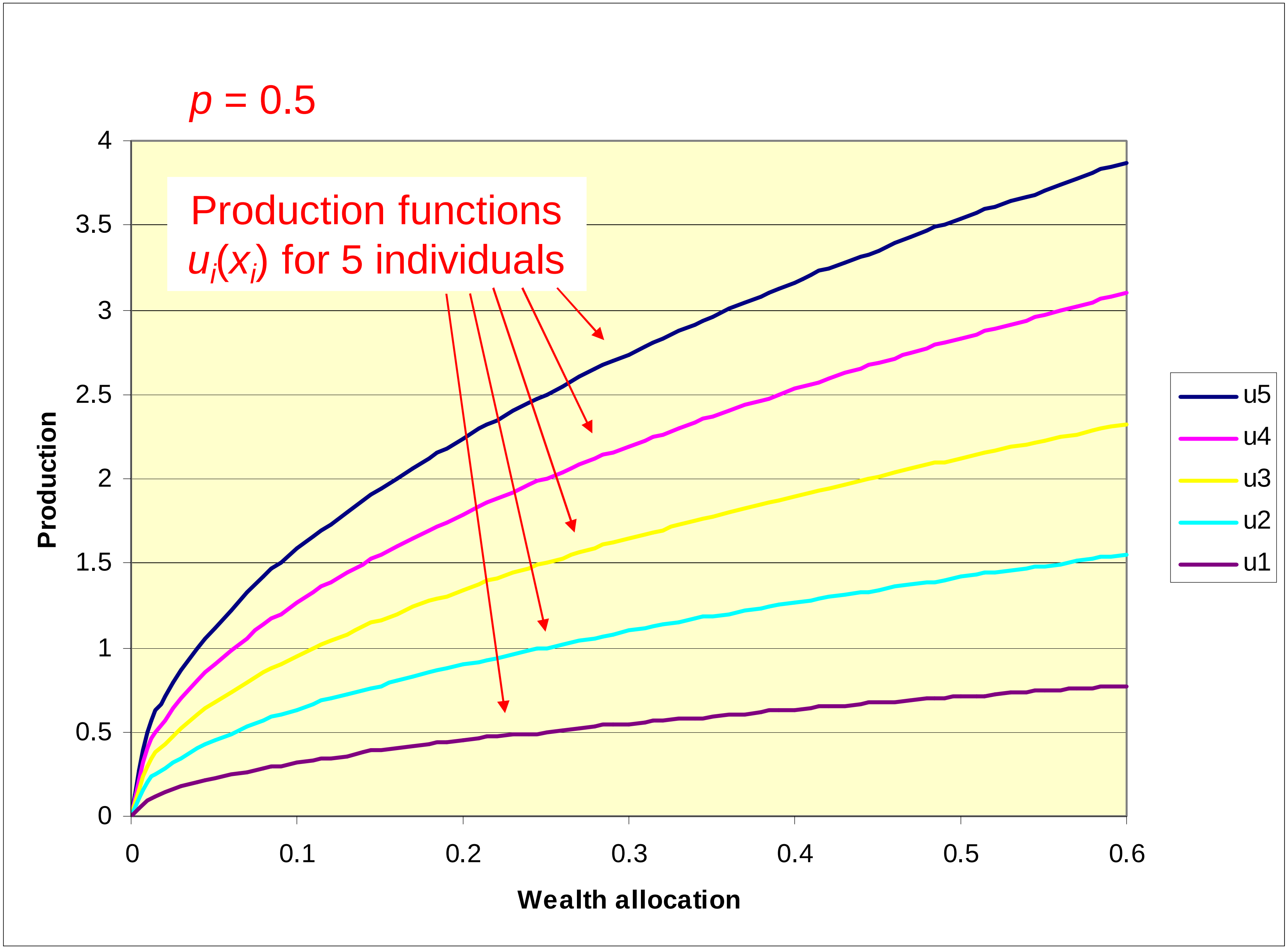
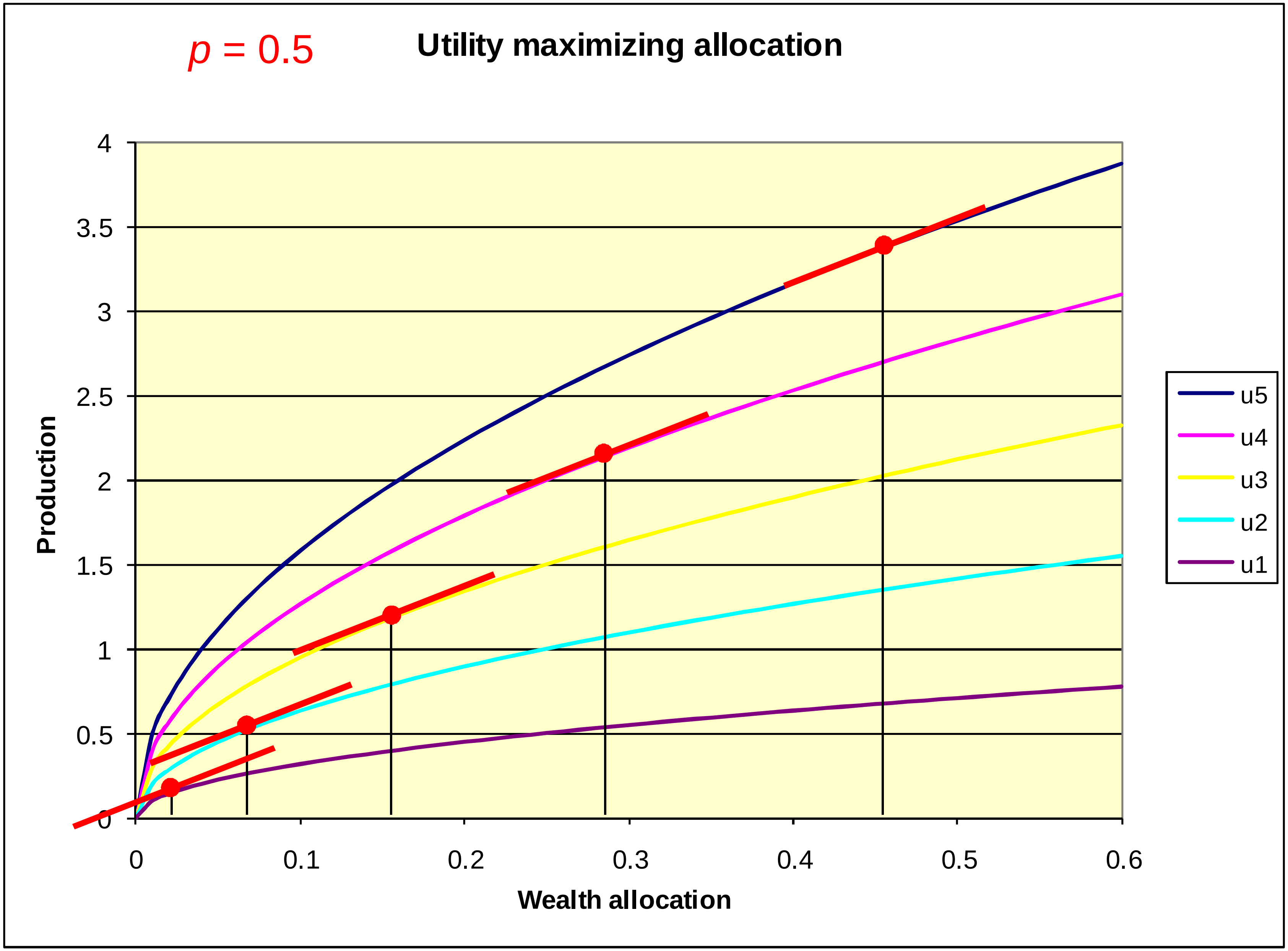
- For convenience assume $u_i(x_i) = c_i x_i^p$


p = 0.5
Production functions
$u_i(x_i)$ for 5 individuals
Production
Wealth allocation
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
u5
u4
u3
u2
u1



p = 0.5
Utility maximizing allocation
Production
Wealth allocation
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.1
0.2
0.3
0.4
0.5
0.6
u5
u4
u3
u2
u1

Utilitarian Model

- Classical utilitarian argument: concave utility functions tend to make the utilitarian solution more **egalitarian**.

Utilitarian Model

- Classical utilitarian argument: concave utility functions tend to make the utilitarian solution more **egalitarian**.

- A **completely** egalitarian allocation $x_1 = \cdots = x_n$ is optimal only when

$$u_1'(1/n) = \cdots = u_n'(1/n)$$

- So, equality is optimal only when everyone has the same marginal productivity in an egalitarian allocation.

Utilitarian Model

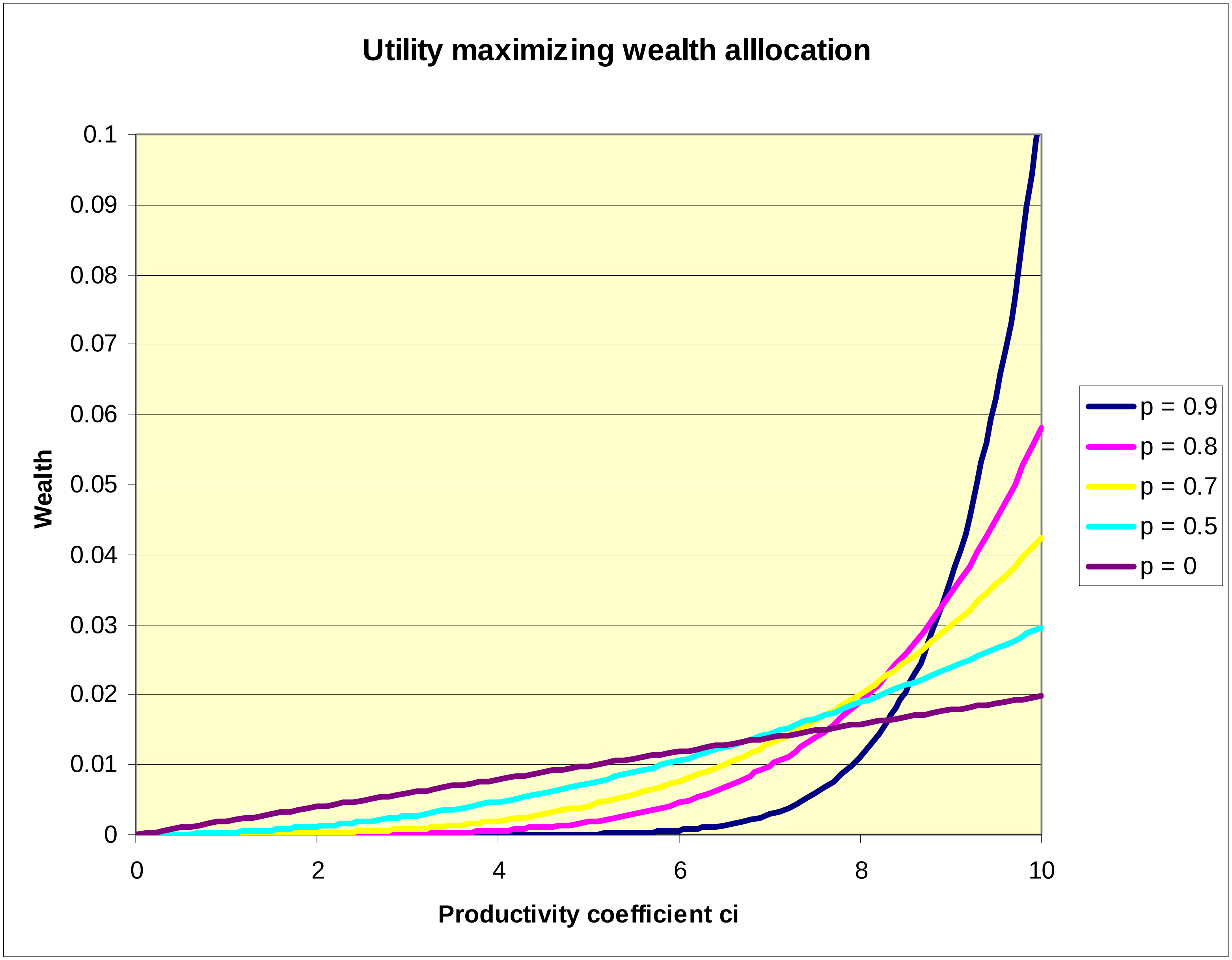
- Recall that $u_i(x_i) = c_i x_i^p$ where $p \geq 0$
- The optimal wealth allocation is

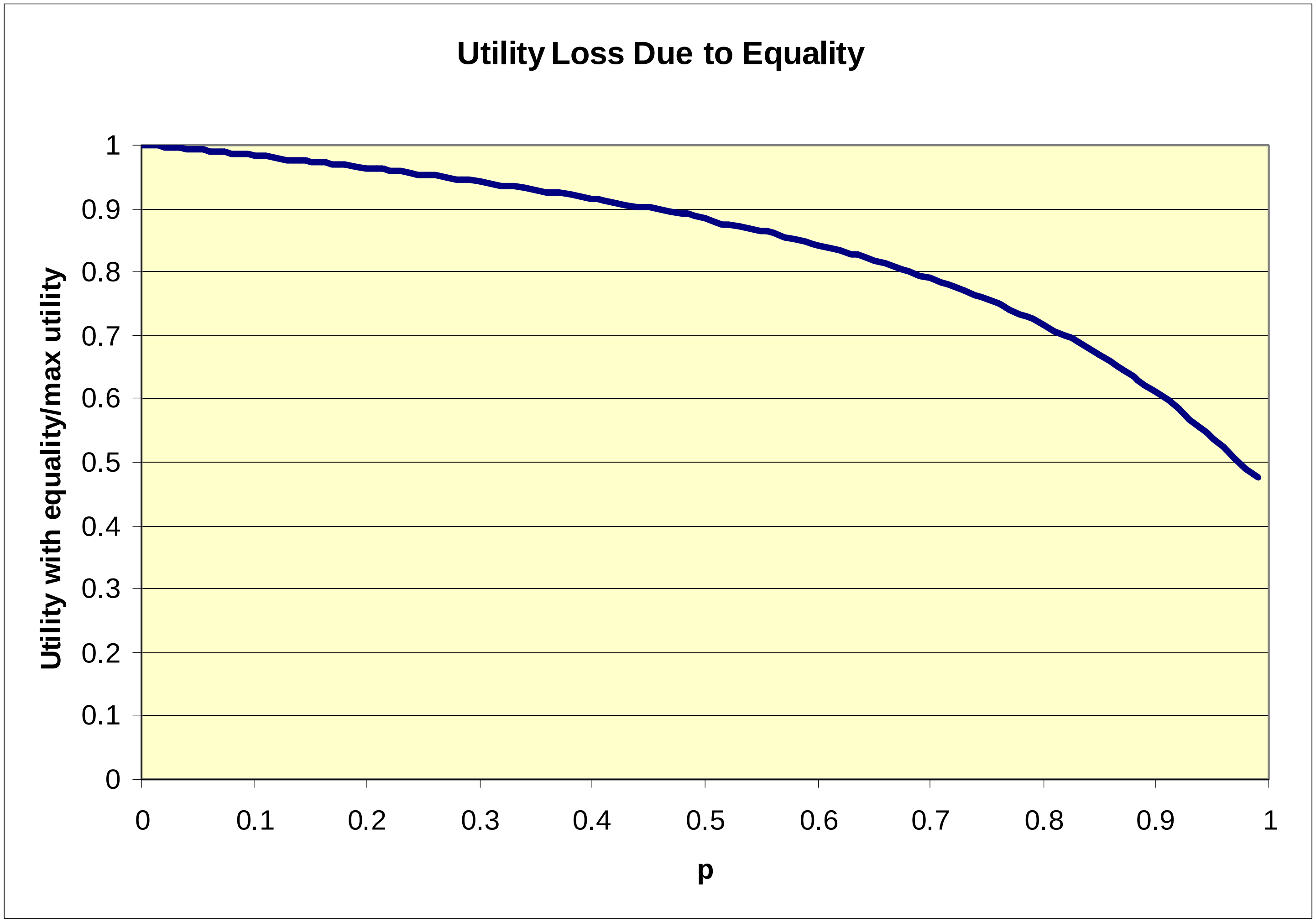
$$x_i = c_i^{\frac{1}{1-p}} \left(\sum_{j=1}^{n} c_j^{\frac{1}{1-p}} \right)^{-1}$$

- When $p < 1$:
  - Allocation is **completely egalitarian** only if $c_1 = \cdots = c_n$
  - Otherwise the **most egalitarian** allocation occurs when $p \to 0$: $x_i = \frac{c_i}{\sum_j c_j}$

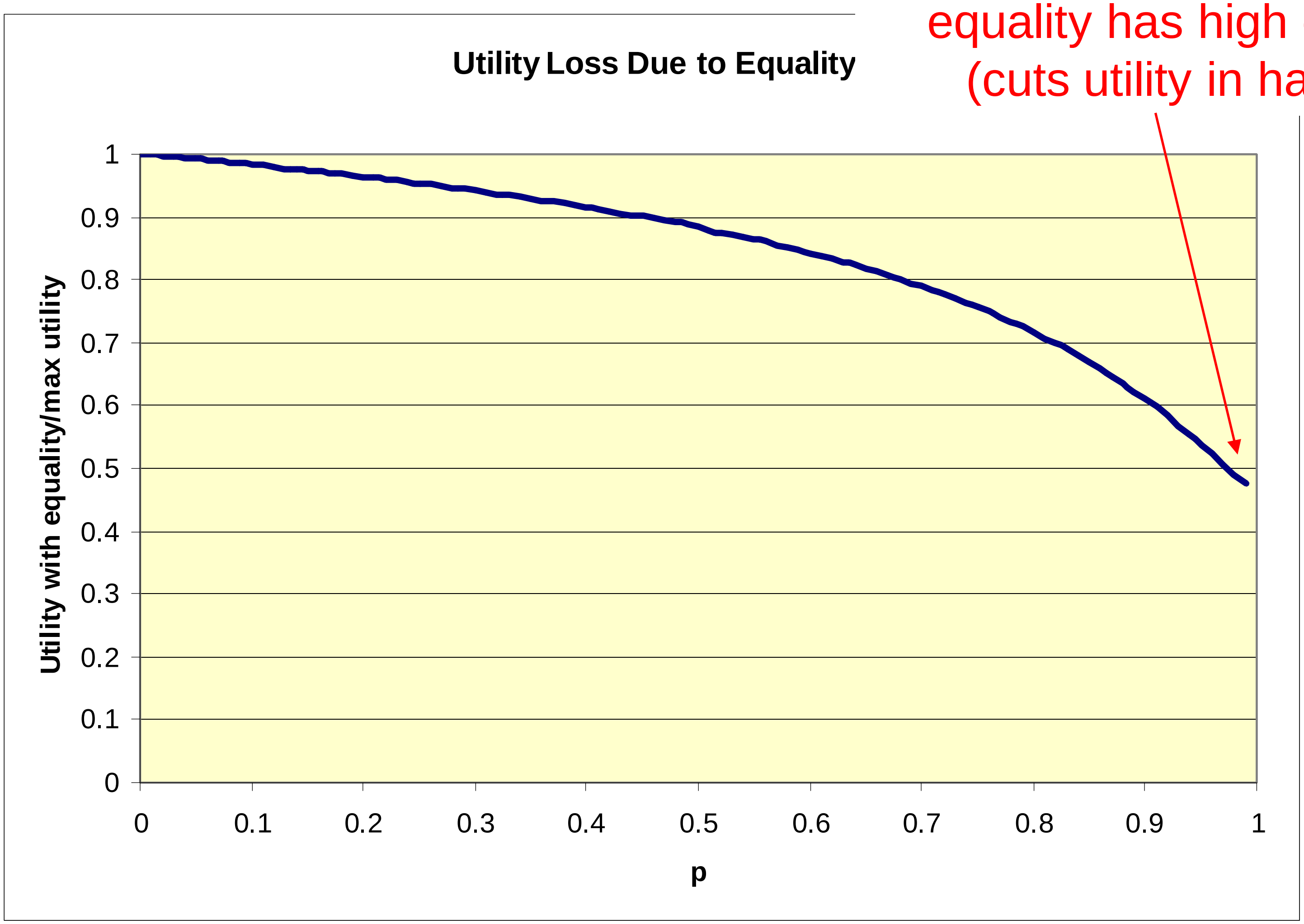
Utilitarian Model

- The **most egalitarian** optimal allocation: people receive wealth in proportion to productivity c_i.
  - And this occurs only when productivity very insensitive to investment ($p \to 0$).
- Allocation can be **very unequal** when p is closer to 1.

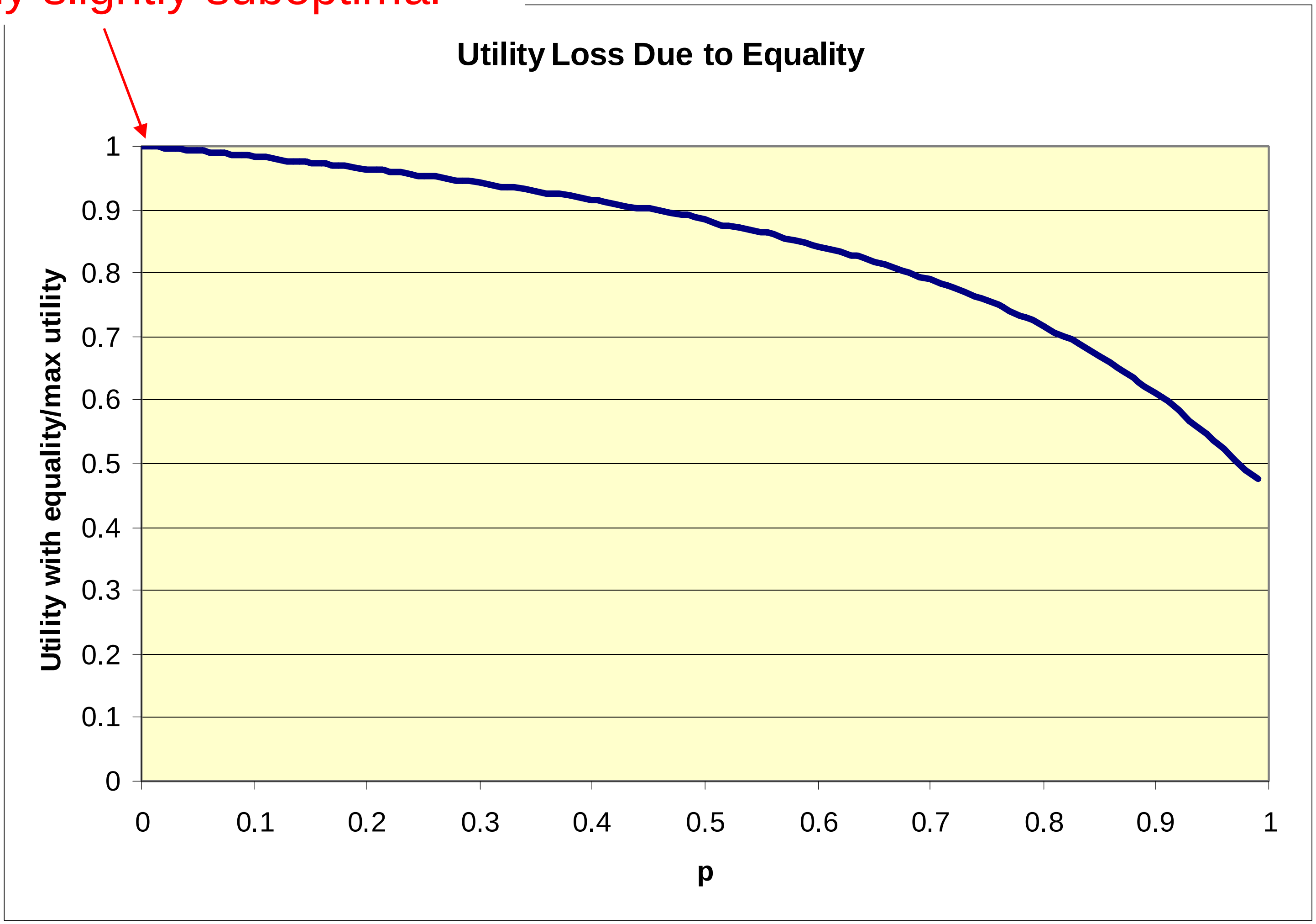

Utility maximizing wealth alllocation
Wealth
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
2
4
6
8
10
Productivity coefficient ci
p = 0.9
p = 0.8
p = 0.7
p = 0.5
p = 0


Utility Loss Due to Equality
Utility with equality/max utility
p
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

When output is proportional to investment, equality has high cost (cuts utility in half)



As $p \rightarrow 0$, optimal utility requires highly unequal allocation, but equal allocation is only slightly suboptimal



Utilitarian Model

- More fundamentally, an egalitarian defense of utilitarianism is based on **contingency**, **not principle**.
    - If we **evaluate** the fairness of utilitarian distribution, then there must be **another standard** of equitable distribution.
- Utilitarianism can endorse:.
    - – Neglect of disabled or nonproductive people.
    - – Meager wage for less talented people who work hard.
    - – Fewer resources for people with less productive jobs. Not all jobs can be equally productive.
- □ if this results in greater total utility.

Rawlsian Difference Principle

- Rawls' **Difference Principle** seeks to maximize the welfare of the worst off.
    - Also known as **maximin** principle.
    - Another formulation: inequality is permissible only to the extent that it is necessary to improve the welfare of those worst off.

$$\max \min_i \{u_i(x_i)\}$$

$$\sum_i u_i(x_i) = 1$$

$$x_i \geq 0, \quad \text{all } i$$

Rawlsian Difference Principle

- The root idea is that when I make a decision for myself, I make a decision for **anyone** in similar circumstances.
  - It doesn't matter who I am.

- Social contract argument
  - I make decisions (formulate a social contract) in an **original position**, behind a **veil of ignorance** as to who I am.
  - I must find the decision acceptable **after** I learn who I am.
  - I cannot rationally assent to a policy that puts me on the bottom, unless I would have been even **worse off** under alternative policies.
  - So the policy must **maximize** the welfare of the **worst off**.

Rawlsian Difference Principle

- Applies only to **basic goods**.
    - Things that people want, no matter what else they want.
    - Salaries, tax burden, medical benefits, etc.
    - For example, salary differentials may satisfy the principle if necessary to make the poorest better off.
- Applies to smallest **groups** for which outcome is predictable.
    - A lottery passes the test even though it doesn't maximize welfare of worst off – the loser is unpredictable.
    - □ unless the lottery participants as a whole are worst off.

Rawlsian Difference Principle

- The difference rule implies a **lexmax** principle**.**
  - If applied recursively.

- **Lexmax (lexicographic maximum) principle**:
  - Maximize welfare of least advantaged class□
  - then next-to-least advantaged class□
  - and so forth.

Lexmax Model

- Applications
  - **Production planning** – Allocate scarce components to products to minimize worst-case delay to a customer.
  - **Location of fire stations** – Minimize worst-case response time.
  - **Workforce management** – Schedule rail crews so as to spread delays equitably over time. Similar for call center scheduling.
  - **Political districting** – Minimize worst-case deviation from proportional representation.
  - **Social planning** – Build a Rawlsian society.

Lexmax Model

- Assume each person's share of total utility is **proportional** to the utility of his/her initial wealth allocation.
  - Thus individuals with more education, salary have greater access to social utility.

- Assume productivity functions $u_i(x_i) = c_i x_i^p$
  - Larger p means productivity more sensitive to investment.

- Assume personal utility function $v(x_i) = x_i^q$
  - Larger q means people care more about getting rich.

Lexmax Model

- The utility maximization problem:

$$\text{lexmax } (y_1, \dots, y_n)$$

$$\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \dots, n$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0, \quad \text{all } i$$

Wealth allocation to person *i* (→ x_i)

Lexmax Model

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Utility allocation to person *i*

Wealth allocation to person *i*

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Utility allocation to person *i*

Wealth allocation to person *i*

Budget

Lexmax Model

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$$x_i \geq 0, \quad \text{all } i$$

Utility allocation to person *i*

y_i's sum to total utility produced

Wealth allocation to person *i*

Budget

Lexmax Model

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$$\text{lexmax } (y_1, \ldots, y_n)$$

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$$\sum_{i=1}^{n} \boxed{y_i} = \sum_{i=1}^{n} u_i(x_i)$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\boxed{x_i} \geq 0, \quad \text{all } i$$

Proportional allocation of total utility

Utility allocation to person *i*

y_i's sum to total utility produced

Wealth allocation to person *i*

Budget

Lexmax Model

- The utility maximization problem:

$$\begin{aligned}
&\text{lexmax } (y_1, \ldots, y_n) \\
&\frac{y_i}{y_1} = \frac{v(x_i)}{v(x_1)}, \quad i = 2, \ldots, n \\
&\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} u_i(x_i) \\
&\sum_{i=1}^{n} x_i = 1 \\
&x_i \geq 0, \quad \text{all } i
\end{aligned}$$

Theorem. If $u_i'(\cdot) \leq u_{i+1}'(\cdot)$ and $v(\cdot)$ is nondecreasing, ← this has an optimal solution in which $y_1 \leq \cdots \leq y_n$

Lexmax Model

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$$\begin{aligned}
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\end{aligned}$$

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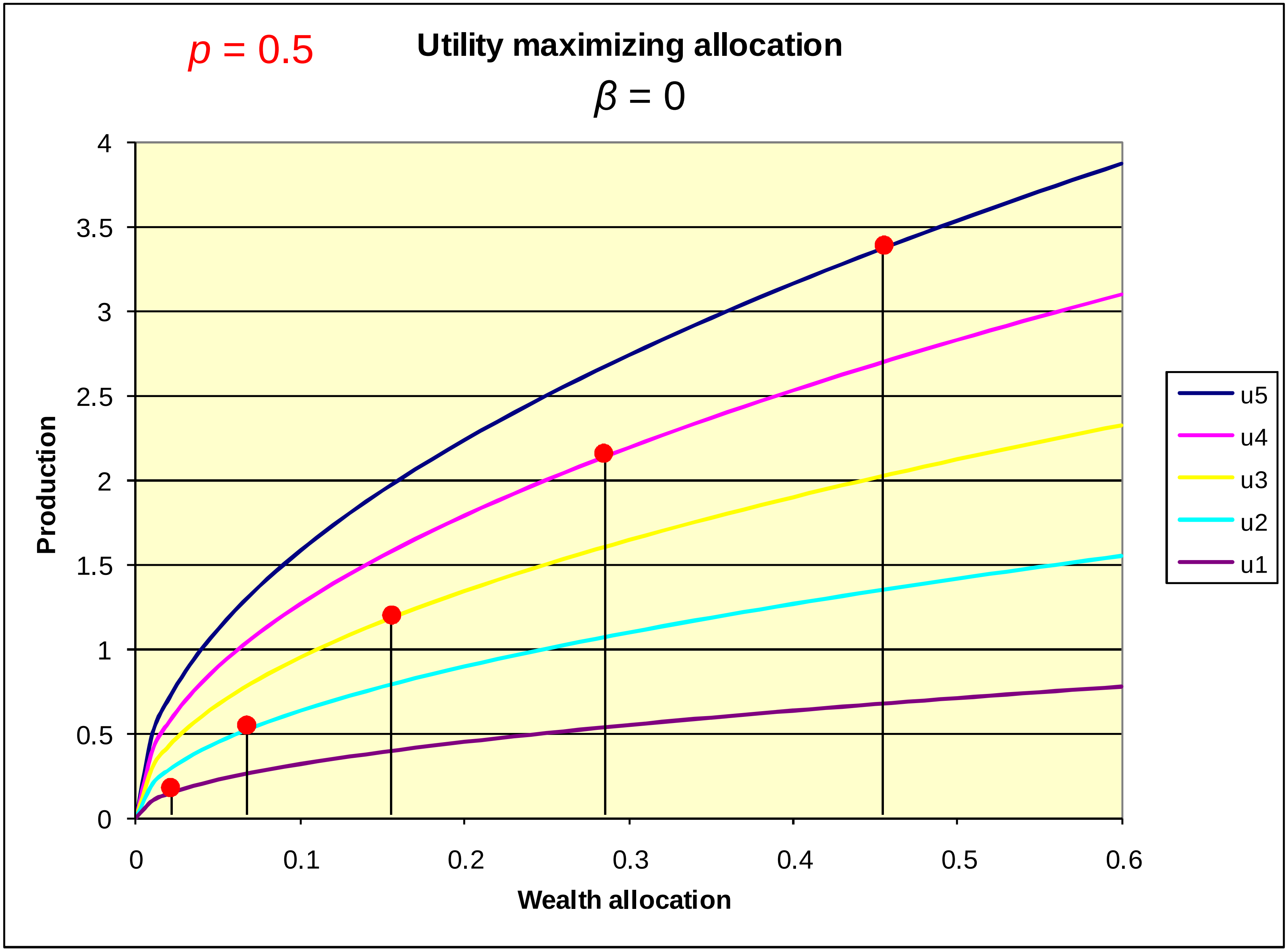
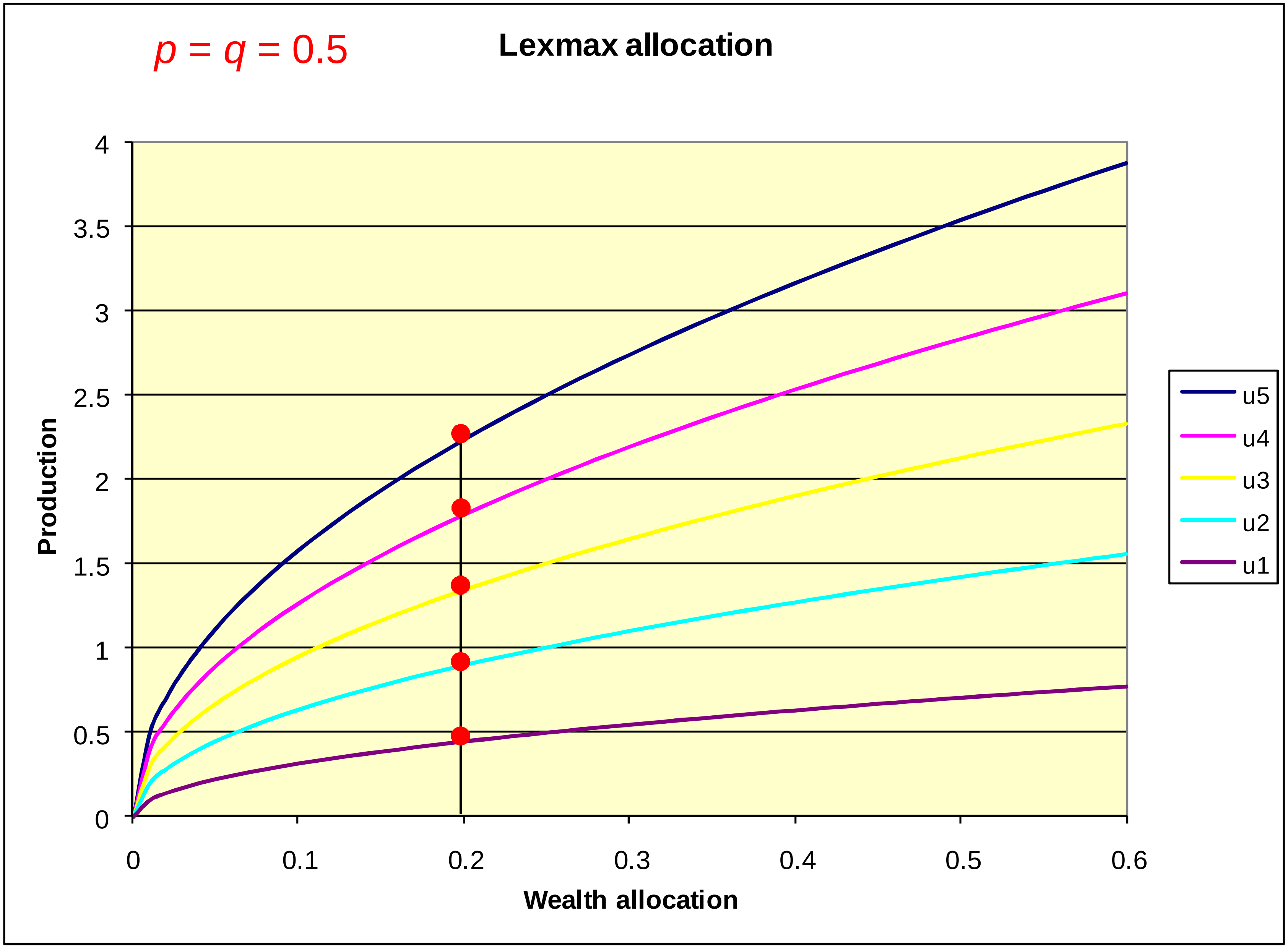
Model now **simplifies**.


p = 0.5
Utility maximizing allocation
β = 0
Production
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
Wealth allocation
u5
u4
u3
u2
u1



p = q = 0.5
Lexmax allocation
Production
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
Wealth allocation
u5
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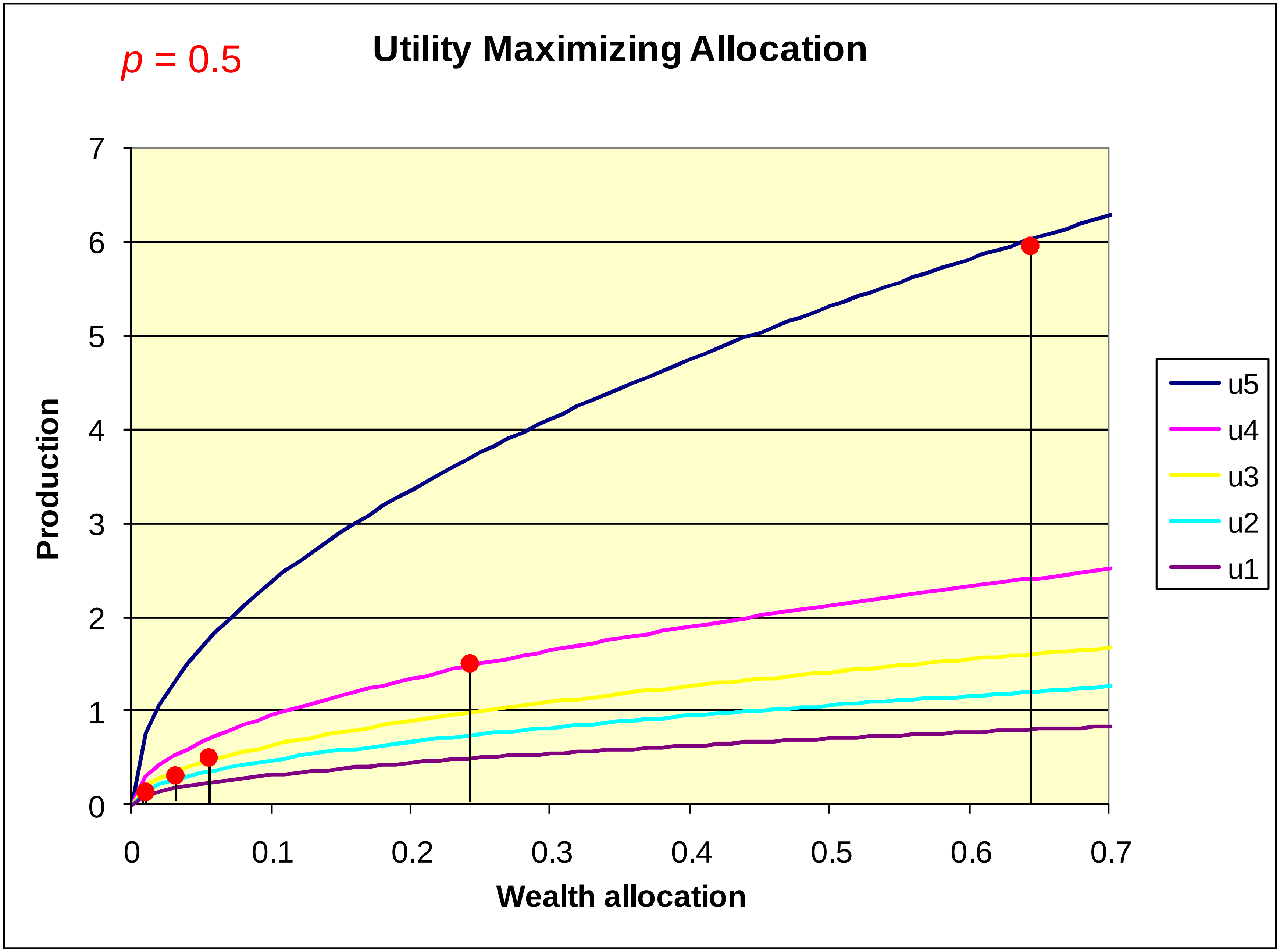
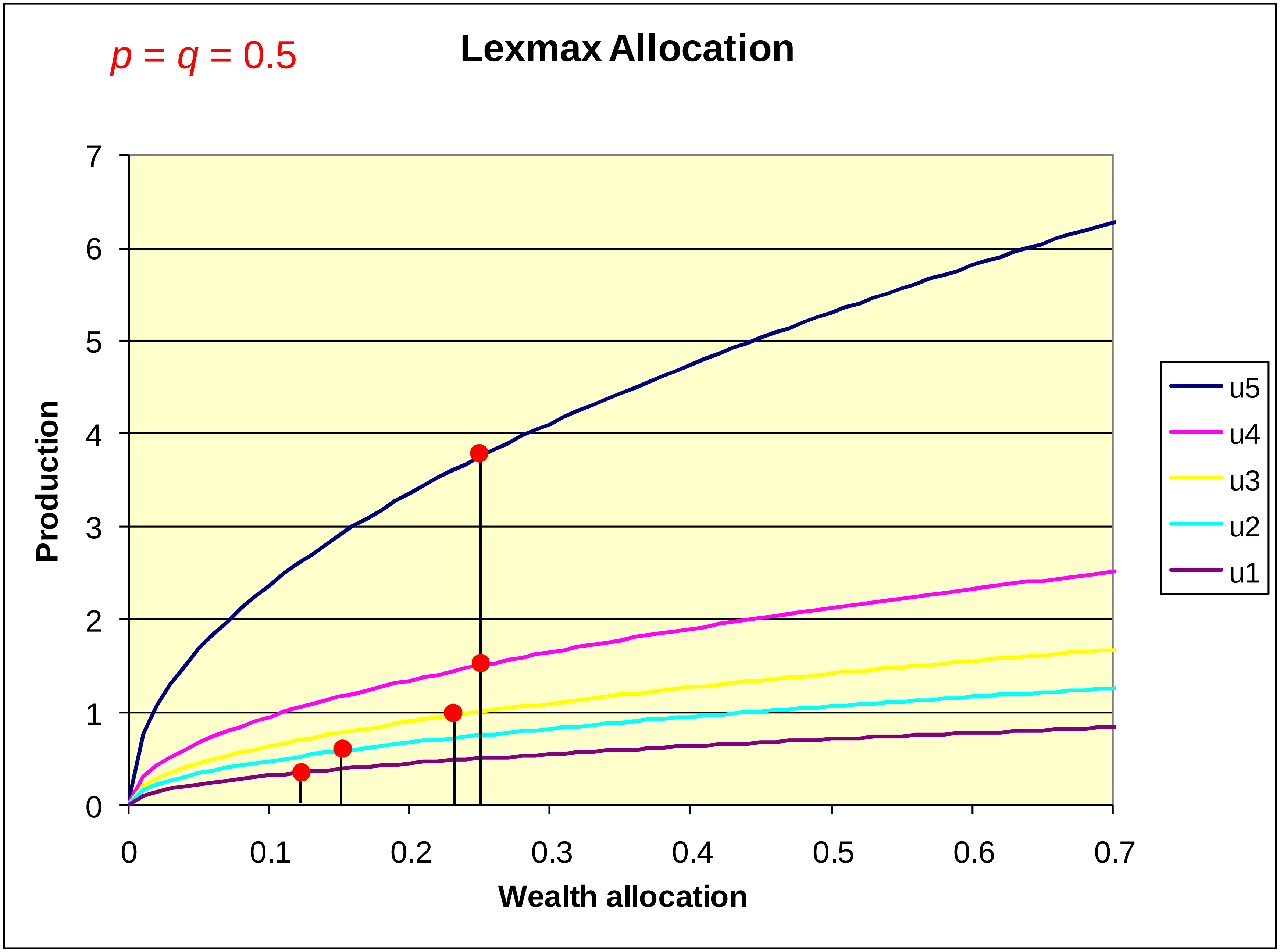


p = 0.5
Utility Maximizing Allocation
Production
Wealth allocation
0
1
2
3
4
5
6
7
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
u5
u4
u3
u2
u1


p = q = 0.5
Lexmax Allocation
Production
7
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Wealth allocation
u5
u4
u3
u2
u1

Lexmax Model

- When does the Rawlsian model result in equality?
  - That is, when do we have $x_1 = \cdots = x_n$ in the solution of the lexmax problem?

Lexmax Model

- Conditions for equality at optimality:

$$\begin{aligned}
&2\mu_1 - \mu_2 = d_1 \\
&\mu_1 + \mu_i - \mu_{i+1} = d_i, \quad i = 2,\ldots,n-2 \\
&\mu_1 + \mu_{n-1} = d_{n-1}
\end{aligned}$$

- with RHS's:

$$d_i = v(x_i)\frac{\sum_i c_i u_i(x_i)}{\sum_i v(x_i)}\left(\frac{v'(x_1)}{v(x_1)} - \frac{u_{i+1}'(x_{i+1}) - u_1'(x_1)}{\sum_i c_i u_i(x_i)} + \frac{v'(x_{i+1}) - v'(x_1)}{\sum_i v(x_i)}\right)$$

- Remarkably, these can be solved in closed form, yielding□

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \leq \frac{q}{p} \cdot \frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n - 1$.

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

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for $k = 1, \ldots, n - 1$.

Average of $n - k$ largest c_i's

Average of k smallest c_i's

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \le \frac{q}{\boxed{p}} \cdot \frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

- Equality is **more likely** to be required when p is small.
  - When investment in an individual yields rapidly decreasing marginal returns.

