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Combinatorics of Pisot Substitutions

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Dedicated to the memory of my father

Patrick Jolivet

1951–2012

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Contents

Introduction

Substitutions everywhere

A substitution is a rule that replaces letters by words. A famous example is the Fibonacci substitution defined by $a \mapsto ab$ and $b \mapsto a$. It can be iterated to produce an infinite word:

$$\begin{array}{l} a \\ ab \\ aba \\ abaab \\ abaababa \\ abaababaabaab \\ abaababaabaababaababa \\ abaababaabaababaababaabaababaabaab\ldots \end{array}$$

Several other notions of substitutions can be considered, where geometrical shapes or multidimensional words are substituted instead of letters. Four examples are given on page 10, and many more can be found on the Tilings Encyclopedia website [**HFd**], in the survey [**Fra08**] or in the book [**GS87**]. Substitutions give rise to highly ordered, self-similar objects which appear in several different domains, both as tools and objects of study.

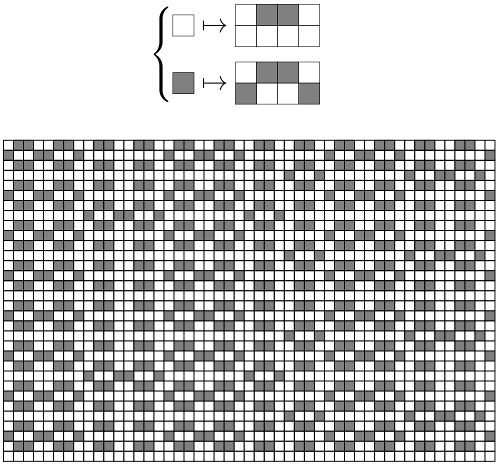
We first briefly survey some of the various areas in which substitutions show up, and then we focus on our main point of interest: one-dimensional Pisot substitutions and some higher-dimensional combinatorial tools that arise in their study.

Substitutions in combinatorics on words Substitutions provide a way to construct words with very specific properties. For example, the Thue-Morse substitution $1 \mapsto 12, 2 \mapsto 21$ generates the infinite word $12212112\ldots$ which has the property that it does not contain any cube (a word of the form www with nonempty w), as proved by Thue [**Thu77**]. Many other properties and historical facts about this sequence are summarized in [**AS99**].

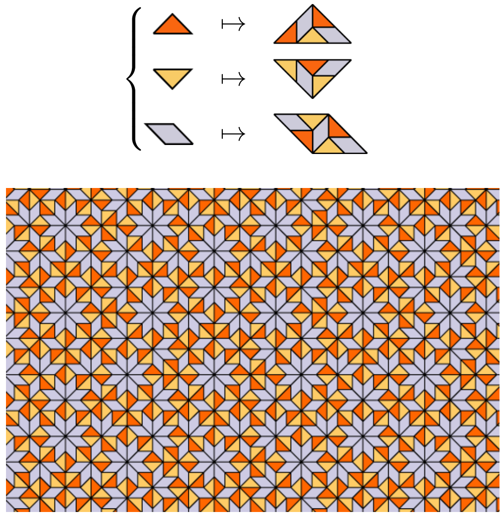
Another example is the class of Sturmian sequences, whose language can be obtained by iterating the substitutions $1 \mapsto 1, 2 \mapsto 21$ and $1 \mapsto 12, 2 \mapsto 2$. These sequences can be characterized in many other ways. They are the infinite sequences of two letters with complexity $n + 1$ (*i.e.*, they have exactly $n + 1$ factors of length n). They also correspond to the codings of irrational rotations on the circle [**MH40**], or equivalently, the codings of discrete lines with irrational slope. See [**PF02**, Chapter 6] and [**Lot97**, Chapter 2] for a survey of their numerous properties.

Substitutions also give rise to interesting language-theoretic decision problems; see for example [**KL03**], or the class of "HD0L" problems, concerning periodicity properties of infinite substitutive words (discussed more in detail in Remark 1.1.4).

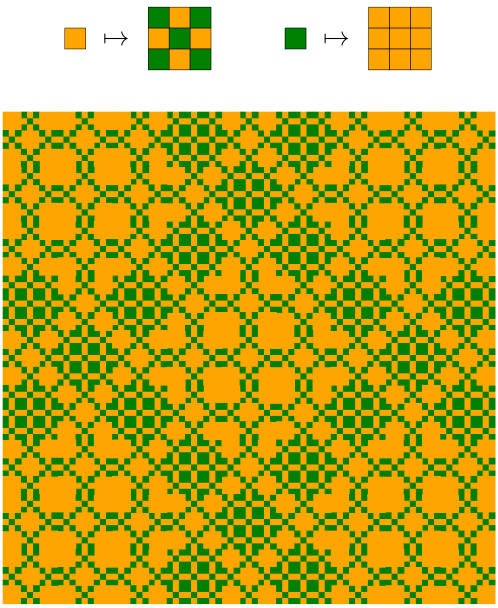
Substitutions of constant length are linked by Cobham's theorem [**Cob69**] to automatic sequences [**AS03**]. A number-theoretical applications of this connection is that every real number whose expansion is automatic is either rational or transcendental [**AB07**].



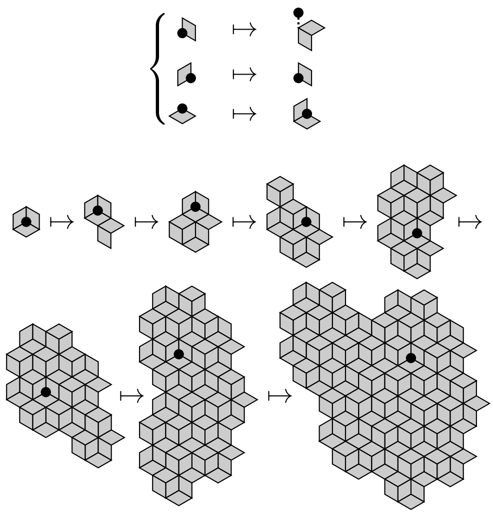
(a) This kind of substitutions is used in higher-dimensional symbolic dynamics, in order to provide suitable "computation zones" for Turing machine simulations. This example is taken from [**AS13b**].



(b) Ammann-Beenker tilings are examples of substitutive, aperiodic, cut-and-project tilings for which simple local matching rules exist, see [**GS87**]. Picture taken from [**HFb**].



(c) This two-dimensional substitution, due to J. Haferman, is an example of a two-dimensional automatic sequence [**AS03**].



(d) Iterating the dual map associated with the substitution $\sigma : 1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 13$. The obtained patterns cover arbitrarily large regions of a discrete plane, and they can be renormalized so that their limit converges to the *Rauzy fractal* of σ.

Aperiodic tile sets, theory of computation Substitutions have been a key ingredient to prove the undecidability of the tiling problem, stated by Wang in [**Wan61**]: "does a given set of tiles admits a tiling of the plane?" Most of the currently known undecidability proofs follow the initial strategy used to solve this problem in [**Ber66, Rob71**], by simulating Turing machines with tilings. In these simulations, some higher-dimensional substitutions are needed for technical reasons, in order to provide suitable "computation space" for the Turing machines. These constructions also gave rise to the first examples of aperiodic tile sets, that is, set of tiles which can tile the plane, but only in a non-periodic way. All the known constructions of aperiodic tile sets use substitutive hierarchical structures, with a few exceptions only [**Kar08, Mon13**].

These techniques are also often used to perform diverse computational tasks in symbolic dynamics, using variants of the Robinson tile set [**Rob71**], for example in the study of simulations between subshifts [**Hoc09, DRS12, AS13b**], or to prove that the tilings induced by an irrational discrete plane can be enforced by local rules if and only if the slope of the plane is computable [**FS12**].

An "inverse question" to the above considerations is asking whether for *every* substitution, the set of tilings it generates can be realized as the set of the valid tilings of a finite set of tiles. It turns out that the answer is positive, in the case of non-deterministic rectangle substitutions in $\mathbb{Z}^2$ [**Moz89**], and in more abstract geometrical settings as well [**GS98, FO10**].

One-dimensional substitutions and symbolic dynamics Substitutions have been used as a tool in dynamics to understand the behaviour of some complicated systems. For example, Morse [**Mor21**] successfully used the Thue-Morse substitution $1 \mapsto 12, 2 \mapsto 21$ to prove the existence of some non-periodic recurrent geodesics on a surface of constant negative curvature. Another example is the use of the Fibonacci substitution $1 \mapsto 12, 2 \mapsto 1$, which gives a coding of the dynamics of the translation $x \mapsto x + (1+\sqrt{5})/2$ in the one-dimensional torus $\mathbb{R}/\mathbb{Z}$ [**MH40**]. A natural question arises:

What kind of dynamics occurs in one-dimensional substitutive systems?

These systems are minimal and have zero entropy, so their dynamics are very constrained among the vast class of symbolic dynamical systems. However, various answers to the above question have been given in terms of geometrical interpretations of the dynamics, as shown by the (non-exhaustive) list of examples below.

- The Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ is the translation $R_\beta : \mathbb{T}^2 \to \mathbb{T}^2$ on the two-dimensional torus, defined by $R_\beta(x) = x + (1/\beta, 1/\beta^2)$, where $\beta \approx 1.839$ is the dominant root of $x^3 - x^2 - x - 1$, which is a cubic Pisot number [**PF02**].
- The Thue-Morse substitution $1 \mapsto 12, 2 \mapsto 21$ and the Rudin-Shapiro $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 42, 4 \mapsto 43$ factors onto the dyadic rotation (*i.e.*, the dynamics of $x \mapsto x+1$ on the group of 2-adic integers) and the Chacon substitution $1 \mapsto 1121, 2 \mapsto 2$ is an exduction of a triadic rotation [**Fer95, PF02**].
- Arnoux-Rauzy sequences are systems generated by arbitrary infinite products of three particular 3-letter substitutions (which are given in Section 1.5). The symbolic dynamics of each product can be realized as an exchange of six intervals on the circle [**AR91**]. In the case of periodic products, they have been proved to be semi-conjugate to two-dimensional toral translations [**AI01, BJS12**]. Some infinite products have been proved to be weak mixing [**CFM08**].

- Some classes of substitutive systems have been proved to be measurable factors of interval exchange maps [**BJ12**].
- Every substitution of *constant length* ℓ (the images all have the same size ℓ) factors onto a translation on the direct product of the ℓ-adic group $\mathbb{Z}_\ell$ with a finite group [**CK71, Mar71, Dek78**].

More details about the above statements can be found in [**Que10, PF02**] and in Section 1.4. A whole class of substitutions which are believed to be semi-conjugate to toral translations are the substitutions satisfying an algebraic restriction, being *Pisot*, which we discuss next.

Pisot substitutions

Rauzy fractals The *Pisot condition* is an algebraic condition on a substitution σ which ensures that its action has a unique expanding direction. (This can be stated in terms of spectral properties of the incidence matrix of σ, see Section 1.1.) Thanks to this property, an infinite broken line can be constructed from σ, with the very special property that it can be projected onto a plane in such a way that the projection is bounded. This allows us to define the *Rauzy fractal* of σ, by taking the closure of this projection. Rauzy fractals can naturally be decomposed into n tiles, where n is the size of the alphabet of σ.