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \le \frac{q}{p}\cdot\frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

- Equality test is **more sensitive** at upper end (large k).
  - Equality is **unlikely** to be required when there is a long upper tail (individuals at the top are very productive).
  - Equality **may be required** even when there is a long **lower** tail (individuals at the bottom are very unproductive).

Lexmax Model

- **Theorem.** The lexmax distribution is egalitarian only if

$$\frac{1}{n-k}\sum_{i=k+1}^{n} c_i - \frac{1}{k}\sum_{i=1}^{k} c_i \le \frac{q}{p}\cdot\frac{n-k}{k}\sum_{i=1}^{n} c_i$$

for $k = 1, \ldots, n-1$.

- Equality is **more likely** to be required when q is large.
  - That is, when greater wealth yields rapidly increasing marginal utility.
  - That is, when people **want to get rich**.

Axiomatics

- Social welfare functions
- Interpersonal comparability
- Deriving the utilitarian criterion
- Deriving the maximin/minimax criterion

Axiomatics

- The economics literature derives social welfare functions from axioms of rational choice.
  - Some axioms are strong and hard to justify.
  - The social welfare function depends on degree of **interpersonal comparability** of utilities.
  - Arrow's impossibility theorem was the first result, but there are many others.

- **Social welfare function**
  - A function $f(u_1, \ldots, u_n)$ of individual utilities.
  - Objective is to maximize $f(u_1, \ldots, u_n)$.

Axiomatics

- Social Preferences
  - Let $u = (u_1, \ldots, u_n)$ be the vector of utilities allocated to individuals.
  - A social welfare function ranks distributions:
    u is preferable to u' if $f(u) > f(u')$.

Interpersonal Comparability

- Unit comparability
  - Suppose each individual's utility u_i is changed to $\beta u_i + \alpha_i$.
  - This doesn't change the utilitarian ranking:

$$\sum_i u_i(x) > \sum_i u_i(y) \text{ if and only if }$$

$$\sum_i \left(\beta u_i(x) + \alpha_i\right) > \sum_i \left(\beta u_i(y) + \alpha_i\right)$$

  - This is **unit comparability**.
  - That is, changing units of measure and giving everyone a different zero point has no effect on ranking.

Interpersonal Comparability

- Unit comparability
  - Unit comparability is enough to make utilitarian calculations **meaningful**.
  - Given certain axioms, along with unit comparability, a utilitarian social welfare function is **necessary**

Axioms

- Anonymity
  - Social preferences are the same if indices of u_is are permuted.
- Strict pareto
  - If $u > u'$, then u is preferred to u'.
- Independence of irrelevant alternatives
  - The preference of u over u' depends only on u and u' and not on what other utility vectors are possible.
- Separability of unconcerned individuals
  - Individuals i for which $u_i = u_i'$ don't affect the ranking of u and u'.

Axiomatics

Theorem

Given **unit comparability**, any social welfare function f that satisfies the axioms has the form $f(u) = \Sigma_i\, a_i u_i$ (**utilitarian**).

Interpersonal Comparability

- Level comparability
  - Suppose each individual's utility u_i is changed to $\phi(u_i)$, where ϕ is a monotone increasing function.
  - This doesn't change the maximin ranking:

$$\min_i \{u(x_i)\} > \min_i \{u(y_i)\} \text{ if and only if } \min_i \{\phi(u(x_i))\} > \min_i \{\phi(u(y_i))\}$$

  - This is **level comparability**.

Axiomatics

- Level comparability
  - Level comparability is enough to make maximin comparisons **meaningful**.

Theorem

Given **level comparability,** any social welfare function that satisfies the axioms leads to a **maximin** or **minimax** criterion.

Axiomatics

- Problem with utilitarian theorem
    - The assumption of unit comparability implies **no more than unit comparability**.
    - This is almost the same as assuming utilitarianism.
    - It rules out a maximin criterion from the start, because the "worst-off" is a meaningless concept.

- Problem with maximin theorem
    - The assumption of level comparability implies **no more than level comparability**.
    - This rules out utilitarianism from the start.

Measures of Inequality

- An example
  - Utilitarian, maximin, and lexmax solution
- Inequality measures
  - Relative range, max, min
  - Relative mean deviation
  - Variance, coefficient of variation
  - McLoone index
  - Gini coefficient
  - Atkinson index
  - Hoover index
  - Theil index

Measures of Inequality

- Assume we wish to **minimize inequality**.
  - We will survey several measures of inequality.
  - They have different strengths and weaknesses.
  - Minimizing inequality may result in less total utility.

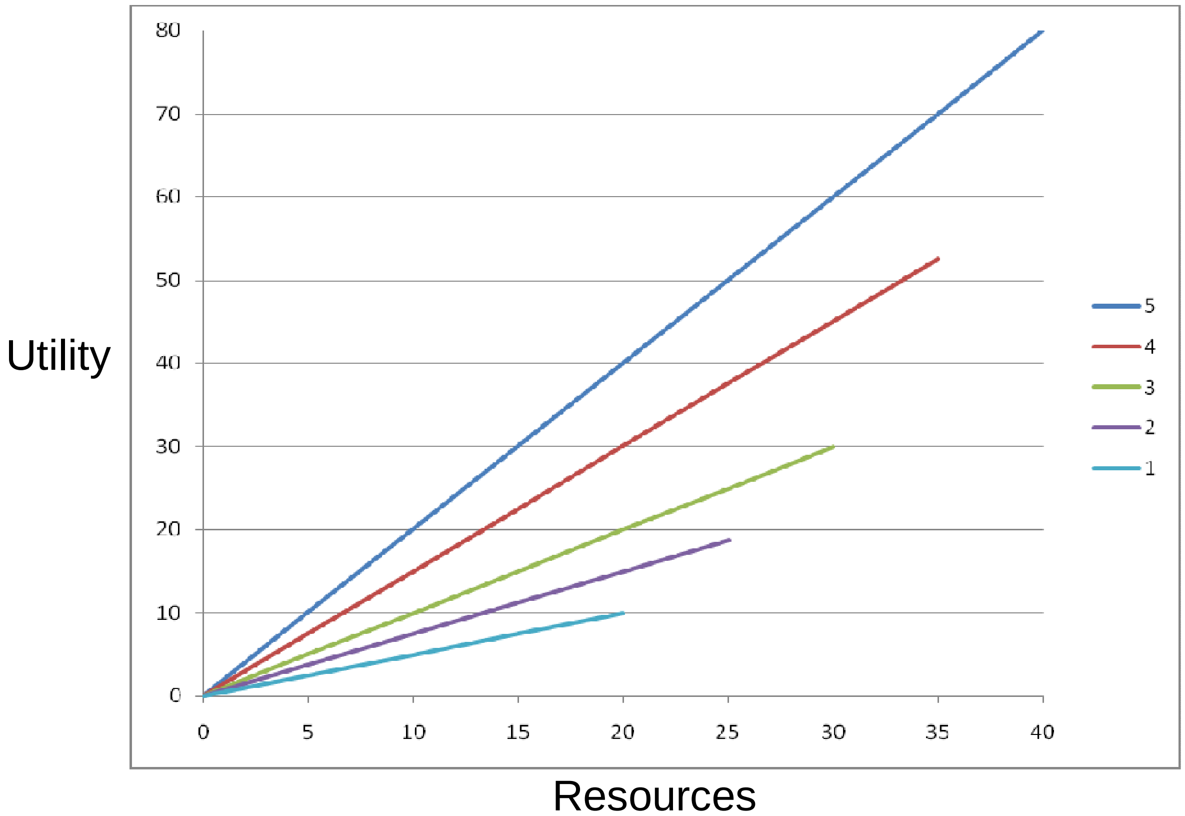
- **Pigou-Dalton** condition.
  - One criterion for evaluating an inequality measure.
  - If utility is transferred from one who is worse off to one who is better off, inequality should increase.

Measures of Inequality

- **Applications**
  - Tax policy
  - Disaster recovery
  - Educational funding
  - Greenhouse gas mitigation
  - Ramp metering on freeways

Example

Production functions for 5 individuals



Utilitarian

$$\max \sum_i u_i$$

LP model: $\max \sum_{i=1}^{5} u_i$

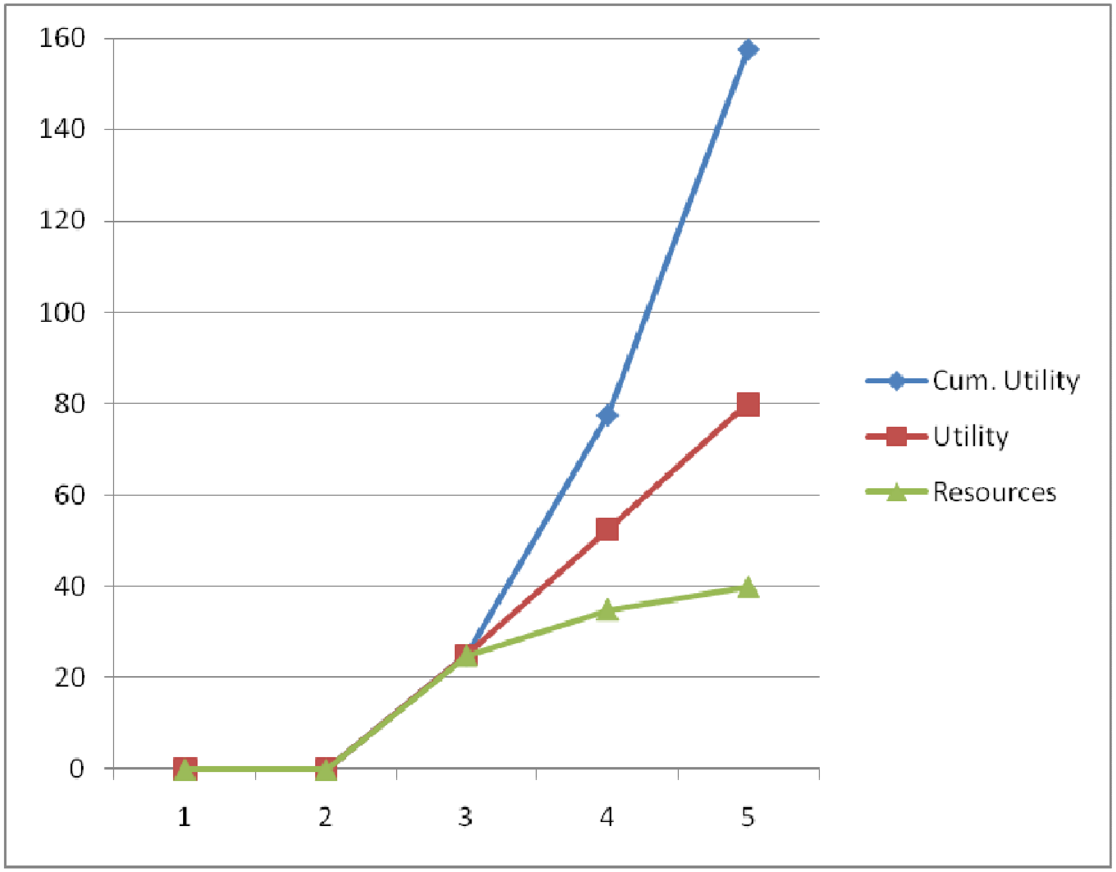
$$u_i = a_i x_i,\ \ 0 \le x_i \le b_i, \text{all } i, \quad \sum_i x_i = B$$

where $(a_1, \ldots a_5) = (0.5, 0.75, 1, 1.5, 2)$

$(b_1, \ldots, b_5) = (20, 25, 30, 35, 40)$

$B = 100$

Utilitarian



Rawlsian

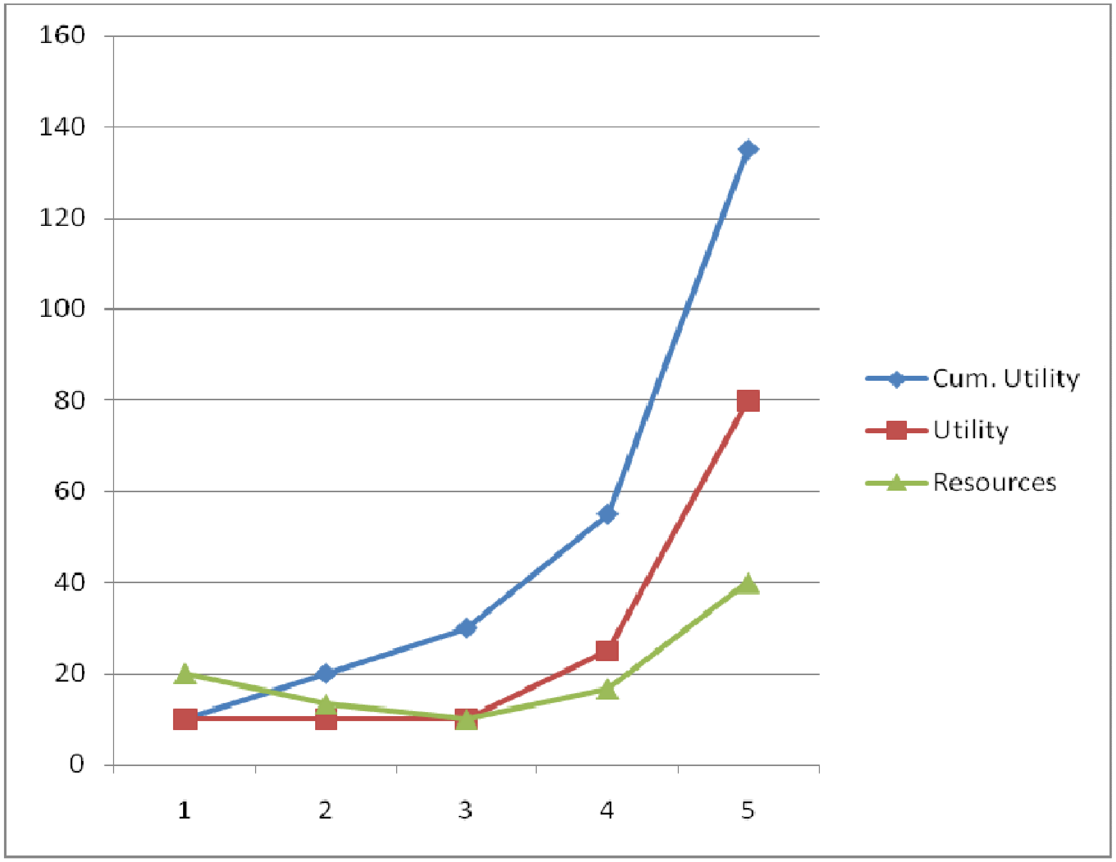
$$\max \left\{ \min_i \{u_i\} \right\}$$

LP model: $\max u_{\min} + \in \sum_i u_i$ ← Ensures that solution is Pareto optimal

$$u_{\min} \le u_i, \;\; \text{all } i$$

$$u_i = a_i x_i, \;\; 0 \le x_i \le b_i, \;\; \text{all } i, \quad \sum_i x_i = B$$

Rawlsian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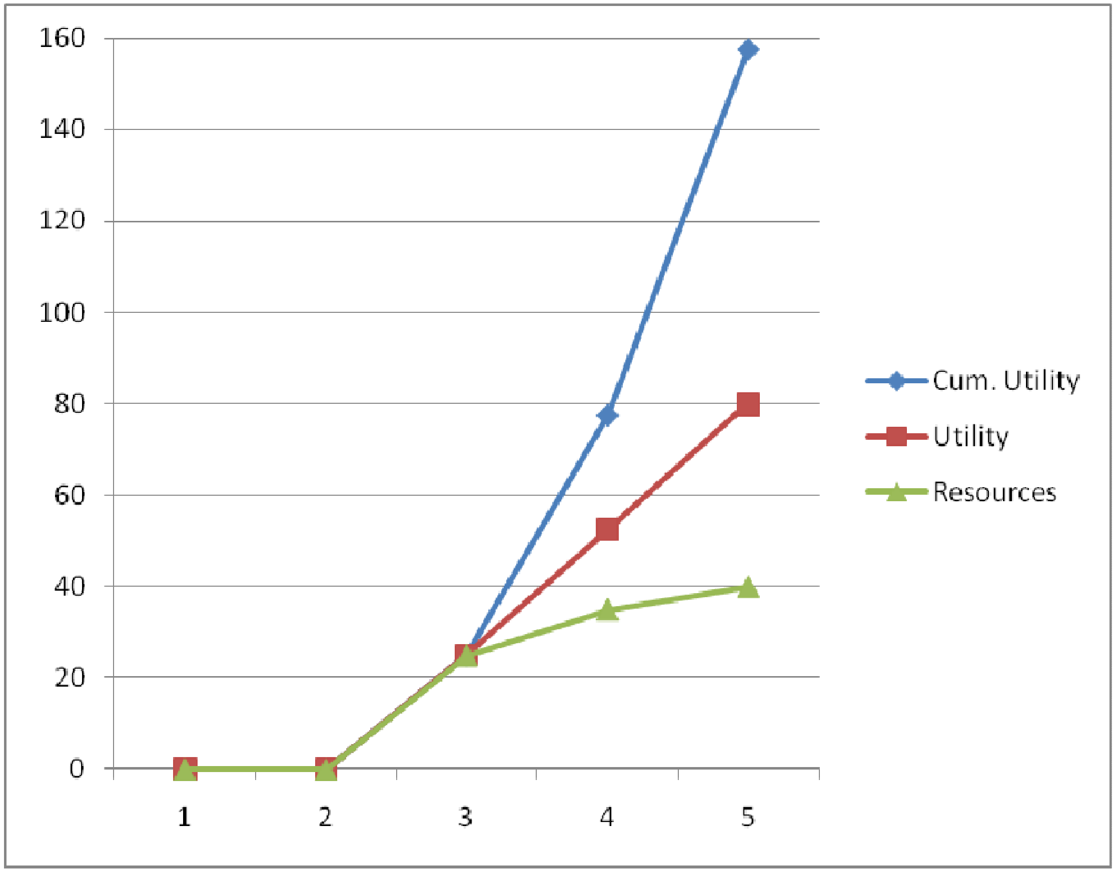
Utilitarian



Lexmax

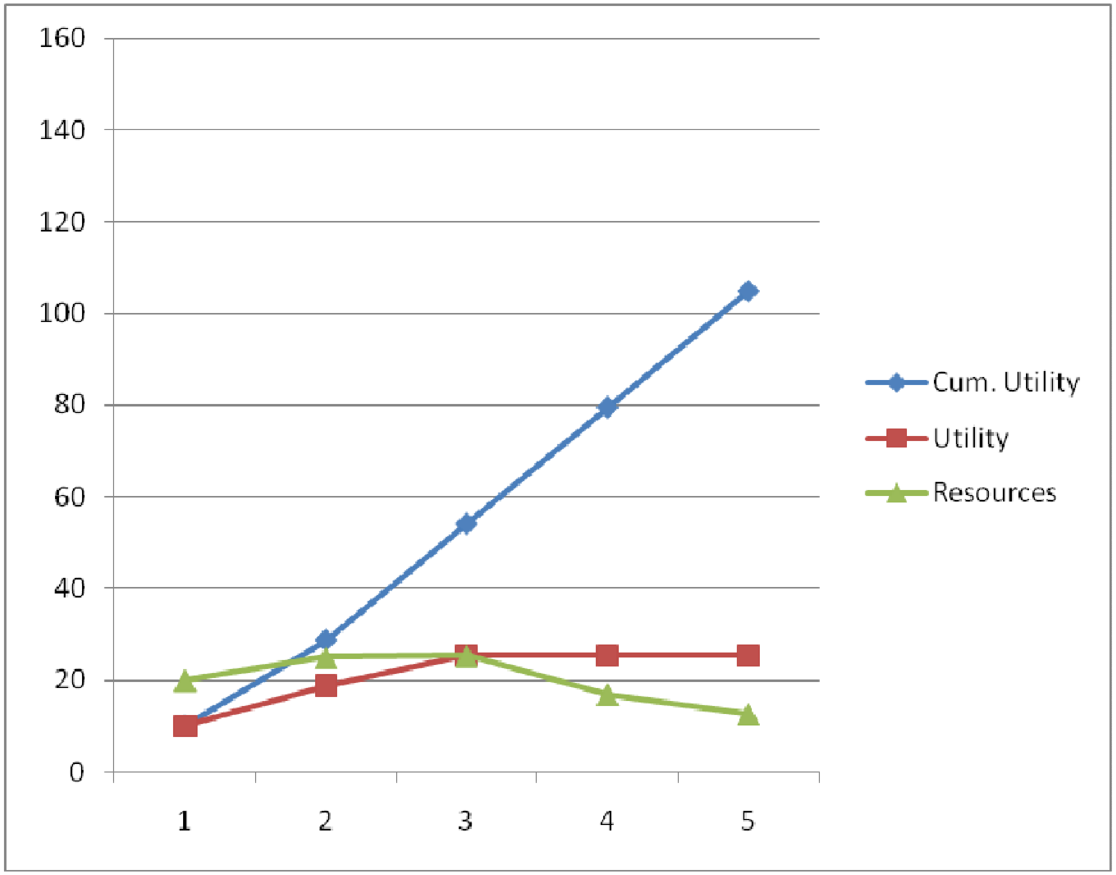
$$\text{lexmax}\,\{u_1,\ldots,u_n\}$$

Sequence of LP models, $k = 1, \square\,, n - 1$:

$$\max u_{\min}$$
$$u_i = u_i^*, \ \text{ all } i < k$$
$$u_{\min} \le u_i, \ \text{ all } i \ge k$$
$$u_i = a_i x_i, \ 0 \le x_i \le b_i, \text{ all } i, \quad \sum_i x_i = B$$

Re-index for each k so that u_i for $i < k$ were fixed in previous iterations.

Lexmax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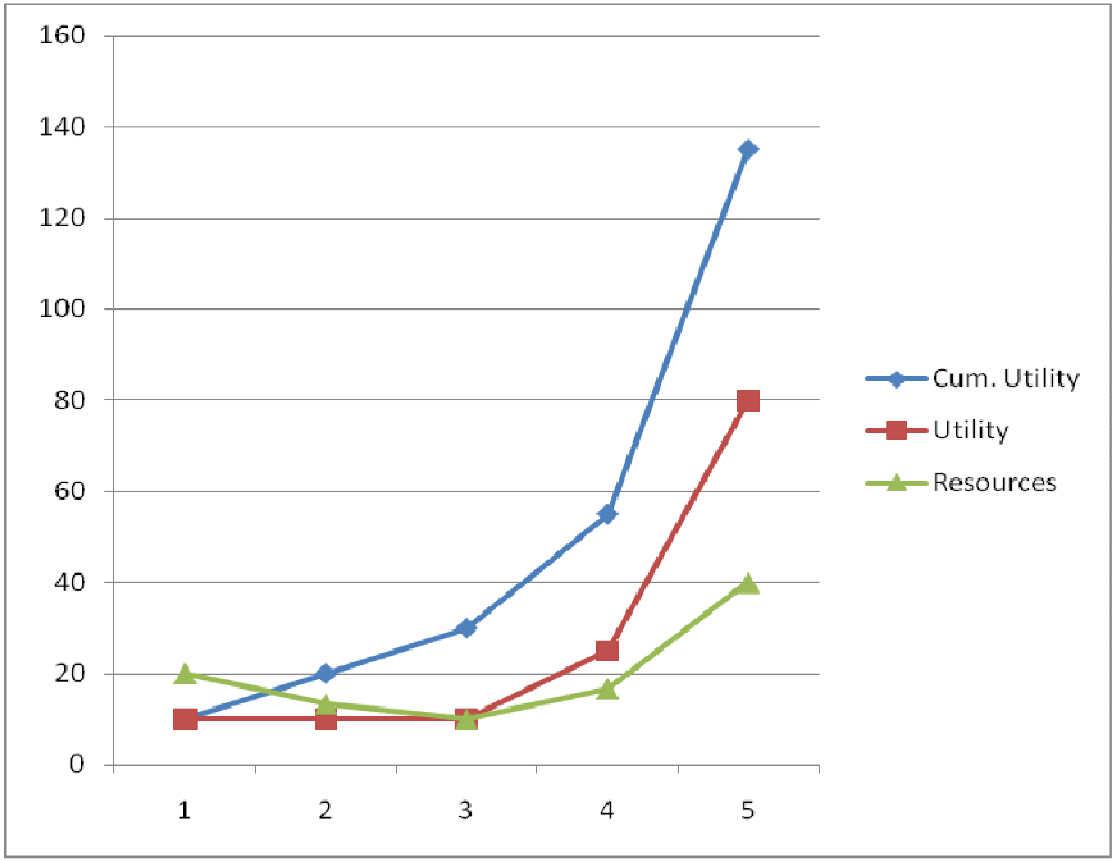
Rawlsian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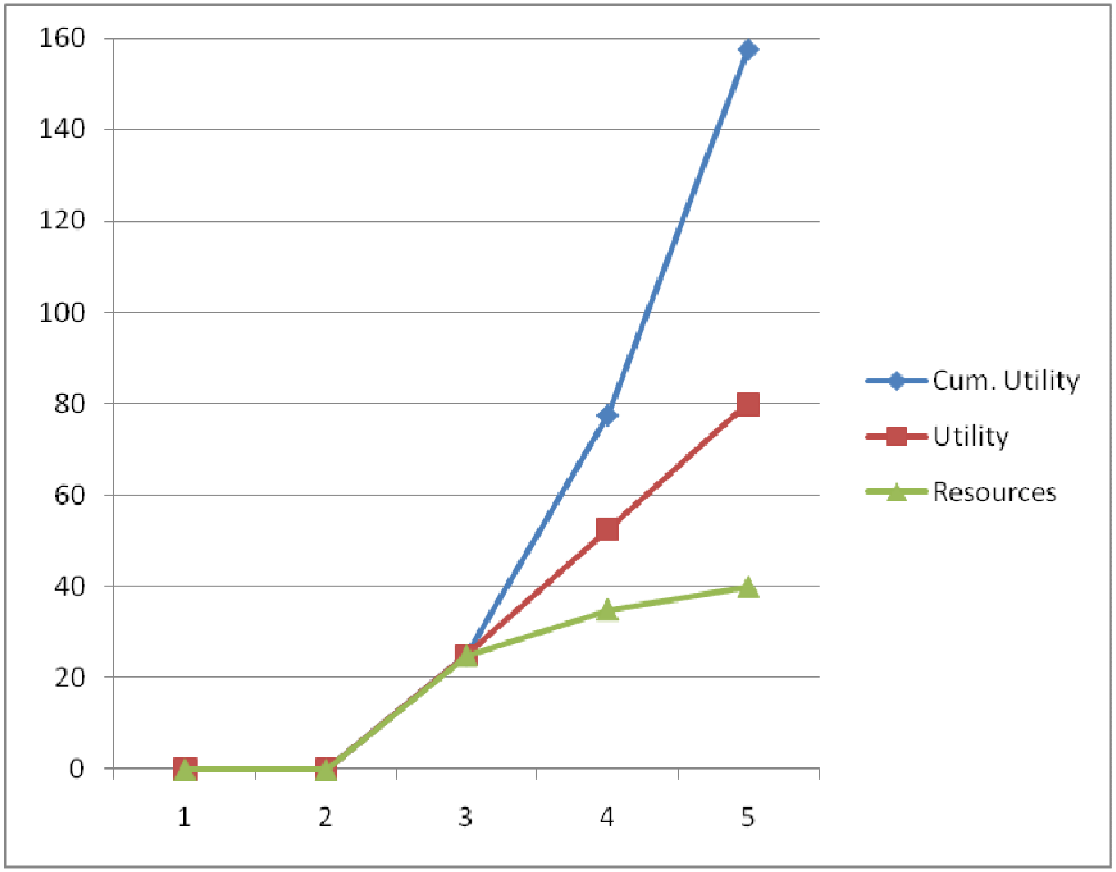
Utilitarian



Relative Range

$$\frac{u_{\max} - u_{\min}}{\bar{u}}$$

where $u_{\max} = \max_i \{u_i\}$ $\quad u_{\min} = \min_i \{u_i\}$ $\quad \bar{u} = (1/n)\sum_i u_i$

Rationale:

- Perceived inequality is relative to the best off.
- A distribution should be judged by the position of the worst-off.
- Therefore, minimize gap between top and bottom.

Problems:

- Ignores distribution between extremes.
- Violates Pigou-Dalton condition

Relative Range

$$\frac{u_{\max} - u_{\min}}{\bar{u}}$$

This is a **fractional linear programming** problem.

Use Charnes-Cooper transformation to an LP. In general,

$$\begin{aligned} &\min \frac{cx + c_0}{dx + d_0} \\ &Ax \geq b \\ &x \geq 0 \end{aligned} \quad \text{becomes} \quad \begin{aligned} &\min cx' + c_0 z \\ &Ax' \geq bz \\ &dx' + d_0 z = 1 \\ &x', z \geq 0 \end{aligned}$$

after change of variable $x = x'/z$ and fixing denominator to 1.

Relative Range

$$\frac{u_{\max} - u_{\min}}{\bar{u}}$$

Fractional LP model:

$$\min \frac{u_{\max} - u_{\min}}{(1/n)\sum_i u_i}$$

$$u_{\max} \geq u_i,\ u_{\min} \leq u_i,\ \text{all } i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{all } i,\quad \sum_i x_i = B$$

LP model:

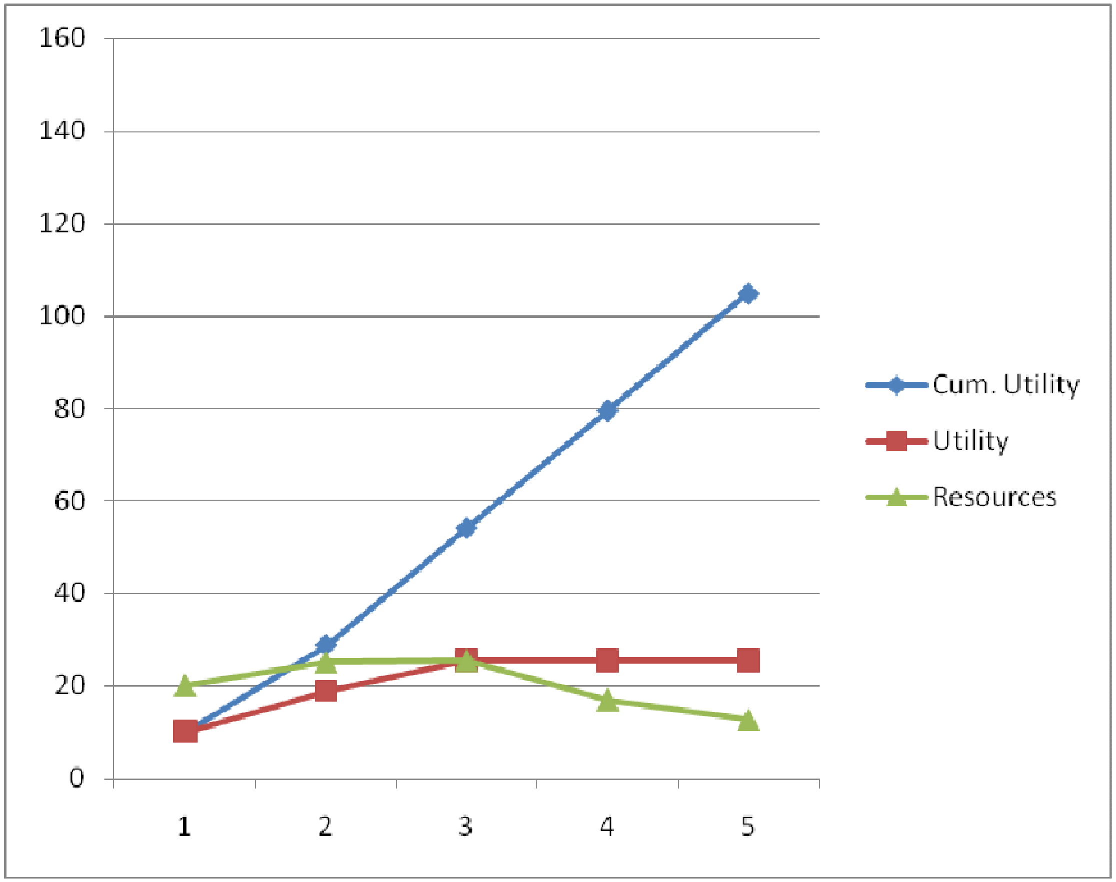
$$\min u_{\max} - u_{\min}$$

$$u_{\max} \geq u_i',\ u_{\min} \leq u_i',\ \text{all } i$$

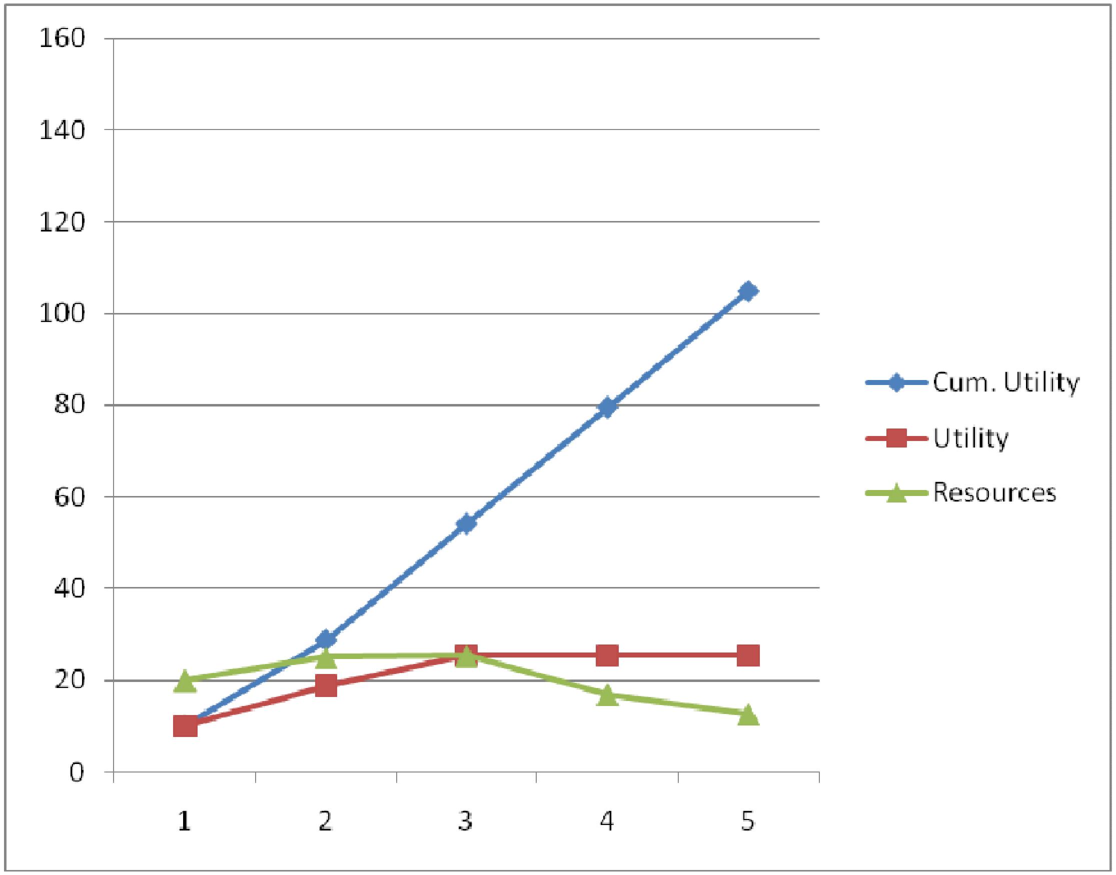
$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

$$(1/n)\sum_i u_i' = 1$$

Relative Range



Lexmax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Cum. Utility
Utility
Resources

Relative Max

$$\frac{u_{\max}}{\bar{u}}$$

Rationale:

- Perceived inequality is relative to the best off.
- Possible application to salary levels (typical vs. CEO)

Problems:

- Ignores distribution below the top.
- Violates Pigou-Dalton condition

Relative Max

$$\frac{u_{\max}}{\bar{u}}$$

Fractional LP model:

$$\min \frac{u_{\max}}{(1/n)\sum_i u_i}$$

$$u_{\max} \geq u_i, \text{ all } i$$

$$u_i = a_i x_i, \; 0 \leq x_i \leq b_i, \; \text{all } i, \; \sum_i x_i = B$$

LP model:

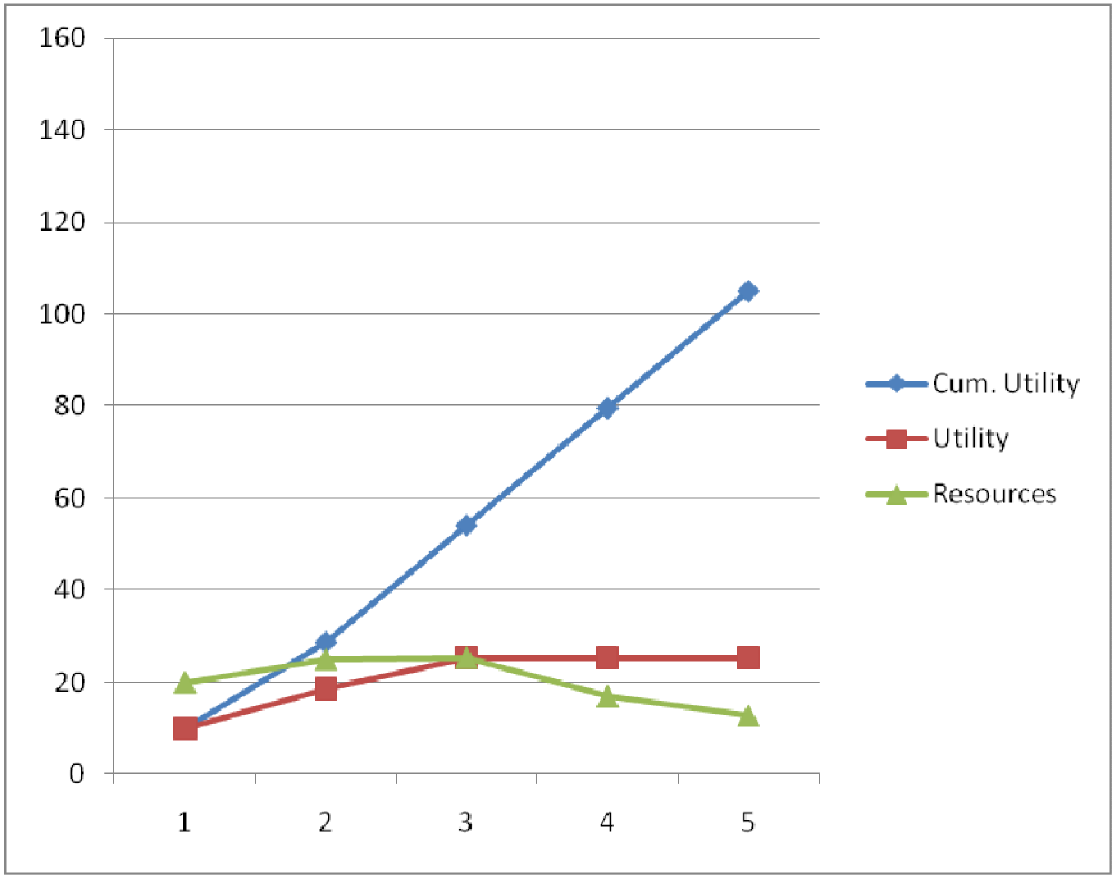
$$\min u_{\max}$$

$$u_{\max} \geq u_i' \text{ all } i$$

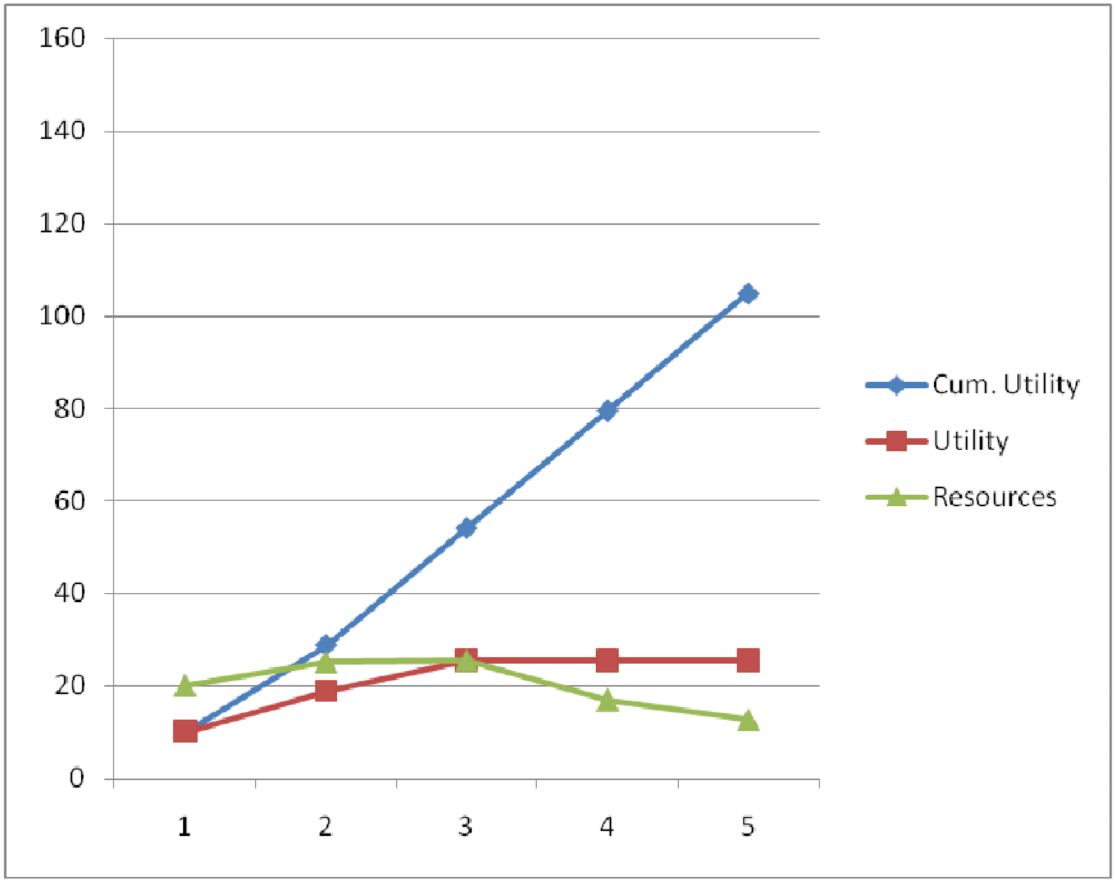
$$u_i' = a_i x_i', \; 0 \leq x_i' \leq b_i z, \; \text{all } i, \; \sum_i x_i' = Bz$$

$$(1/n)\sum_i u_i' = 1$$

Relative Max



Relative Range



Relative Min

$$\frac{u_{\min}}{\bar{u}}$$

Rationale:

- Measures adherence to Rawlsian Difference Principle.
- □ relativized to mean

Problems:

- Ignores distribution above the bottom.
- Violates Pigou-Dalton condition

Relative Min

$$\frac{u_{\min}}{\bar{u}}$$

Fractional LP model:

$$\max \frac{u_{\min}}{(1/n)\sum_i u_i}$$

$$u_{\min} \le u_i, \text{ all } i$$

$$u_i = a_i x_i, \ 0 \le x_i \le b_i, \ \text{ all } i, \quad \sum_i x_i = B$$

LP model:

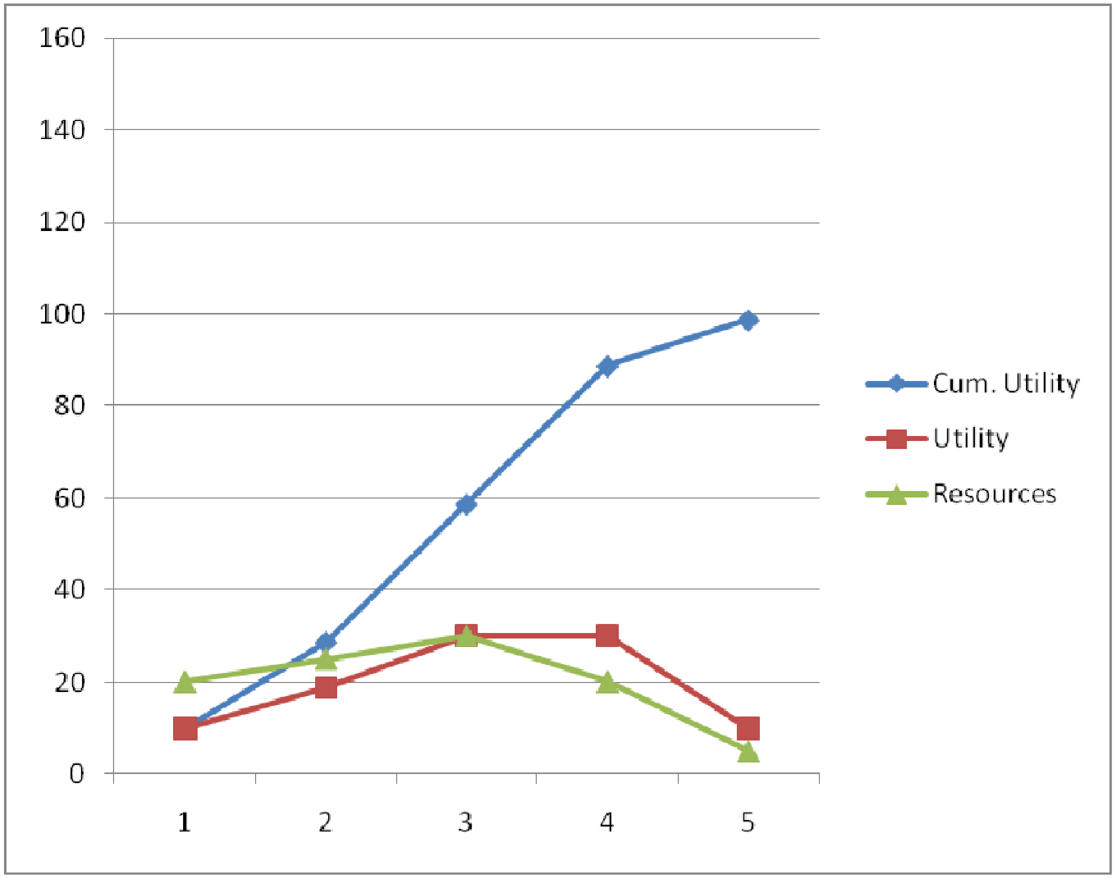
$$\max u_{\min}$$

$$u_{\min} \ge u_i' \text{ all } i$$

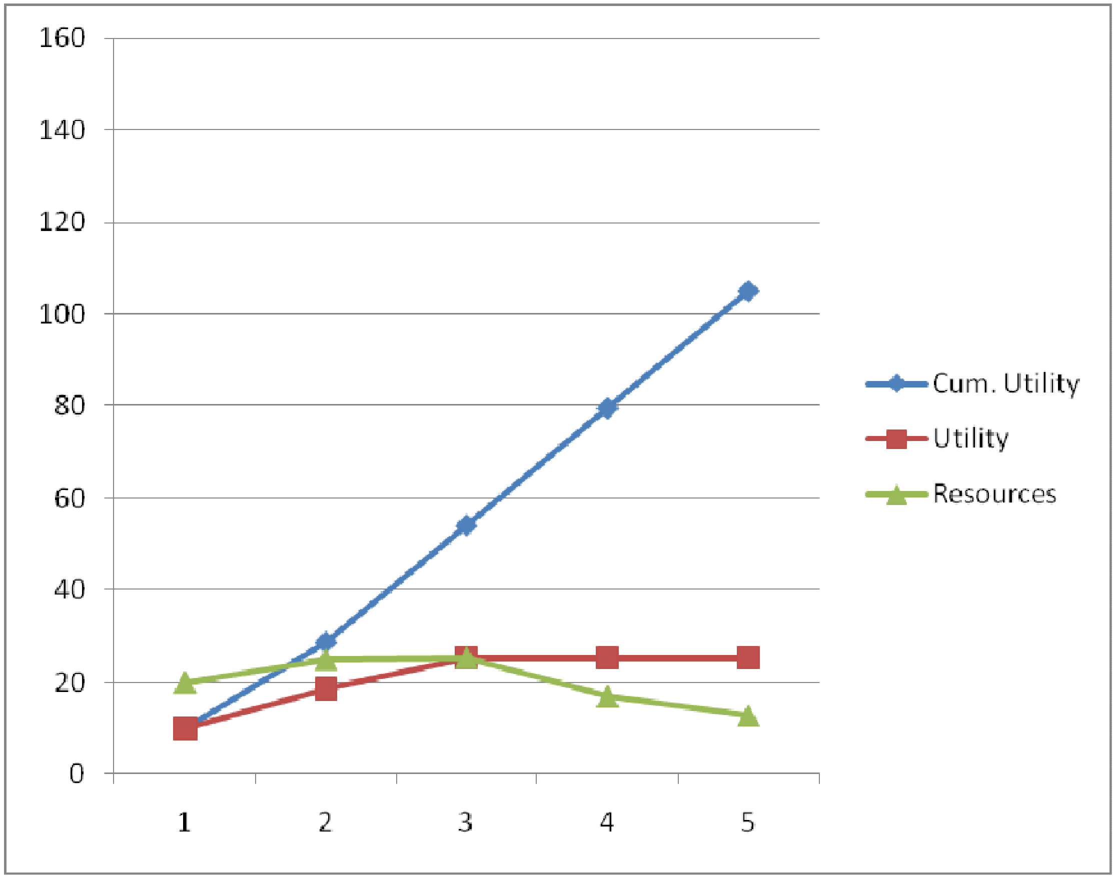
$$u_i' = a_i x_i', \ 0 \le x_i' \le b_i z, \ \text{ all } i, \quad \sum_i x_i' = Bz$$

$$(1/n)\sum_i u_i' = 1$$

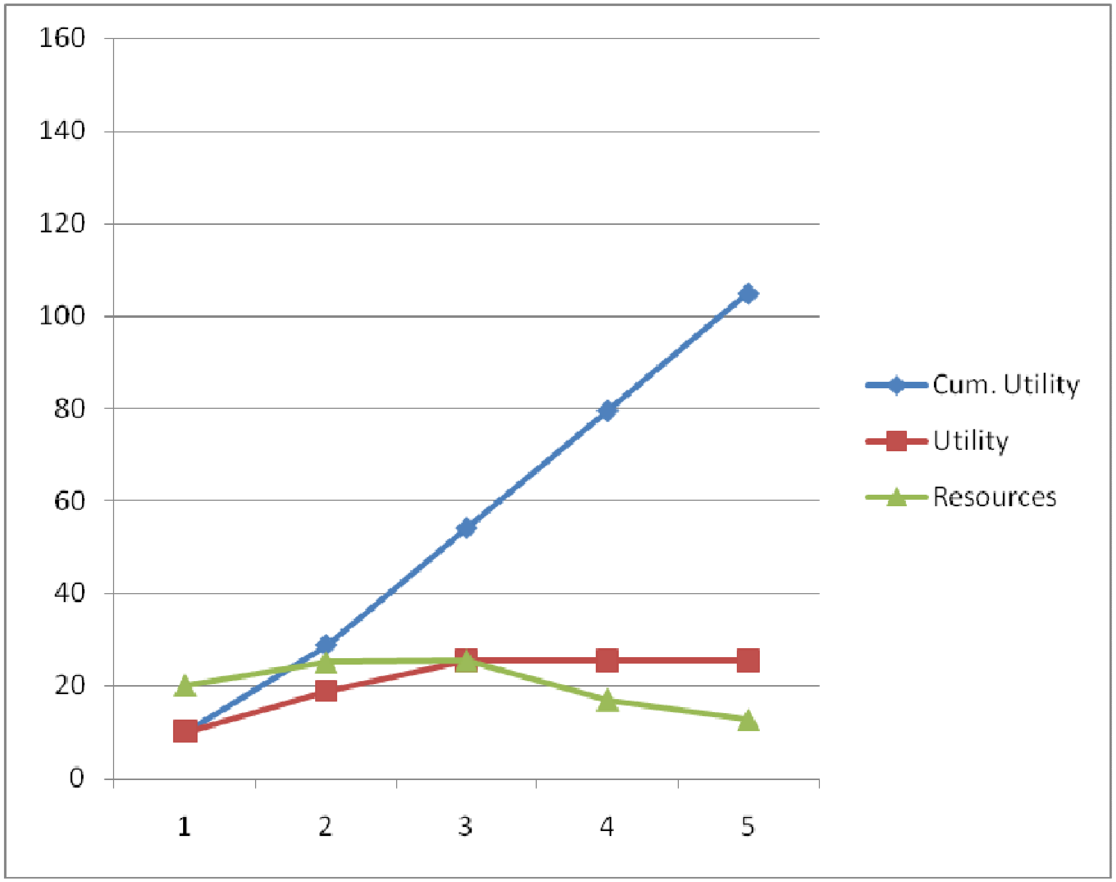
Relative Min



Relative Max



Relative Range



Relative Mean Deviation

$$\frac{\sum_i |u_i - \bar{u}|}{\bar{u}}$$

Rationale:

- Perceived inequality is relative to average.
- Entire distribution should be measured.

Problems:

- Violates Pigou-Dalton condition
- Insensitive to transfers on the same side of the mean.
- Insensitive to placement of transfers from one side of the mean to the other.

Relative Mean Deviation

$$\frac{\sum_i |u_i - \bar{u}|}{\bar{u}}$$

Fractional LP model: $\max \frac{\sum_i (u_i^+ + u_i^-)}{\bar{u}}$

$$u_i^+ \geq u_i - \bar{u},\ u_i^- \geq \bar{u} - u_i,\ \text{all } i$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{all } i,\quad \sum_i x_i = B$$

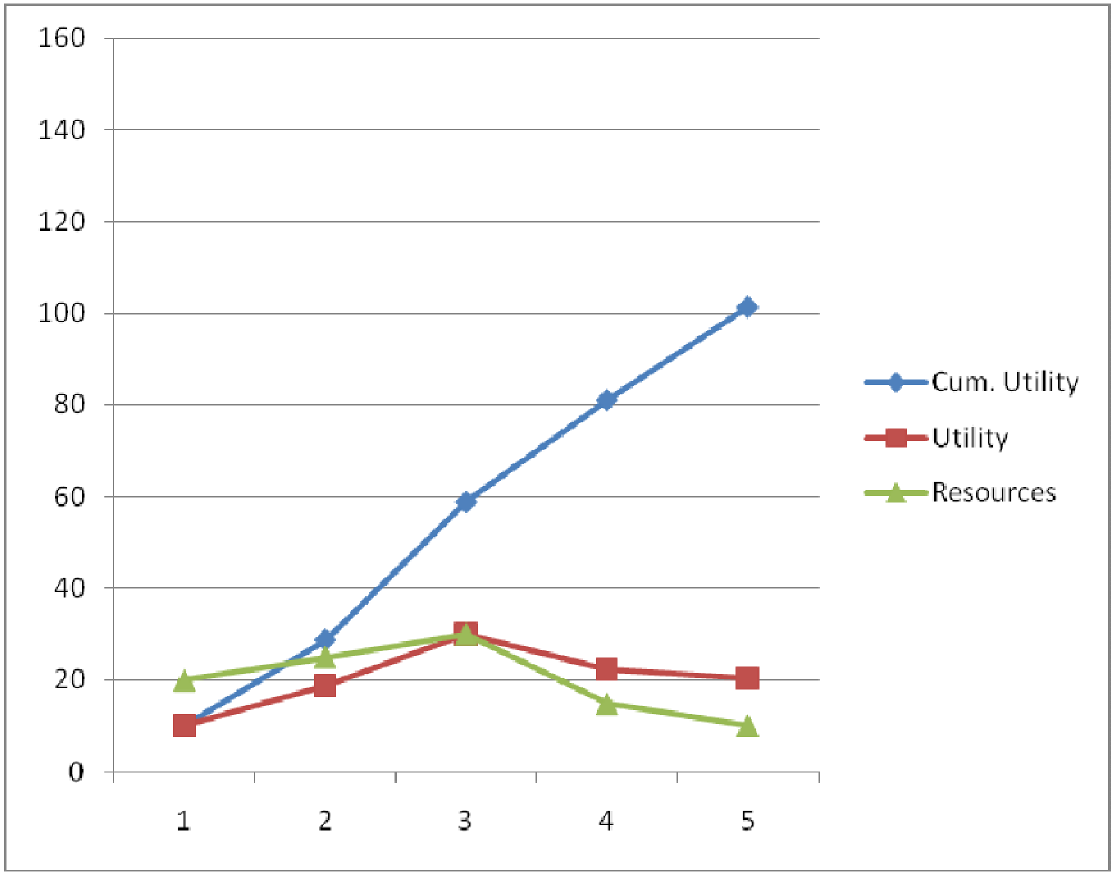
LP model: $\max \sum_i (u_i^+ + u_i^-)$

$$u_i^+ \geq u_i' - 1,\ u_i^- \leq u_i' - 1,\ \text{all } i$$

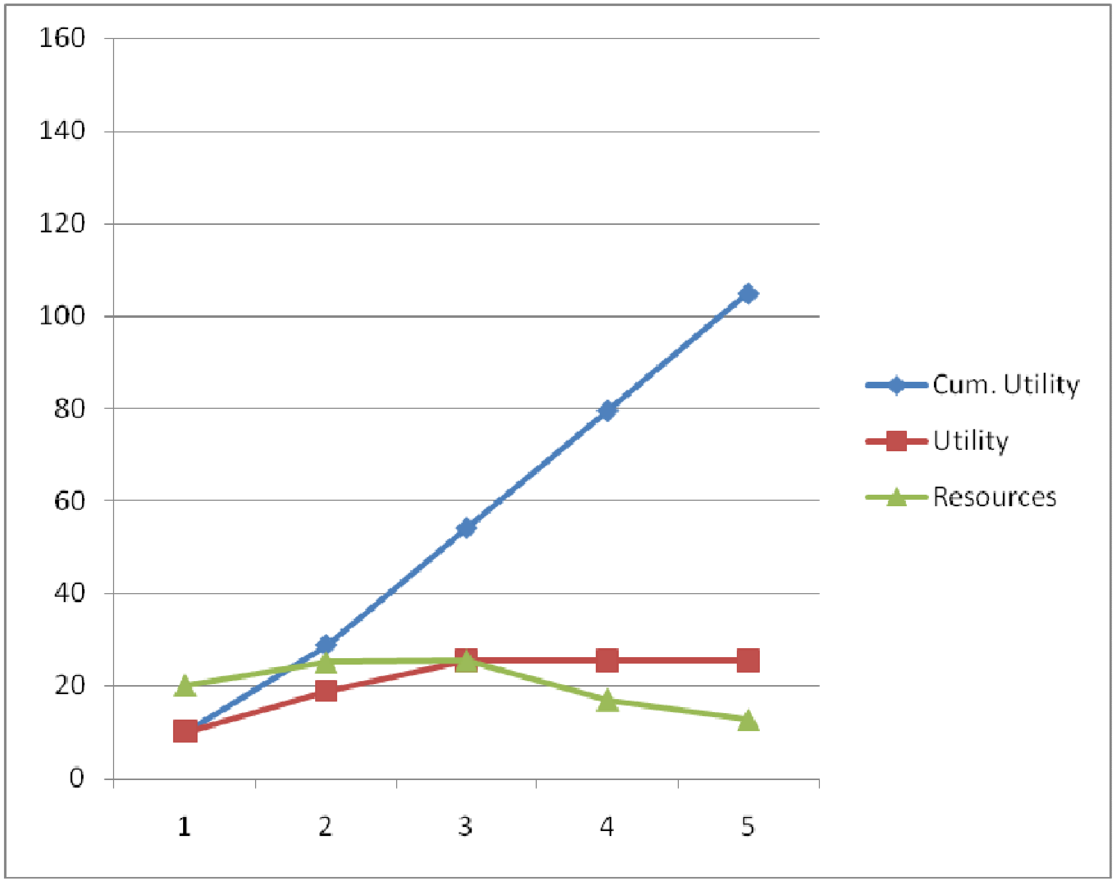
$$(1/n)\sum_i u_i' = 1$$

$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

Relative Mean Deviation



Relative Range



Variance

$$(1/n)\sum_i (u_i - \bar{u})^2$$

Rationale:

- Weight each utility by its distance from the mean.
- Satisfies Pigou-Dalton condition.
- Sensitive to transfers on one side of the mean.
- Sensitive to placement of transfers from one side of the mean to the other.

Problems:

- Weighting is arbitrary?
- Variance depends on scaling of utility.

Variance

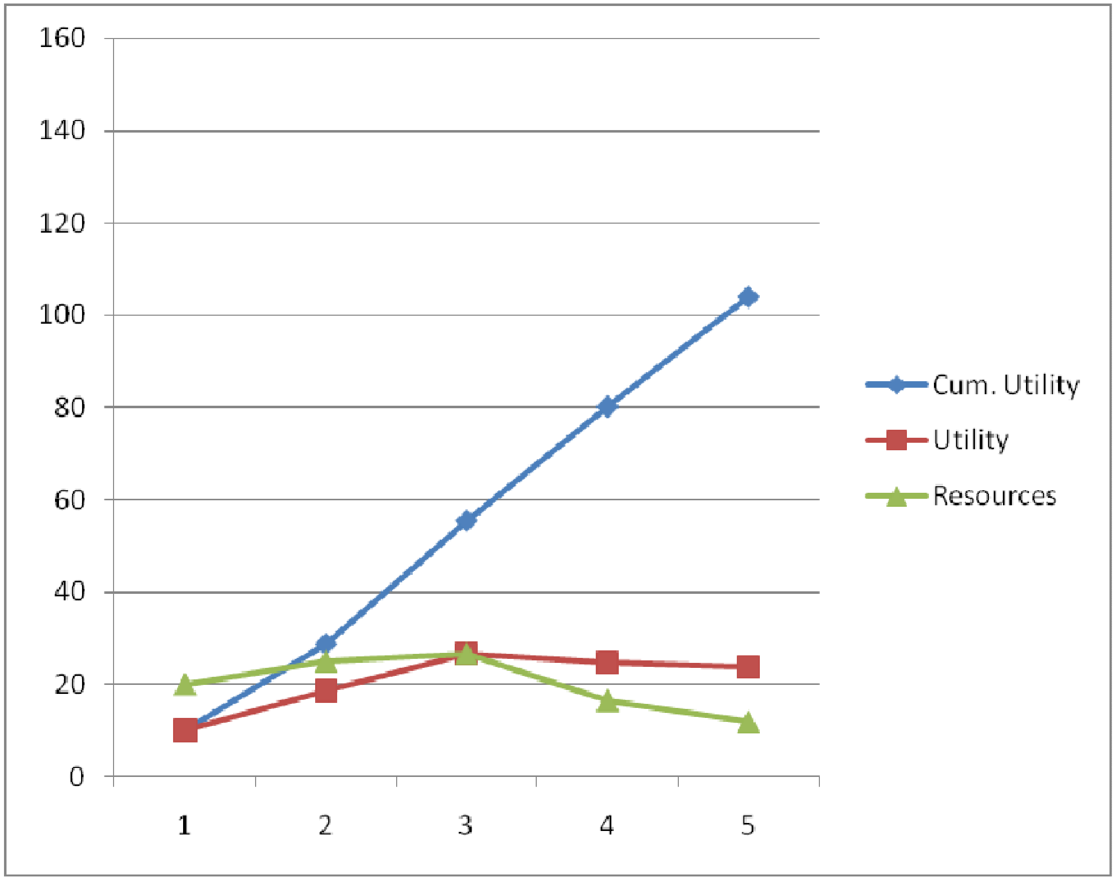
$$(1/n)\sum_i (u_i - \bar{u})^2$$

Convex nonlinear model: $\min (1/n)\sum_i (u_i - \bar{u})^2$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ \ 0 \le x_i \le b_i,\ \text{all } i,\quad \sum_i x_i = B$$

Variance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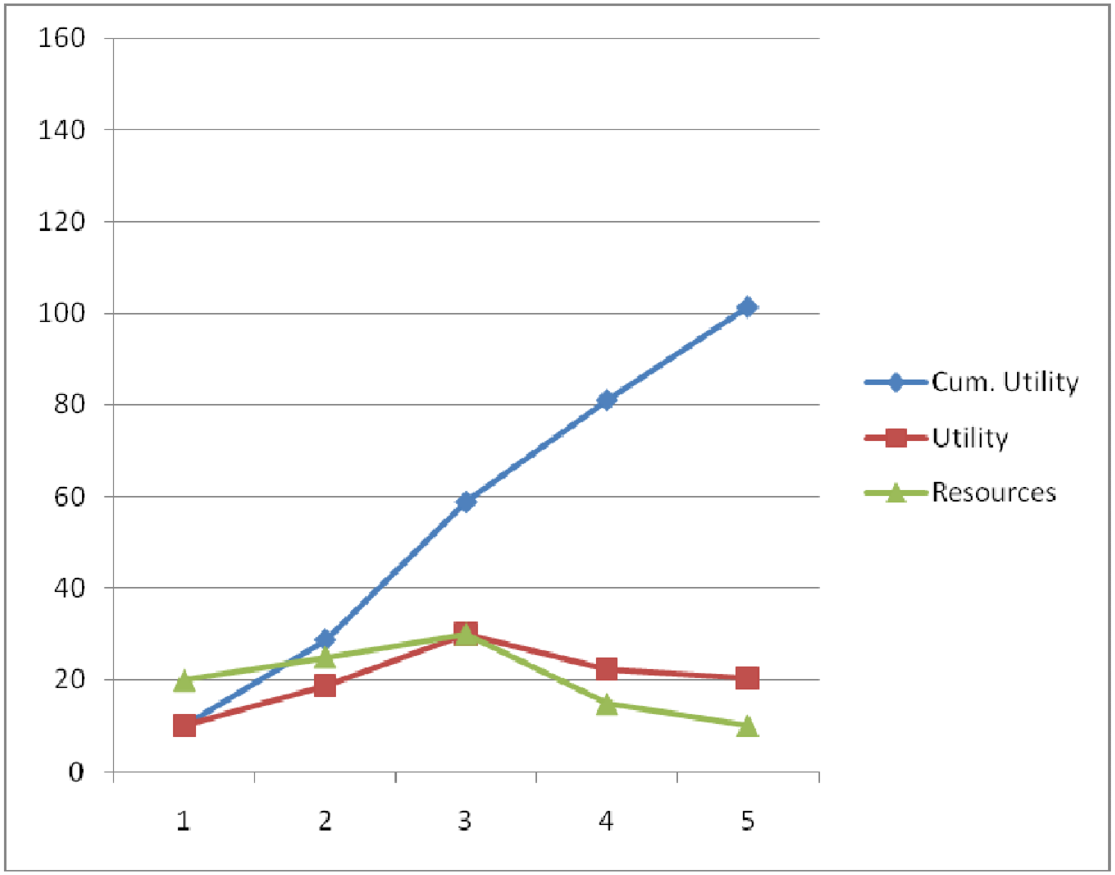
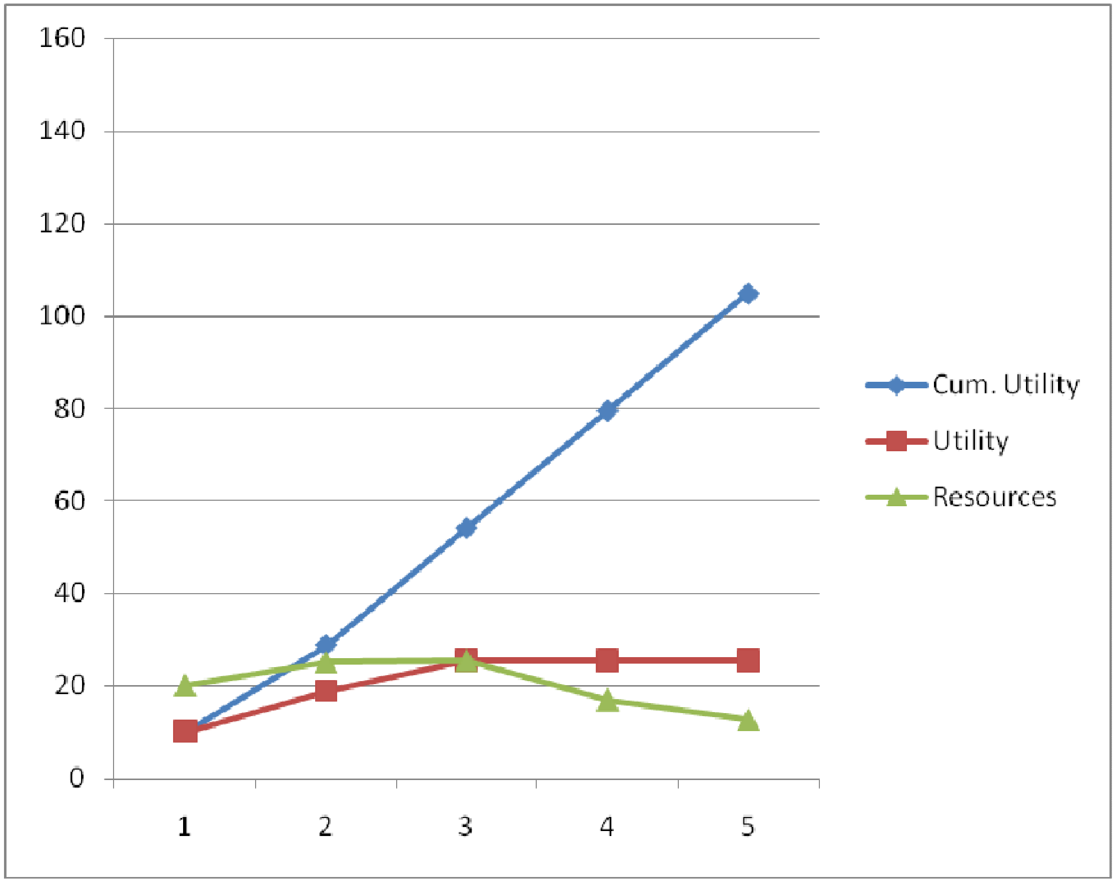
Relative Mean Deviation



Relative Range



Coefficient of Variation

$$\frac{\left((1/n)\sum_i (u_i - \bar{u})^2 \right)^{1/2}}{\bar{u}}$$

Rationale:

- Similar to variance.
- Invariant with respect to scaling of utilities.

Problems:

- When minimizing inequality, there is an incentive to reduce average utility.
- Should be minimized only for fixed total utility.

Coefficient of Variation

$$\frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}}$$

Again use change of variable $u = u'/z$ and fix denominator to 1.

$$\min \frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}}$$
$$Au \geq b$$
$$u \geq 0$$

becomes

$$\min \left((1/n)\sum_i (u_i' - 1)^2\right)^{1/2}$$
$$Au' \geq bz$$
$$(1/n)\sum_i u_i' = 1$$
$$u' \geq 0$$

Can drop exponent to make problem convex

Coefficient of Variation

$$\frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}}$$

Fractional nonlinear model:

$$\max \frac{\left((1/n)\sum_i (u_i - \bar{u})^2\right)^{1/2}}{\bar{u}}$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \le x_i \le b_i,\ \text{all } i,\quad \sum_i x_i = B$$

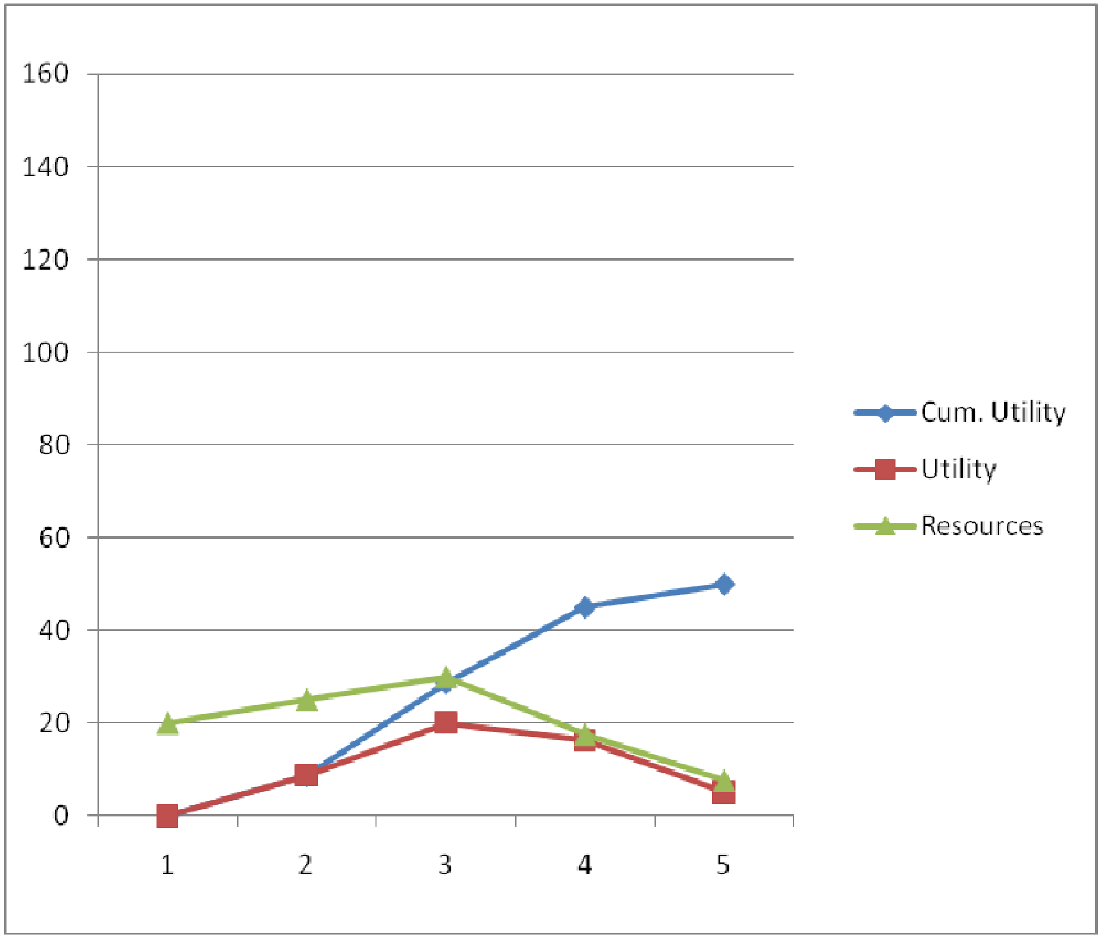
Convex nonlinear model:

$$\min\ (1/n)\sum_i (u_i' - 1)^2$$

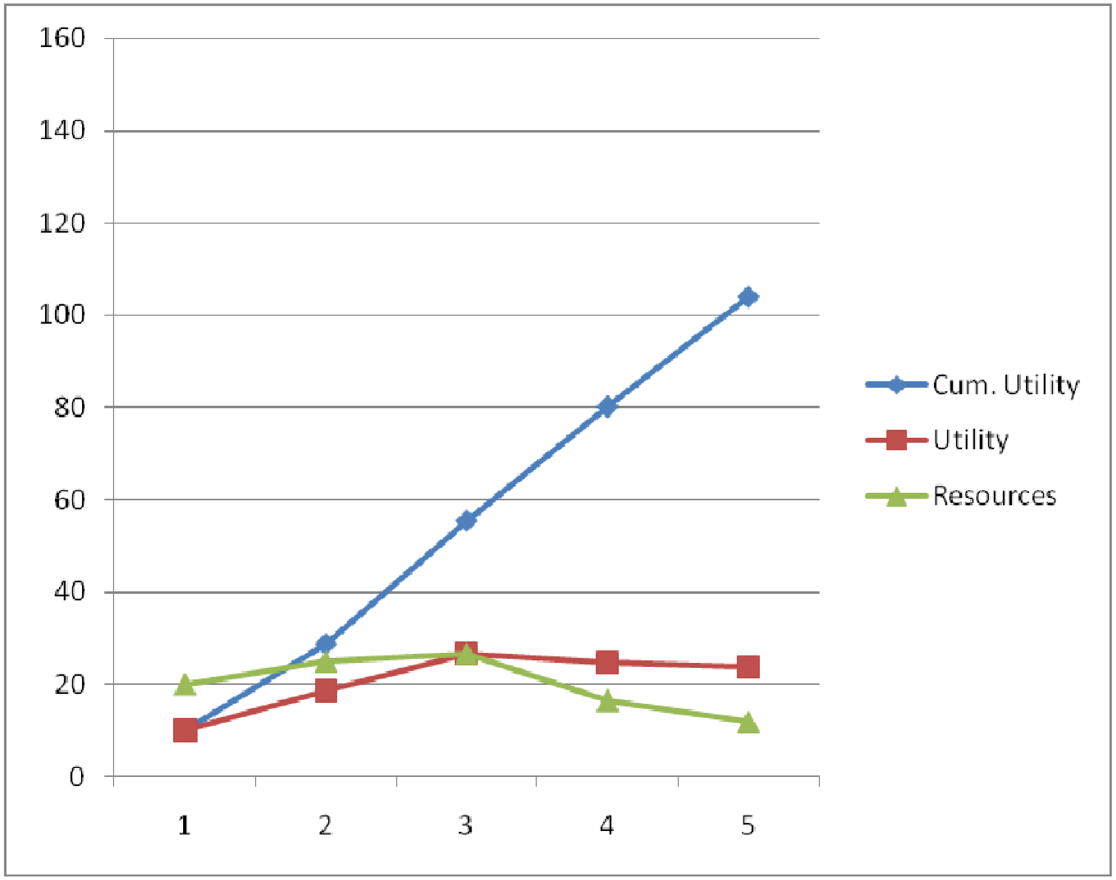
$$(1/n)\sum_i u_i' = 1$$

$$u_i' = a_i x_i',\ 0 \le x_i' \le b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

Coefficient of Variation



Variance



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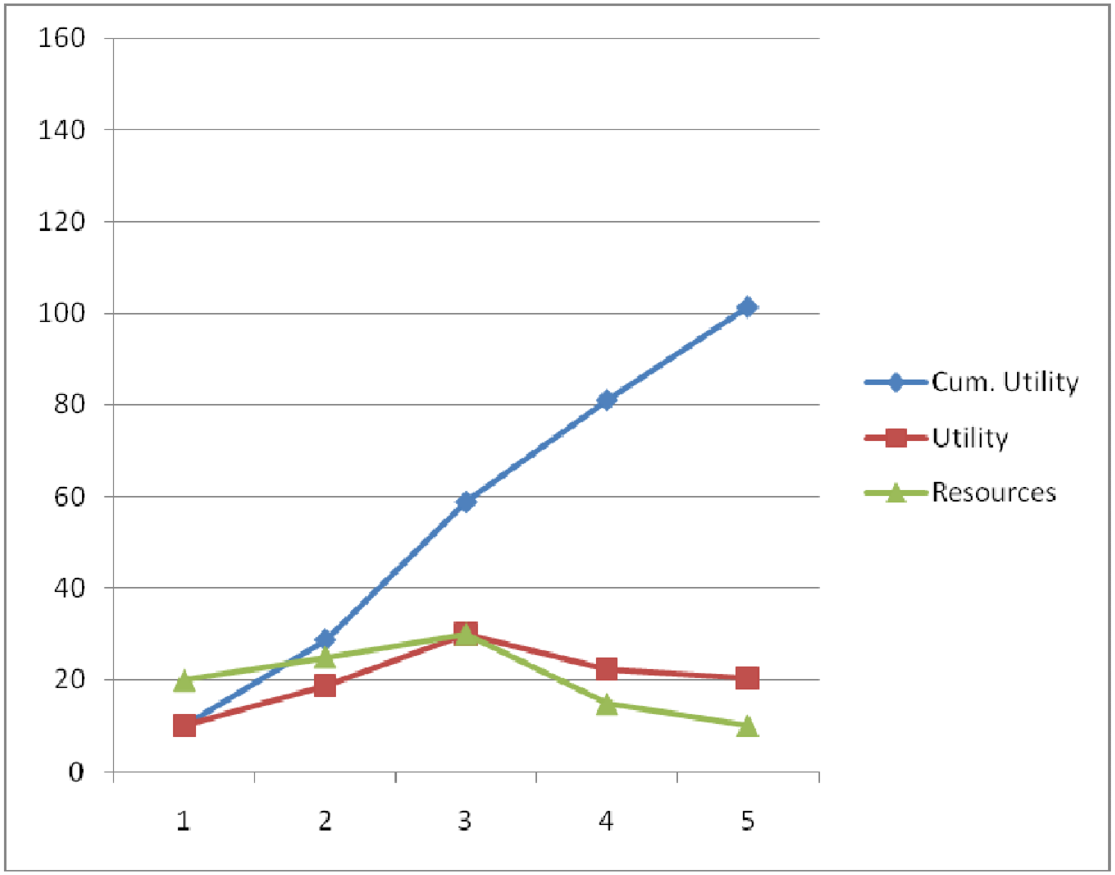
Relative Mean Deviation



McLoone Index

$$\frac{(1/2)\sum_{i:u_i<m} u_i}{\bar{u}}$$

Rationale:

- Ratio of average utility below median to overall average.
- No one wants to be “below average.”
- Pushes average up while pushing inequality down.

Problems:

- Violates Pigou-Dalton condition.
- Insensitive to upper half.

McLoone Index

$$\frac{(1/2)\sum_{i:u_i<m} u_i}{\bar{u}}$$

Fractional MILP model:

$$\max \frac{\sum_i v_i}{\sum_i u_i}$$

Defines median m → $m - My_i \le u_i \le m + M(1-y_i), \;\text{all } i$

Defines $v_i = u_i$ if u_i is below median → $v_i \le u_i, v_i \le My_i, \;\text{all } i$

Half of utilities are below median → $\sum_i y_i < n/2$

$u_i = a_i x_i, \; 0 \le x_i \le b_i, \;\text{all } i, \quad \sum_i x_i = B$

Selects utilities below median → $y_i \in \{0,1\}, \;\text{all } i$

McLoone Index

$$\frac{(1/2)\sum_{i:u_i<m} u_i}{\bar{u}}$$

MILP model:

$$\max \sum_i v_i'$$

$$m' - My_i \le u_i' \le m' + M(1-y_i), \text{ all } i$$

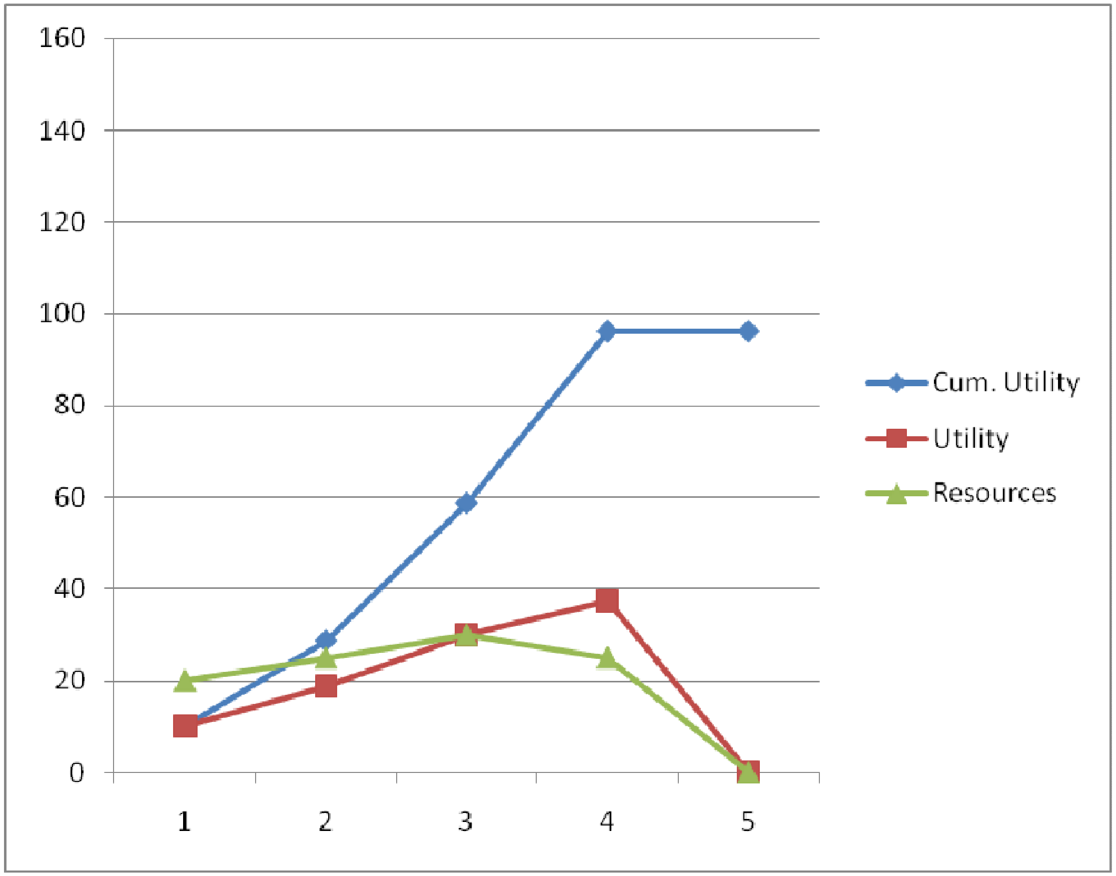
$$v_i' \le u_i', v_i' \le My_i, \text{ all } i$$

$$\sum_i y_i < n/2$$

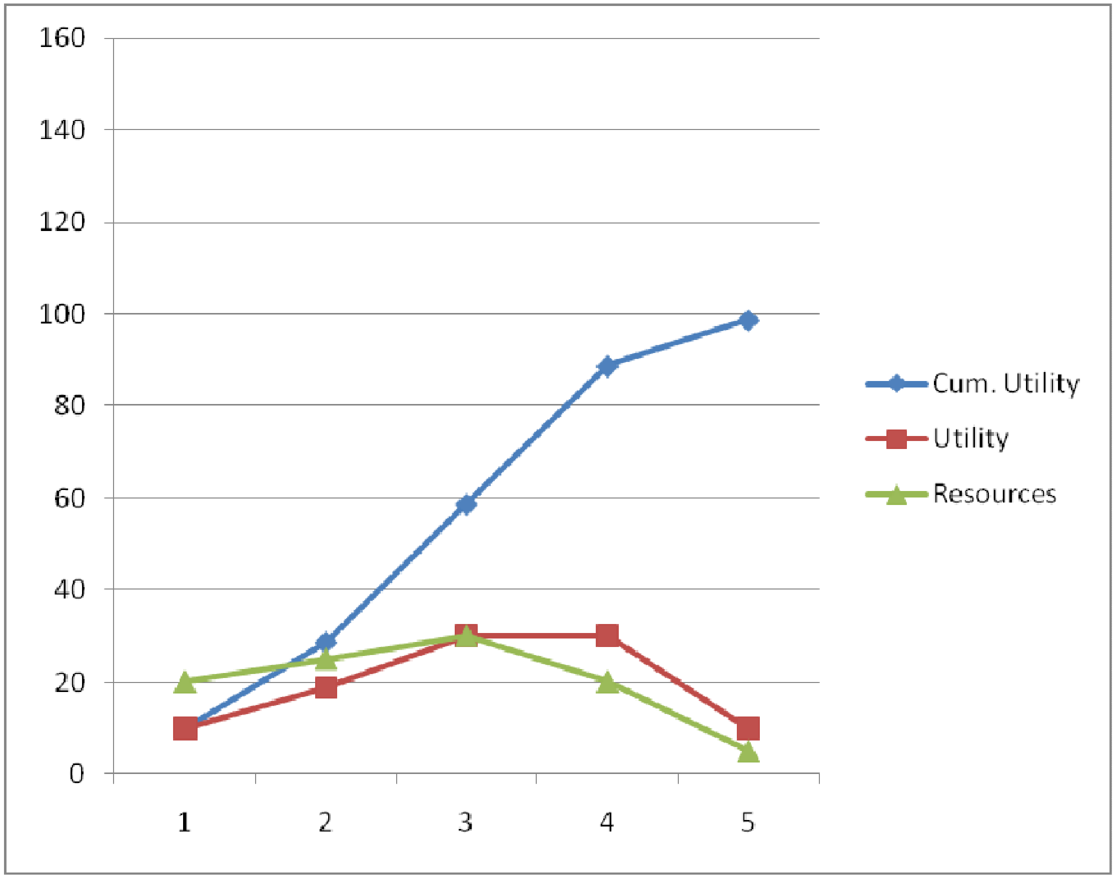
$$u_i' = a_i x_i', \; 0 \le x_i' \le b_i z, \text{ all } i, \quad \sum_i x_i' = Bz$$

$$y_i \in \{0,1\}, \text{ all } i$$

McLoone Index



Relative Min



Gini Coefficient

$$\frac{(1/n^2)\sum_{i,j}\left|u_i - u_j\right|}{2\bar{u}}$$

Rationale:

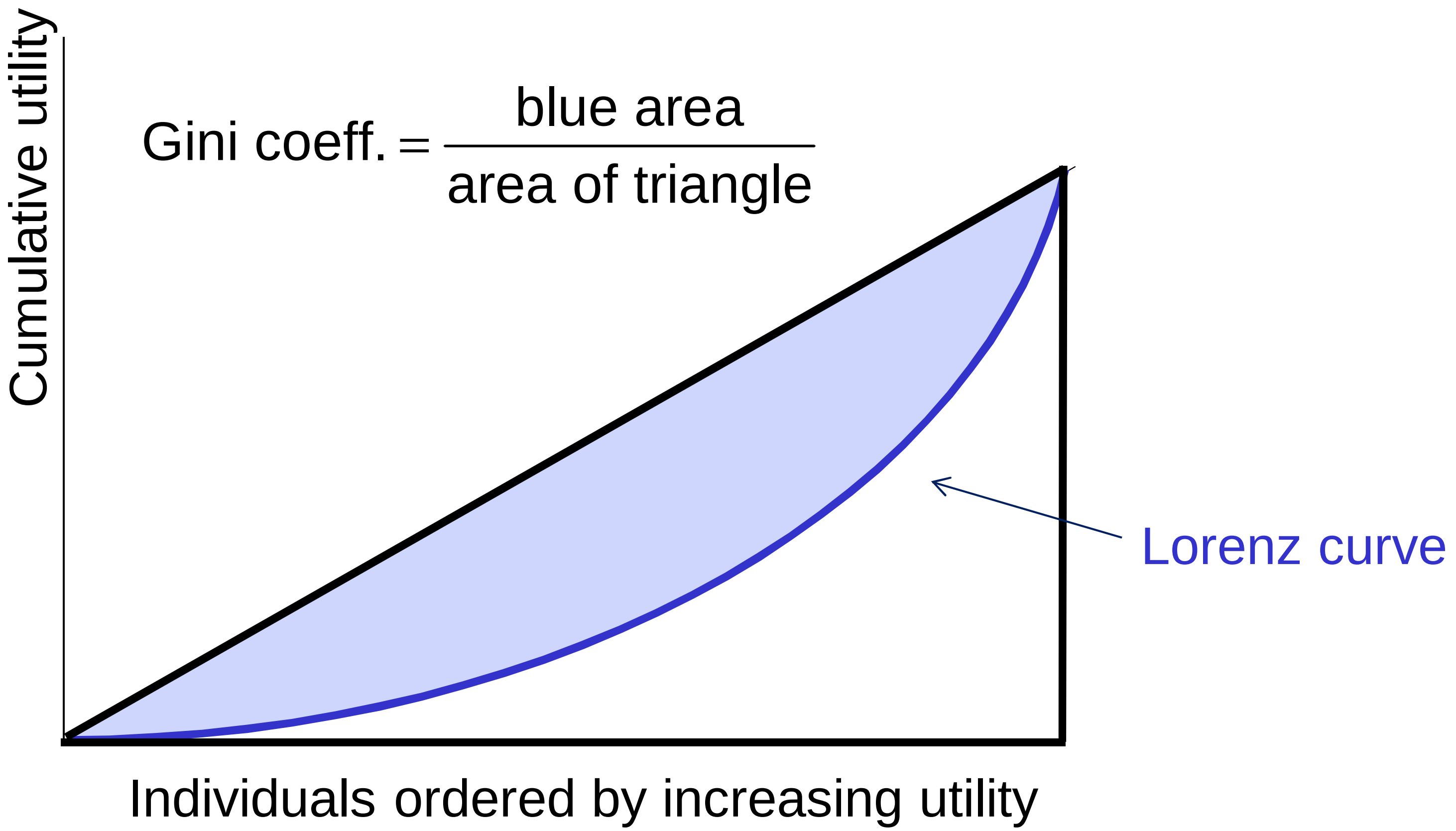
- Relative mean difference between all pairs.
- Takes all differences into account.
- Related to area above cumulative distribution (Lorenz curve).
- Satisfies Pigou-Dalton condition.

Problems:

- Insensitive to shape of Lorenz curve, for a given area.

Gini Coefficient

$$\frac{(1/n^2)\sum_{i,j}\left|u_i - u_j\right|}{2\bar{u}}$$



Gini Coefficient

$$\frac{(1/n^2)\sum_{i,j}\left|u_i - u_j\right|}{2\bar{u}}$$

Fractional LP model:

$$\max \frac{(1/2n^2)\sum_{ij}(u_{ij}^+ + u_{ij}^-)}{\bar{u}}$$

$$u_{ij}^+ \geq u_i - u_j,\ u_{ij}^- \geq u_j - u_i,\ \text{all } i,j$$

$$\bar{u} = (1/n)\sum_i u_i$$

$$u_i = a_i x_i,\ 0 \leq x_i \leq b_i,\ \text{all } i,\quad \sum_i x_i = B$$

LP model:

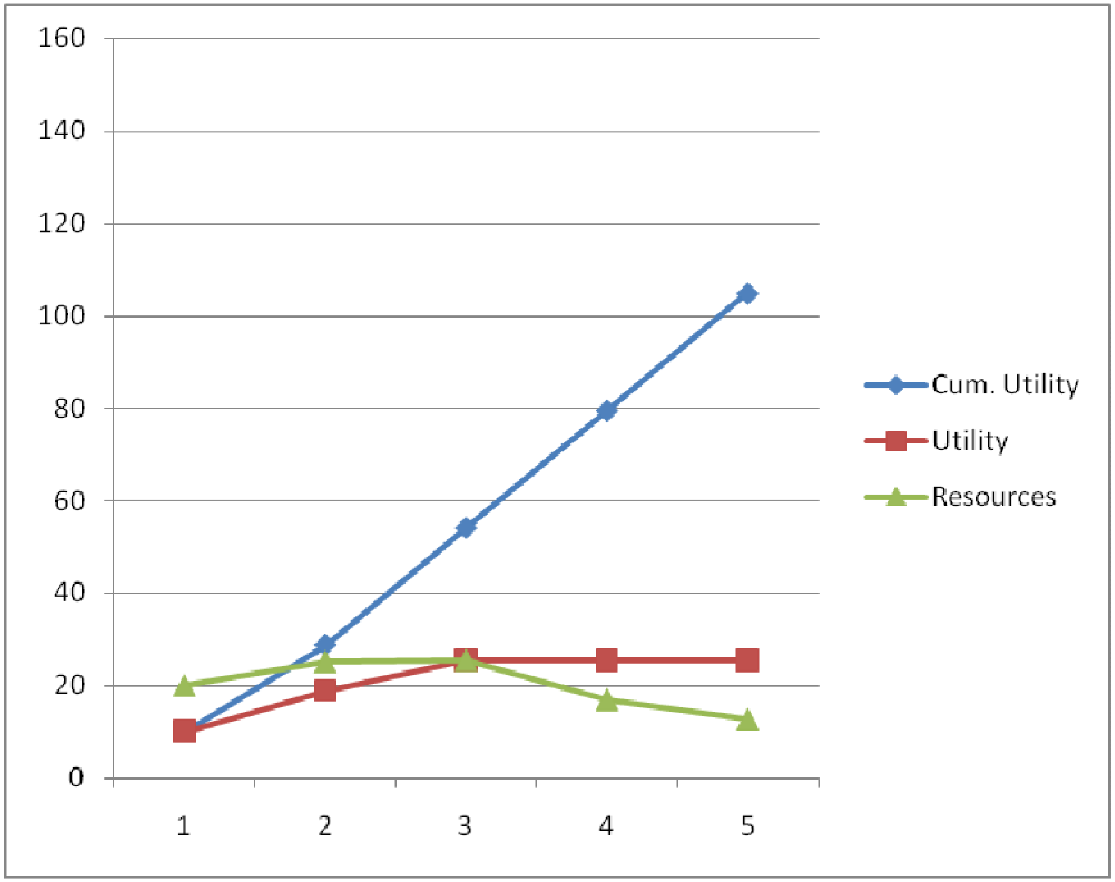
$$\max\ (1/2n^2)\sum_{ij}(u_{ij}^+ + u_{ij}^-)$$

$$u_{ij}^+ \geq u_i' - u_j',\ u_{ij}^- \geq u_j' - u_i',\ \text{all } i,j$$

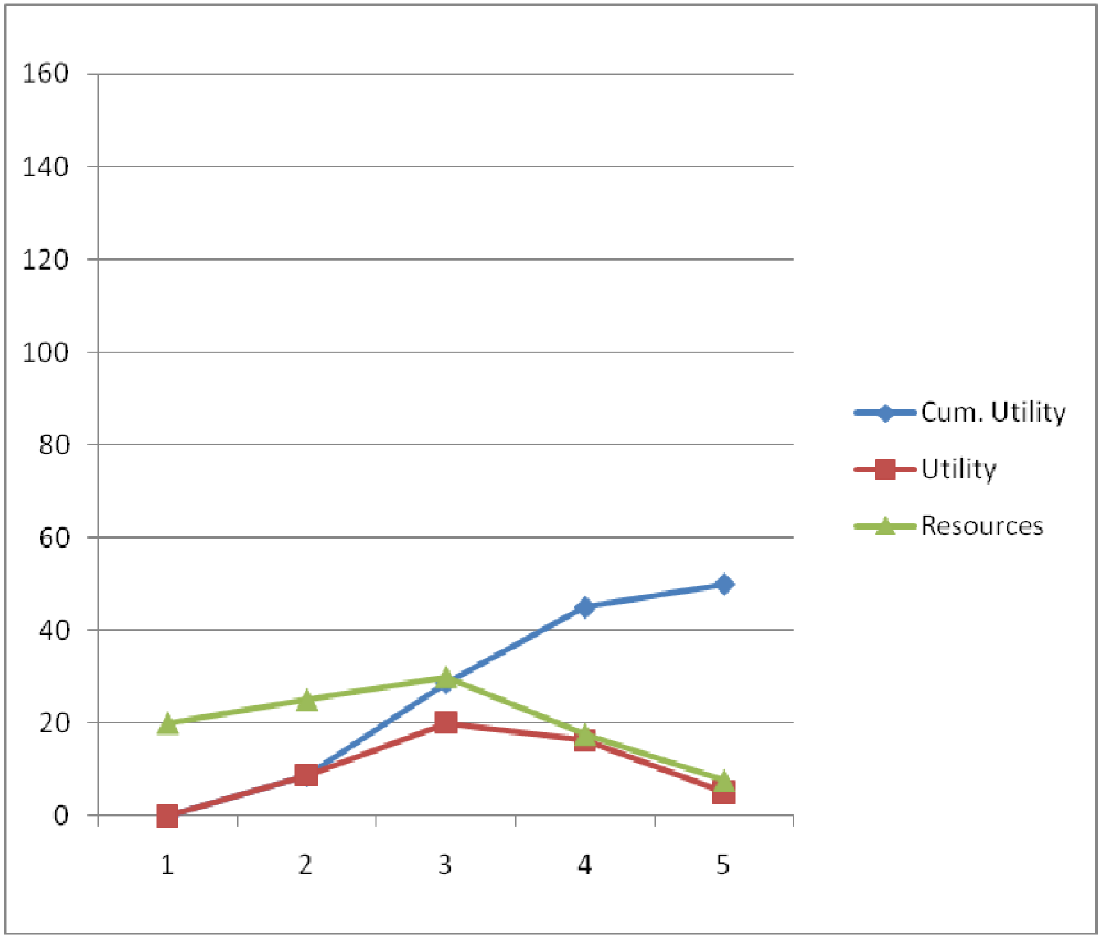
$$(1/n)\sum_i u_i' = 1$$

$$u_i' = a_i x_i',\ 0 \leq x_i' \leq b_i z,\ \text{all } i,\quad \sum_i x_i' = Bz$$

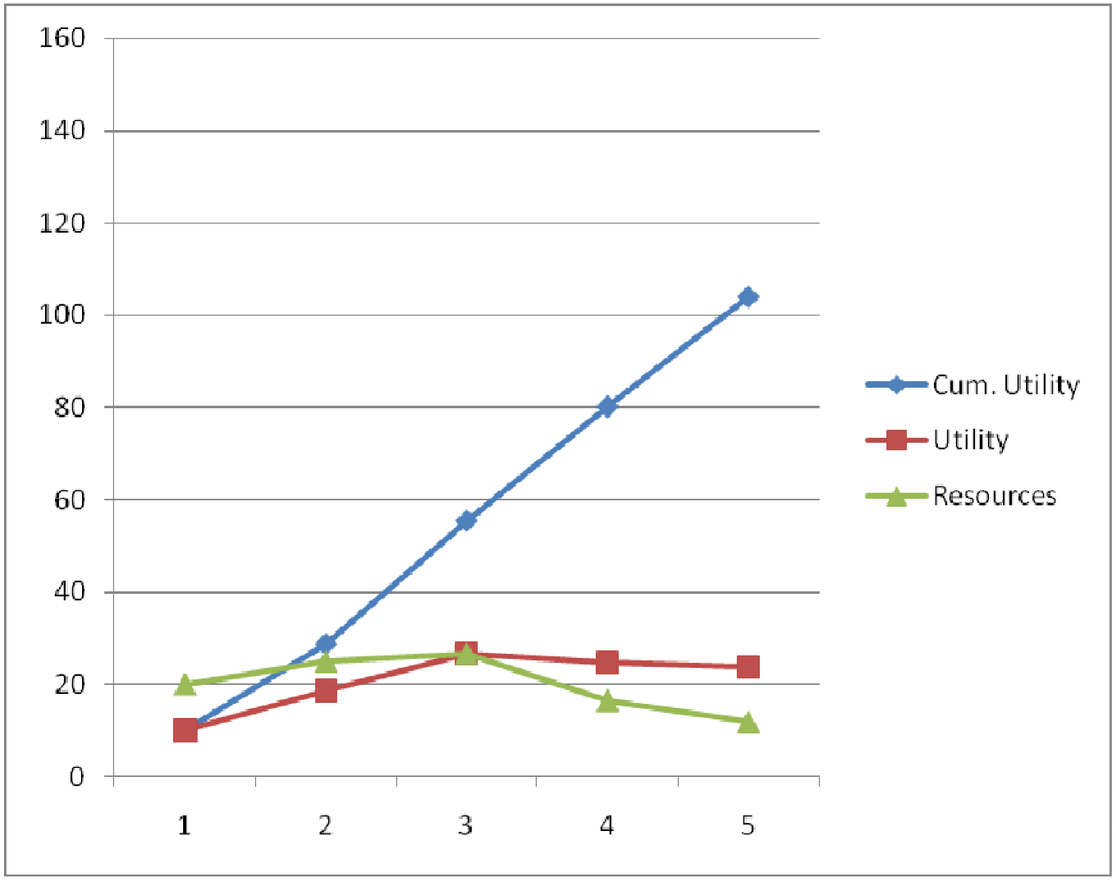
Gini Coefficient



Coefficient of Variation



Variance



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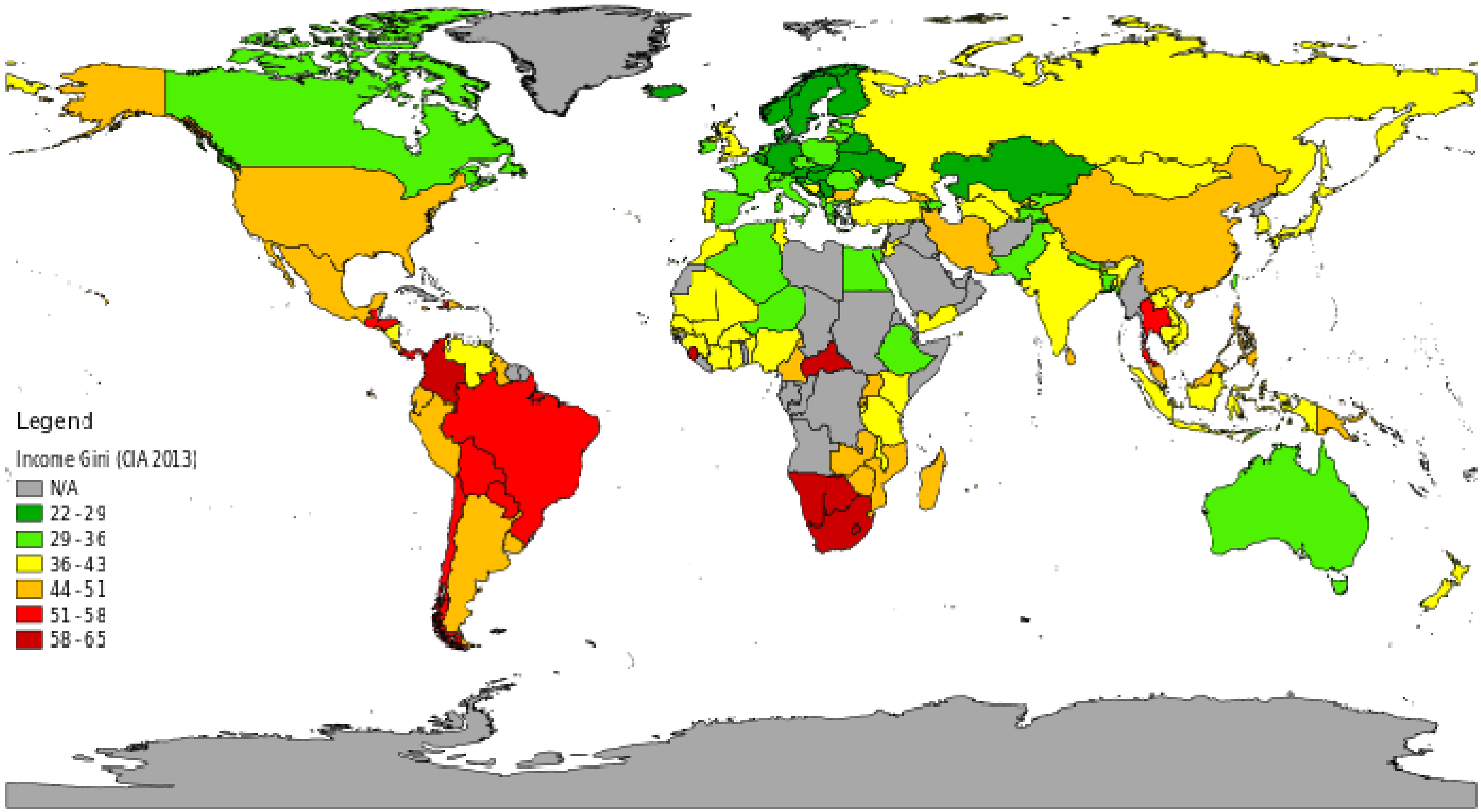
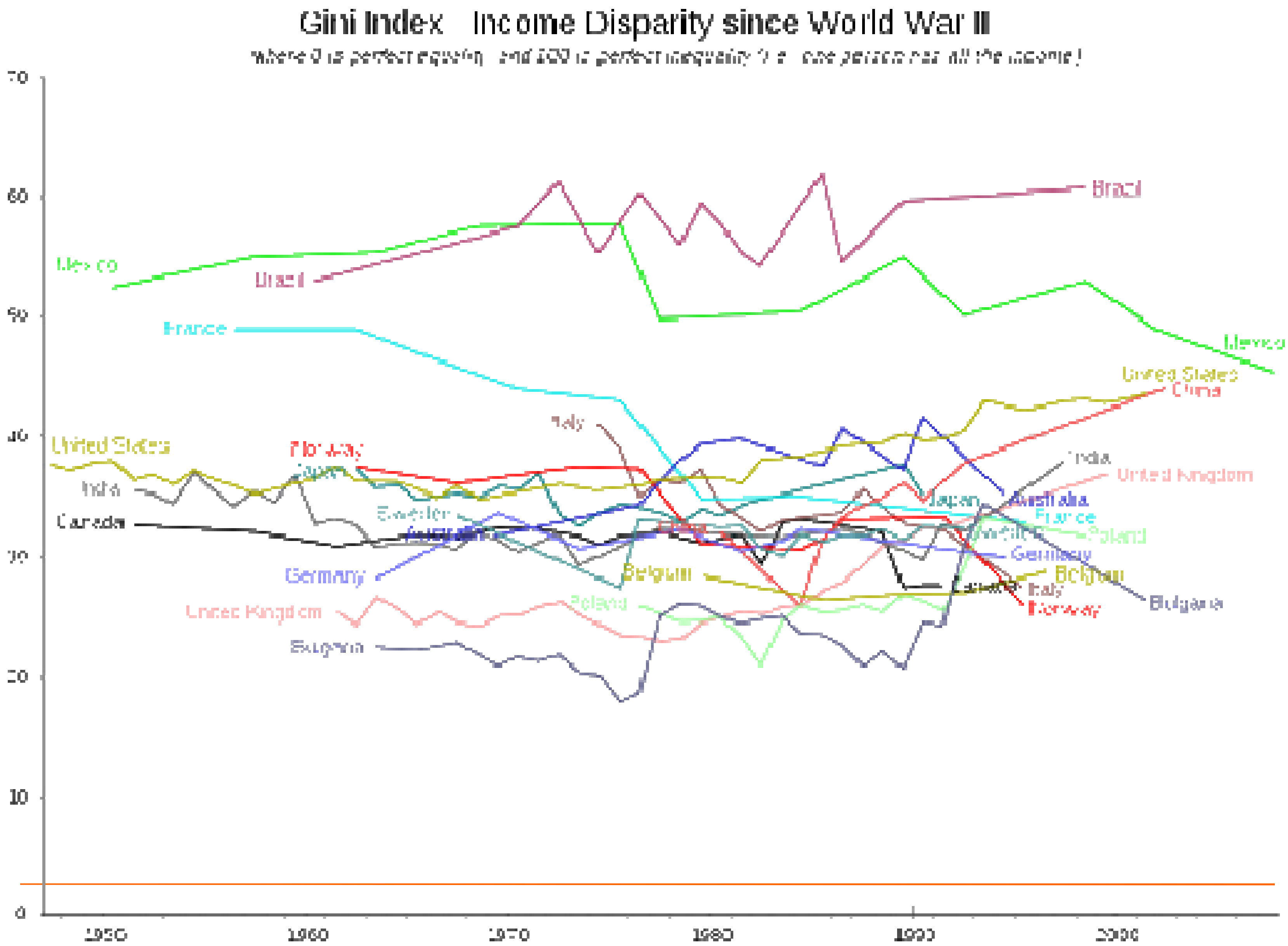
Gini Coefficient by Country (2013)



Historical Gini Coefficient, 1945-2010



Atkinson Index

$$1-\left((1/n)\sum_i \left(\frac{x_i}{\overline{x}}\right)^p \right)^{1/p}$$

Rationale:

- Best seen as measuring inequality of **resources** x_i.
- Assumes allotment y of resources results in utility y^p
- This is average utility per individual.

Atkinson Index

$$1-\left((1/n)\sum_i \left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}$$

Rationale:

- Best seen as measuring inequality of **resources** x_i.
- Assumes allotment y of resources results in utility y^p
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.

Atkinson Index

$$1-\left((1/n)\sum_i \left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}$$

Rationale:

- Best seen as measuring inequality of **resources** x_i.
- Assumes allotment y of resources results in utility y^p
- This is average utility per individual.
- This is equal resource allotment to each individual that results in same total utility.
- This is additional resources per individual necessary to sustain inequality.

Atkinson Index

$$1-\left((1/n)\sum_i\left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}$$

Rationale:

- *p* indicates "importance" of equality.
- Similar to L_p norm
- $p = 1$ means inequality has no importance
- $p = 0$ is Rawlsian (measures utility of worst-off individual).

Problems:

- Measures utility, not equality.
- Doesn't evaluate distribution of utility, only of resources.
- *p* describes utility curve, not importance of equality.

Atkinson Index

$$1-\left((1/n)\sum_i \left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}$$

To minimize index, solve fractional problem

$$\max \sum_i \left(\frac{x_i}{\bar{x}}\right)^p = \frac{\sum_i x_i^p}{\bar{x}^p}$$

$$Ax \geq b, \;\; x \geq 0$$

After change of variable $x_i = x_i'/z$, this becomes

$$\max \sum_i x_i'^p$$

$$(1/n)\sum_i x_i' = 1$$

$$Ax' \geq bz, \;\; x' \geq 0$$

Atkinson Index

$$1-\left((1/n)\sum_i\left(\frac{x_i}{\bar{x}}\right)^p\right)^{1/p}$$

Fractional nonlinear model:

$$\max \frac{\sum_i x_i^p}{\bar{x}^p}$$

$$\bar{x}=(1/n)\sum_i x_i$$

$$\sum_i x_i = B,\ \ x\geq 0$$

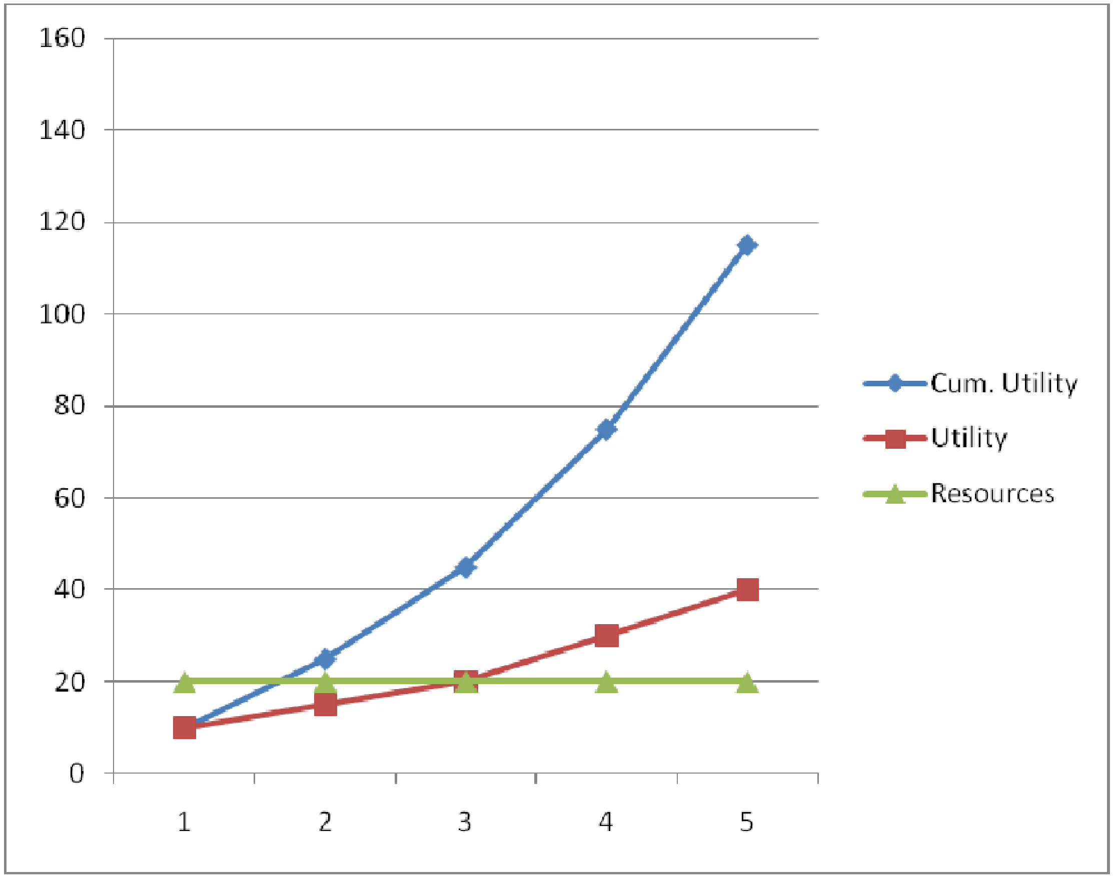
Concave nonlinear model:

$$\max \sum_i x_i'^p$$

$$(1/n)\sum_i x_i' = 1$$

$$\sum_i x_i' = Bz,\ \ x'\geq 0$$

Atkinson index



Hoover Index

$$(1/2)\frac{\sum_i |u_i - \bar{u}|}{\sum_i u_i}$$

Rationale:

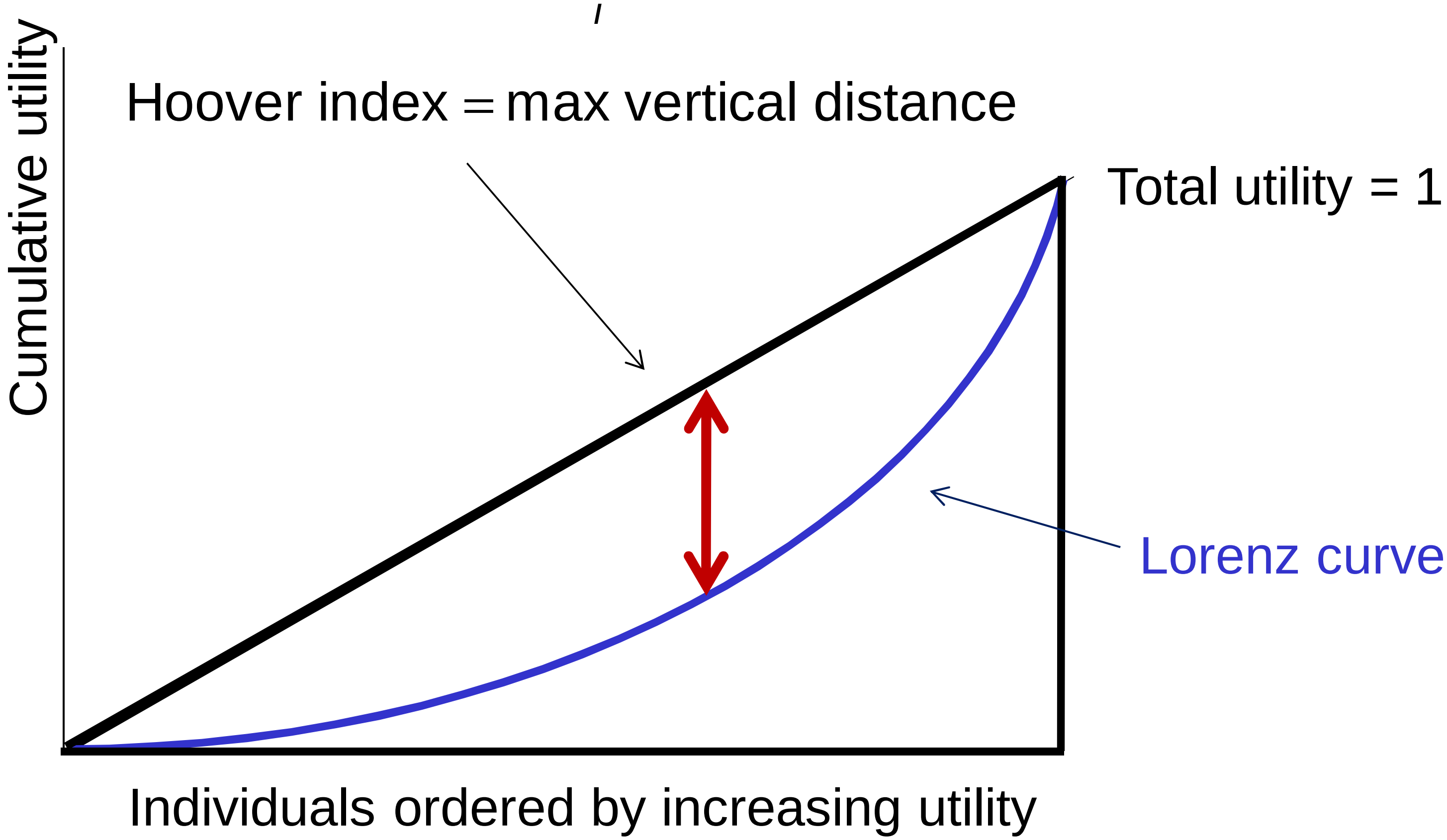
- Fraction of total utility that must be redistributed to achieve total equality.
- Proportional to maximum vertical distance between Lorenz curve and 45^o line.
- Originated in regional studies, population distribution, etc. (1930s).
- Easy to calculate.

Problems:

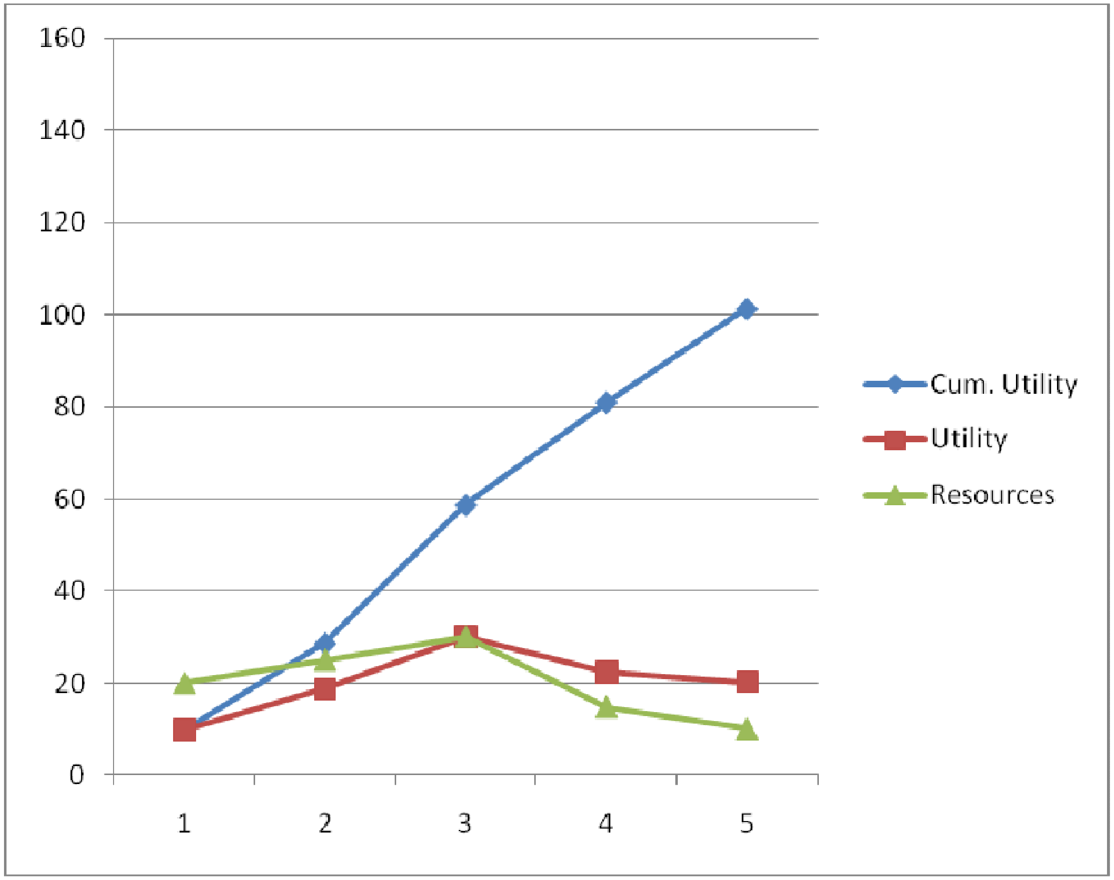
- Less informative than Gini coefficient?

Hoover Index

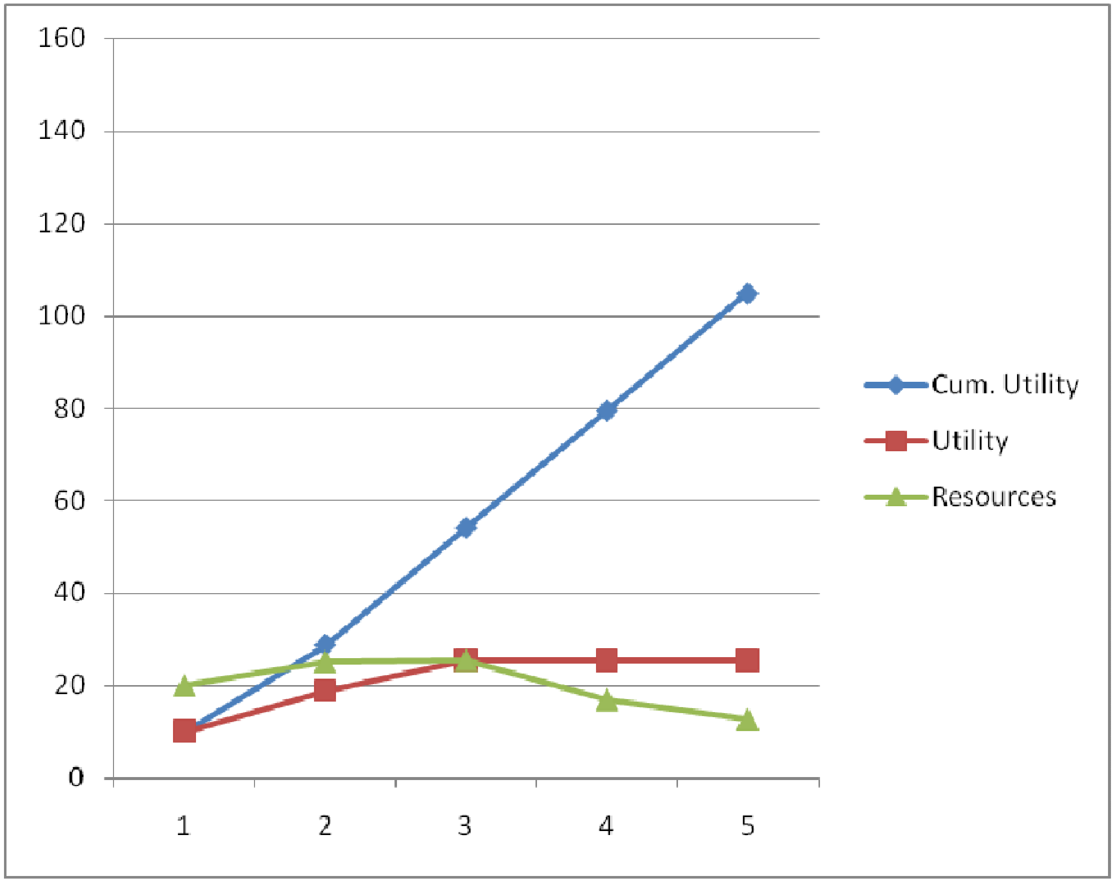
$$(1/2)\frac{\sum_i \left| u_i - \bar{u} \right|}{\sum_i u_i}$$



Hoover Index



Gini Coefficient



Theil Index

$$(1/n)\sum_i \left(\frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)$$

Rationale:

- One of a family of entropy measures of inequality.
- Index is zero for complete equality (maximum entropy)
- Measures nonrandomness of distribution.
- Described as stochastic version of Hoover index.

Problems:

- Motivation unclear.
- A. Sen doesn’t like it.

Theil Index

$$(1/n)\sum_i \left(\frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)$$

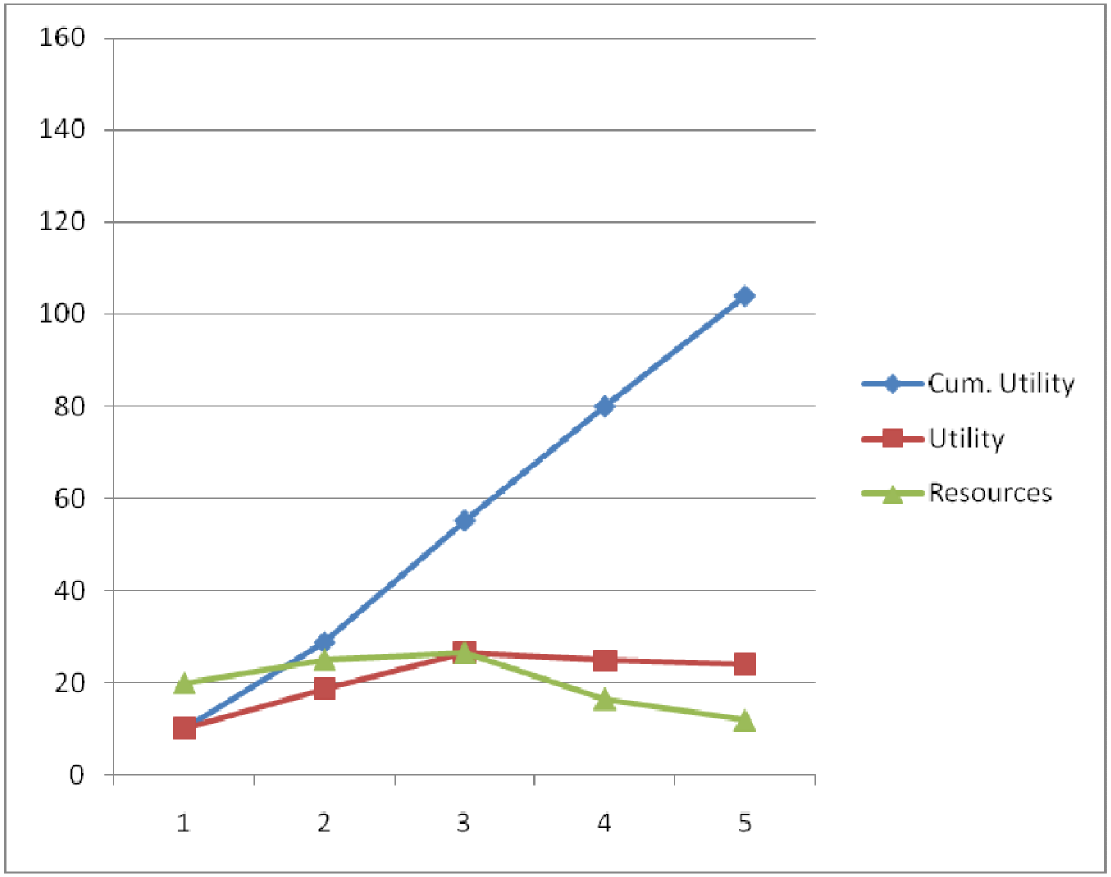
Nasty nonconvex model:

$$\min\, (1/n)\sum_i \left(\frac{u_i}{\bar{u}} \ln \frac{u_i}{\bar{u}} \right)$$

$$\bar{u} = (1/n)\sum_i u_i$$

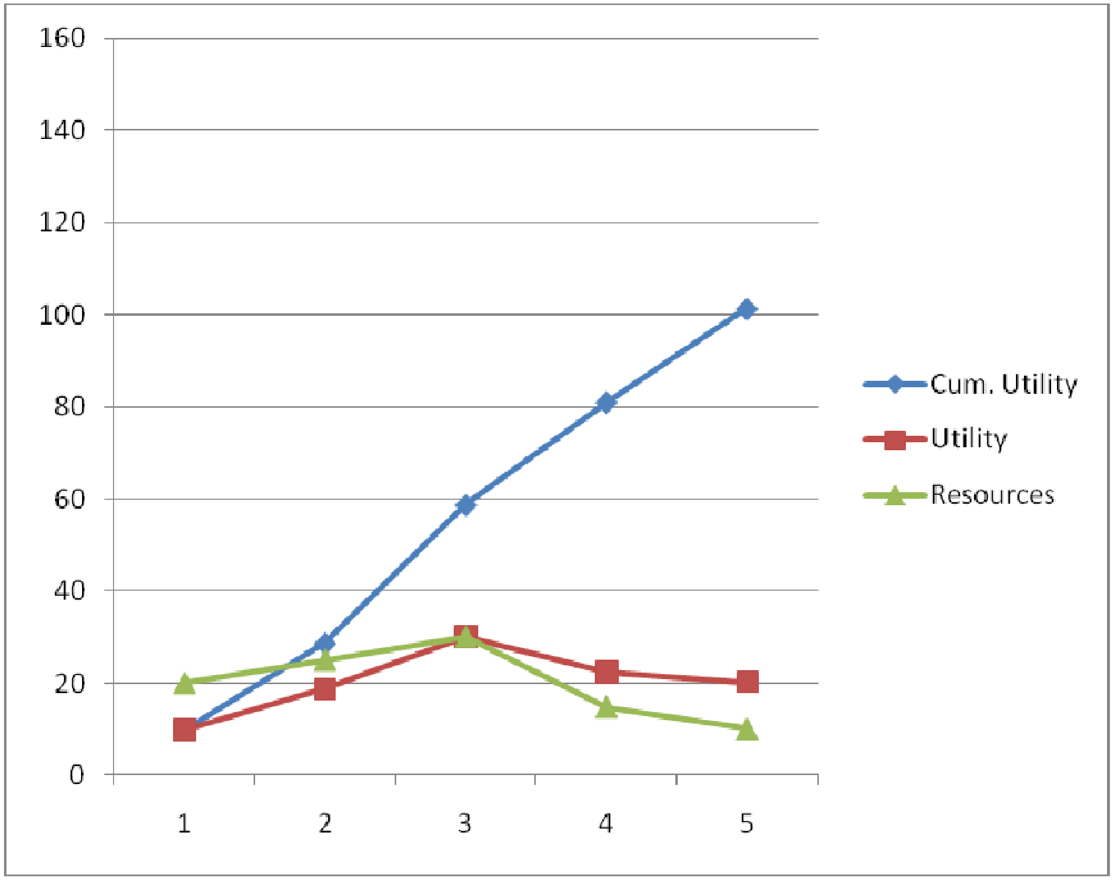
$$u_i = a_i x_i,\ 0 \le x_i \le b_i,\ \text{all } i,\quad \sum_i x_i = B$$

Theil Index

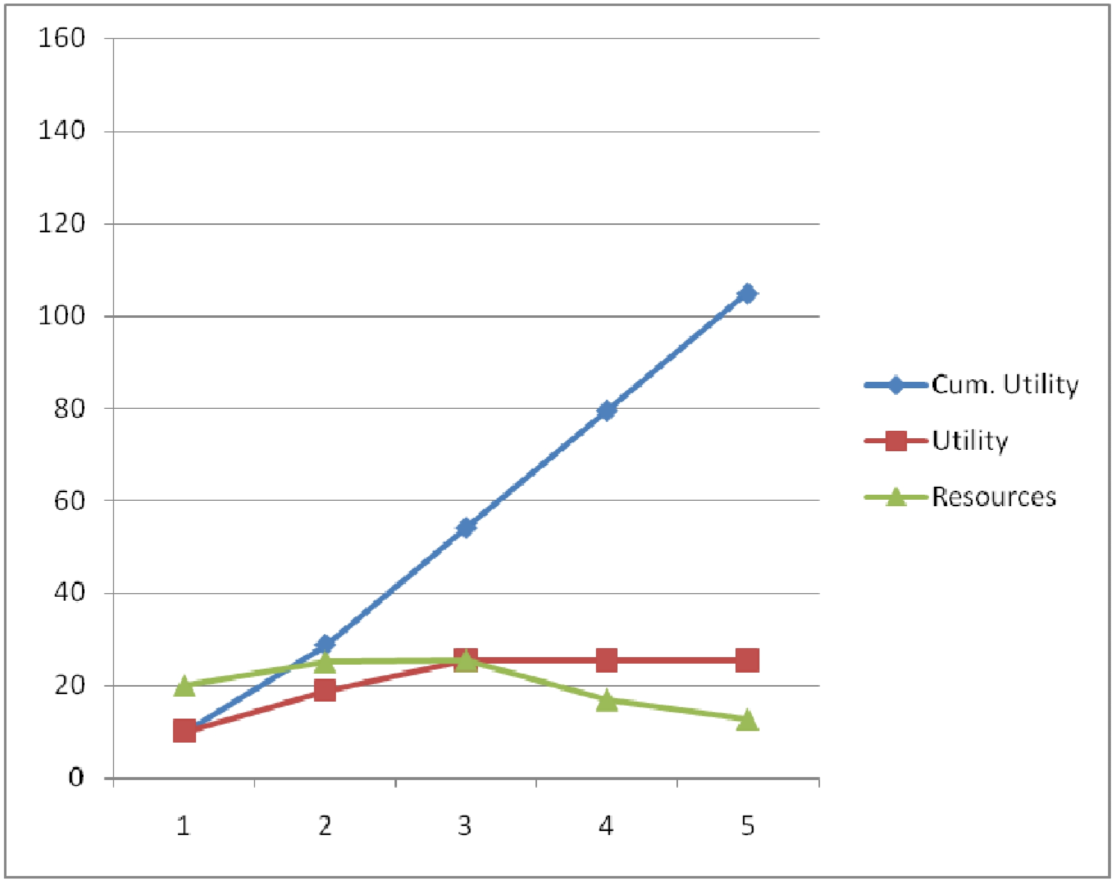


160
140
120
100
80
60
40
20
0
1
2
3
4
5
Cum. Utility
Utility
Resources

Hoover Index



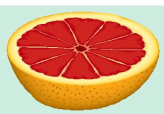
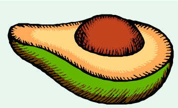
Gini Coefficient



An Allocation Problem

- From Yaari and Bar-Hillel, 1983.
- **12 grapefruit** and **12 avocados** are to be divided between **Jones** and **Smith**.
- How to divide justly?