These fractals owe their name to Gérard Rauzy, who introduced them in order to study properties of the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ [**Rau82**]. They are formally defined in Section 1.3, and an example is pictured in Figure 0.1.

Using Rauzy fractals in dynamics Let us illustrate our main point on the particular 3-letter substitution $\sigma : 1 \mapsto 12, 2 \mapsto 1312, 3 \mapsto 112$, which is an ordinary example a Pisot substitution. It can be iterated to produce an infinite word u satisfying $\sigma(u) = u$, a *fixed point* of σ:

$$u = 121312121121213121213121212131212131212\ldots$$

The most natural way to associate a dynamical system with σ is to define the set $X_\sigma \subseteq \{1, 2, 3\}^{\mathbb{Z}}$ of all the bi-infinite sequences that have the same language as the fixed point u, and the symbolic shift S which shifts a sequence $x \in X_\sigma$ by one step: $S(x) = (x_{n+1})_{n\in\mathbb{Z}}$. In the cases we will consider, X_σ is an uncountable set and it does not depend on the chosen fixed point u (see Section 1.4 for more details). This symbolic dynamical system is minimal, has zero entropy and does not contain any shift-periodic point.

Now we want to express (X_σ, S) as a translation on the two-dimensional torus $\mathbb{T}^2$. The classical approach from symbolic dynamics is to find a 3-set partition of $\mathbb{T}^2$ and a translation $R : \mathbb{T}^2 \to \mathbb{T}^2$ such that every orbit in $(\mathbb{T}^2, R)$ can be encoded by a sequence in X_σ, by recording the history of the orbit with respect to the 3-set partition. Hence there is correspondence between the orbits of the shift S in X_σ and the orbits of R in the torus.

There are many possible ways to build such a partition *a priori*, but one fruitful approach can be made via Rauzy fractals, as initiated by Rauzy [**Rau82**]. In the case of our example σ, the Rauzy fractal admits a periodic tiling of the plane, so its three tiles yield a natural partition of the torus $\mathbb{T}^2$. Using this partition, it is possible to find a translation R such that the systems (X_σ, S) and $(\mathbb{T}^2, R)$ have the same dynamics, by using the three tiles $1, 2, 3$ to code the orbits of R in $\mathbb{T}^2$. This is illustrated in Figure 0.1.

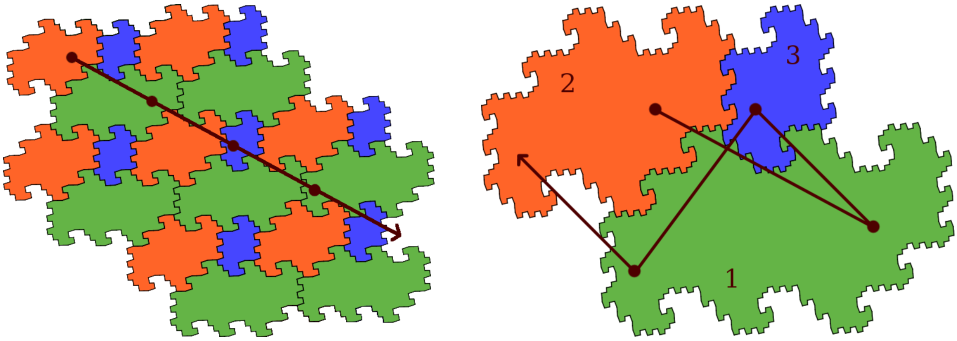


Figure 0.1: Using the Rauzy fractal of σ to give a geometrical interpretation of the dynamics of (X_σ, S). Four iterations of the translation $R : \mathbb{T}^2 \to \mathbb{T}^2$ are shown above, whose corresponding coding is 21312. On the left it represented in the periodic tiling of the plane, and on the right it is represented within the fundamental domain.

The construction described above can be carried out for any Pisot substitution [**AI01, CS01b**]. However, in general, it could happen that the resulting geometrical realization is only a factor of the original system (X_σ, S). Proving that the two systems are always equivalent (semi-conjugate) is one of the most famous open problems in this area, known as the *Pisot conjecture*. More detailed statements can be found in Sections 1.3 and 1.4.

Dual substitutions, combinatorial substitutions A powerful tool for the study of one-dimensional Pisot substitutions has been provided by Arnoux and Ito in [**AI01**], where a one-dimensional substitution σ is expressed as a higher-dimensional substitution (called *dual substitution*, illustrated in Figure (d) page 10). Many properties of σ can be reformulated in terms of geometric combinatorial properties of its dual substitution, as we will see later.

This viewpoint is the main guideline of the work done in this thesis: we establish some properties of one-dimensional substitutions using this tool, and then we study the general class of higher-dimensional combinatorial substitutions stemming from the examples provided by dual substitutions.

Contributions of this thesis

In the first five chapters, we establish several new properties of one-dimensional Pisot substitutions. Most of the results that currently exist concern a *single* substitution, but we obtain results about some infinite families of substitutions defined by taking products over a given finite set.

In the last two chapters, we “step back” from the Pisot assumption to study some more general objects arising from the combinatorial tools used in the previous chapters, focusing on some computational (un)decidability questions.

We will start by introducing some background notions in **Chapter 1**, which will be needed in the first five chapters.