Utility provided by one fruit of each kind

	Jones	Smith
	100	50
	0	50

An Allocation Problem

The optimization problem:

Social welfare function

$$\max\ f(u_1, u_2)$$

$$u_1 = 100x_{11},\ \ u_2 = 50x_{12} + 50x_{22}$$

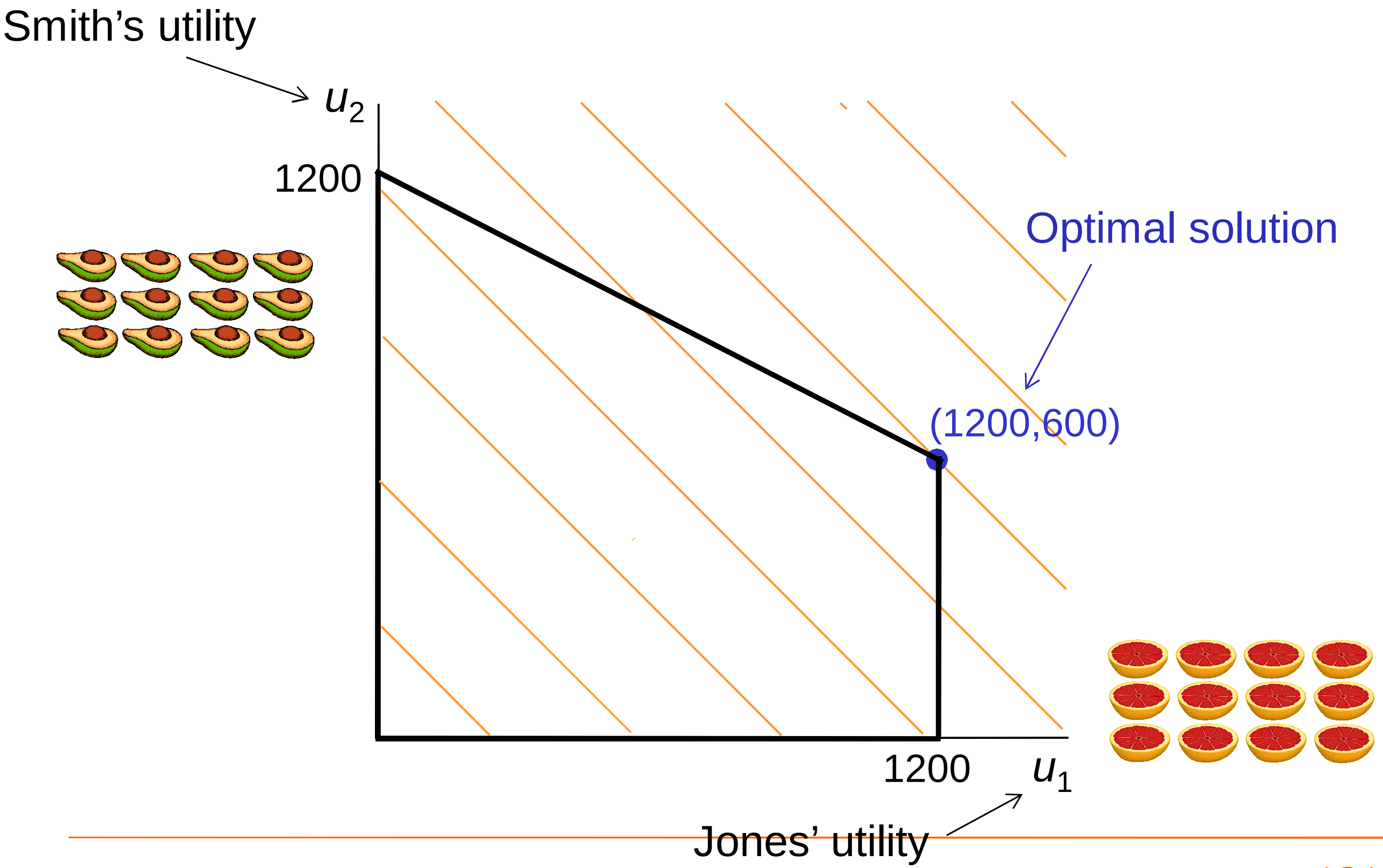
$$x_{i1} + x_{i2} = 12,\ \ i = 1,2$$

$$x_{ij} \geq 0,\ \ \text{all } i, j$$

where u_i = utility for person i (Jones, Smith)
x_{ij} = allocation of fruit i (grapefruit, avocados) to person j

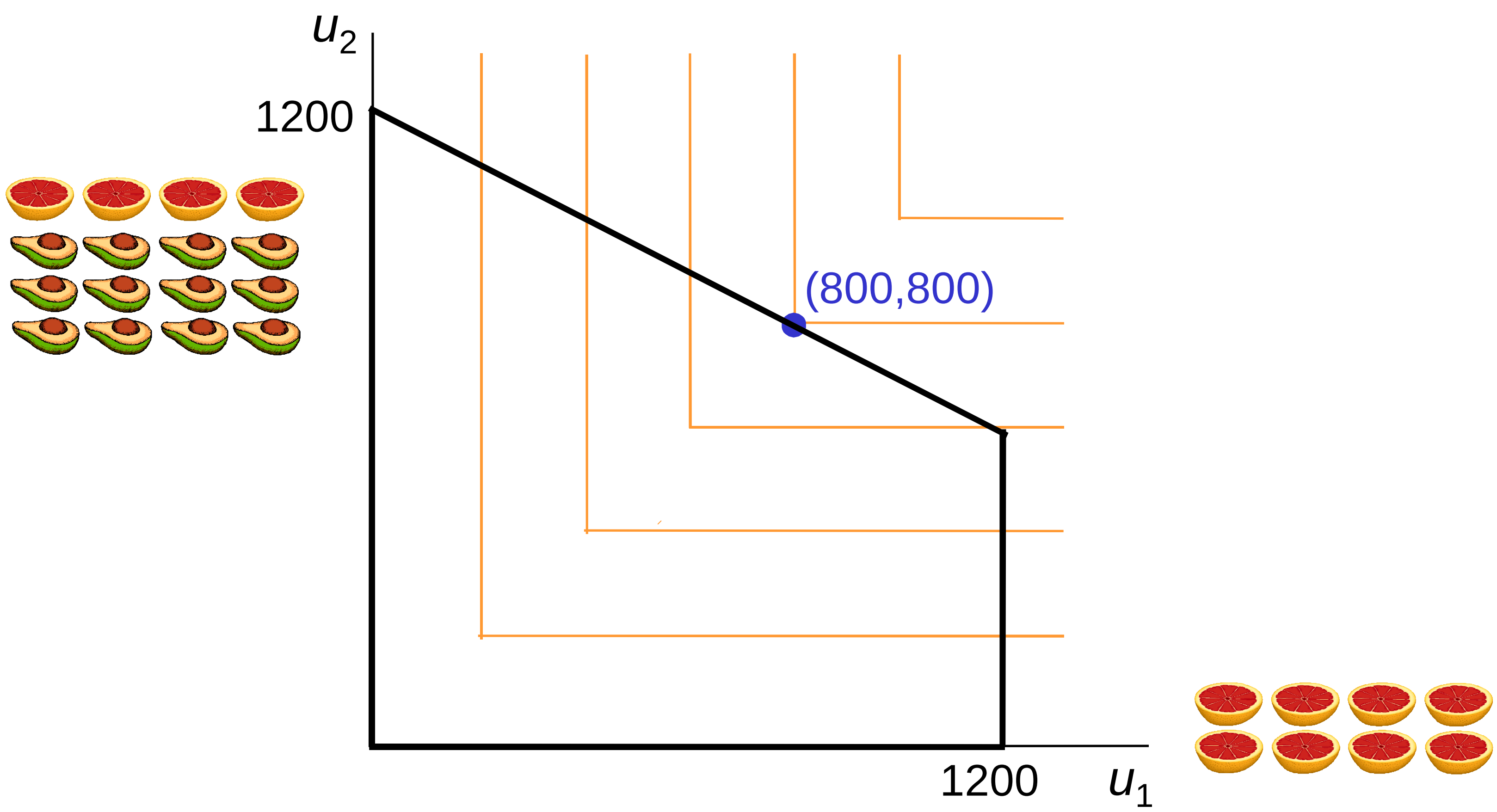
Utilitarian Solution

$$f(u_1, u_2) = u_1 + u_2$$



Rawlsian (maximin) solution

$$f(u_1,u_2) = \min\{u_1,u_2\}$$



Bargaining Solutions

- Nash Bargaining Solution
  - Example
  - Axiomatic justification
  - Bargaining justification
- Raiffa-Kalai-Smorodinsky Solution
  - Example
  - Axiomatic justification
  - Bargaining jusification

Bargaining Solutions

- A **bargaining solution** is an equilibrium allocation in the sense that none of the parties wish to bargain further.

  - Because all parties are "satisfied" in some sense, the outcome may be viewed as "fair."

  - Bargaining models have a **default** outcome, which is the result of a failure to reach agreement.

  - The default outcome can be seen as a **starting point**.

Bargaining Solutions

- Several proposals for the default outcome (starting point):
  - **Zero** for everyone. Useful when only the resources being allocated are relevant to fairness of allocation.
  - **Equal split**. Resources (not necessarily utilities) are divided equally. May be regarded as a “fair” **starting point**.
  - **Strongly pareto set**. Each party receives resources that can benefit no one else. Parties can always agree on this.

Nash Bargaining Solution

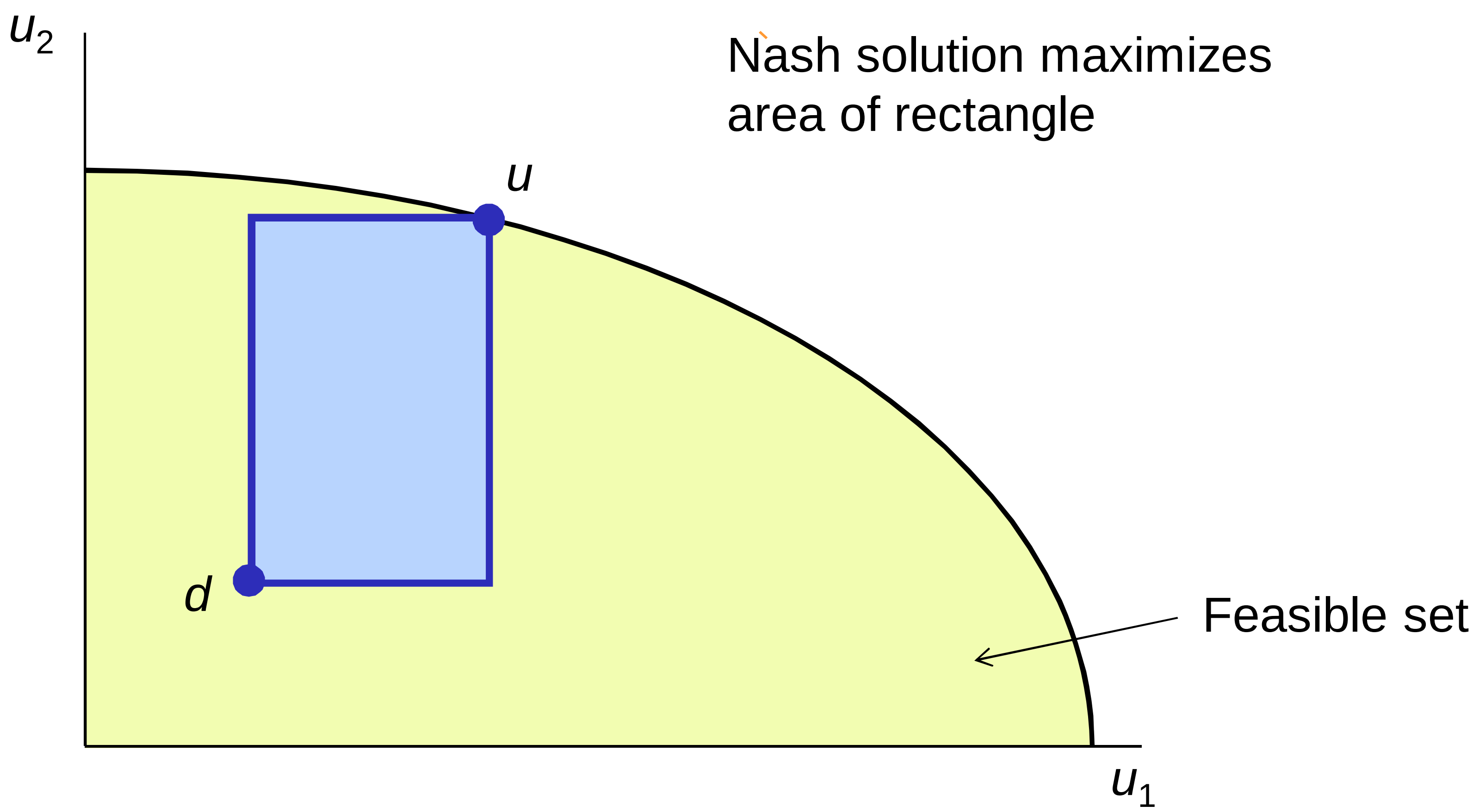
- The **Nash bargaining solution** maximizes the social welfare function

$$f(u) = \prod_i (u_i - d_i)$$

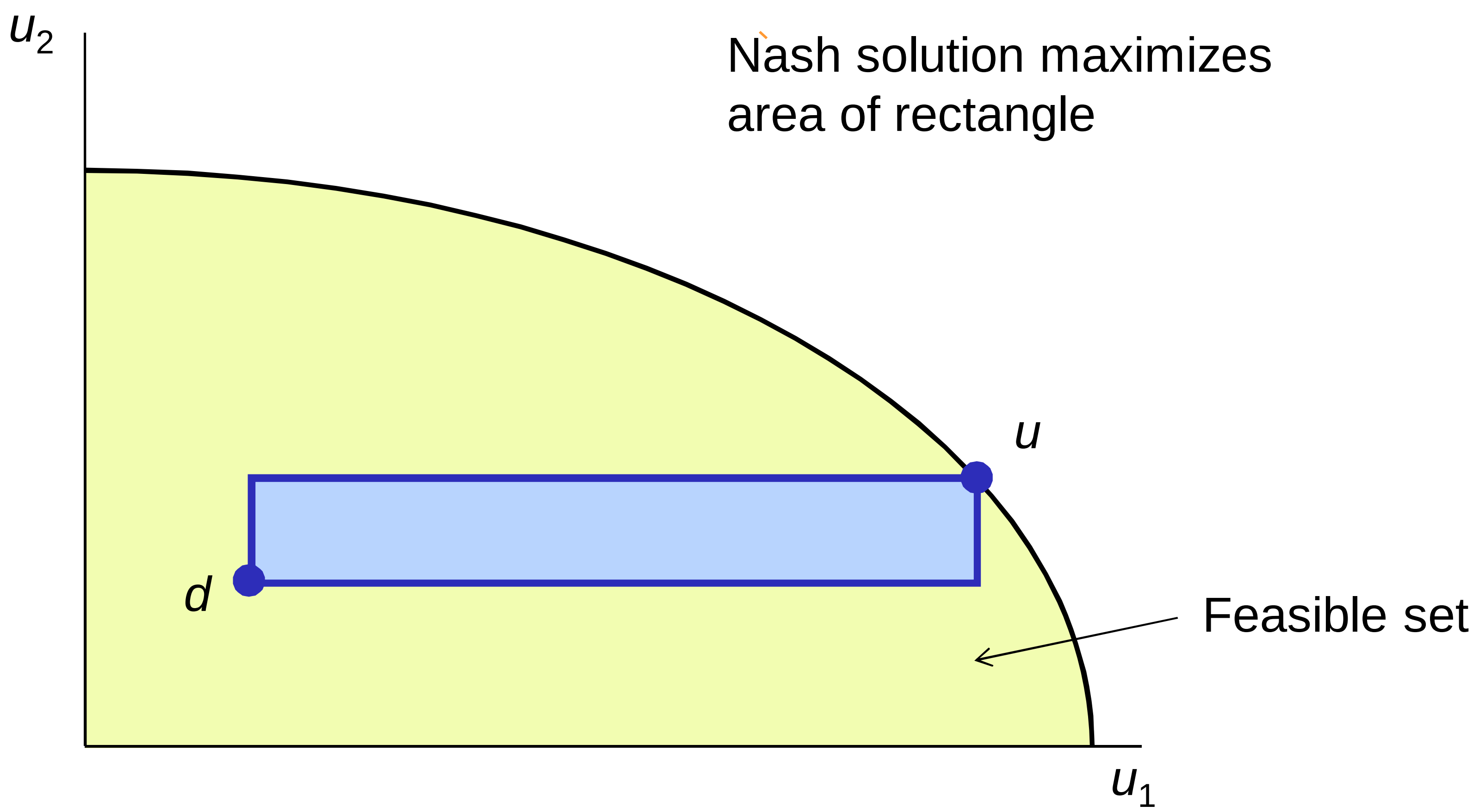
where *d* is the default outcome.

  - **Not** the same as **Nash equilibrium**.
  - It maximizes the **product of the gains** achieved by the bargainers, relative to the fallback position.
  - Assume feasible set is **convex**, so that Nash solution is unique (due to strict concavity of *f*).

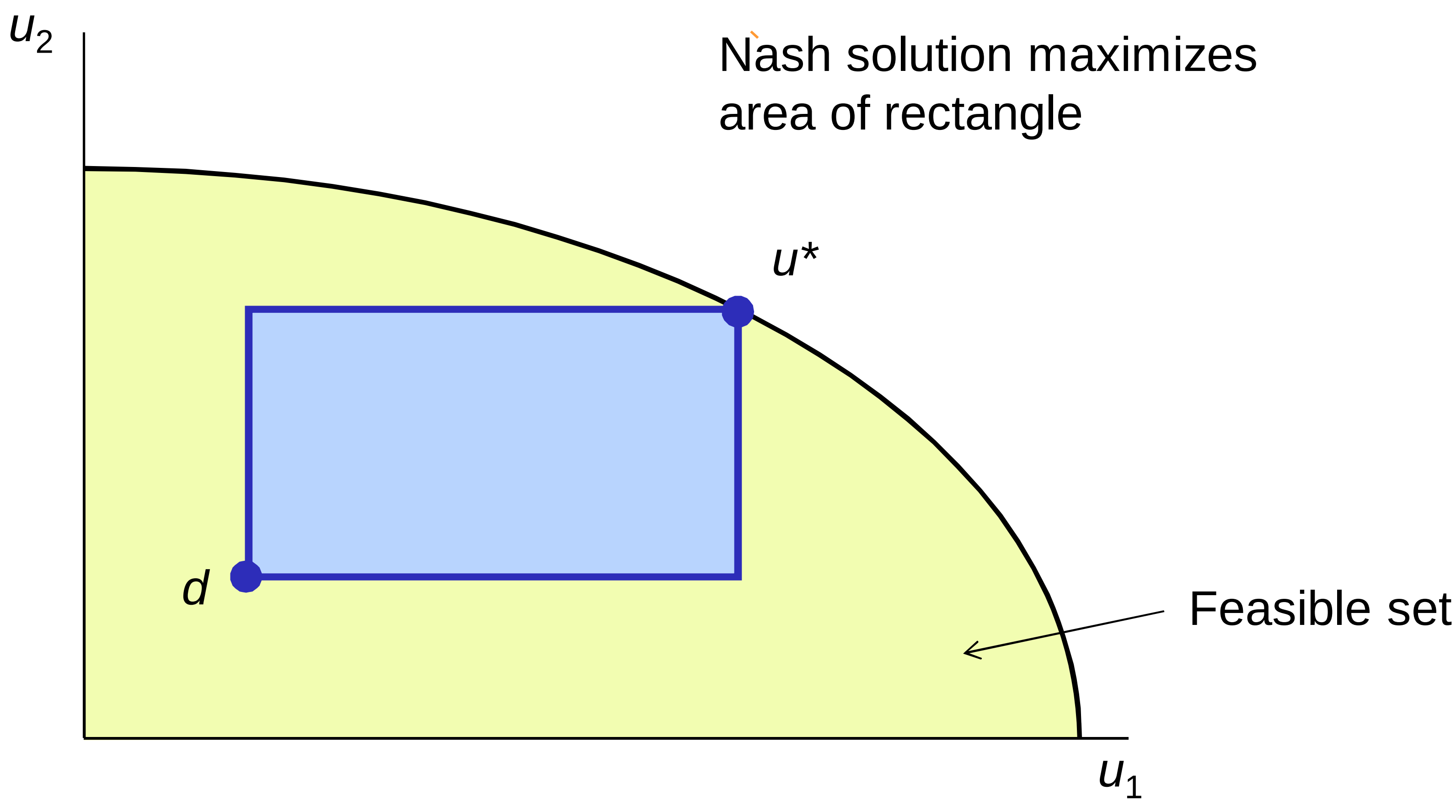
Nash Bargaining Solution



Nash Bargaining Solution



Nash Bargaining Solution



Nash Bargaining Solution

- Major **application** to telecommunications.
  - Where it is known as **proportional fairness**
  - u is proportionally fair if for all feasible allocations u'

$$\sum_i \frac{u_i' - u_i}{u_i} \leq 0$$

  - Here, u_i is the utility of the packet flow rate assigned user i.
  - **Maximin** criterion also used.

Nash Bargaining Solution

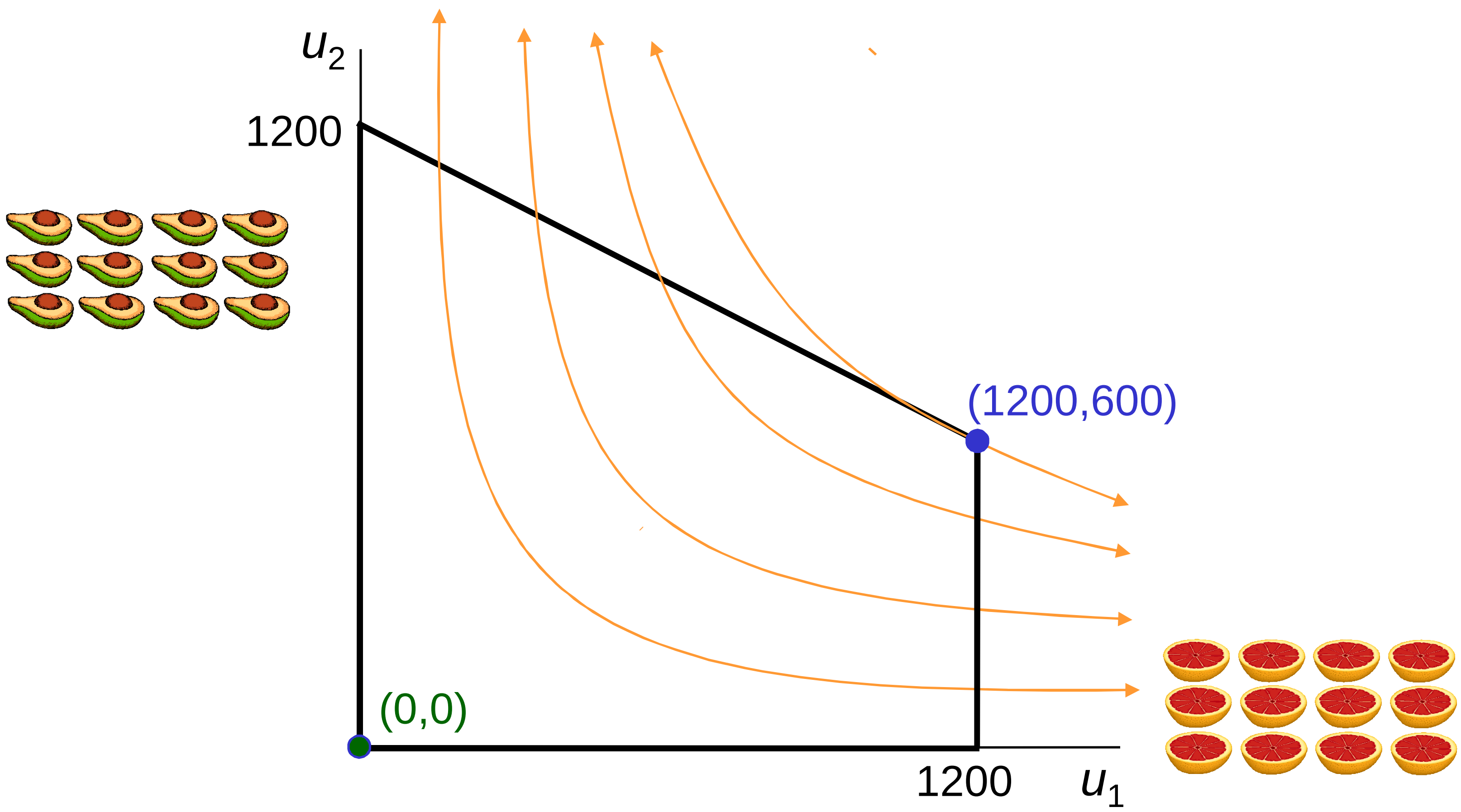
- The **optimization problem** has a concave objective function if we maximize log $f(u)$.

$$\max \log \prod_i (u_i - d_i) = \sum_i \log(u_i - d_i)$$
$$u \in S$$

- Problem is relatively easy if feasible set S is convex.

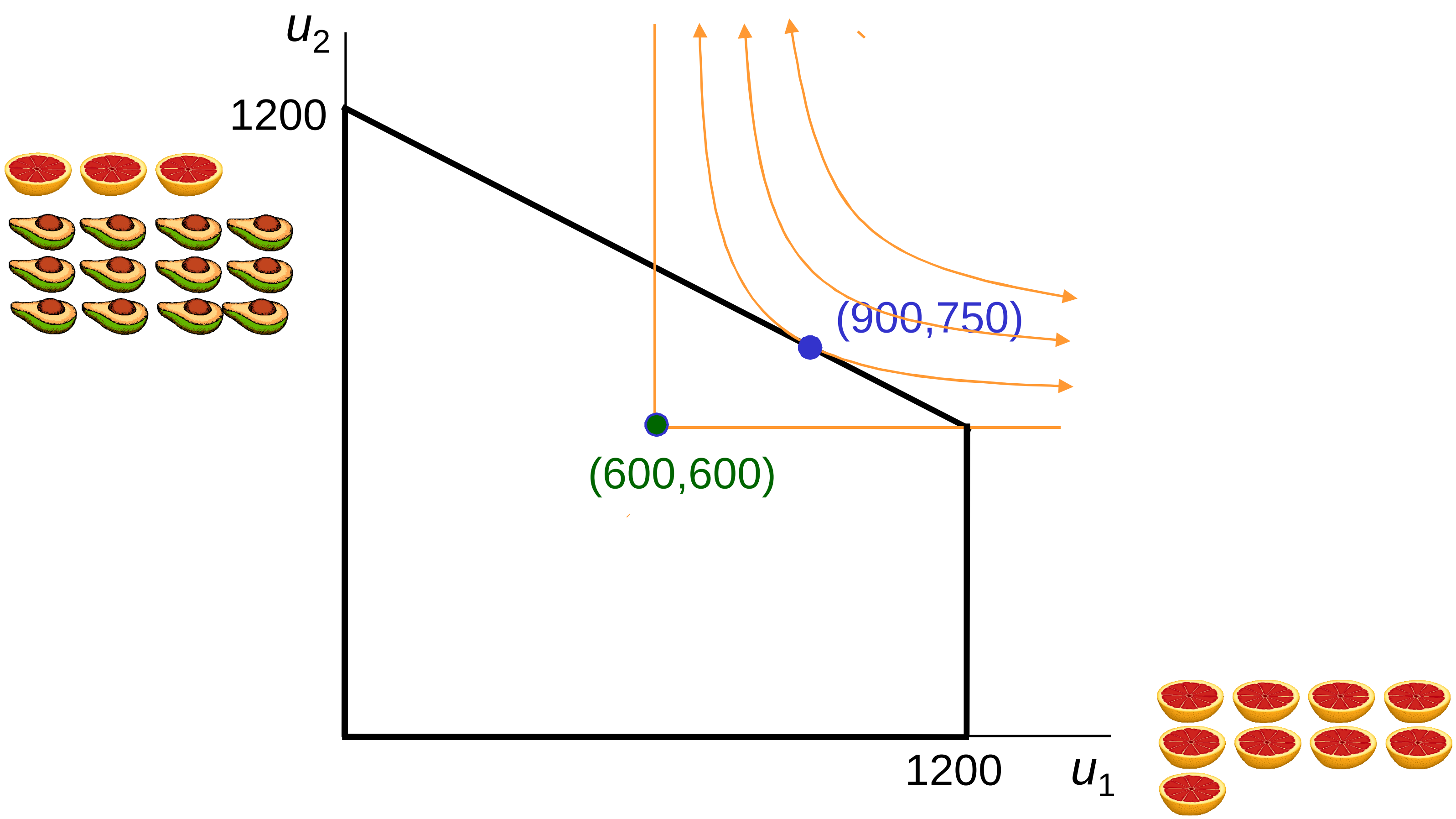
Nash Bargaining Solution

From Zero



Nash Bargaining Solution

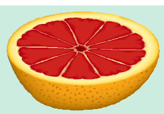
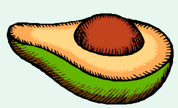
From Equality



Nash Bargaining Solution

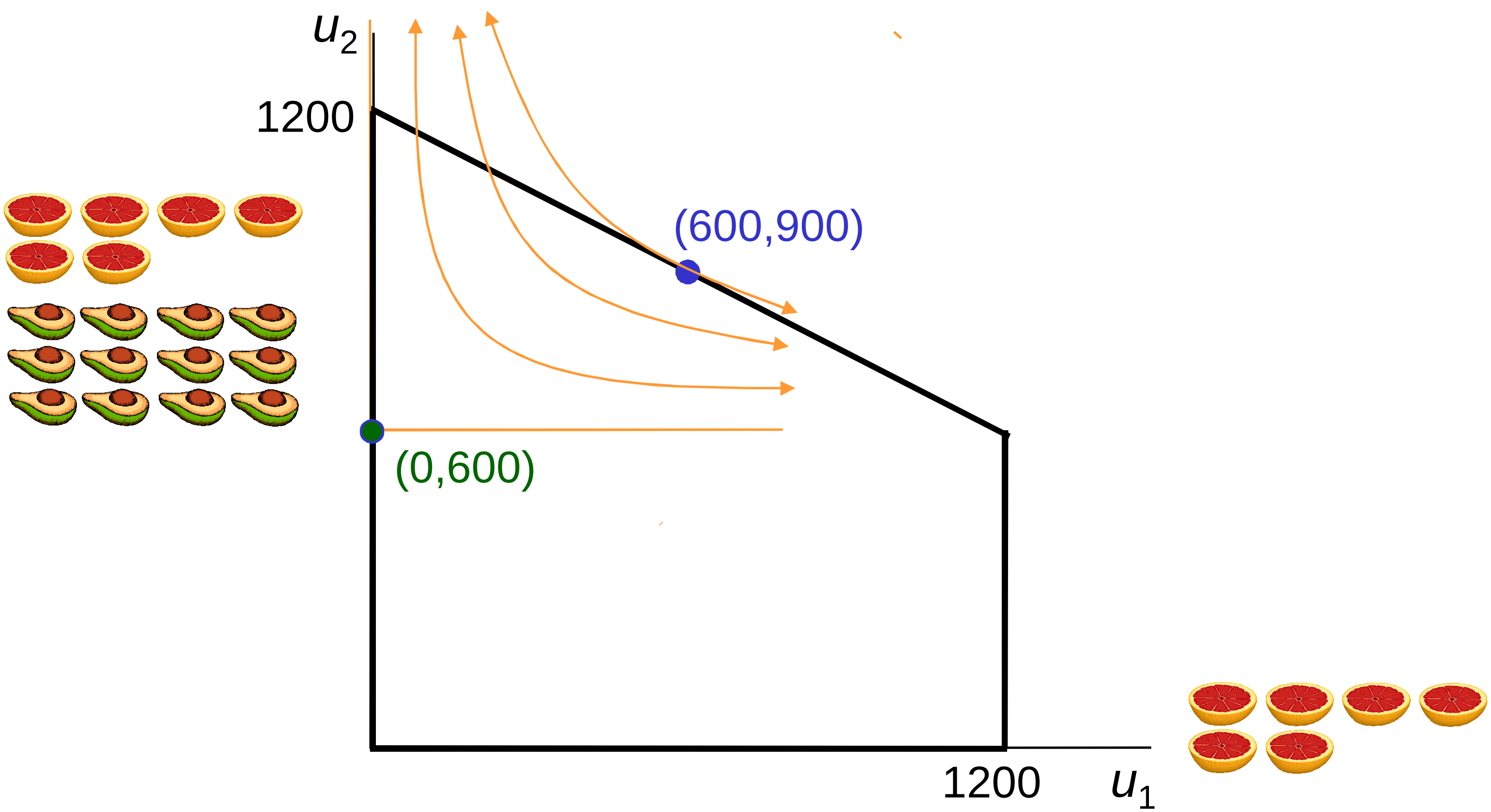
- **Strongly pareto set** gives Smith all 12 avocados.
  - Nothing for Jones.
  - Results in utility $(u_1, u_2) = (0{,}600)$

Utility provided by one fruit of each kind

	Jones	Smith
	100	50
	0	50

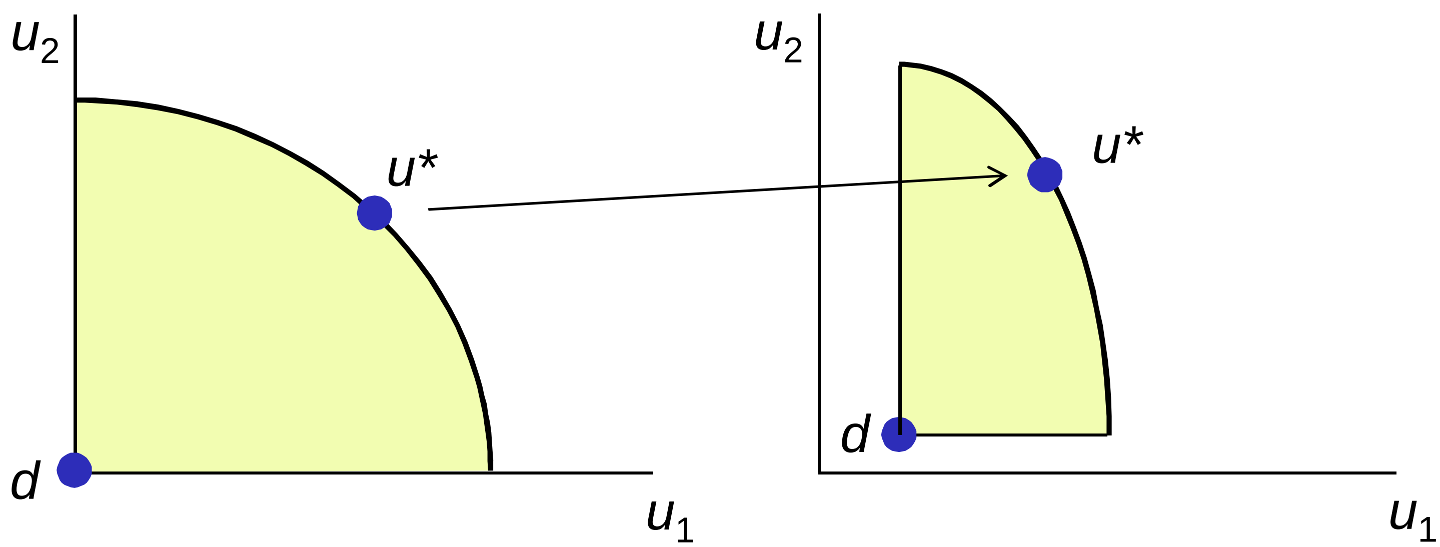
Nash Bargaining Solution

From Strongly Pareto Set



Axiomatic Justification

- **Axiom 1**. Invariance under translation and rescaling.
  - If we map $u_i \rightarrow a_i u_i + b_i$, $d_i \rightarrow a_i d_i + b_i$, then bargaining solution $u_i^* \rightarrow a_i u_i^* + b_i$.



This is **cardinal noncomparability**.

Axiomatic Justification

- **Axiom 1**. Invariance under translation and rescaling.
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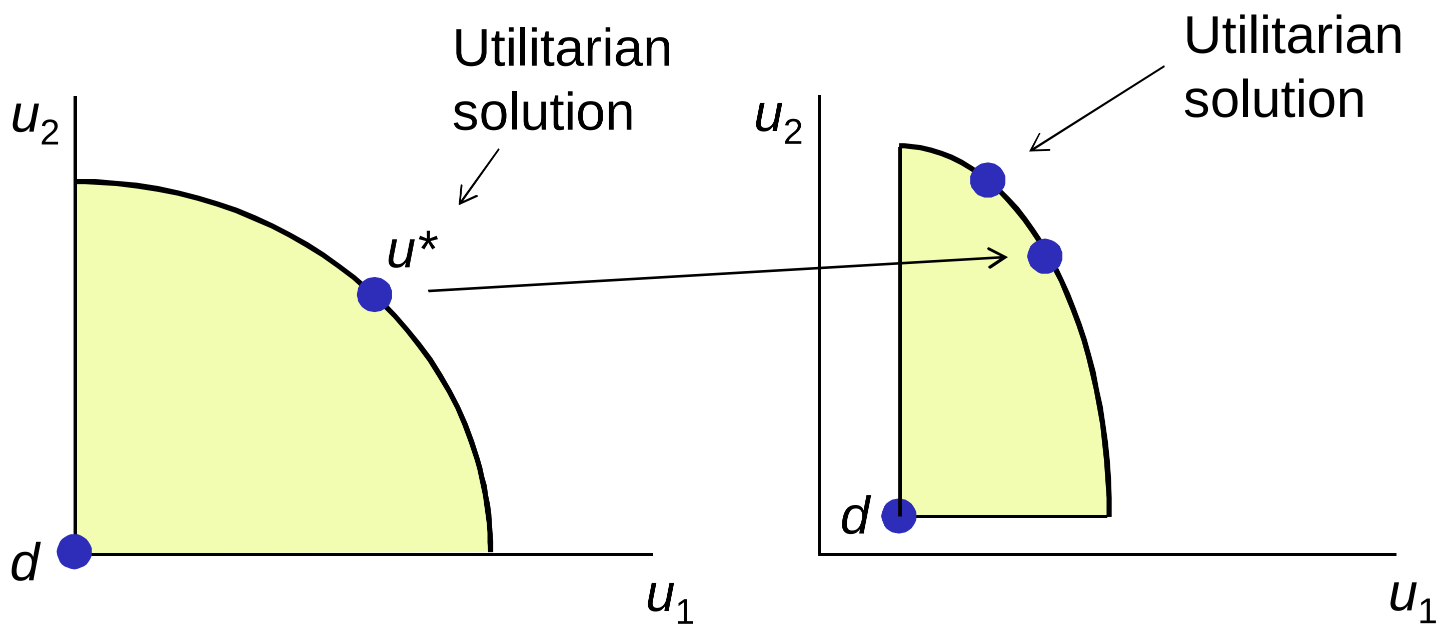
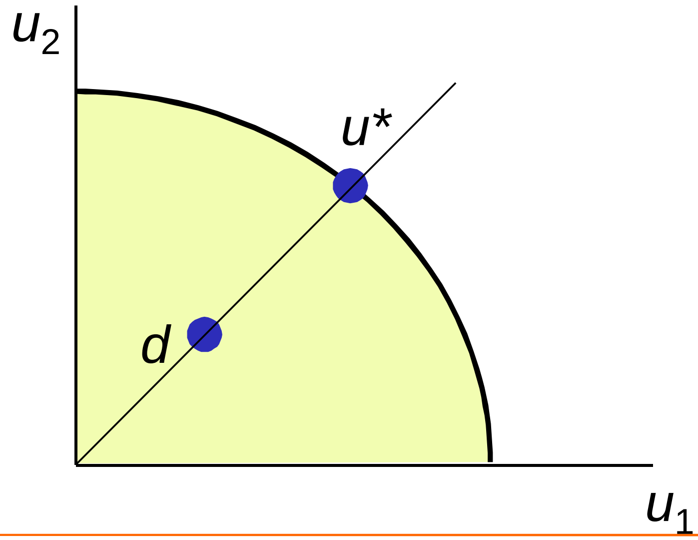


  - **Strong assumption** – failed, e.g., by utilitarian welfare function

Axiomatic Justification

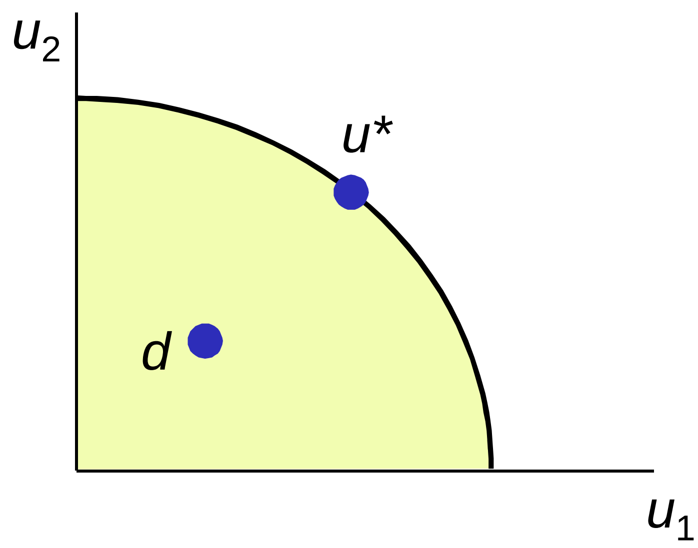
- **Axiom 2**. Pareto optimality.
  - Bargaining solution is pareto optimal.

- **Axiom 3.** Symmetry.
  - If all d_is are equal and feasible set is symmetric, then all u_i*s are equal in bargaining solution.



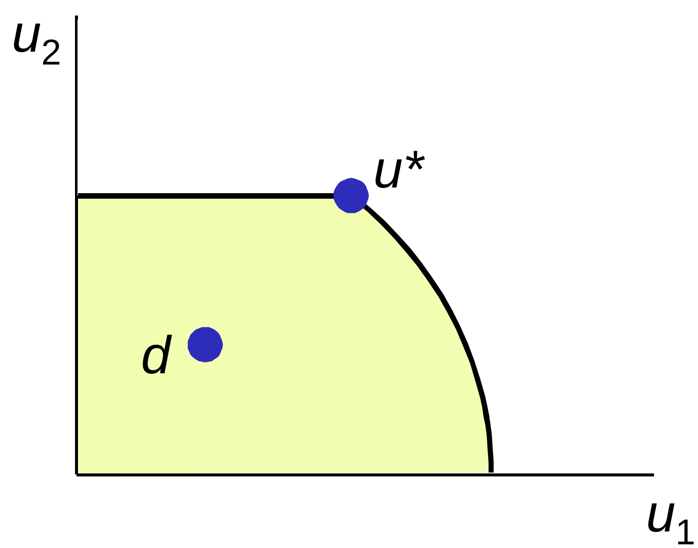
Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow's axiom.
  - If *u** is a solution with respect to *d*



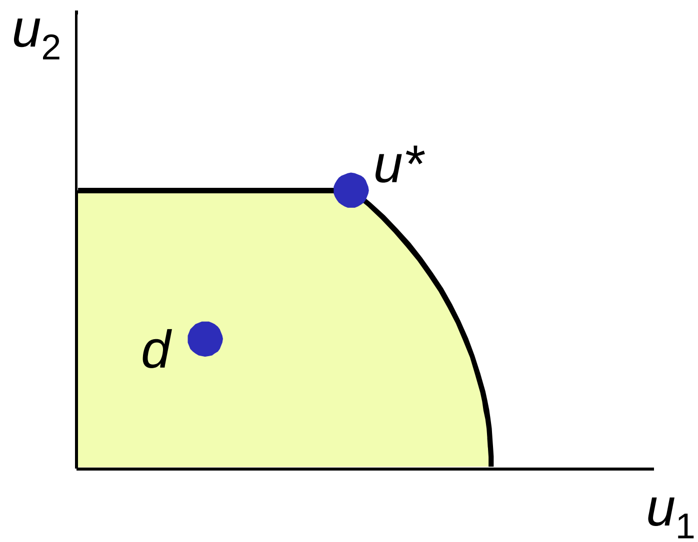
Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow's axiom.
  - If *u** is a solution with respect to *d*, then it is a solution in a smaller feasible set that contains *u** and *d*.



Axiomatic Justification

- **Axiom 4.** Independence of irrelevant alternatives.
  - Not the same as Arrow's axiom.
  - If *u** is a solution with respect to *d*, then it is a solution in a smaller feasible set that contains *u** and *d*.
  - This basically says that the solution behaves like an **optimum**.



Axiomatic Justification

Theorem. Exactly one solution satisfies Axioms 1-4, namely the Nash bargaining solution.

Proof (2 dimensions).

First show that the Nash solution satisfies the axioms.

Axiom 1. Invariance under transformation. If

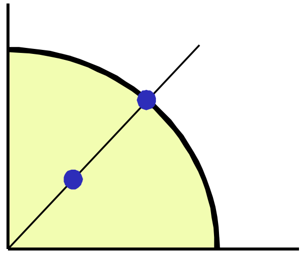
$$\prod_i (u_i^* - d_1) \geq \prod_i (u_i - d_1)$$

then

$$\prod_i \left((a_i u_i^* + b_i) - (a_i d_i + b_i)\right) \geq \prod_i \left((a_i u_i + b_i) - (a_i d_i + b_i)\right)$$

Axiomatic Justification

Axiom 2. Pareto optimality. Clear because social welfare function is strictly monotone increasing.



Axiom 3. Symmetry. Obvious.

Axiom 4. Independence of irrelevant alternatives. Follows from the fact that *u** is an optimum.

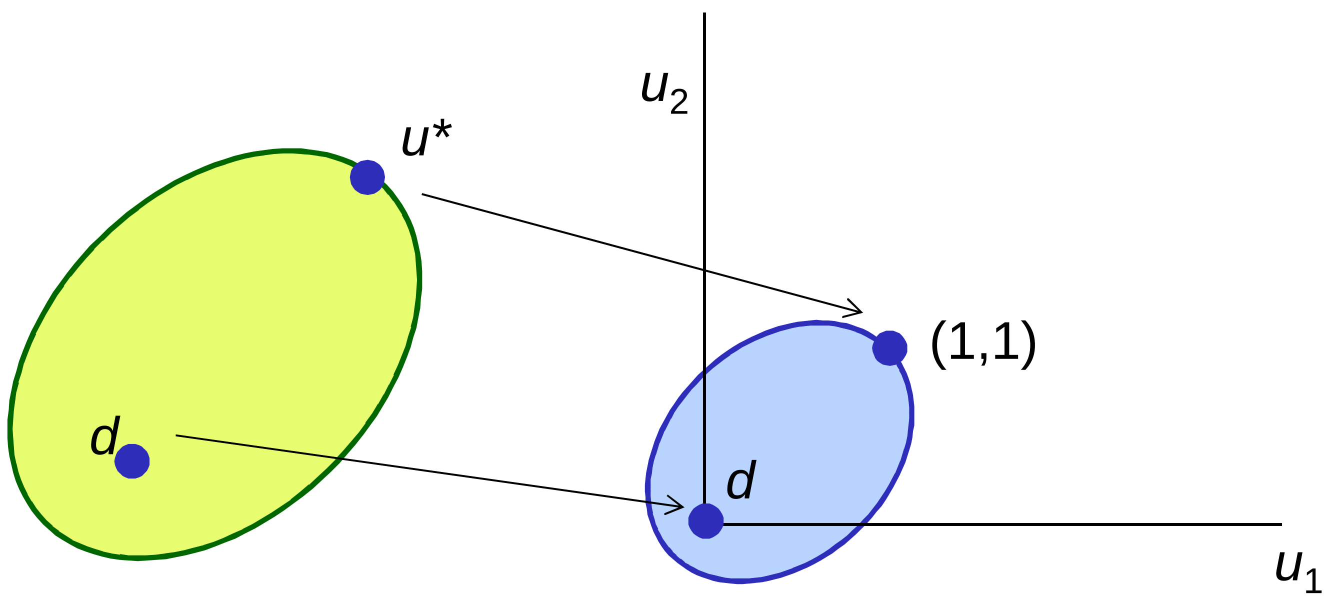
Now show that **only** the Nash solution satisfies the axioms

Axiomatic Justification

Let u^* be the Nash solution for a given problem. Then it satisfies the axioms with respect to d. Select a transformation that sends

$$(u_1,u_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has Nash solution (1,1), by Axiom 1:



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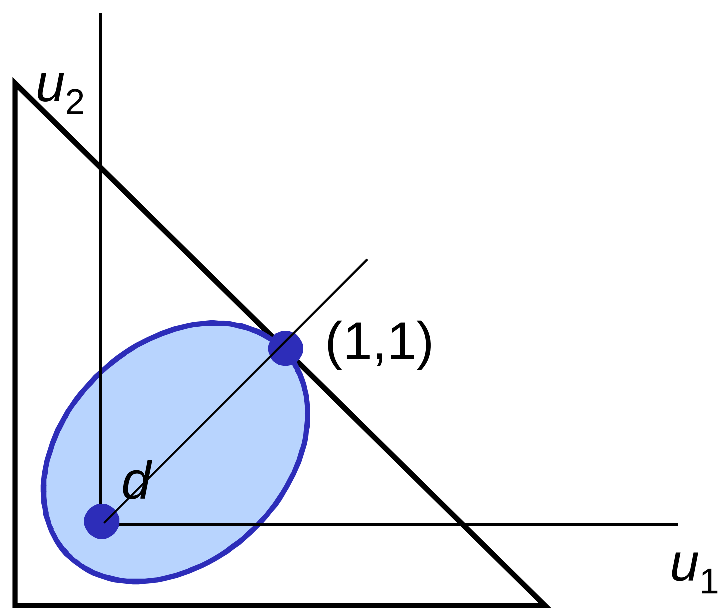
By Axioms 2 & 3, (1,1) is the **only** bargaining solution in the triangle:



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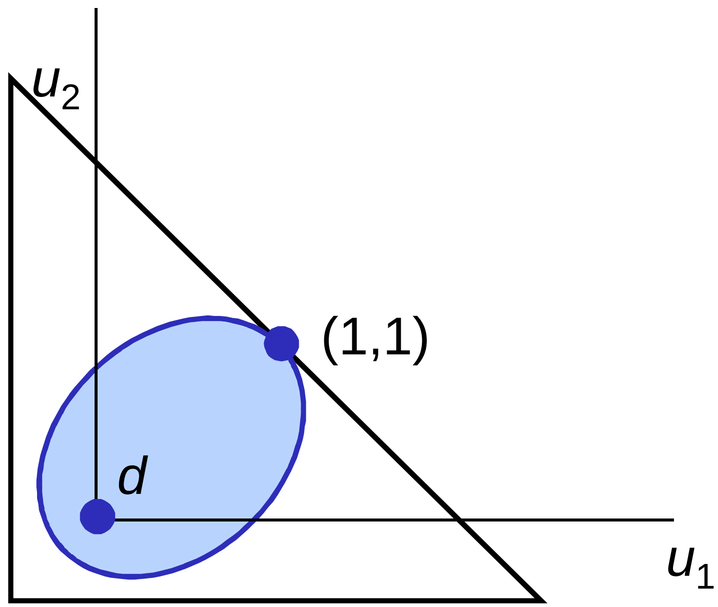

So by Axiom 4, (1,1) is the only bargaining solution in blue set.

Axiomatic Justification

Let u^* be the Nash solution for a given problem. Then it satisfies the axioms with respect to d. Select a transformation that sends

$$(u_1,u_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

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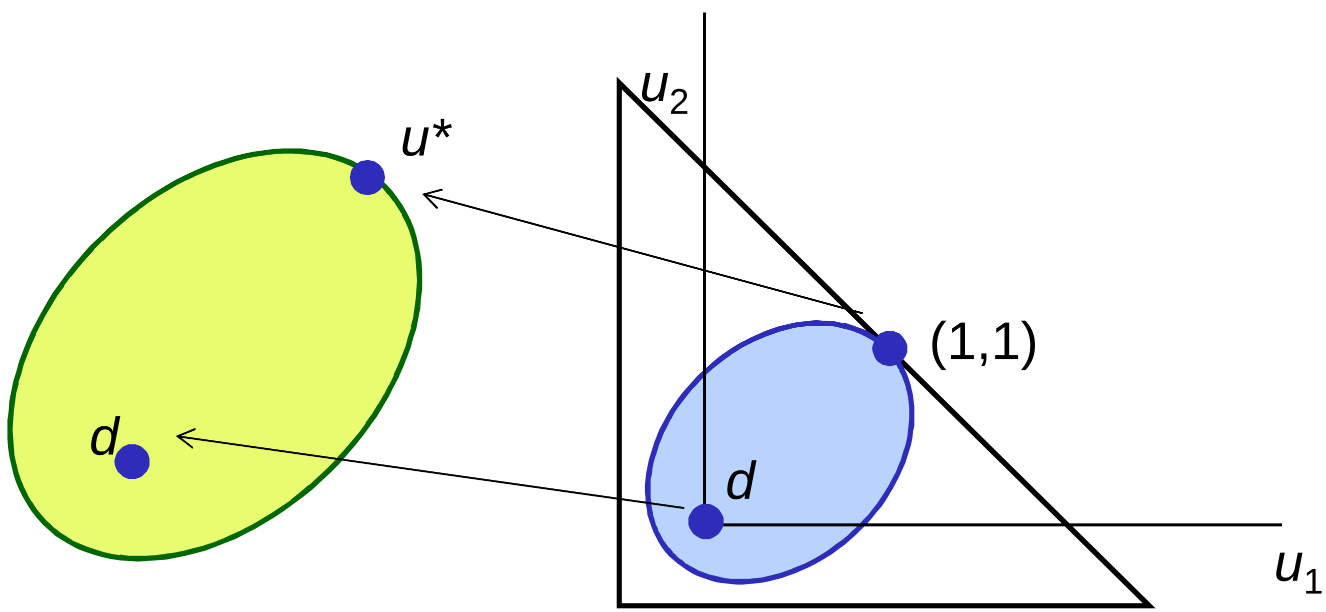
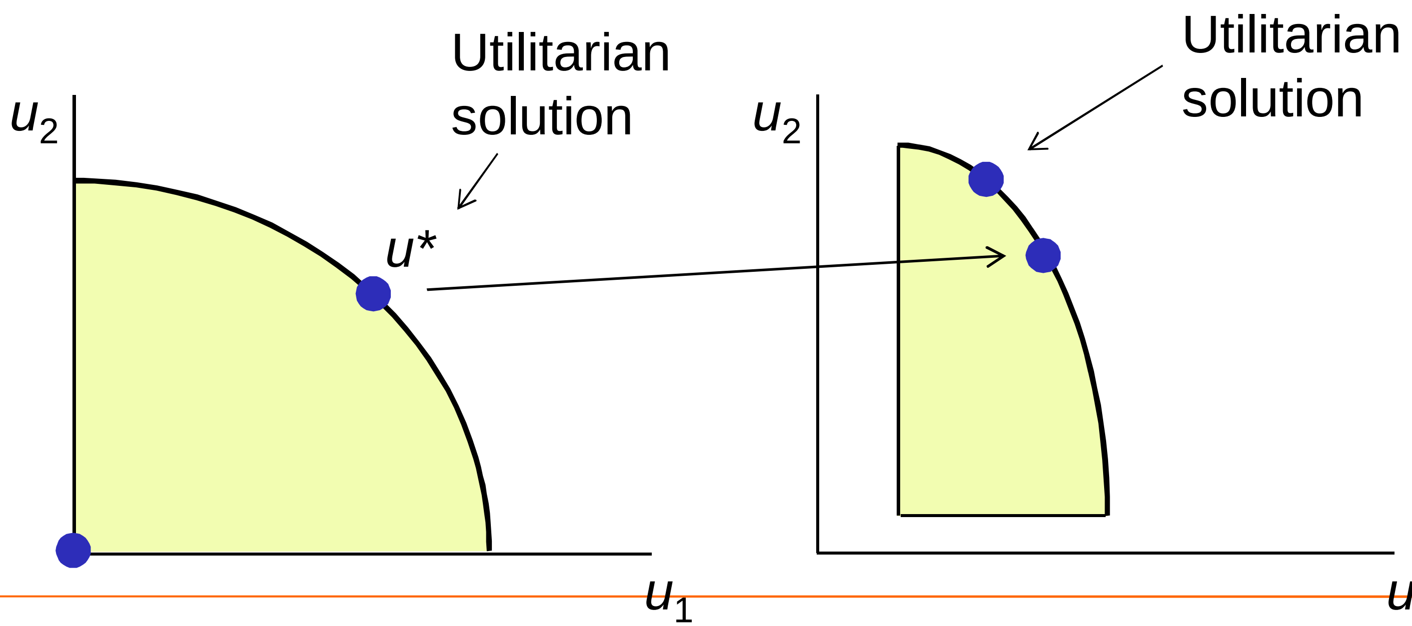


So by Axiom 4, (1,1) is the only bargaining solution in blue set.

By Axiom 1, u^* is the only bargaining solution in the original problem.

Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** (invariance under transformation) is very strong.
  - Axiom 1 denies **interpersonal comparability**.
  - So how can it reflect moral concerns?



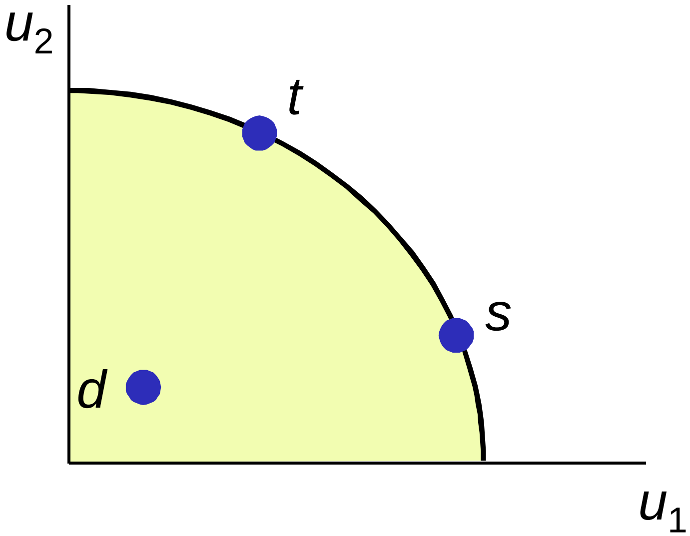
Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** (invariance under transformation) is very strong.
  - Axiom 1 denies **interpersonal comparability**.
  - So how can it reflect moral concerns?

- Most attention has been focused on **Axiom 4** (independence of irrelevant alternatives).
  - Will address this later.

Bargaining Justification

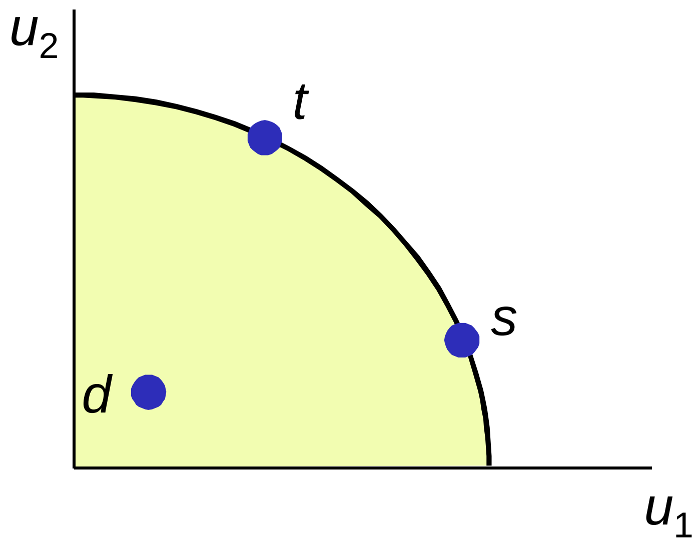
Players 1 and 2 make offers *s*, *t*.



Bargaining Justification

Players 1 and 2 make offers *s*, *t*.

Let $p = P$(player 2 will reject *s*), as estimated by player 1.



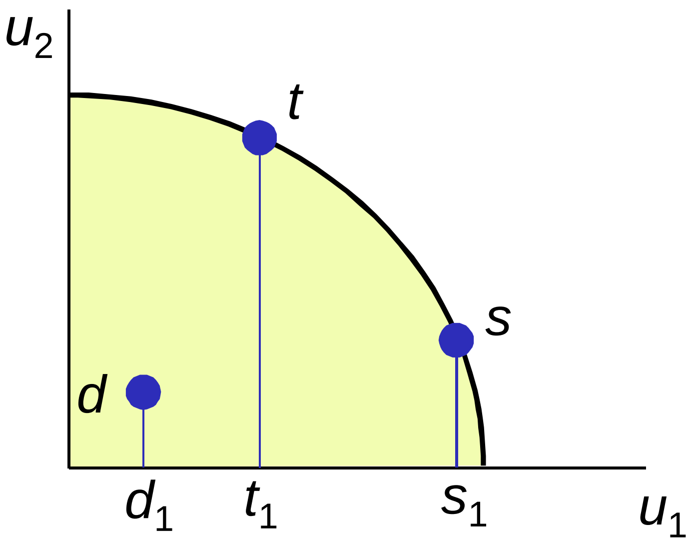
Bargaining Justification

Players 1 and 2 make offers *s*, *t*.

Let $p = P(\text{player 2 will reject } s)$, as estimated by player 1.

Then player 1 will stick with *s*, rather than make a counteroffer, if

$$(1-p)s_1 + pd_1 \geq t_1$$



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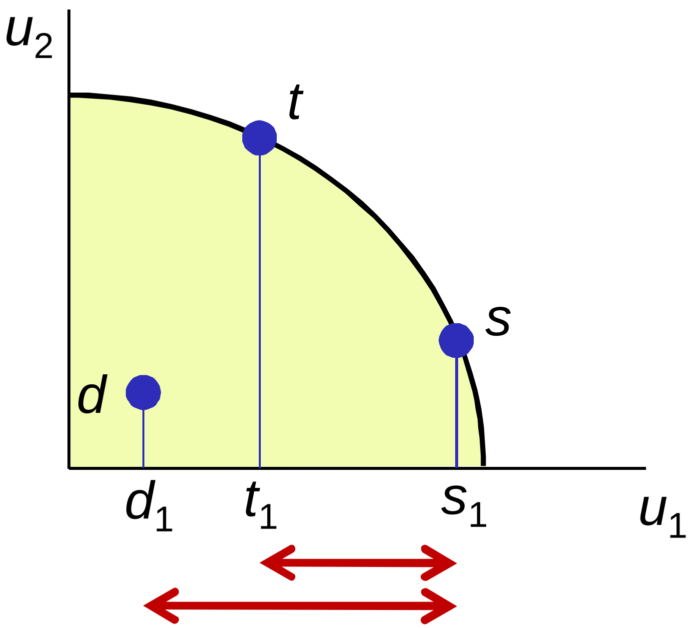


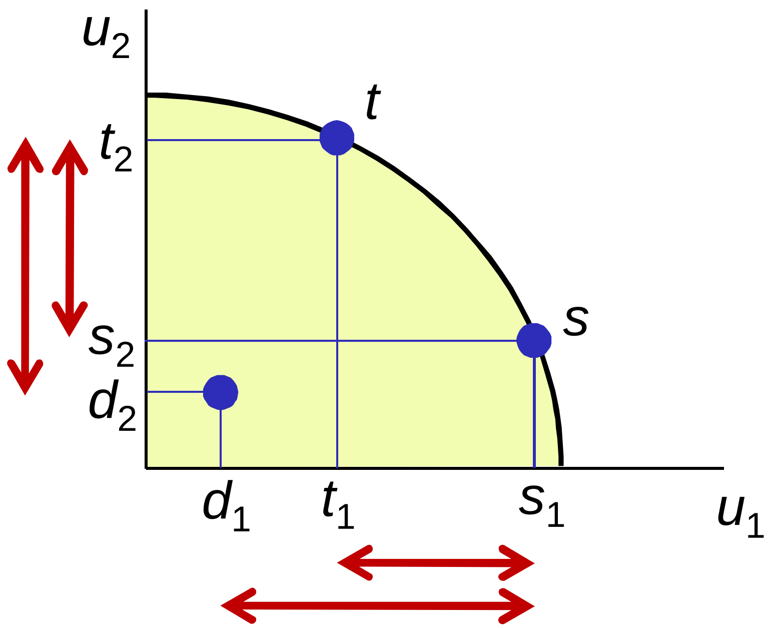
So player 1 will stick with *s* if

$$p \leq \frac{s_1 - t_1}{s_1 - d_1} = r_1$$

Bargaining Justification

It is rational for player 1 to make a counteroffer s', rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$



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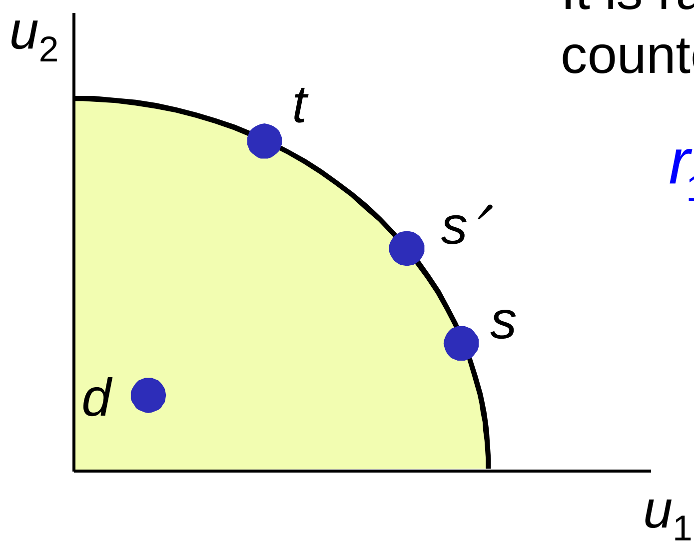


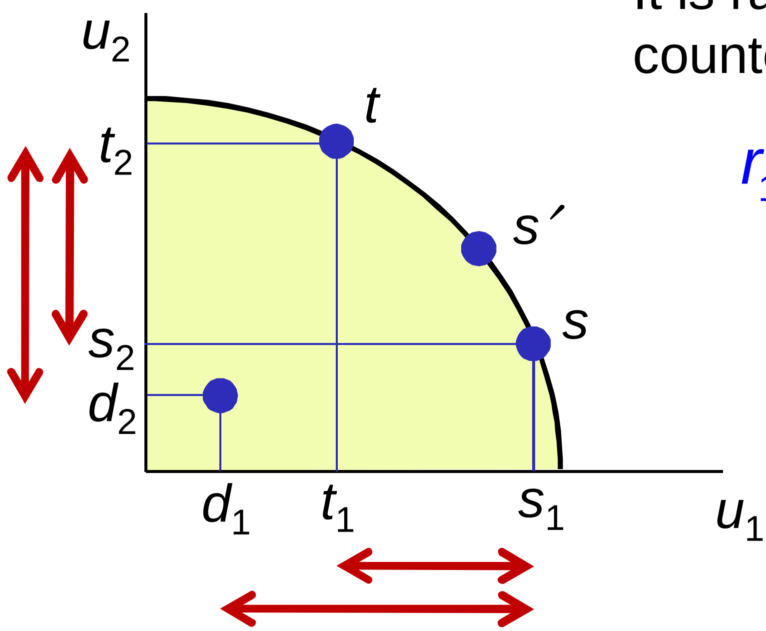
It is rational for player 2 to make the next counteroffer if

$$r_1' = \frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2} = r_2'$$

Bargaining Justification

It is rational for player 1 to make a counteroffer s', rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2} = r_2$$



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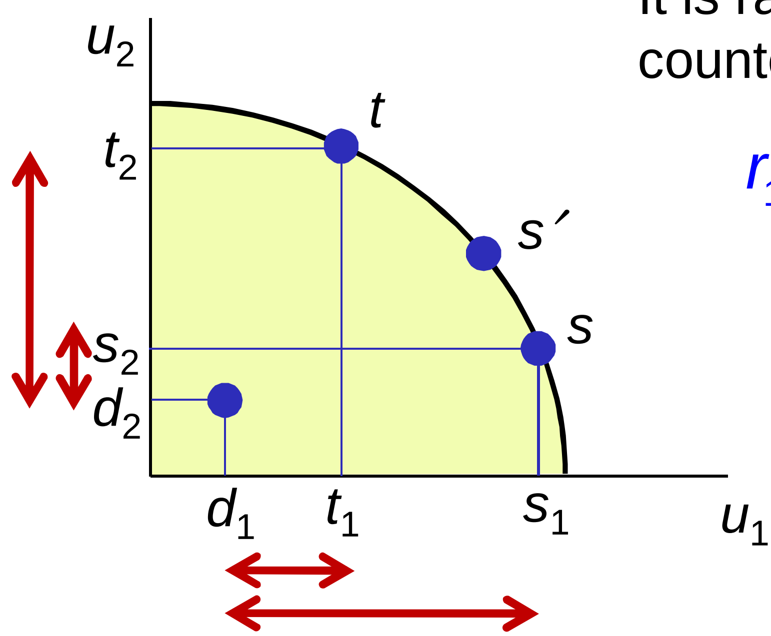
$$r_1' = \frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2} = r_2'$$

But $$\frac{s_1 - t_1}{s_1 - d_1} \leq \frac{t_2 - s_2}{t_2 - d_2}$$

Bargaining Justification

It is rational for player 1 to make a counteroffer s', rather than player 2, if

$$r_1 = \frac{s_1 - t_1}{s_1 - d_1} \le \frac{t_2 - s_2}{t_2 - d_2} = r_2$$



It is rational for player 2 to make the next counteroffer if

$$r_1' = \frac{s_1' - t_1}{s_1' - d_1} \ge \frac{t_2 - s_2'}{t_2 - d_2} = r_2'$$

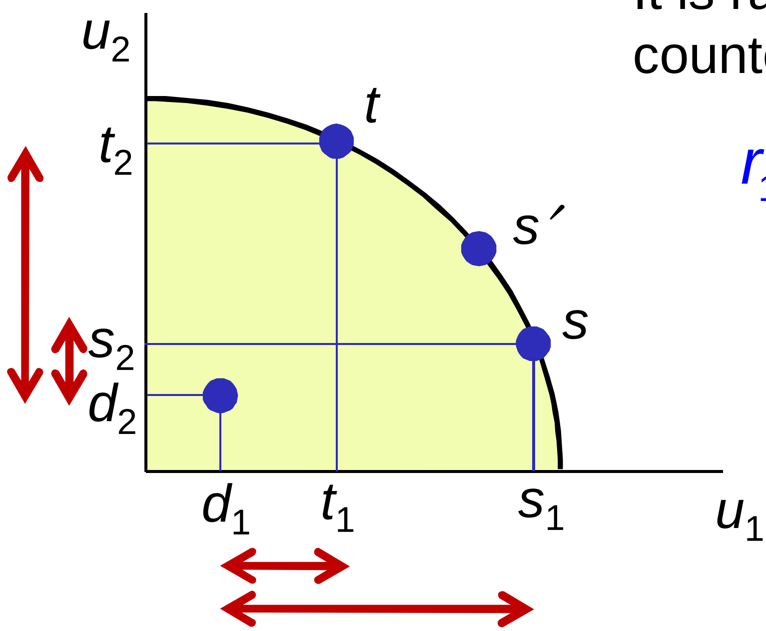
But

$$\frac{s_1 - t_1}{s_1 - d_1} \le \frac{t_2 - s_2}{t_2 - d_2}$$

$$\Longleftrightarrow \frac{t_1 - d_1}{s_1 - d_1} \ge \frac{s_2 - d_2}{t_2 - d_2}$$

Bargaining Justification

So we have $(s_1 - d_1)(s_2 - d_2) \le (t_1 - d_1)(t_2 - d_2)$



It is rational for player 2 to make the next counteroffer if

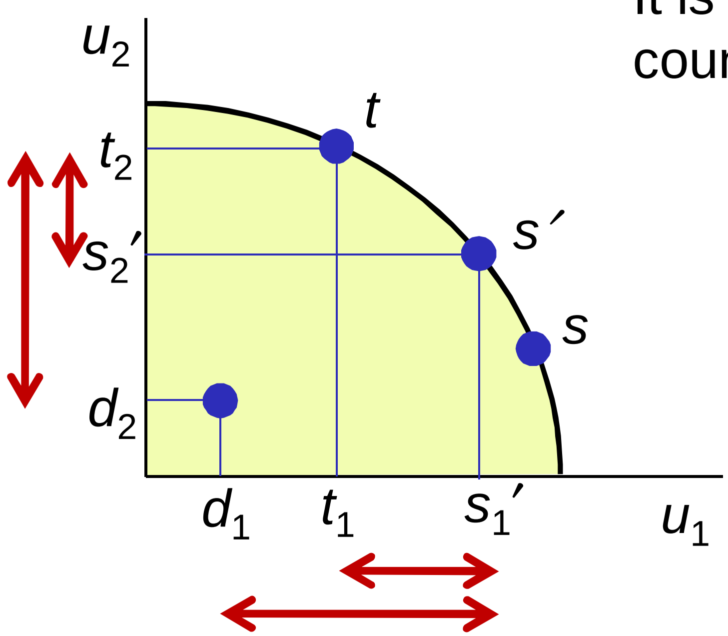
$$r_1' = \frac{s_1' - t_1}{s_1' - d_1} \ge \frac{t_2 - s_2'}{t_2 - d_2} = r_2'$$

But
$$\frac{s_1 - t_1}{s_1 - d_1} \le \frac{t_2 - s_2}{t_2 - d_2}$$

$$\Longleftrightarrow \frac{t_1 - d_1}{s_1 - d_1} \ge \frac{s_2 - d_2}{t_2 - d_2}$$

Bargaining Justification

So we have $$(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)$$



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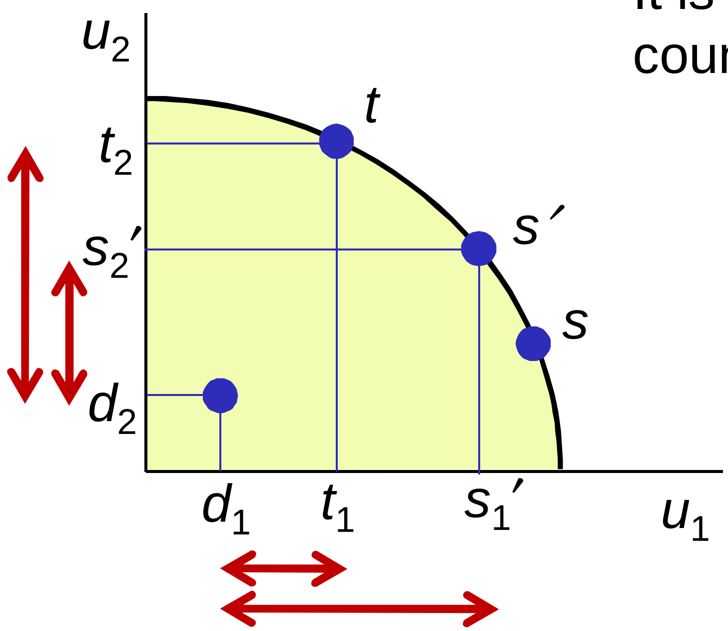

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So we have $$(s_1 - d_1)(s_2 - d_2) \leq (t_1 - d_1)(t_2 - d_2)$$

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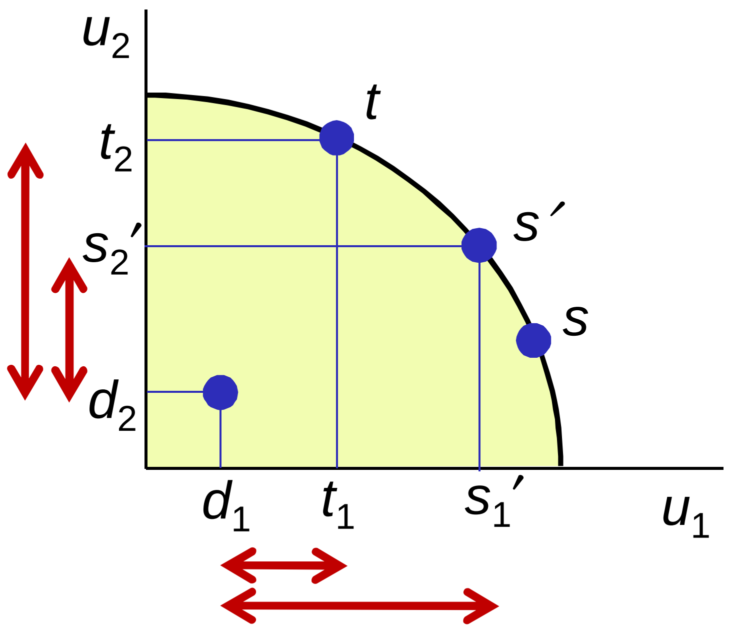

Similarly $$\frac{s_1' - t_1}{s_1' - d_1} \geq \frac{t_2 - s_2'}{t_2 - d_2}$$

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Bargaining Justification

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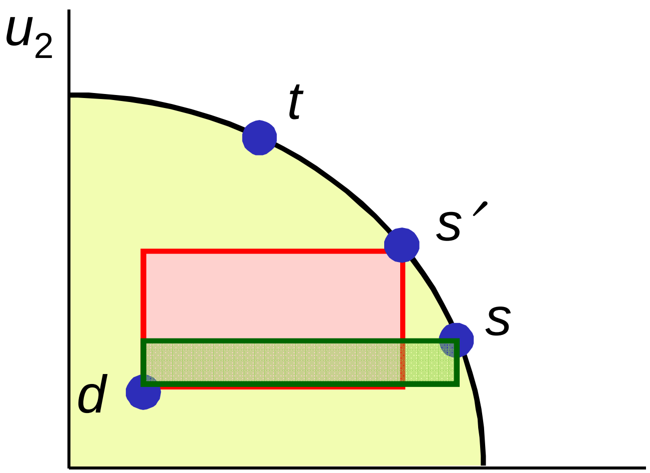
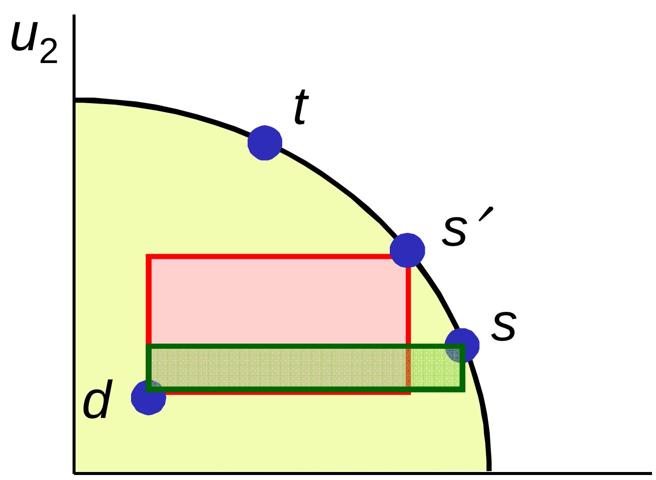


This implies an improvement in the Nash social welfare function

Bargaining Justification

So we have $$(s_1 - d_1)(s_2 - d_2) \le (t_1 - d_1)(t_2 - d_2)$$

and we have $$(t_1 - d_1)(t_2 - d_2) \le (s'_1 - d_1)(s'_2 - d_2)$$

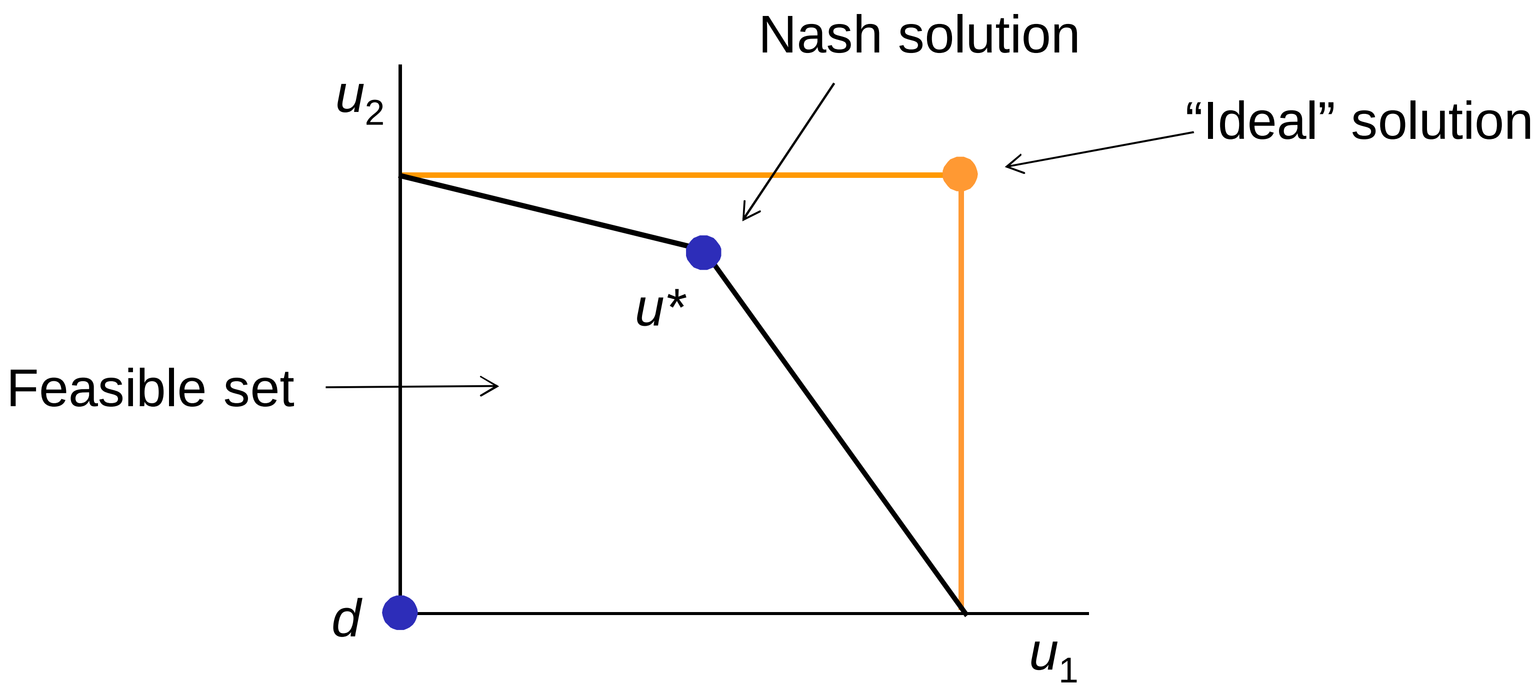


This implies an improvement in the Nash social welfare function.

Given a minimum distance between offers, continued bargaining converges to Nash solution.

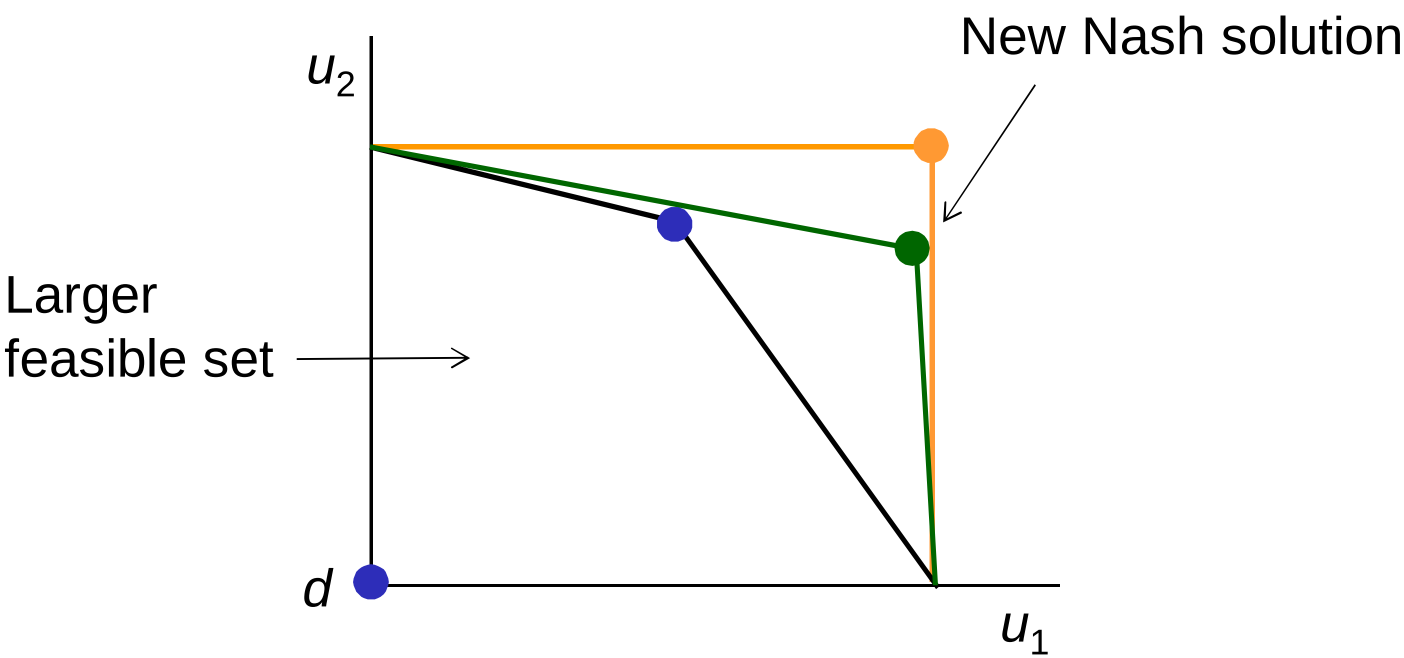
Raiffa-Kalai-Smorodinsky Bargaining Solution

- This approach begins with a critique of the Nash bargaining solution.



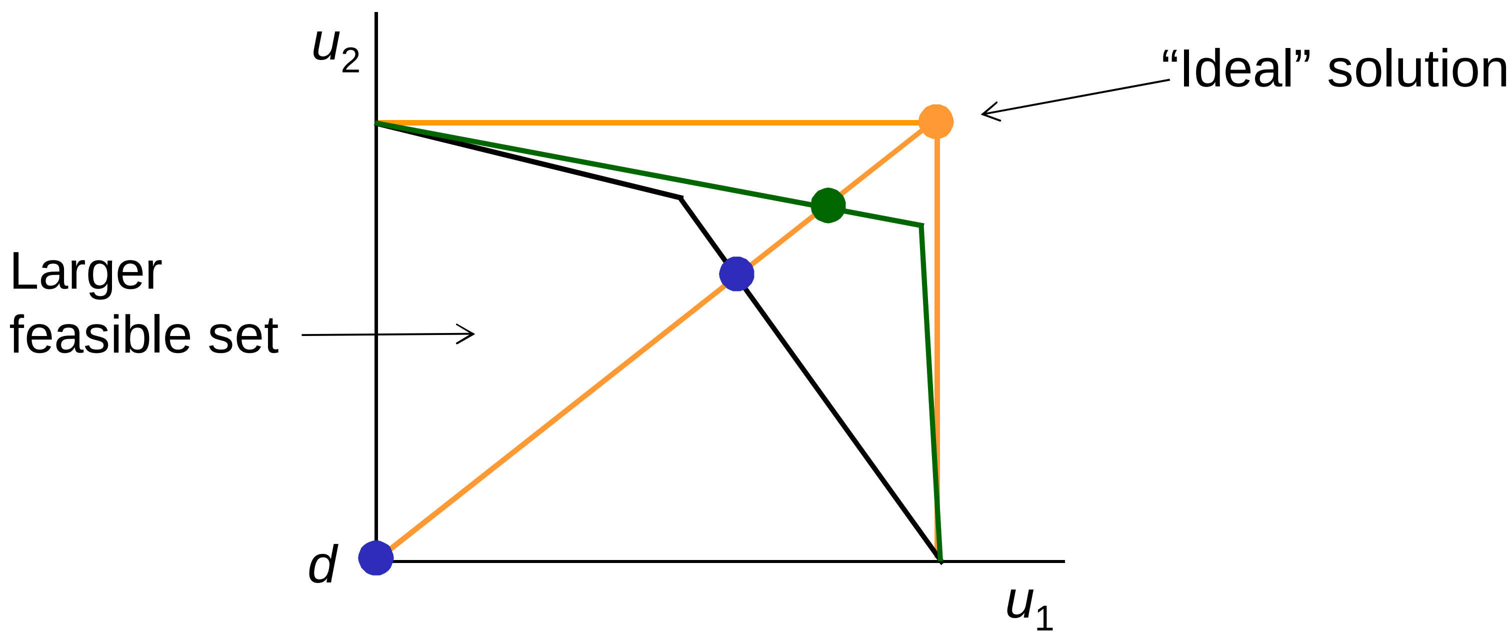
Raiffa-Kalai-Smorodinsky Bargaining Solution

- This approach begins with a critique of the Nash bargaining solution.
  - The new Nash solution is worse for player 2 even though the feasible set is larger.



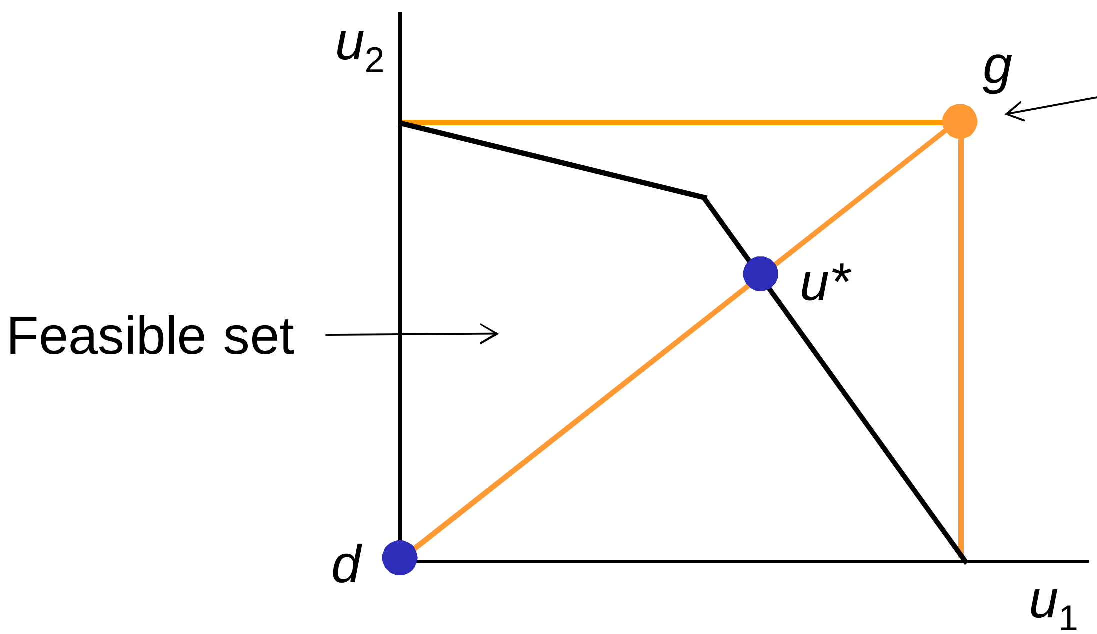
Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from *d* to ideal solution.