Infinite families of substitutions Our first object of study is the infinite family of substitutions obtained by taking arbitrary finite products $\sigma_{i_1} \cdots \sigma_{i_n}$ over a given finite set of substitutions $\{\sigma_1, \ldots, \sigma_k\}$. Given such a family we tackle the following questions:

- *Do the substitutions obtained in this way have good dynamical properties? In particular, are they semi-conjugate to toral translations?*
- *What properties of products $\sigma_{i_1} \cdots \sigma_{i_n}$ can be "translated" into a language-theoretical characterization of words $i_1 \cdots i_n$?*

In **Chapter 2** we will introduce some combinatorial tools that will allow us to give some answers to the above questions in **Chapter 3**.

Since we are dealing with Pisot substitutions, we must make sure that the products $\sigma_{i_1} \cdots \sigma_{i_n}$ we consider are Pisot. However, even if each σ_i is Pisot, it is not guaranteed that their products are also Pisot, because spectral properties of matrix products behave in non-predictable ways (see Remark 1.1.3).

We will hence restrict ourselves to three specific families of substitutions, all acting on 3-letter words $\{1, 2, 3\}^\star$, which have nice properties under composition: the *Arnoux-Rauzy* substitutions, the *Brun* substitutions and the *Jacobi-Perron* substitutions. These families of substitutions (defined in Section 1.5) provide interesting objects of study. Arnoux-Rauzy sequences are well-studied in dynamics and combinatorics on words (some of their properties are surveyed in Section 1.5), and the Brun and Jacobi-Perron are substitutions associated with some classical multidimensional continued fraction algorithms.

We answer the above questions in the following way for the three families of substitutions described above.

- For every admissible finite product σ, the symbolic dynamical system (X_σ, S) is semi-conjugate to a translation on a two-dimensional torus (Theorem 3.1.1).
- The language of the words $i_1 \cdots i_n$ such that the origin is an interior point of the Rauzy fractal of $\sigma_{i_1} \cdots \sigma_{i_n}$ can be characterized in a simple language-theoretic way (Theorem 3.2.3). For example in the Brun case, the characterization is given by a rational language (more precisely, the set of edge labellings of the cycles in a finite directed graph of four vertices).

We will also prove that all the Rauzy fractals considered above are connected (Theorem 3.2.1). Note that there are many works devoted to decide such properties for a *single* substitution σ (many algorithms exist, see for example [**BR10**, Chapter 5]), but that only a few works are devoted to the study of some infinite classes. Such works include a study of the Rauzy fractals of the class $1 \mapsto 1^a 2, 2 \mapsto 1^b 3, 3 \mapsto 1$ parametrized by integers a, b [**LMST13**], and the use of some homological methods [**BŠW13**], which gives an alternative of Theorem 3.2.3 for Arnoux-Rauzy substitutions.

Combinatorial tools, dual substitutions The goal of **Chapter 2** is to develop combinatorial tools in order to prove the results stated above. We take an approach initiated in [**IO93, IO94, AI01**]: every unimodular substitution $\sigma : \{1, 2, 3\}^\star \to \{1, 2, 3\}^\star$ can be associated with a *dual substitution* $\mathbf{E}_1^\star(\sigma)$, which does not act on $\{1, 2, 3\}^\star$, but on two-dimensional patterns consisting of a union of unit cube faces in $\mathbb{R}^3$. (This is illustrated in Figure (d) page 10, and more details are given in Section 1.2.)

The main advantage of working with dual substitutions is that some of the properties of σ we are interested in can be reformulated in terms of qualitative properties of the growth of the patterns $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$, obtained by iterating $\mathbf{E}_1^\star(\sigma)$ from a "seed" $\mathcal{U} =$ . For instance, these patterns cover discs of arbitrarily large radius if and only if (X_σ, S) is semi-conjugate to a toral translation [**IR06**].

Our main result is that for *any* sequence $(\sigma_{i_n})_{n\in\mathbb{N}}$, the patterns $\mathbf{E}_1^\star(\sigma_{i_1})\cdots\mathbf{E}_1^\star(\sigma_{i_n})(\mathcal{U})$ cover discs of arbitrarily large radius (Theorem 2.5.5), in the case of Arnoux-Rauzy, Brun and Jacobi-Perron substitutions. This not only proves the desired result for arbitrary finite products $\sigma_{i_1}\cdots\sigma_{i_n}$ (by considering the periodic sequence $(i_1\cdots i_n)^\infty$), but also opens new grounds for the study of systems generated by *arbitrary infinite* products (also known as S-adic systems, see [**BST13**]). We will prove similar results in Theorems 2.5.1, 2.5.2 and 2.5.4, which will be needed for the other applications in Chapter 3.

Our approach to prove the "arbitrarily large radius" property mentioned above is based on the radically innovative work initiated by Ito and Ohtsuki in [**IO94**], whose strategy is proving that more and more "thick annuli" are generated (and preserved) around $\mathcal{U}$ by iterating dual substitutions (more details are given in Section 2.1). This strategy is used in [**IO94**] to the study the Jacobi-Perron substitutions. However, several difficulties (which are not considered in [**IO94**]) have to be dealt with for this approach to work in practice. We address these difficulties by introducing several new combinatorial tools, as we describe below.

- It can happen that the image of a connected pattern by a dual substitution is disconnected, which is a problem if we want the image of an annulus to remain an annulus. One way around this is the notion of $\mathcal{L}$-*covering* (introduced in [**IO94**]), which intuitively means that a pattern is "path connected" by paths of contiguous connected patterns (the elements of $\mathcal{L}$). We have needed to extend this notion by considering patterns in $\mathcal{L}$ of more than two faces to be able to find suitable sets $\mathcal{L}$ for the substitutions we study. (In particular, the set of 7 patterns given in [**IO94**] for Jacobi-Perron substitutions is not sufficient.) See Section 2.2.
- Contrary as stated in [**IO94**], $\mathcal{L}$-coverings alone are not sufficient to ensure that the image of an annulus remains an annulus. We thus introduce a stronger condition (*strong* $\mathcal{L}$-coverings) under which the desired annulus property holds. Checking that such a condition holds for our particular families of substitutions involves some rather heavy computations, for which computer algebra is needed. See Sections 2.2 and 2.3.
- Finally, we must prove that some annuli are eventually generated around the seed we start with. This was originally done in [**IO94**] by studying a large graph whose vertices are patterns, which tracks all the possible surroundings of the seeds. The problem with this approach is that the obtained graphs are too large and difficult to analyse in some particular cases like the Brun substitutions. We thus introduce a new tool, *generation graphs*, which yield much smaller graphs (the vertices are single faces and not patterns), which can be constructed algorithmically, and which contain the information that we need about the annulus generation properties around seeds. See Section 2.4.

This strategy is powerful, as witnessed by the many applications that we will give in Chapter 3, and by the fact that we are able to adapt it easily to another family of substitutions to answer a question from discrete geometry in Section 3.4. This approach is also used in [**FIY13**] to study the substitutions associated with another (multiplicative) continued fraction algorithm.

Discrete plane generation The fact that the Brun (or the Jacobi-Perron) substitutions are associated with a continued fraction algorithm allows us to get a general strategy to generate a discrete plane using dual substitutions, given *any* normal vector: it suffices to expand the vector using the continued fraction algorithm, and to apply successively the corresponding dual substitutions starting from $\mathcal{U}$ (a key property of dual substitutions is that they generate patterns belonging to discrete planes).

In the case of the Brun or Jacobi-Perron substitutions, we will see that the seed $\mathcal{U}$ is not always enough: sometimes only an infinite strict subset of the discrete plane is obtained. However, our results (Theorem 2.5.1) imply that there exist some *finite* seeds starting from which this strategy generates a full discrete plane. It is important to note that the seeds do *not* depend on the choice of the initial vector, and that the existence of such finite seeds was not guaranteed *a priori*.

Fernique [**Fer07b**] gave a strategy to generate discrete planes with a rational normal vectors using similar tools. Our results extend his results in the case where the normal vector is totally irrational. This result can also be interpreted as a two-dimensional generalization of the fact that the language of Sturmian sequences (codings of discrete lines in $\{1,2\}^{\mathbb{Z}}$) is the language generated by the two substitutions $1 \mapsto 1, 2 \mapsto 21$ and $1 \mapsto 12, 2 \mapsto 2$. We can then claim the following:

> *The language of two-dimensional Sturmian sequences (codings of discrete planes in $\{1,2,3\}^{\mathbb{Z}^2}$) is the language generated by the three dual substitutions $\Sigma_1^{\mathsf{Brun}}, \Sigma_2^{\mathsf{Brun}}, \Sigma_3^{\mathsf{Brun}}$ associated with the Brun algorithm (defined in Section 1.5).*

Other applications In **Chapter 3** we will give some other applications of the results established in the second chapter. The first three are of number-theoretical nature. (1) Thanks to some properties of the Jacobi-Perron algorithm, we are able to associate Rauzy fractal dynamics with every cubic real field in Theorem 3.3.1. (2) We prove some convergence results about some of the considered continued fraction algorithms in Theorem 3.3.2. (3) The language-theoretical characterization of the Rauzy fractals with zero interior point can be reinterpreted in the context of β-numeration, in terms of *finiteness* properties of digital expansions in non-integer bases. We obtain in Section 3.2 a characterization of the products of substitutions for which the "extended (F) property" holds (which is a generalization of the classical (F) property in β-numeration). More detailed statements of the above results and more references are given in Chapter 3.

A last application of our discrete plane generation results will be given in Theorem 3.4.8, where we answer a question from discrete geometry, about the "thickness" used to define a discrete approximation of a two-dimensional plane. This application is joint work with Valérie Berthé, Damien Jamet and Xavier Provençal [**BJJP13**].

The results of Chapters 2 and 3 (and their applications to products of Pisot substitutions) have been obtained in collaboration with Valérie Berthé, Jérémie Bourdon and Anne Siegel in [**BJS13, BJS12, BBJS13**].

Fundamental groups of Rauzy fractals One of the many topics in Rauzy fractal topology is the study of their fundamental group, which intuitively measures how many "holes" a fractal has. All the currently known examples of Rauzy fractals either have a trivial fundamental group (the fractal is simply connected), or have a non-free uncountable fundamental group (the fractal has infinitely many holes at arbitrarily small scales). Several criteria to determine if a given Rauzy fractal has a trivial or uncountable fundamental group have been given in [**ST09**].

The above facts suggest that there is a "dichotomy" between trivial and uncountable fundamental groups. A Rauzy fractal with nontrivial but countable fundamental group would correspond to Rauzy fractal with "finitely many holes". It is then natural to ask:

> *What kind of groups arise as the fundamental group of a Rauzy fractal? In particular, can such groups be nontrivial but countable?*

In **Chapter 4** we answer the second question above completely: we prove that every free group of finite rank can be realized as the fundamental group of a planar Rauzy fractal (Theorem 4.5.3). This answer is complete in the countable case because such groups are the only ones that can appear in this case (Proposition 4.2.5). These results are obtained by defining some symbolic operations on substitutions (*symbol splittings* and *conjugacy by free group automorphisms*), which naturally translate into topological manipulations of the subtiles of Rauzy fractals. This is work done in collaboration with Jun Luo and Benoît Loridant, published in [**JLL13**].