Raiffa-Kalai-Smorodinsky Bargaining Solution

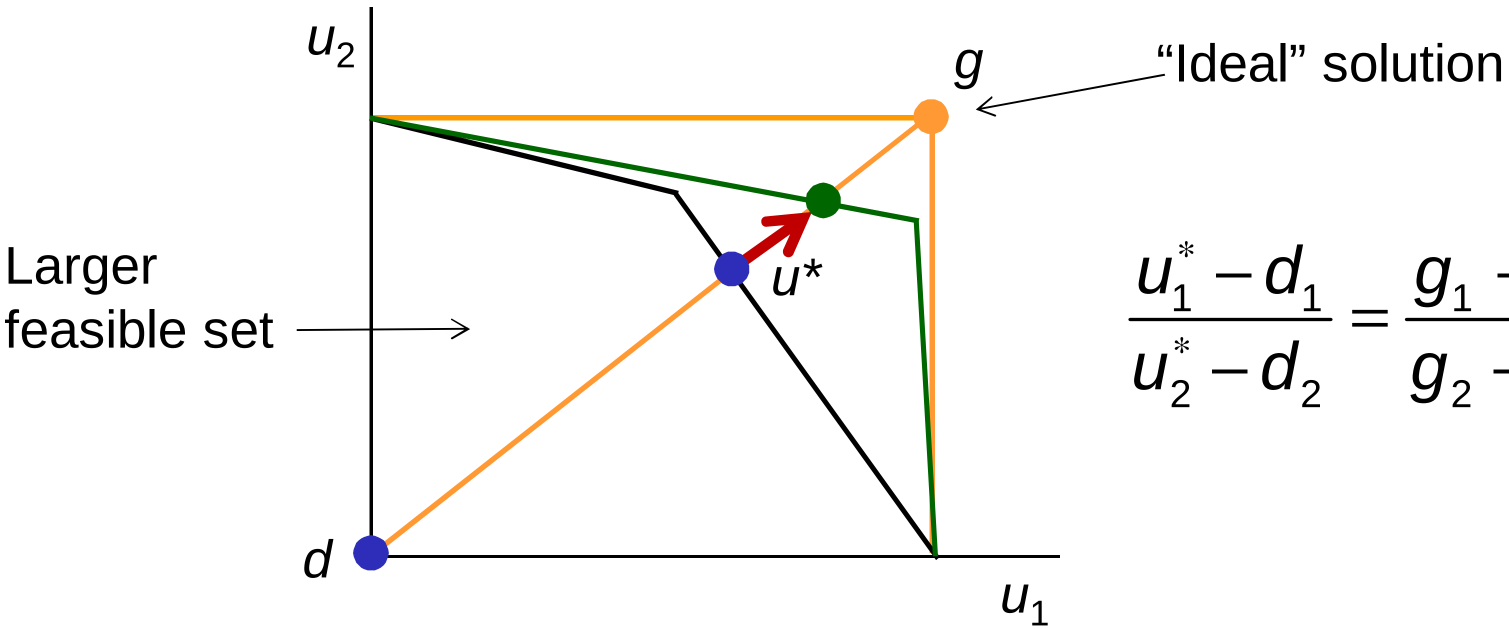
- **Proposal**: Bargaining solution is pareto optimal point on line from *d* to ideal solution.
  - The players receive an equal fraction of their possible utility gains.



$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Proposal**: Bargaining solution is pareto optimal point on line from *d* to ideal solution.
  - Replace Axiom 4 with **Axiom 4′ (Monotonicity)**: A larger feasible set with same ideal solution results in a bargaining solution that is better (or no worse) for all players.



$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Applications**
  - Allocation of wireless capacity.
  - Allocation of cloud computing resources.
  - Datacenter resource scheduling
    (also dominant resource fairness)
  - Resource allocation in visual sensor networks
  - Labor-market negotiations

Raiffa-Kalai-Smorodinsky Bargaining Solution

- **Optimization model**.
  - Not an optimization problem over original feasible set (we gave up Axiom 4).
  - But it is an optimization problem (pareto optimality) over the line segment from *d* to ideal solution.

$$\max \sum_i u_i$$

$$(g_1 - d_1)(u_i - d_i) = (g_i - d_i)(u_1 - d_1), \quad \text{all } i$$

$$u \in S$$

$$\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{g_1 - d_1}{g_2 - d_2}$$

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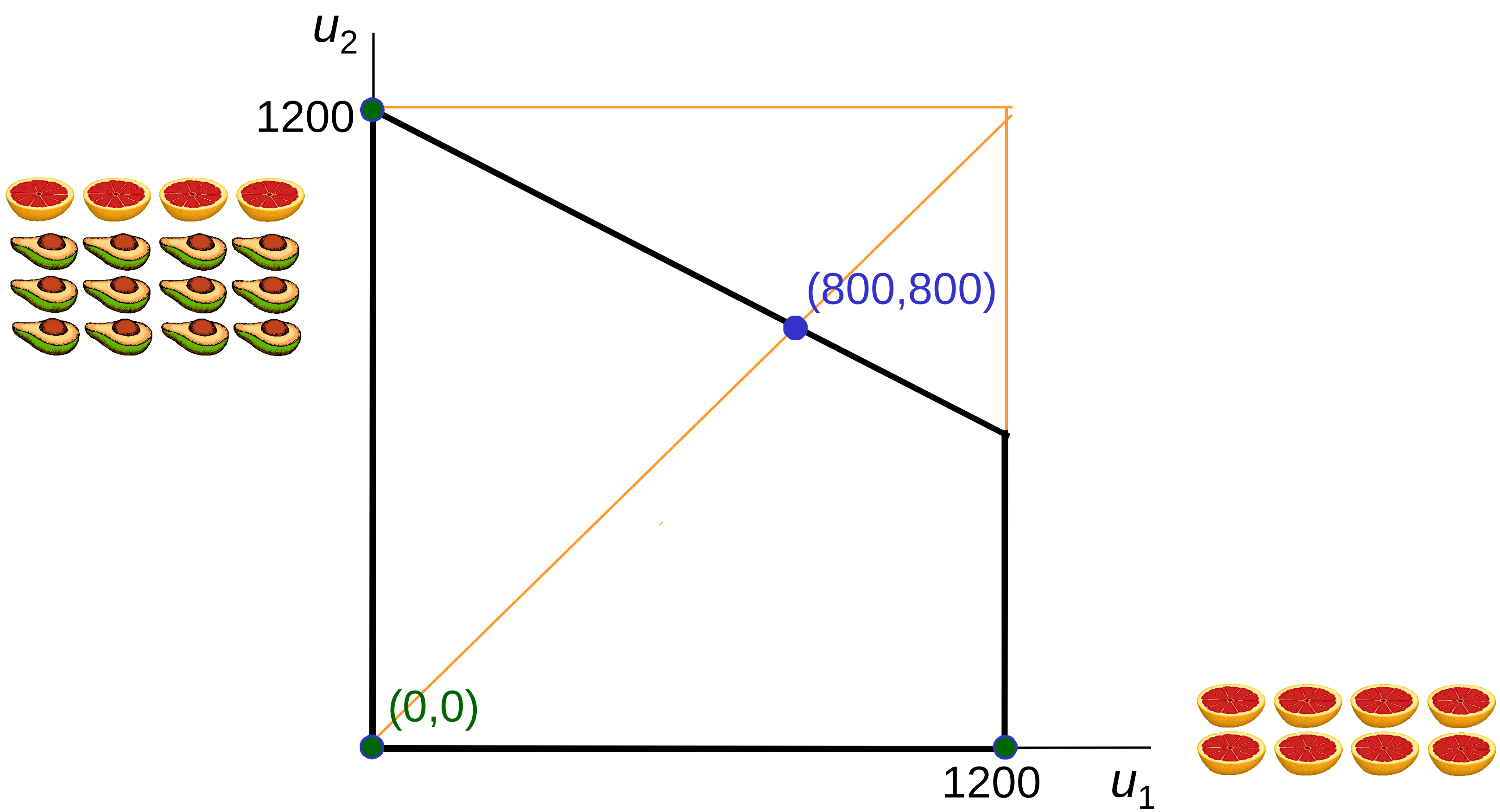
Linear constraint

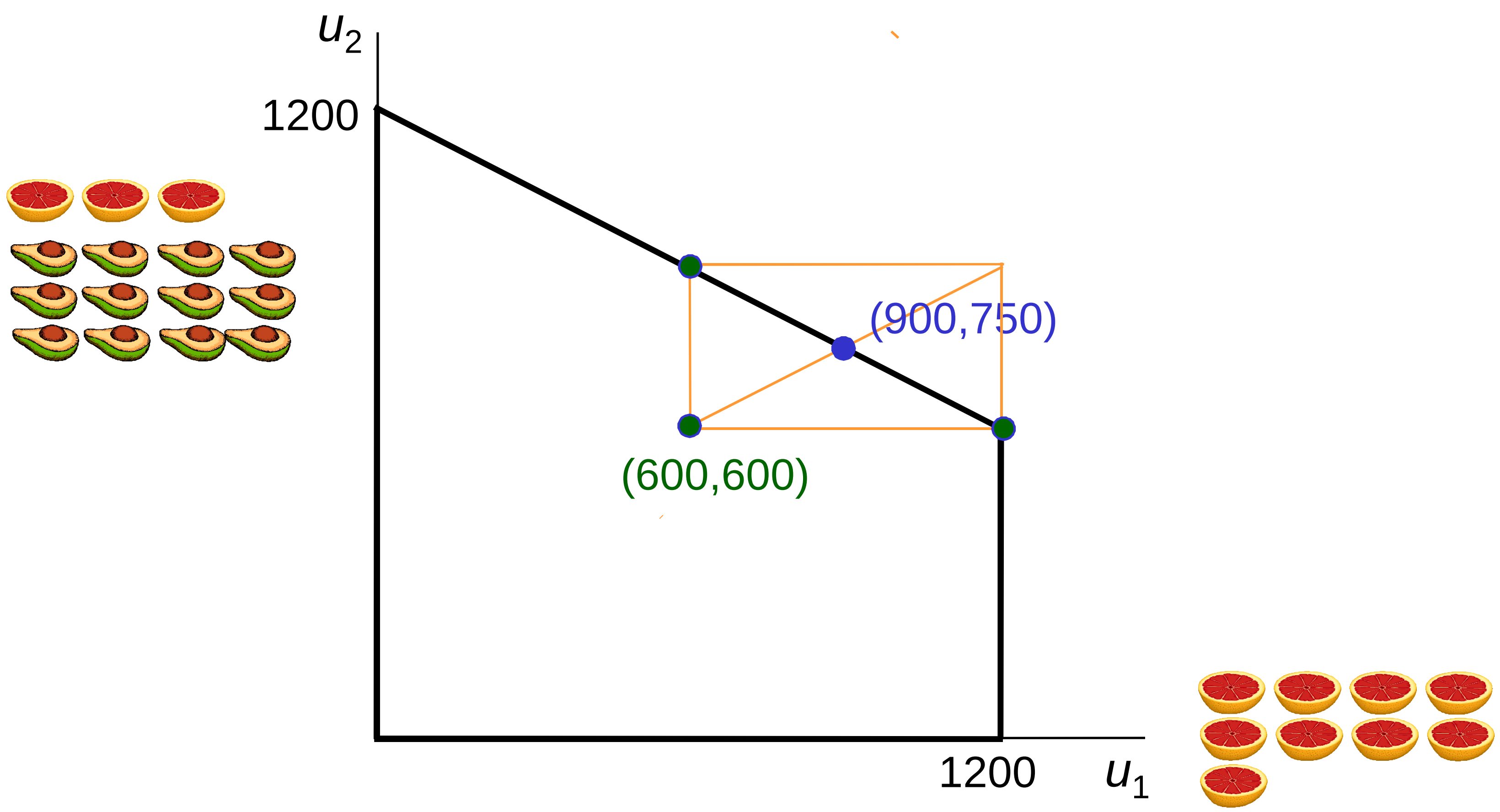
Raiffa-Kalai-Smorodinsky Bargaining Solution

From Zero



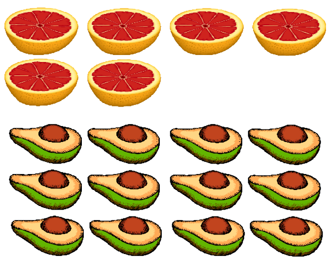
Raiffa-Kalai-Smorodinsky Bargaining Solution

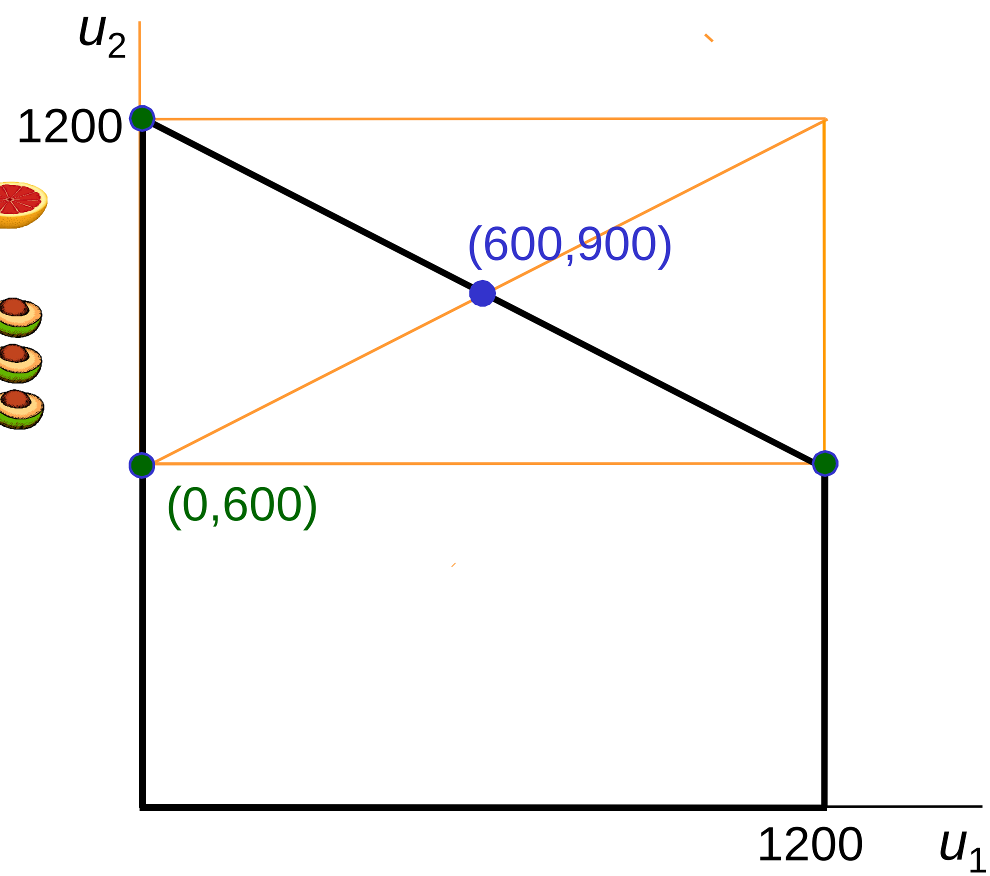
From Equality

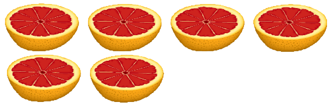


Raiffa-Kalai-Smorodinsky Bargaining Solution

From Strong Pareto Set







Axiomatic Justification

- **Axiom 1**. Invariance under transformation.
- **Axiom 2.** Pareto optimality.
- **Axiom 3.** Symmetry.
- **Axiom 4′.** Monotonicity.

Axiomatic Justification

Theorem. Exactly one solution satisfies Axioms 1-4′, namely the RKS bargaining solution.

Proof (2 dimensions).

Easy to show that RKS solution satisfies the axioms.

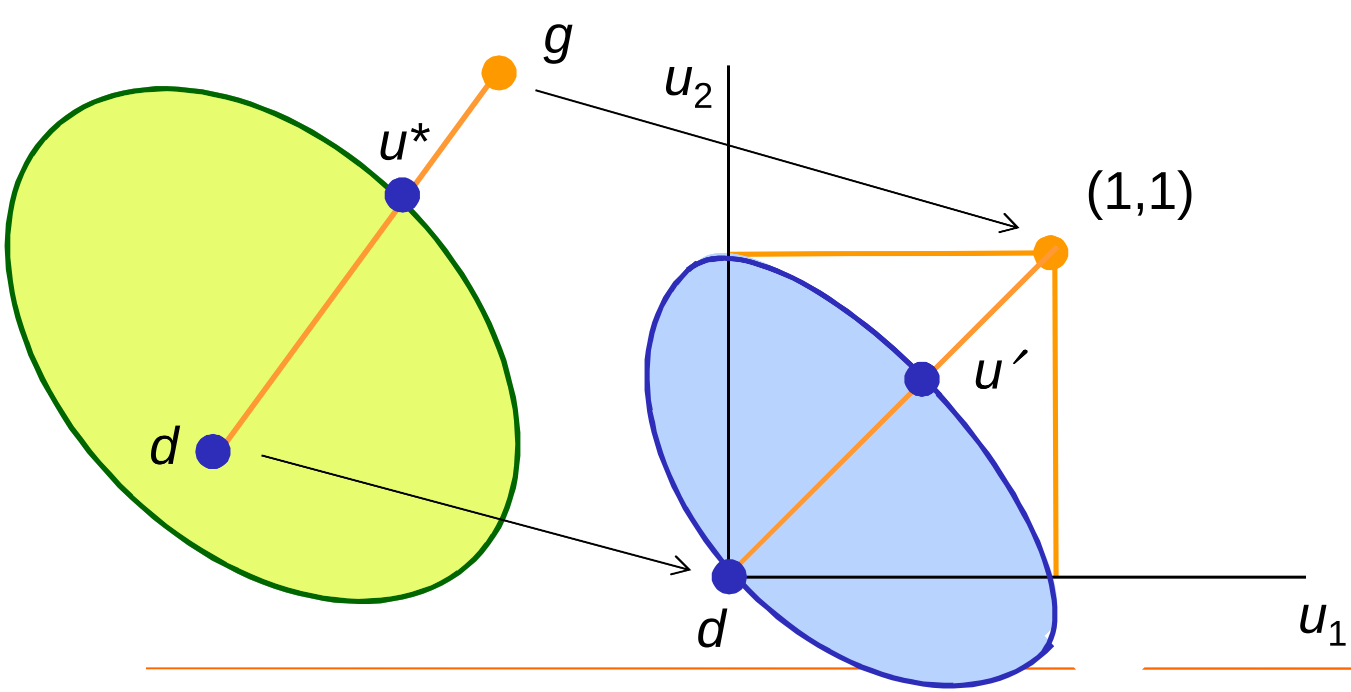
Now show that **only** the RKS solution satisfies the axioms.

Axiomatic Justification

Let *u** be the RKS solution for a given problem. Then it satisfies the axioms with respect to *d*. Select a transformation that sends

$$(g_1,g_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution *u′*, by Axiom 1:



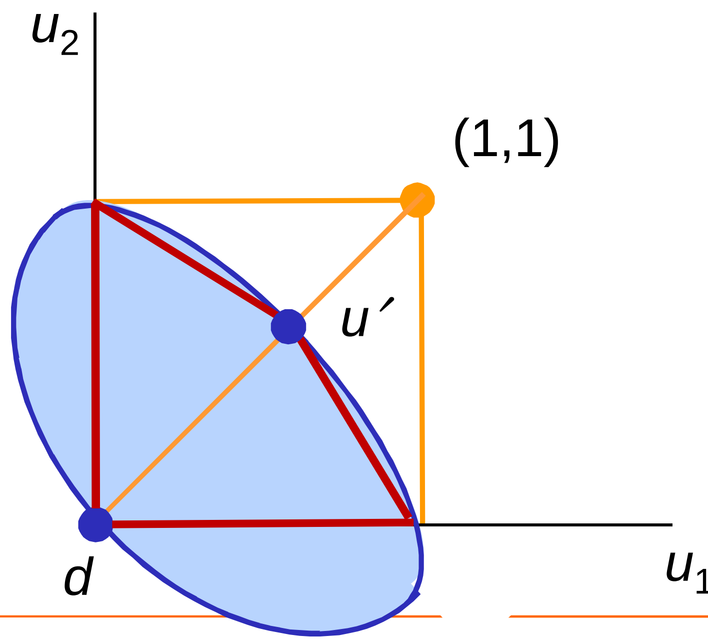
Axiomatic Justification

Let u^* be the RKS solution for a given problem. Then it satisfies the axioms with respect to d. Select a transformation that sends

$$(g_1,g_2) \rightarrow (1,1), \quad (d_1,d_2) \rightarrow (0,0)$$

The transformed problem has RKS solution u', by Axiom 1:

By Axioms 2 & 3, u' is the **only** bargaining solution in the red polygon:



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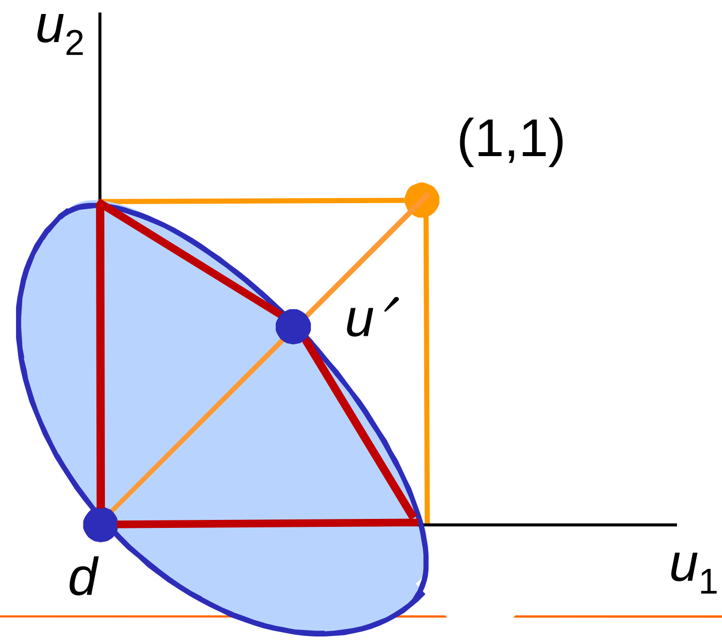

The red polygon lies inside blue set. So by Axiom 4′, its bargaining solution is no better than bargaining solution on blue set. So u' is the only bargaining solution on blue set.

Axiomatic Justification

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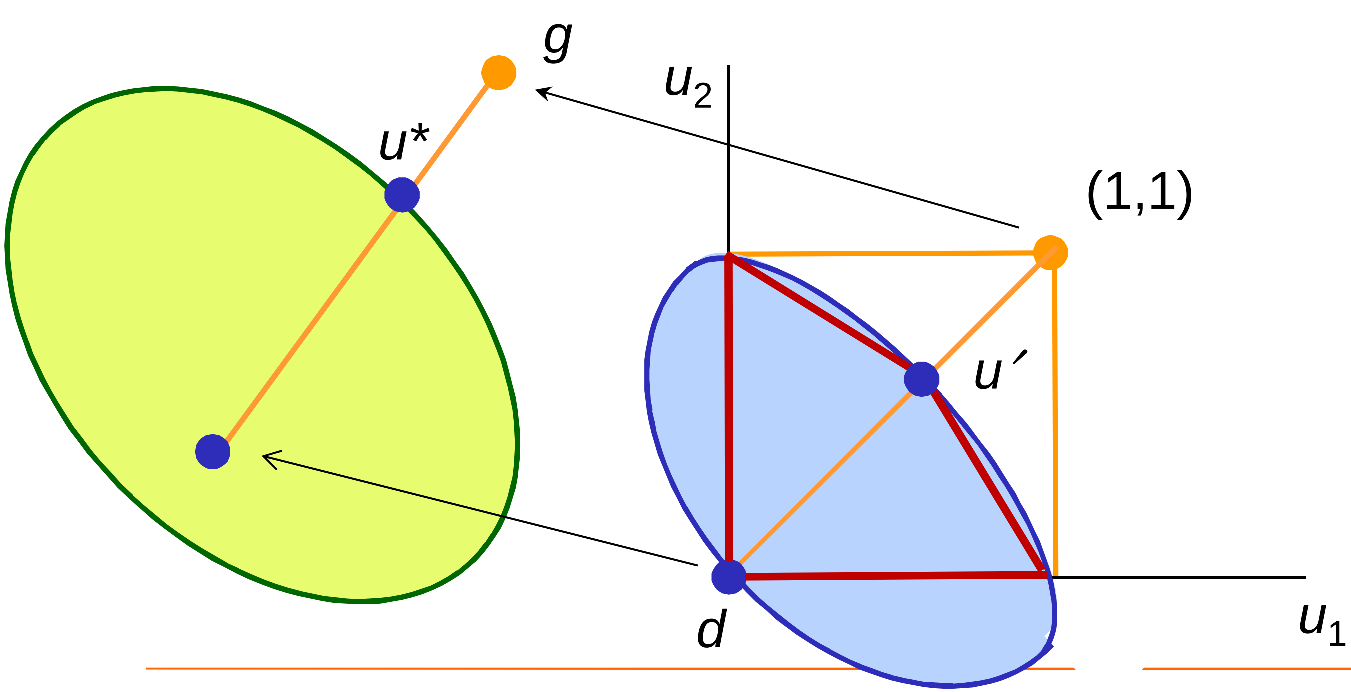


By Axiom 1, u^* is the only bargaining solution in the original problem.

Axiomatic Justification

- **Problems** with axiomatic justification.
  - **Axiom 1** is still in effect.
  - It denies **interpersonal comparability**.
  - Dropping Axiom 4 sacrifices optimization of a social welfare function.
  - This may not be necessary if Axiom 1 is rejected.
  - Needs modification for > 2 players (more on this shortly).

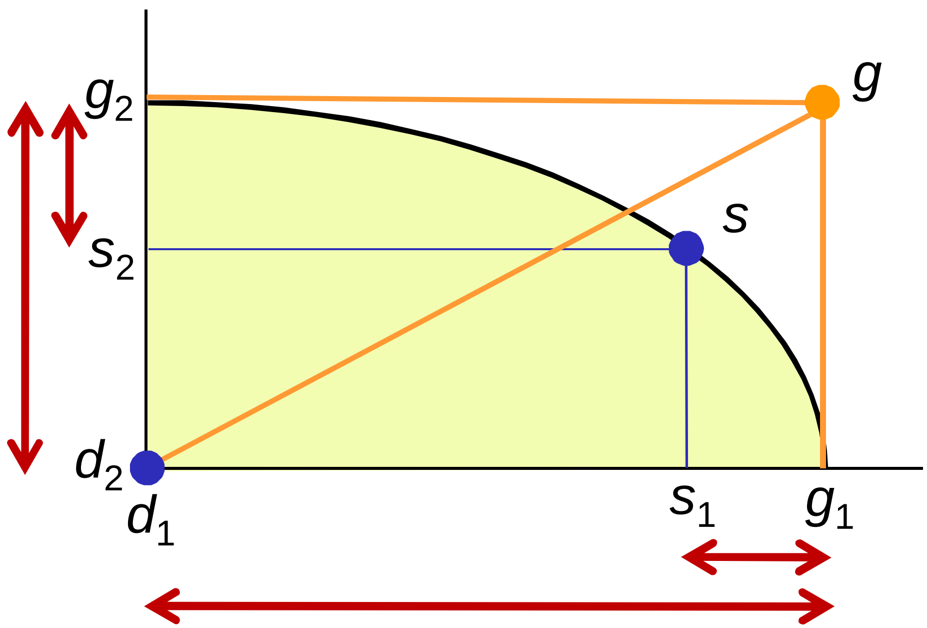
Bargaining Justification

Resistance to an agreement *s* depends on sacrifice relative to sacrifice under no agreement. Here, player 2 is making a larger relative sacrifice:

$$\frac{g_1 - s_1}{g_1 - d_1} \le \frac{g_2 - s_2}{g_2 - d_2}$$

Minimizing resistance to agreement requires minimizing

$$\max_i \left\{ \frac{g_i - s_i}{g_i - d_i} \right\}$$



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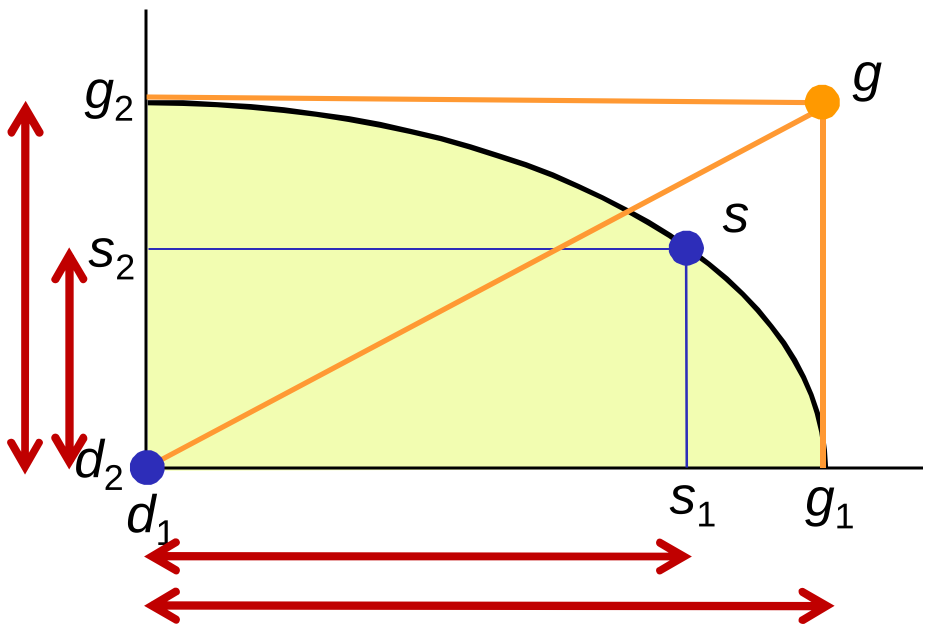

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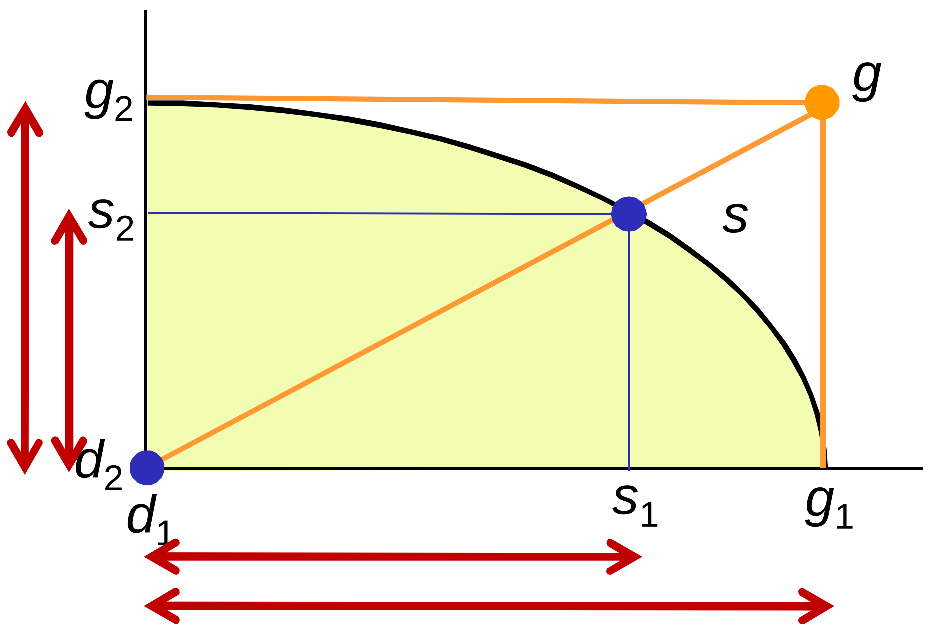

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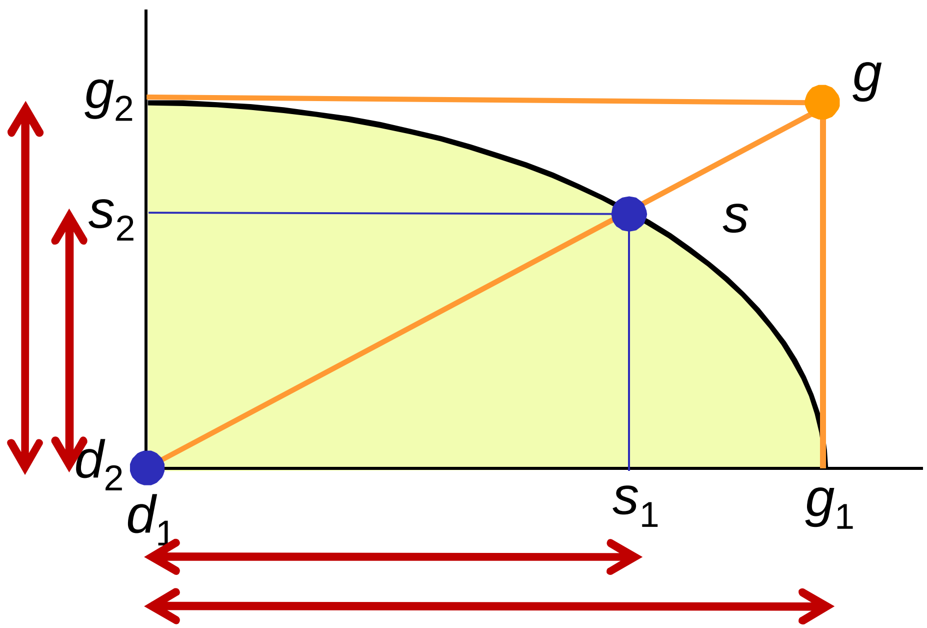
which is achieved by RKS point.

Bargaining Justification

This is the **Rawlsian social contract** argument applied to **gains relative to the ideal**.



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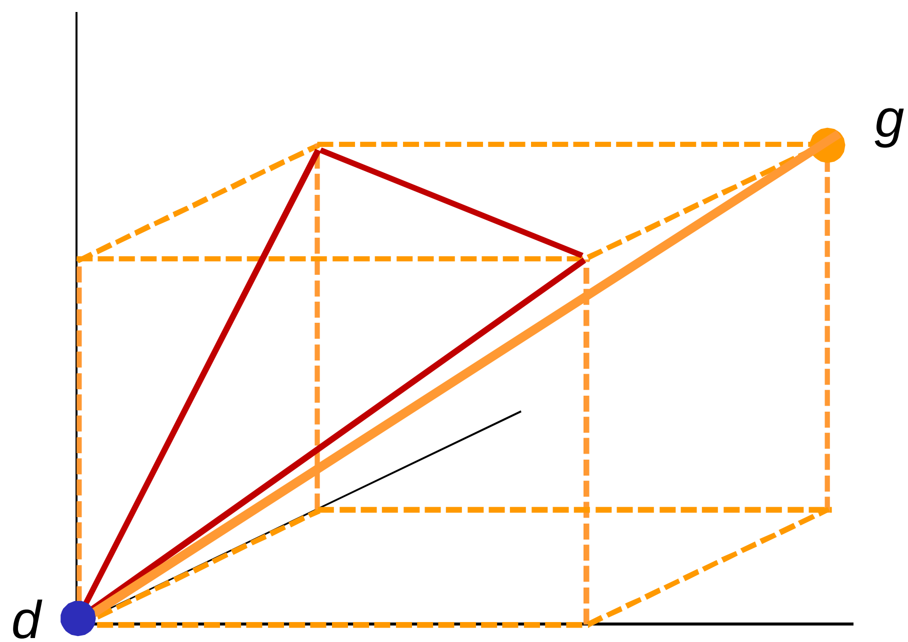
Problem with RKS Solutioon

- However, the RKS solution is Rawlsian only for **2 players**.
  - In fact, RKS leads to counterintuitive results for 3 players.



Red triangle is feasible set.

RKS point is *d* !

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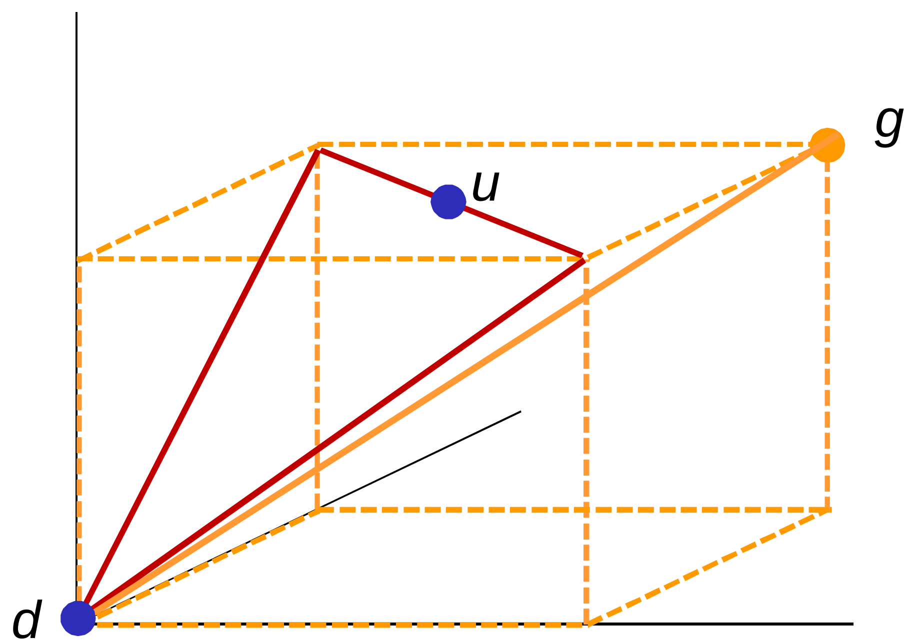
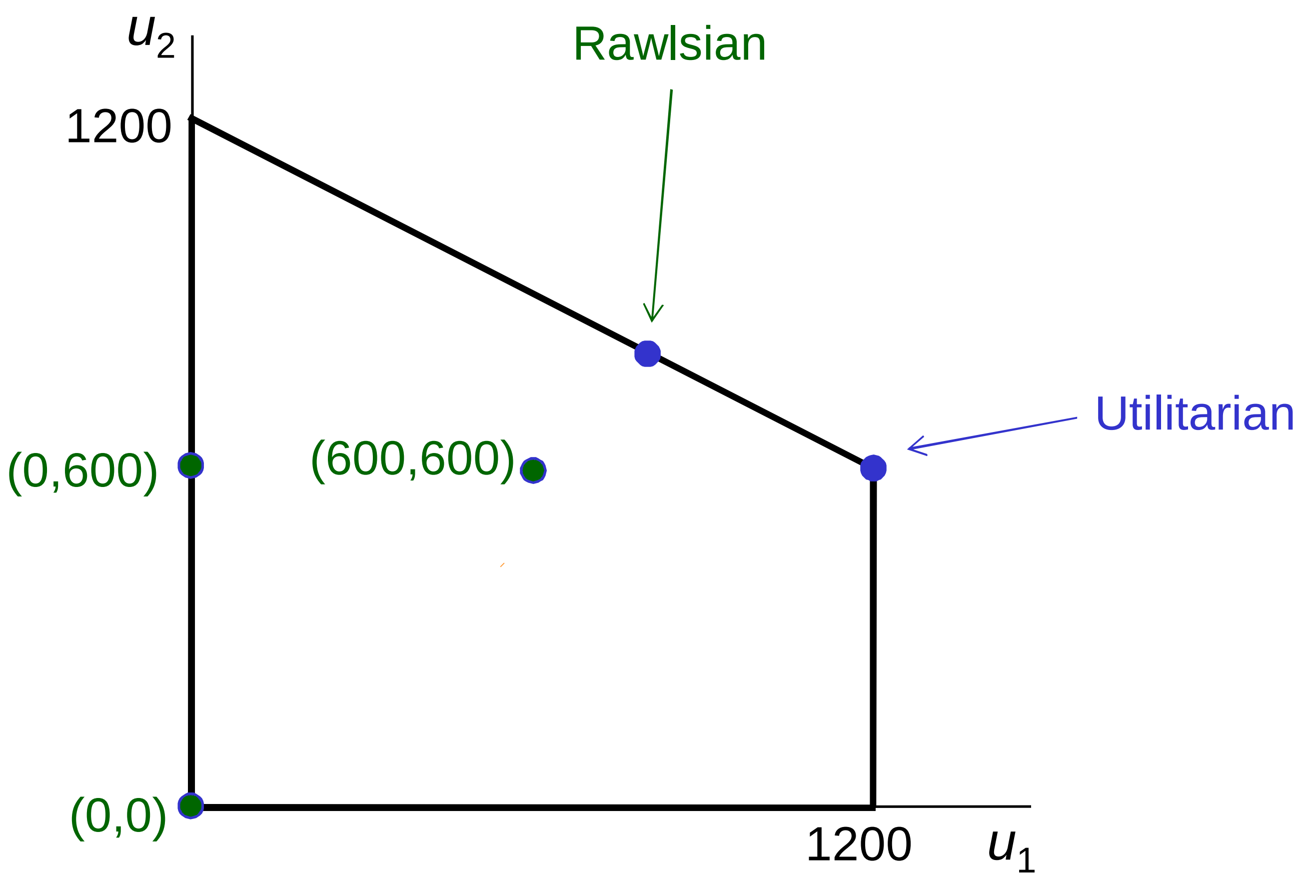


Red triangle is feasible set.

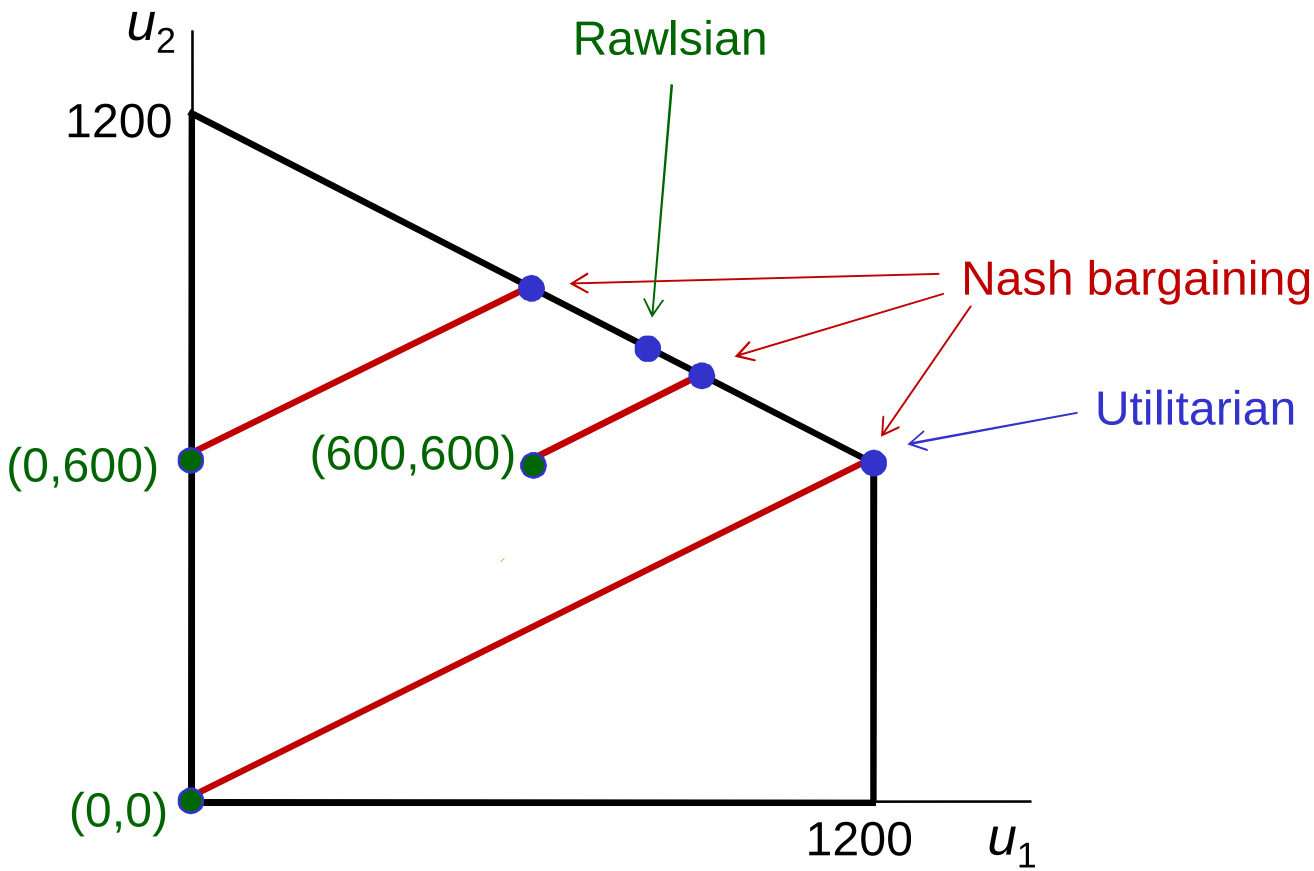
RKS point is *d* !