Results about topological properties of Rauzy fractal are interesting in their own right, but they also have implications in several other domains, for example in number theory where some particular topological properties can be interpreted in terms of properties about a number system. More details about this connection are given in Section 3.3.

Chapter 5 is devoted to some examples of Rauzy fractals with diverse properties. This short chapter will serve as a transition from the previous chapters to the last two chapters, in which the focus will not be directly on one-dimensional Pisot substitutions anymore.

General notions of multidimensional substitutions The idea behind the notion of $\mathcal{L}$-covering used in Chapter 2 is to express dual $\mathbf{E}_1^\star$ substitutions using "concatenation rules": instead of directly computing the image of a pattern by computing the position of the image of each face (using Definition 1.2.3), we compute the image by *concatenating* the images one after the other (this procedure is described more in details in Section 6.1).

The aim of **Chapter 6** is to study the more general class of *combinatorial substitutions*, which are defined by specifying how patterns must be concatenated in the images. The motivating examples are the substitutions obtained from $\mathbf{E}_1^\star$ substitutions, as described above. We will see that many problems arise when defining such substitutions in full generality: we must make sure that the resulting maps are consistent, and that the images of the cells of a pattern do not overlap. It will be proved that checking these properties is algorithmically undecidable for two-dimensional combinatorial substitutions (Theorems 6.3.1 and 6.3.3). We use classical tools from computability theory, by reducing some undecidable problems about tilings to our decision problems.

On the other hand, we will prove that consistency and non-overlapping can be algorithmically verified in some cases (Theorems 6.4.1, 6.4.2 and 6.4.4). These decidability results will allow us to answer some questions raised in [**ABS04**], asking for generic ways to prove the consistency and non-overlapping of some combinatorial substitutions obtained from dual substitutions, without using the tools inherent to the theory of dual substitutions.

Decidability questions for self-affine sets When considering all the algorithmic tools that exist for the study of Rauzy fractals [**ST09**], one can wonder whether the same can be achieved for a broader class of fractals: self-affine sets.

Iterated function systems are a classical way to define fractals: for every finite family of contracting mappings $f_1, \dots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ there exists a unique compact set $X \subseteq \mathbb{R}^d$ such that $X = \bigcup_{i=1}^{n} f_i(X)$ [**Hut81**]. "Fractal sets" can be defined as the sets X thus obtained. When all the f_i are affine mappings, we say that X is a *self-affine* set. In particular, every Rauzy fractal is the solution of an affine (graph-directed) iterated function system (see Section 4.2). A large body of the literature on fractals is devoted to the study of self-affine sets; see the references given in Section 7.1.

In **Chapter 7** we will prove that some properties which are decidable for Rauzy fractals (for example, the disjointness of the interior of their subtiles) are undecidable for self-affine sets specified by maps with rational coefficients (Theorem 7.4.3). Other properties, such as having nonempty interior, are also proved to be undecidable (Theorems 7.4.1 and 7.4.2).

These results are obtained by studying a particular class of self-affine sets associated with *multitape automata* (defined in Section 7.2). Using standard reduction techniques from computability theory, we first establish the undecidability of some language-theoretical properties of such automata, which can then be translated to undecidability results about their associated self-affine sets.

The class of fractals for which our undecidability results hold falls into the category of "box-like" self-affine sets, which is a widely studied class of fractals. (See Section 7.1 for more details.) This is joint work with Jarkko Kari [**JK13**].

Articles included in this thesis

[**BJS13**] **Connectedness of the fractals associated with Arnoux-Rauzy substitutions**
with Valérie Berthé and Anne Siegel (work done during my master thesis)
- *RAIRO Theor. Inform. Appl. (2013), to appear*

[**BJS12**] **Substitutive Arnoux-Rauzy sequences have pure discrete spectrum**
with Valérie Berthé and Anne Siegel
- *Unif. Distrib. Theory 7 (2012), no. 1, 173–197*

[**JK12**] **Consistency of multidimensional combinatorial substitutions**
with Jarkko Kari
- *Theoret. Comput. Sci. 454 (2012), 178–188*
- *Conference version in CSR 2012, proceedings LNCS 7353, 205–216*

[**BJJP13**] **Critical connectedness of thin arithmetical discrete planes**
with Valérie Berthé, Damien Jamet and Xavier Provençal
- *Journal preprint submitted for publication*
- *Conference version in DGCI 2013, proceedings LNCS 7749, 107–118*

[**BBJS13**] **A combinatorial approach to products of Pisot substitutions**
with Valérie Berthé, Jérémie Bourdon and Anne Siegel
- *Journal preprint submitted for publication*
- *Conference version in WORDS 2013, proceedings LNCS 8079, 58–70*, under the title *Generating discrete planes with substitutions*

[**JLL13**] **Rauzy fractals with countable fundamental group**
with Benoît Loridant and Jun Luo
- *Preprint submitted for publication*

[**JK13**] **Undecidable properties of self-affine sets**
with Jarkko Kari
- *Preprint submitted for publication*

Computer experiments and computer assisted proofs

Many of the results presented in this thesis orginate from computer experiments, and several proofs are computer assisted, in the sense that they rely on systematic enumeration of finitely many (but many) cases using a computer. All have been performed using the Sage mathematics software [**Sag**].

Computer experiments This thesis contains many examples of Rauzy fractals with various properties, for example in Chapter 5. They have been produced using the `WordMorphism.rauzy_fractal_plot` method, which has been integrated into Sage. Almost all of the results proved in this thesis have been intuited by performing explicit computations, also using Sage. We have tried to provide explicit examples arising from these experiments along the statements and proofs.