Rawlsian point is *u.*

Summary

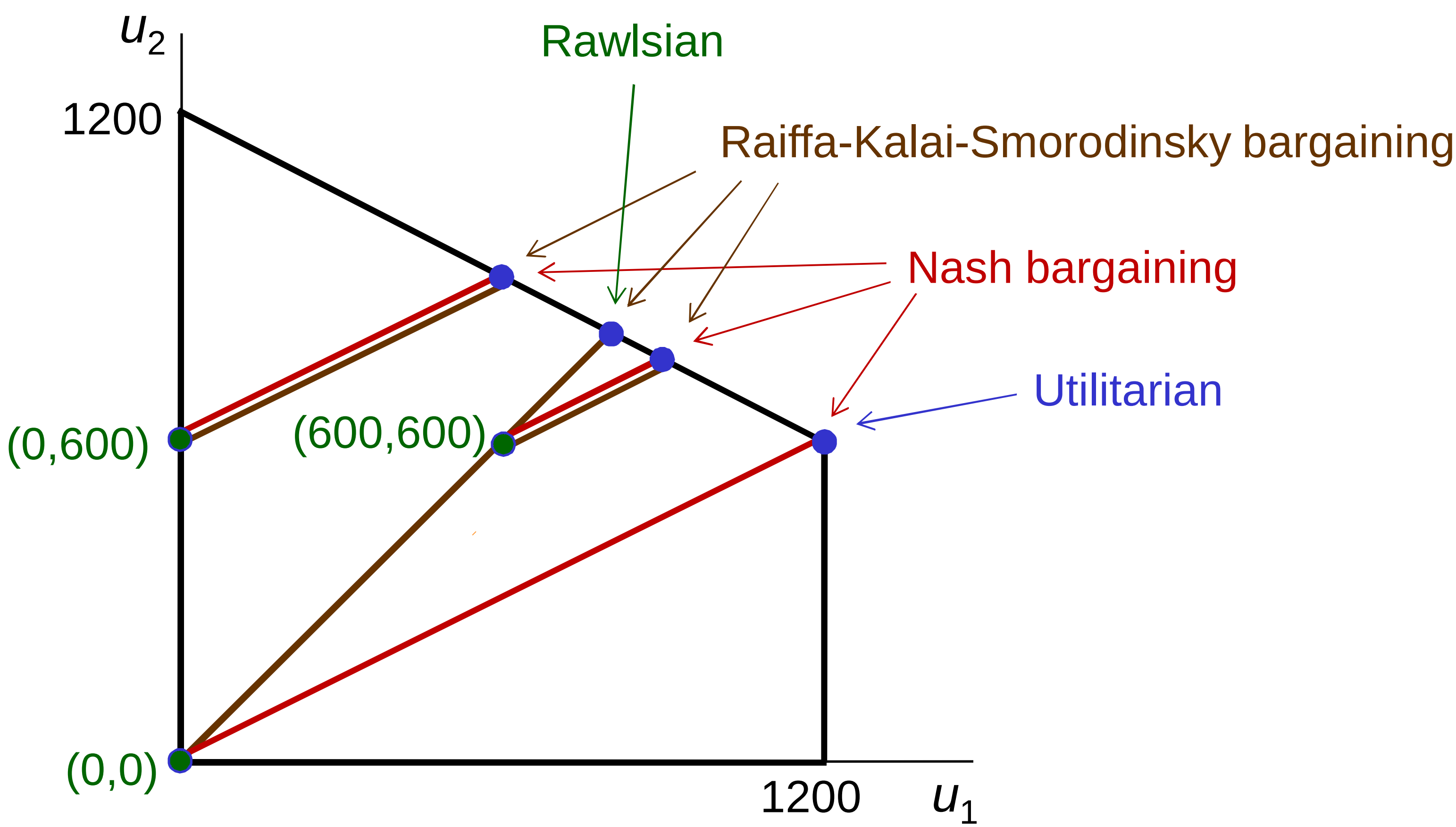


Summary



u_2
1200
Rawlsian
Nash bargaining
Utilitarian
(0,600)
(600,600)
(0,0)
1200
u_1

Summary



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1200
u_1

Combining Equity and Efficiency

- A proposed model
- Health care application

Combining Equity and Efficiency

- Utilitarian and Rawlsian distributions seem **too extreme** in practice.
  - How to combine them?

- **One proposal:**
  - Maximize welfare of **worst off** (Rawlsian)...
  - □ until this requires **undue sacrifice** from others
  - Seems appropriate in **health care** allocation.

Combining Equity and Efficiency

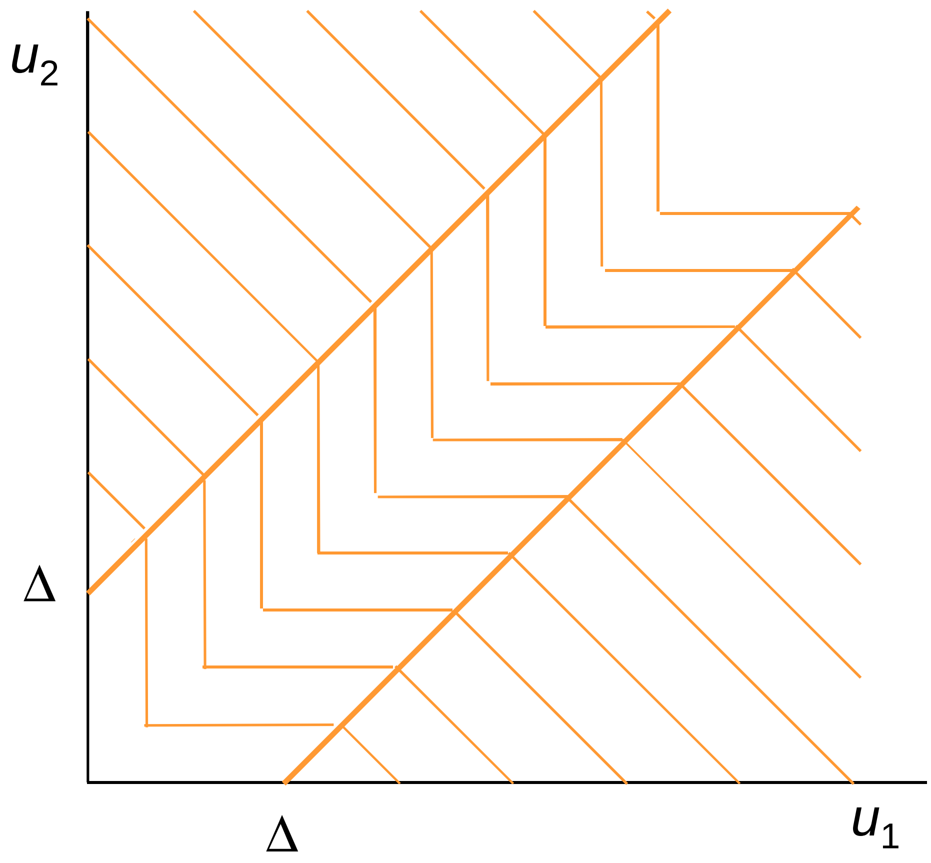
- In particular:
  - Switch from **Rawlsian** to **utilitarian** when **inequality** exceeds Δ.

Combining Equity and Efficiency

- In particular:
  - Switch from **Rawlsian** to **utilitarian** when **inequality** exceeds Δ.
  - Build mixed integer programming model.
  - Let u_i = utility allocated to person i
- For 2 persons:
  - Maximize $\min_i \{u_1, u_2\}$ (Rawlsian) when $|u_1 - u_2| \leq \Delta$
  - Maximize $u_1 + u_2$ (utilitarian) when $|u_1 - u_2| > \Delta$

Two-person Model

Contours of **social welfare function** for 2 persons.



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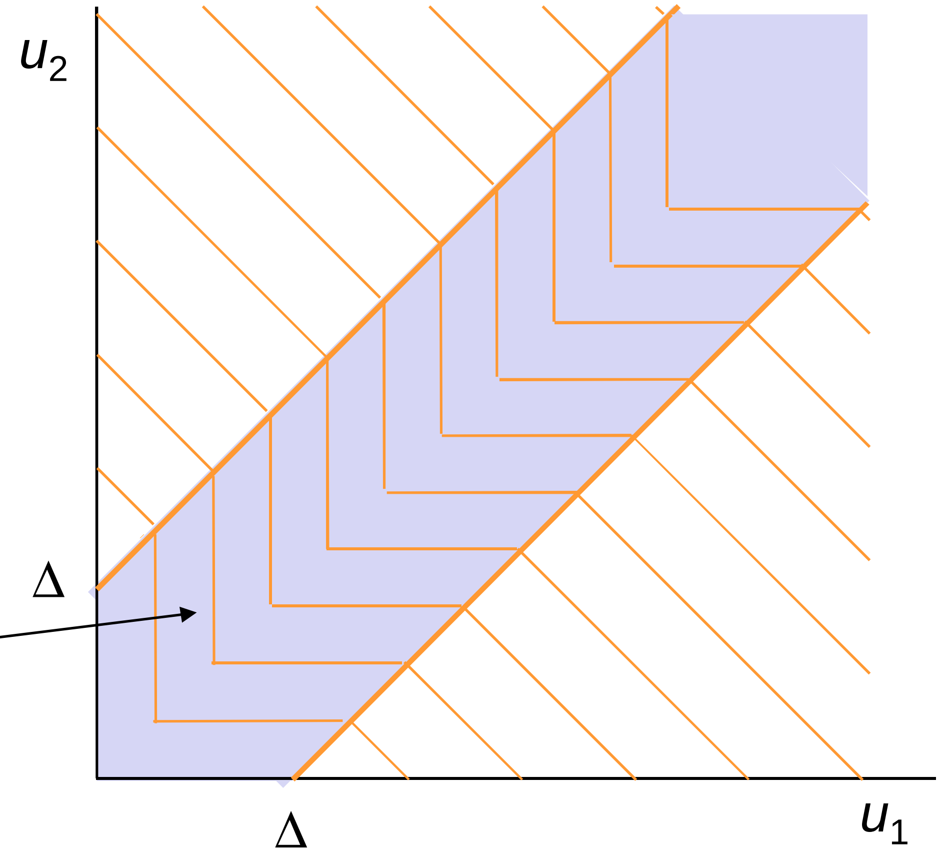


Rawlsian region

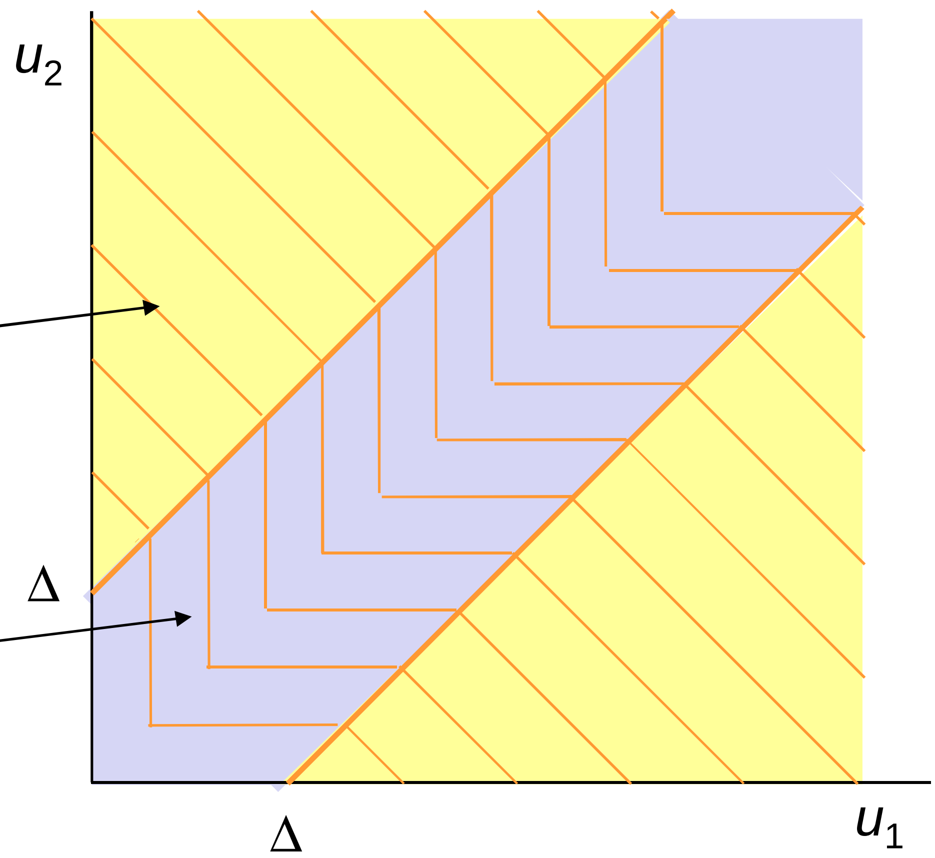
$\min\{u_1, u_2\}$

Two-person Model

Contours of **social welfare function** for 2 persons.

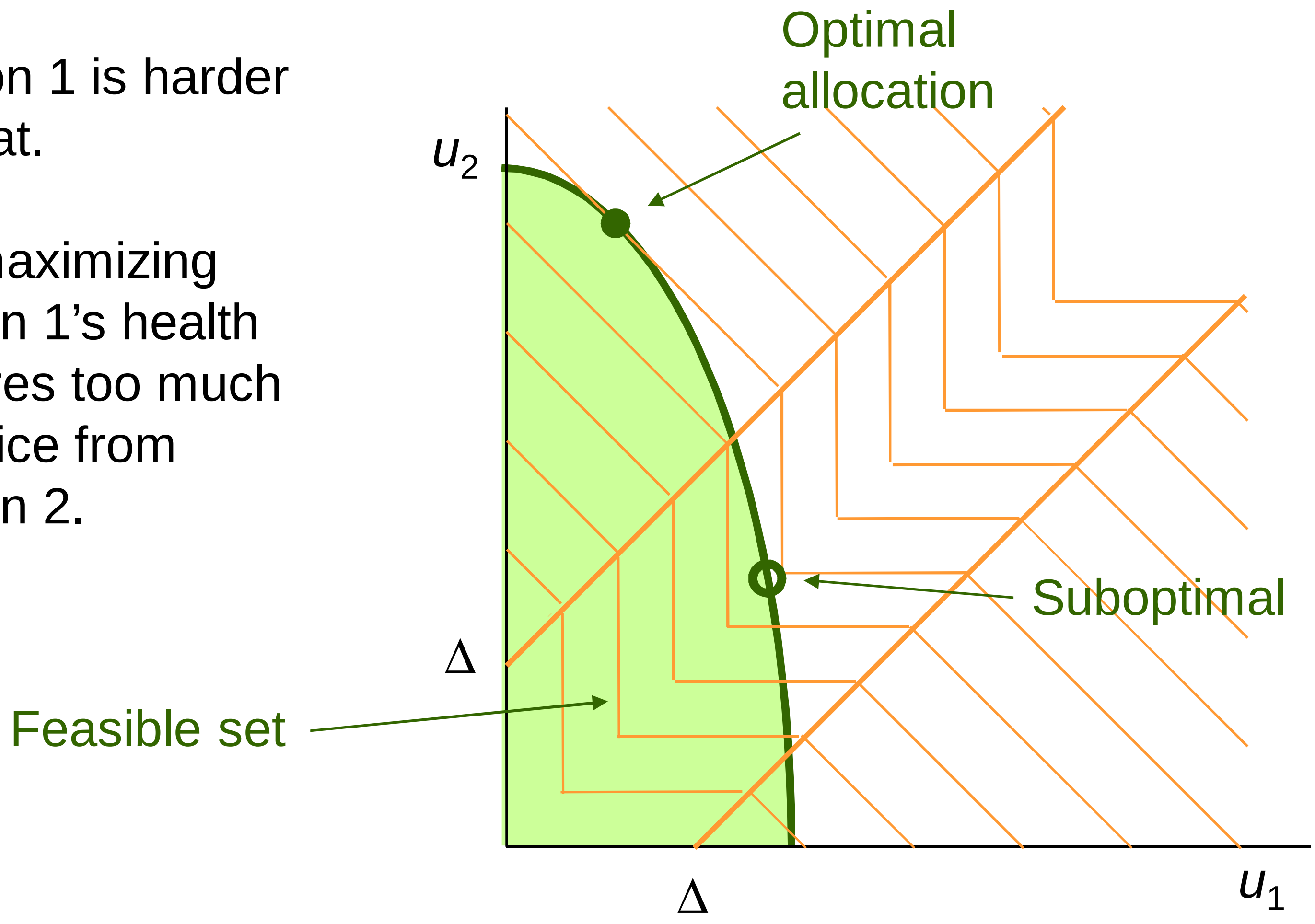
Utilitarian region
$u_1 + u_2$

Rawlsian region
$\min\{u_1, u_2\}$



Person 1 is harder to treat.

But maximizing person 1's health requires too much sacrifice from person 2.

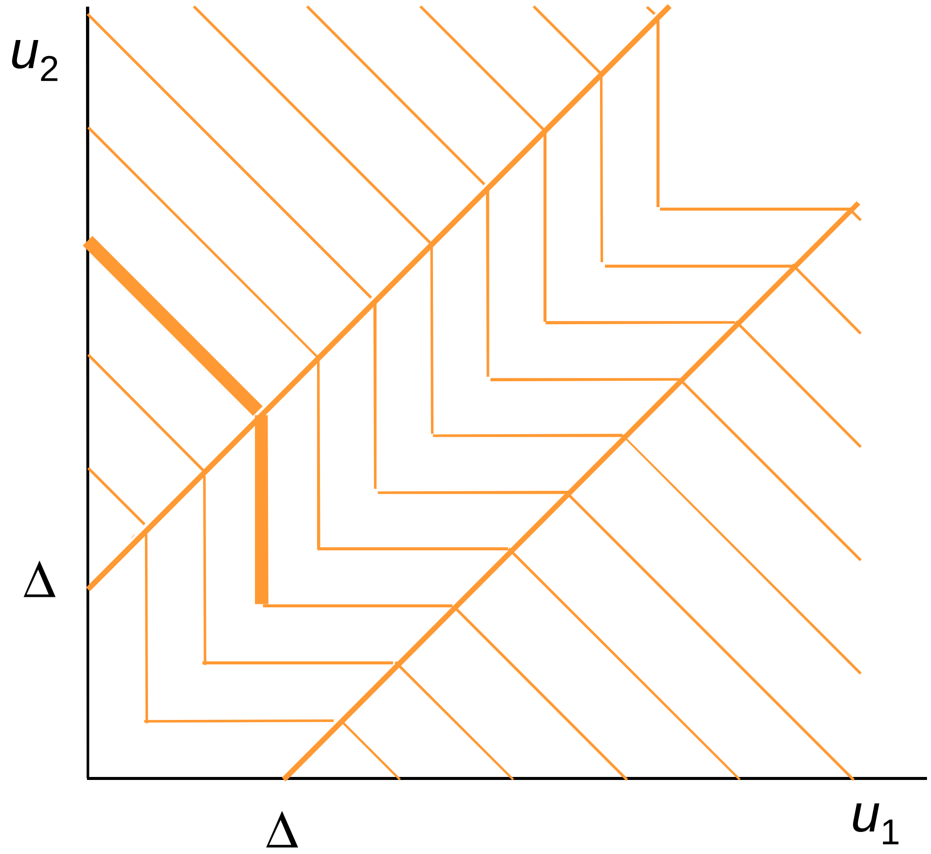


Advantages

- Only **one parameter** Δ
  - **Focus** for debate.
  - Δ has **intuitive meaning** (unlike weights)
  - Examine **consequences** of different settings for Δ
  - Find **least objectionable** setting
  - Results in a **consistent** policy

Social Welfare Function

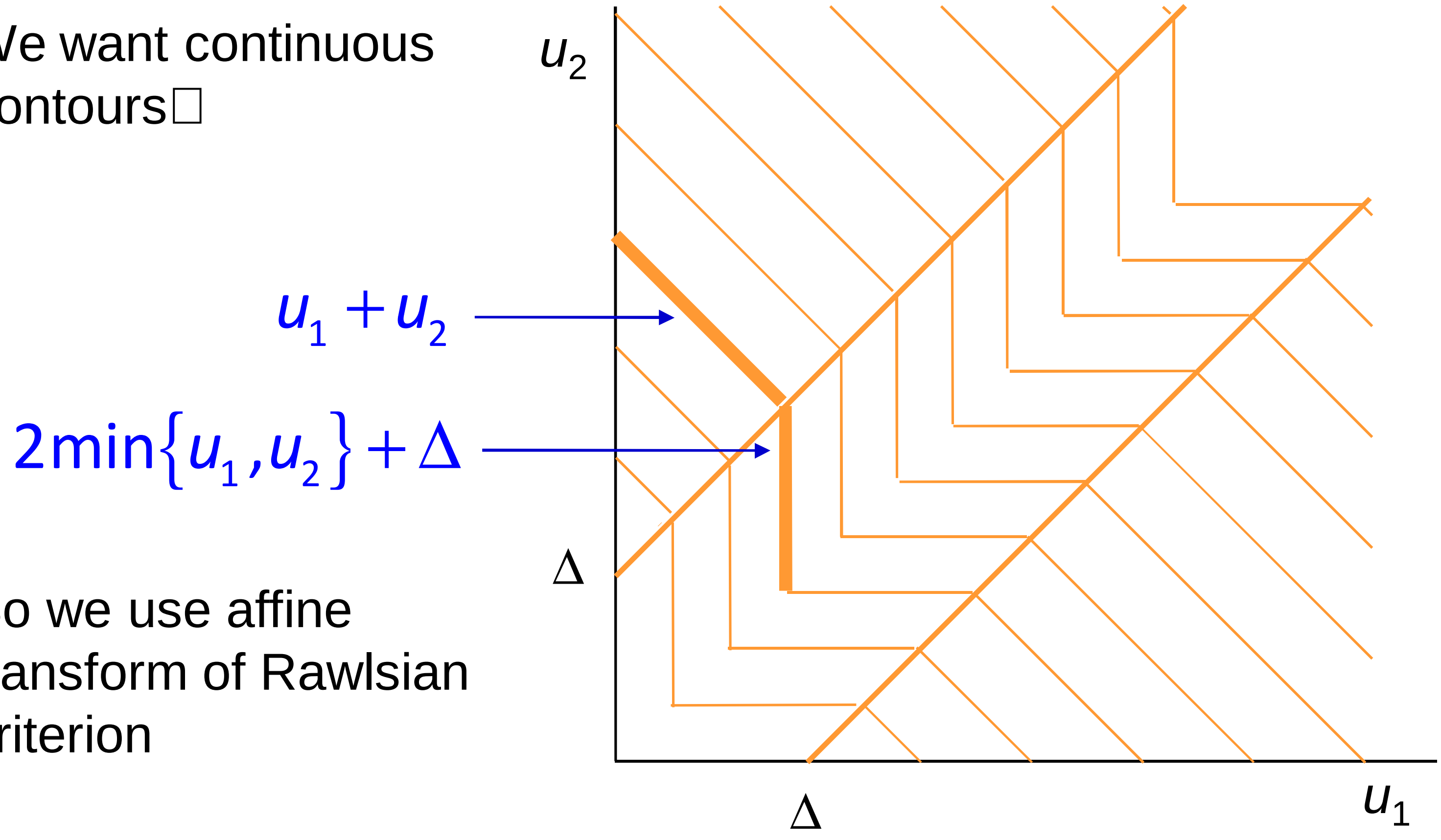
We want continuous contours□



Social Welfare Function

We want continuous contours

So we use affine transform of Rawlsian criterion



Social Welfare Function

The social welfare problem becomes

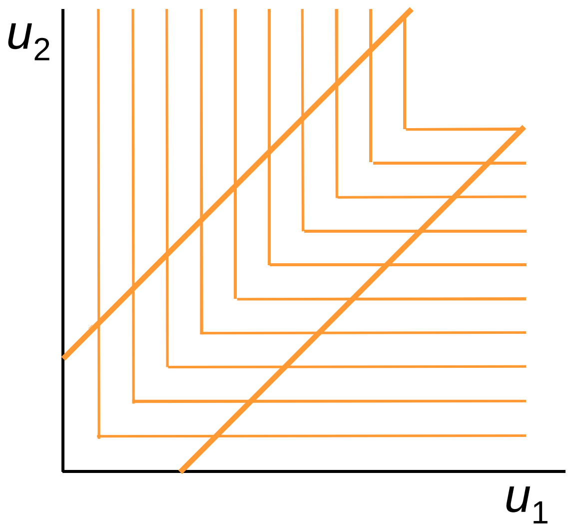
$$\max\ z$$

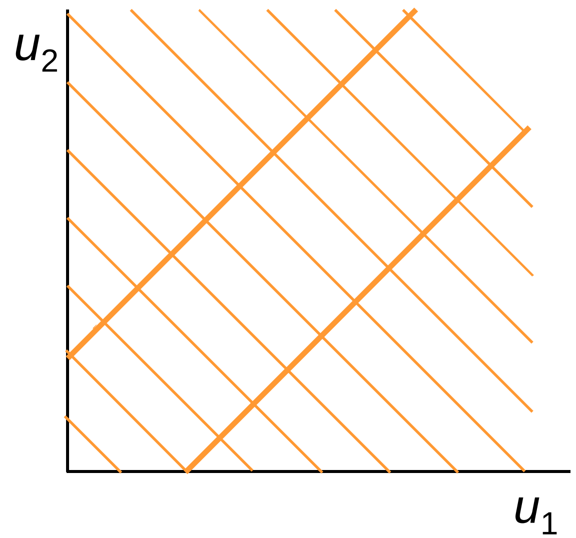
$$z \le \left\{ \begin{array}{ll} 2\min\{u_1, u_2\} + \Delta, & \text{if } |u_1 - u_2| \le \Delta \\ u_1 + u_2, & \text{otherwise} \end{array} \right\}$$

constraints on feasible set

MILP Model

Epigraph is union of 2 polyhedra.

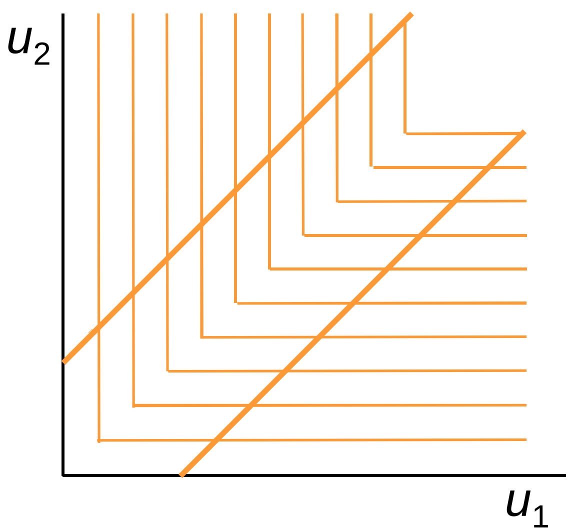


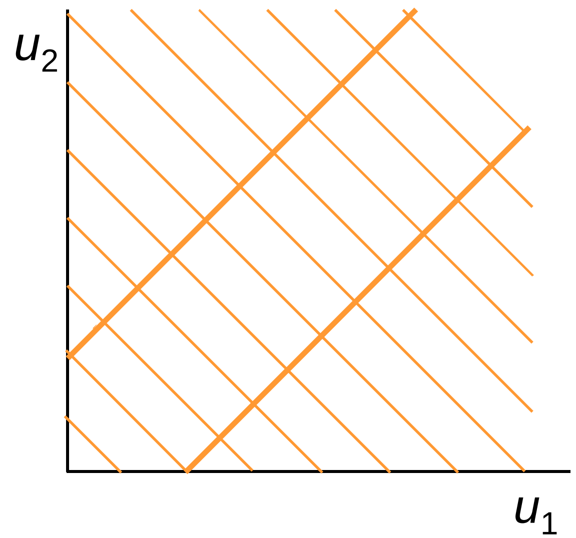


MILP Model

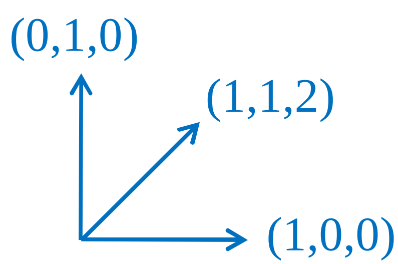
Epigraph is union of 2 polyhedra.

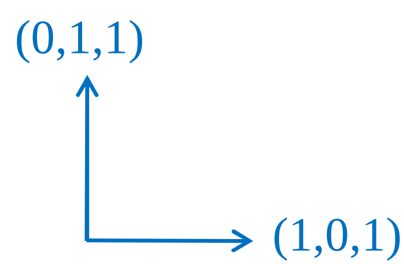
Because they have **different recession cones**, there is no MILP model.





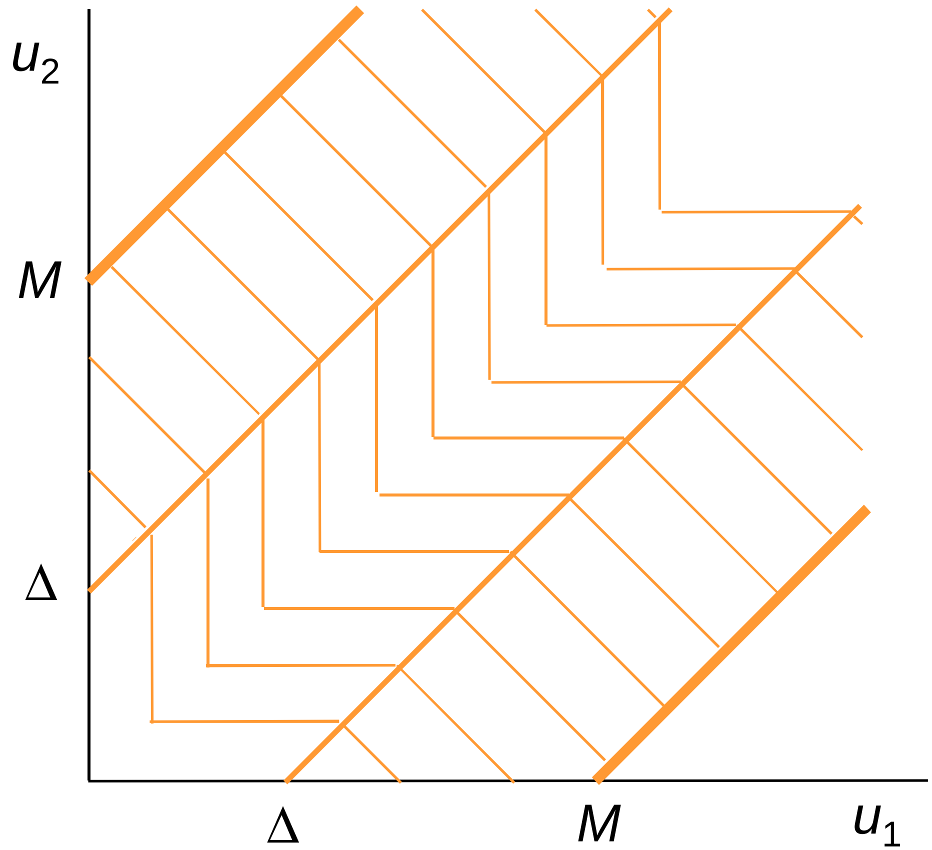
Recession directions (u_1,u_2,z)





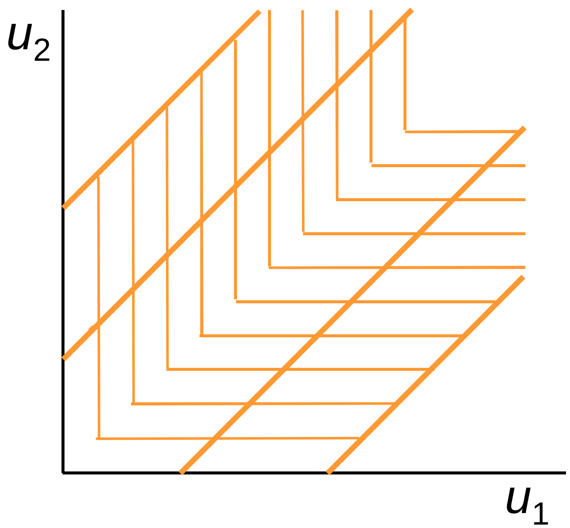
MILP Model

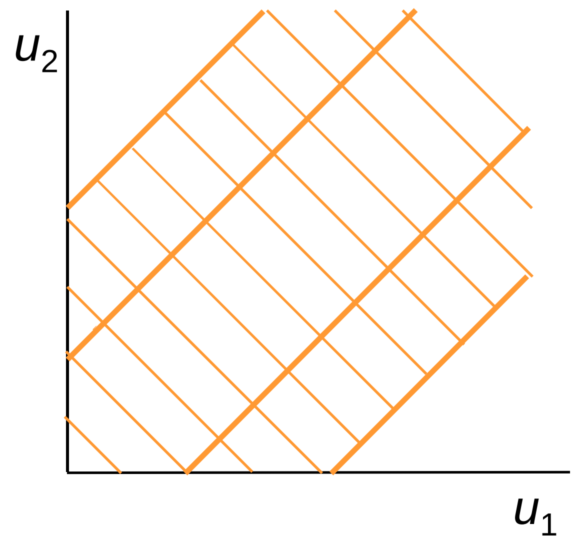
Impose constraints $|u_1 - u_2| \leq M$



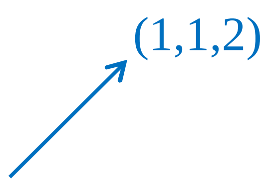
MILP Model

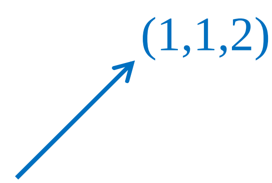
This equalizes recession cones.





Recession directions (u_1,u_2,z)





MILP Model

We have the model□

$$\max\ z$$

$$z \le 2u_i + \Delta + (M - \Delta)\delta, \quad i = 1,2$$

$$z \le u_1 + u_2 + \Delta(1 - \delta)$$

$$u_1 - u_2 \le M, \quad u_2 - u_1 \le M$$

$$u_1, u_2 \ge 0$$

$$\delta \in \{0,1\}$$

constraints on feasible set

u_1

MILP Model

We have the model□

$$\max\ z$$

$$z \le 2u_i + \Delta + (M-\Delta)\delta, \quad i = 1,2$$

$$z \le u_1 + u_2 + \Delta(1-\delta)$$

$$u_1 - u_2 \le M, \quad u_2 - u_1 \le M$$

$$u_1, u_2 \ge 0$$

$$\delta \in \{0,1\}$$

u_1

This is a **convex hull** formulation.

n-person Model

Rewrite the 2-person social welfare function as□

$$\Delta + 2u_{\min} + (u_1 - u_{\min} - \Delta)^+ + (u_2 - u_{\min} - \Delta)^+$$

$\min\{u_1, u_2\}$

$\alpha^+ = \max\{0, \alpha\}$

n-person Model

Rewrite the 2-person social welfare function as

$$\Delta + 2u_{\min} + (u_1 - u_{\min} - \Delta)^+ + (u_2 - u_{\min} - \Delta)^+$$

$\min\{u_1, u_2\}$ (annotating $u_{\min}$)

$\alpha^+ = \max\{0, \alpha\}$ (annotating $^+$)

This can be generalized to *n* persons:

$$(n-1)\Delta + nu_{\min} + \sum_{j=1}^{n} (u_j - u_{\min} - \Delta)^+$$

n-person Model

Rewrite the 2-person social welfare function as□

$$\Delta + 2u_{\min} + \left(u_1 - u_{\min} - \Delta\right)^+ + \left(u_2 - u_{\min} - \Delta\right)^+$$

$\min\{u_1, u_2\}$ $\qquad$ $\alpha^+ = \max\{0, \alpha\}$

This can be generalized to *n* persons:

$$(n-1)\Delta + nu_{\min} + \sum_{j=1}^{n} \left(u_j - u_{\min} - \Delta\right)^+$$

Epigraph is a union of $n!$ polyhedra with same recession direction $(u,z) = (1, \quad ,1,n)$ if we require $|u_i - u_j| \le M$

So there is an MILP model□

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$\max\ z$$

$$z \le u_i + \sum_{j \ne i} w_{ij},\ \text{all } i$$

$$w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \ne j$$

$$w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \ne j$$

$$u_i - u_j \le M,\ \text{all } i, j$$

$$u_i \ge 0,\ \text{all } i$$

$$\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j$$

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$
\begin{aligned}
&\max\ z \\
&z \le u_i + \sum_{j \neq i} w_{ij},\ \text{all } i \\
&w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \neq j \\
&w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \neq j \\
&u_i - u_j \le M,\ \text{all } i, j \\
&u_i \ge 0,\ \text{all } i \\
&\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \neq j
\end{aligned}
$$

u_1

Theorem. The model is correct (not easy to prove).

n-person MILP Model

To avoid $n!$ 0-1 variables, add auxiliary variables w_{ij}

$$\max\ z$$
$$z \le u_i + \sum_{j \ne i} w_{ij},\ \text{all } i$$
$$w_{ij} \le \Delta + u_i + \delta_{ij}(M - \Delta),\ \text{all } i, j \text{ with } i \ne j$$
$$w_{ij} \le u_j + (1 - \delta_{ij})\Delta,\ \text{all } i, j \text{ with } i \ne j$$
$$u_i - u_j \le M,\ \text{all } i, j$$
$$u_i \ge 0,\ \text{all } i$$
$$\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j$$

u_1

Theorem. The model is correct (not easy to prove).

Theorem. This is a convex hull formulation (not easy to prove).

n-group Model

In practice, funds may be allocated to groups of different sizes

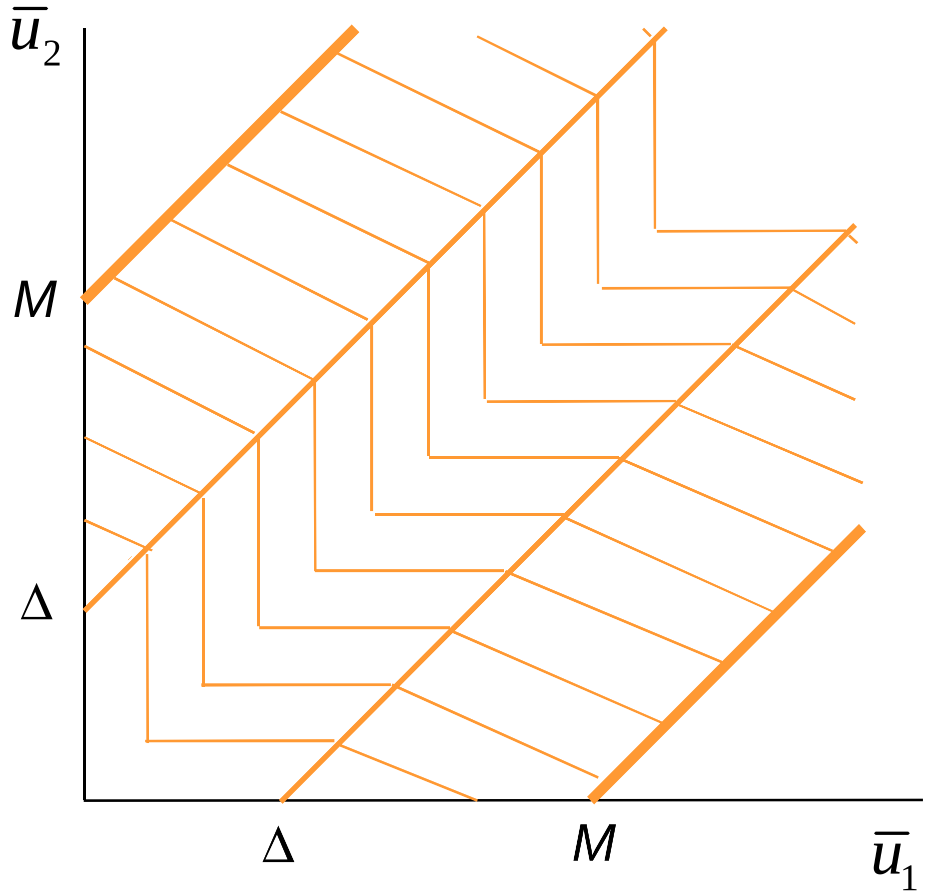
For example, disease/treatment categories.

Let $\bar{u}_i$ = average utility gained by a person in group *i*

n_i = size of group *i*

n-group Model

2-person case with $n_1 < n_2$. Contours have slope $-n_1/n_2$



n-group MILP Model

Again add auxiliary variables w_{ij}

$$
\begin{aligned}
&\max\ z \\
&z \le (n_i - 1)\Delta + n_i \bar{u}_i + \sum_{j \ne i} w_{ij},\ \text{all } i \\
&w_{ij} \le n_j(\bar{u}_i + \Delta) + \delta_{ij} n_j (M - \Delta),\ \text{all } i, j \text{ with } i \ne j \\
&w_{ij} \le \bar{u}_j + (1 - \delta_{ij}) n_j \Delta,\ \text{all } i, j \text{ with } i \ne j \\
&\bar{u}_i - \bar{u}_j \le M,\ \text{all } i, j \\
&\bar{u}_i \ge 0,\ \text{all } i \\
&\delta_{ij} \in \{0,1\},\ \text{all } i, j \text{ with } i \ne j
\end{aligned}
$$

Theorem. The model is correct.

Theorem. This is a convex hull formulation.

Health Example

Measure utility in QALYs (quality-adjusted life years).

QALY and cost data based on Briggs & Gray, (2000) etc.

Each group is a disease/treatment pair.

Treatments are discrete, so group funding is all-or-nothing.

Divide groups into relatively homogeneous subgroups.

Health Example

Add constraints to define feasible set

$$\max \ z$$

$$z \le (n_i - 1)\Delta + n_i \bar{u}_i + \sum_{j \ne i} w_{ij}, \ \text{all } i$$

$$w_{ij} \le n_j(\bar{u}_i + \Delta) + \delta_{ij} n_j (M - \Delta), \ \text{all } i, j \text{ with } i \ne j$$

$$w_{ij} \le \bar{u}_j + (1 - \delta_{ij}) n_j \Delta, \ \text{all } i, j \text{ with } i \ne j$$

$$\bar{u}_i - \bar{u}_j \le M, \ \text{all } i, j$$

$$\bar{u}_i \ge 0, \ \text{all } i$$

$$\delta_{ij} \in \{0,1\}, \ \text{all } i, j \text{ with } i \ne j$$

$$\bar{u}_i = q_i y_i + \alpha_i$$

$$\sum_i n_i c_i y_i \le \text{ budget}$$

$$y_i \in \{0,1\}, \ \text{all } i$$

y_i indicates whether group *i* is funded

QALY & cost data

Part 1

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
Pacemaker for atrioventricular heart block					
Subgroup A	3500	3	1167	13	35
Subgroup B	3500	5	700	10	45
Subgroup C	3500	10	350	5	35
Hip replacement					
Subgroup A	3000	2	1500	3	45
Subgroup B	3000	4	750	4	45
Subgroup C	3000	8	375	5	45
Valve replacement for aortic stenosis					
Subgroup A	4500	3	1500	2.5	20
Subgroup B	4500	5	900	3	20
Subgroup C	4500	10	450	3.5	20
CABG[1] for left main disease					
Mild angina	3000	1.25	2400	4.75	50
Moderate angina	3000	2.25	1333	3.75	55
Severe angina	3000	2.75	1091	3.25	60
CABG for triple vessel disease					
Mild angina	3000	0.5	6000	5.5	50
Moderate angina	3000	1.25	2400	4.75	55
Severe angina	3000	2.25	1333	3.75	60
CABG for double vessel disease					
Mild angina	3000	0.25	12,000	5.75	60
Moderate angina	3000	0.75	4000	5.25	65
Severe angina	3000	1.25	2400	4.75	70

QALY & cost data

Part 2

Intervention	Cost per person c_i (£)	QALYs gained q_i	Cost per QALY (£)	QALYs without intervention α_i	Subgroup size n_i
	22,500	4.5	5000	1.1	2
Kidney transplant					
Subgroup A	15,000	4	3750	1	8
Subgroup B	15,000	6	2500	1	8
Kidney dialysis					
Less than 1 year survival					
Subgroup A	5000	0.1	50,000	0.3	8
1-2 years survival					
Subgroup B	12,000	0.4	30,000	0.6	6
2-5 years survival					
Subgroup C	20,000	1.2	16,667	0.5	4
Subgroup D	28,000	1.7	16,471	0.7	4
Subgroup E	36,000	2.3	15,652	0.8	4
5-10 years survival					
Subgroup F	46,000	3.3	13,939	0.6	3
Subgroup G	56,000	3.9	14,359	0.8	2
Subgroup H	66,000	4.7	14,043	0.9	2
Subgroup I	77,000	5.4	14,259	1.1	2
At least 10 years survival					
Subgroup J	88,000	6.5	13,538	0.9	2
Subgroup K	100,000	7.4	13,514	1.0	1
Subgroup L	111,000	8.2	13,537	1.2	1

Results

Total budget £3 million

Δ	Pace-	Hip	Aortic	CABG			Heart	Kidney	Kidney dialysis				
range	maker	repl.	valve	L	3	2	trans.	trans.	< 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Utilitarian solution

Δ	Pace-	Hip	Aortic	CABG			Heart	Kidney	Kidney dialysis				
range	maker	repl.	valve	L	3	2	trans.	trans.	< 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Rawlsian solution

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Fund for all Δ

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

More dialysis with larger Δ, beginning with longer life span

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Abrupt change at Δ = 5.60

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Come and go together

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

In-out-in

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Most rapid change. Possible range for politically acceptable compromise

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	Kidney dialysis 1-2	Kidney dialysis 2-5	Kidney dialysis 5-10	Kidney dialysis > 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

32 groups, 1089 integer variables
Solution time (CPLEX 12.2) is negligible

Δ range	Pace-maker	Hip repl.	Aortic valve	CABG L	CABG 3	CABG 2	Heart trans.	Kidney trans.	Kidney dialysis < 1	1-2	2-5	5-10	> 10
0–3.3	111	111	111	111	111	111	1	11	0	0	000	0000	000
3.4–4.0	111	111	111	111	111	111	0	11	1	0	000	0000	000
4.0–4.4	111	111	111	111	111	111	0	01	1	0	000	0000	001
4.5–5.01	111	011	111	111	111	111	1	01	1	0	000	0000	011
5.02–5.55	111	011	011	111	111	111	0	01	1	0	000	0001	011
5.56–5.58	111	011	011	111	111	011	0	01	1	0	000	0001	111
5.59	111	011	011	110	111	111	0	01	1	0	000	0001	111
5.60–13.1	111	111	111	101	000	000	1	11	1	0	111	1111	111
13.2–14.2	111	011	111	011	000	000	1	11	1	1	111	1111	111
14.3–15.4	111	111	111	011	000	000	1	11	1	1	101	1111	111
15.5–up	111	011	111	011	001	000	1	11	1	0	011	1111	111

Results

Table 3: Solution times in seconds for m groups and different values of Δ. Instances with more than a few hundred groups seem very unlikely to occur in practice.

	Δ							
m	0	1	2	3	4	5	6	∞
330	0.02	1.2	0.67	0.56	0.50	0.30	0.03	0.02
660	0.03	4.1	1.6	1.6	0.92	0.80	0.05	0.02
990	0.02	5.2	3.1	3.6	1.5	1.5	0.08	0.02
1320	0.00	15	4.3	4.2	2.7	3.0	0.09	0.02
1980	0.02	24	11	11	11	5.4	0.14	0.02
2640	0.00	32	19	14	8.6	8.8	0.19	0.02
3300	0.17	51	43	44	34	13	0.25	0.02