Computer assisted proofs Most of the results of Section 2.6 have been proved using computer algebra. The `E1Star` class (a Sage implementation of dual substitutions) has been used to perform exhaustive enumerations of small patterns, in order to prove some properties of the families of substitutions under study: enumerating minimal annuli in Section 2.6.1, proving some (strong) covering properties in Section 2.6.2 and proving the Property A for some families of substitution in Section 2.6.3.

Another type of results for which computer algebra was used are the construction of generation graphs, which make use of the directed graphs implementations found in Sage (the `DiGraph` class). This is done in Section 2.6.5 for Brun substitutions and in Section 2.6.6 for Jacobi-Perron substitutions. We also mention that the directed graph used in Section 2.6.7 in the proof of Lemma 2.6.18 has *not* been obtained algorithmically (it was constructed by hand), but doing so required many computations which have also been performed using the `E1Star` class.

Chapter 1

Preliminaries

We introduce the main objects studied in the first chapters, that is, Pisot substitutions and related notions such as dual substitutions, Rauzy fractals or multidimensional continued fraction algorithms.

Contents

1.1 Substitutions

Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^\star$ be the set of finite words over $\mathcal{A}$. Elements of $\mathcal{A}$ will be referred to as ***letters*** or ***symbols***. The ith letter of a word $w \in \mathcal{A}^\star$ is denoted by w_i.

Definition 1.1.1. Let $\mathcal{A} = \{1, \dots, n\}$ be a finite alphabet. A ***substitution*** is a function $\sigma : \mathcal{A}^\star \to \mathcal{A}^\star$ such that $\sigma(uv) = \sigma(u)\sigma(v)$ for all words $u, v \in \mathcal{A}^\star$, and such that $\sigma(a)$ is nonempty for every $a \in \mathcal{A}$. Equivalently, σ is a non-erasing morphism of the free monoid generated by $\mathcal{A}$.

The domain of definition of a substitution can naturally be extended to infinite sequences. If $x \in \mathcal{A}^\mathbb{N}$ then $\sigma(x)$ is defined as the infinite word $\sigma(x_1)\sigma(x_2)\dots$ obtained by concatenation, and if $x \in \mathcal{A}^\mathbb{Z}$ then $\sigma(x)$ is defined as the infinite word $\sigma(x_{-2})\sigma(x_{-1}) \,.\, \sigma(x_0)\sigma(x_1)\dots$ where the dot stands for the position between indices -1 and 0.

Let $\mathbf{P} : \mathcal{A}^\star \to \mathbb{Z}^n$ be the ***Abelianization map*** defined by $\mathbf{P}(w) = (|w|_1, \dots, |w|_n)$, where $|w|_i$ denotes the number of occurrences of i in w. The ***incidence matrix*** $\mathbf{M}_\sigma$ of σ is the matrix of size $n \times n$ whose ith column is equal to $\mathbf{P}(\sigma(i))$ for every $i \in \mathcal{A}$. It can be verified that the relation $\mathbf{M}_\sigma \mathbf{M}_\tau = \mathbf{M}_{\sigma\tau}$ holds for every substitutions σ, τ.

All the substitutions that we will encounter will be assumed to be ***primitive***: there exists k such that the entries of $\mathbf{M}_\sigma^k$ are all positive. The classical Perron-Frobenius theory [**Per07, Fro12, Wik**] then guarantees the existence of a unique largest eigenvalue of $\mathbf{M}_\sigma$ and a corresponding eigenvector with strictly positive entries.

Algebraic assumptions A primitive substitution σ is:

- ***unimodular*** if $\det \mathbf{M}_\sigma = \pm 1$;
- ***irreducible*** if the algebraic degree of the dominant eigenvalue of σ is equal to the size of the alphabet of σ;
- ***Pisot*** if the dominant eigenvalue of σ is a Pisot number, that is, an algebraic integer larger than 1 whose other conjugates are smaller than 1 in modulus.

Note that every Pisot irreducible substitution is primitive. This can be proved using classical positive matrix theory [**CS01b**].

Most of the Pisot substitutions we will consider in this thesis will be unimodular and irreducible, for example in Chapters 2 and 3. Some reducible substitutions will be encountered in Chapters 4 and 5. These algebraic assumptions will be exploited in Section 1.3 in order to define Rauzy fractals.

Example 1.1.2. A famous example is the *Tribonacci substitution* $\sigma : 1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. Its incidence matrix

$$\mathbf{M}_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is unimodular and admits a dominant cubic eigenvalue $\beta \approx 1.839$ which is a Pisot number (with its two conjugates $\beta', \beta'' \approx -0.42i \pm 0.61i$), so σ is a unimodular irreducible substitution. Another example the substitution $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$, whose incidence matrix has a dominant cubic Pisot eigenvalue $\beta \approx 1.325$. This substitution is unimodular and Pisot but it is not irreducible.

Remark 1.1.3. The Pisot property does not behave well under products of substitutions. For example, let $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ and $\tau : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 13$. These two substitutions are both irreducible and Pisot (with cubic dominant eigenvalues), and the product $\tau\sigma$ is irreducible Pisot as well. However the product $\tau^2\sigma : 1 \mapsto 313, 2 \mapsto 3213, 3 \mapsto 3$ is not Pisot because the characteristic polynomial of $\mathbf{M}_{\tau^2\sigma}$ is $(x-1)^3$.

Fixed points of substitutions A ***fixed point*** of σ is an infinite word $x \in \mathcal{A}^\mathbb{N}$ such that $\sigma(x) = x$. A ***periodic point*** of σ is an infinite word $x \in \mathcal{A}^\mathbb{N}$ such that $\sigma^k(x) = x$ for some $k \geqslant 0$. It can be proved that every substitution admits a periodic point [**Que10**].

For example, the substitution $\sigma : 1 \mapsto 12, 2 \mapsto 1$ admits the infinite fixed point $1211212112112\dots$ obtained by iterating σ on 1. The substitution $\sigma : 1 \mapsto 212, 2 \mapsto 11$ does not admit any fixed point, but admits the infinite periodic point $11212111121211\dots$ of period 2.

Remark 1.1.4 (Periodicity of substitutive fixed points). One can wonder whether an infinite fixed point u of a given substitution is (ultimately) ***shift-periodic***, that is, if there exist $k \geqslant 1$ and $m \geqslant 0$ such that $u_n = u_{n+k}$ for all $n \geqslant m$. This is an important question because substitutive fixed points are used to define symbolic dynamical systems (see Section 1.4). This problem has been proved to be decidable ([**HL86, Pan86**], see also [**Hon08**]), and more general versions have also been proved to be decidable [**Dur12, Dur13, Mit11**]. Such problems are also known as D0L or HD0L periodicity problem.

Note that a result by Holton and Zamboni [**HZ98**] states that if σ is a primitive substitution such that $\mathbf{M}_\sigma$ has a nonzero eigenvalue of modulus less than one, then σ does not admit any shift-periodic fixed point. This result has been improved by Adamczewski in [**Ada03**].

1.2 Dual substitutions and discrete planes

We now define dual substitutions, which will be our main tool for Chapters 2 and 3. We must first define the objects on which dual substitutions act: faces and discrete planes. The definitions in this section can be given in arbitrary dimension, but we restrict to the case of 3-letter substitutions, since the below tools will be only applied to study 3-letter substitutions.

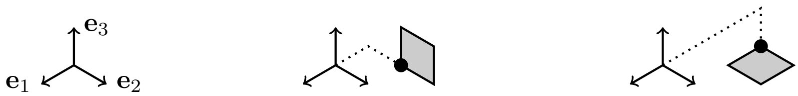
We denote by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the canonical basis of $\mathbb{R}^3$. A ***face*** $[\mathbf{x}, i]^\star$ is defined by

$$[\mathbf{x}, 1]^\star = \{\mathbf{x} + \lambda\mathbf{e}_2 + \mu\mathbf{e}_3 : \lambda, \mu \in [0,1]\} =$$


$$[\mathbf{x}, 2]^\star = \{\mathbf{x} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_3 : \lambda, \mu \in [0,1]\} =$$


$$[\mathbf{x}, 3]^\star = \{\mathbf{x} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2 : \lambda, \mu \in [0,1]\} =$$


where $i \in \{1, 2, 3\}$ is the ***type*** of $[\mathbf{x}, i]^\star$, and $\mathbf{x} \in \mathbb{Z}^3$ is the ***vector*** of $[\mathbf{x}, i]^\star$. Two examples of faces are plotted below.



$[(-1, 1, 0), 1]^\star$ $\qquad$ $[(-3, 0, -1), 3]^\star$

The translation of a face by a vector will be denoted by $\mathbf{x} + [\mathbf{y}, i]^\star = [\mathbf{x} + \mathbf{y}, i]^\star$. We will refer to collections of faces as *unions* of faces. By abuse of language we will say that a face f *belongs* to a union of faces even if f is in fact *included* in it. We now define discrete planes, introduced in [**Rev91**]. Denote by $\langle \cdot, \cdot \rangle$ the usual scalar product.

Definition 1.2.1. Let $\mathbf{v} \in \mathbb{R}^3_{>0}$. The ***discrete plane of normal vector*** $\mathbf{v}$ is defined by

$$\Gamma_\mathbf{v} = \{[\mathbf{x}, i]^\star : i \in \{1, 2, 3\}, \mathbf{x} \in \mathbb{Z}^3 \text{ such that } 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle\}.$$

More intuitively, $\Gamma_\mathbf{v}$ can be seen as the boundary of the union of the unit cubes with integer coordinates that intersect the lower half-space $\{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{x}, \mathbf{v} \rangle < 0\}$. The set of the vertices of $\Gamma_\mathbf{v}$ in $\mathbb{Z}^3$ corresponds to the classical notion of standard arithmetic discrete plane in discrete geometry. See Section 3.4, where some other definitions of discrete planes are introduced. An illustration is given in Figure 1.1.

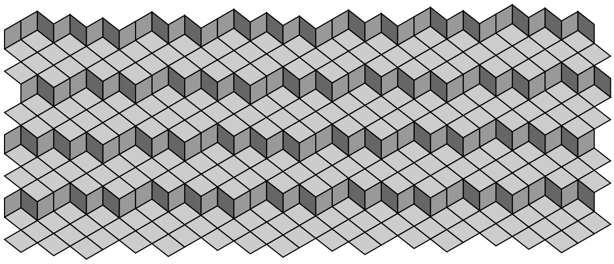


Figure 1.1: A portion of the discrete plane $\Gamma_{(1,\sqrt{2},\sqrt{17})}$.

Remark 1.2.2. The pattern $\mathcal{U} = [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star =$  is included in every discrete plane, because the coordinates of the normal vector $\mathbf{v}$ of a discrete plane $\Gamma_\mathbf{v}$ are always assumed to be positive in Definition 1.2.1.

Definition 1.2.3 ([AI01]). Let σ be a unimodular substitution. The ***dual substitution*** $\mathbf{E}_1^\star(\sigma)$ is defined for any face $[\mathbf{x}, i]^\star$ as

$$\mathbf{E}_1^\star(\sigma)([\mathbf{x}, i]^\star) \;=\; \bigcup_{(p,j,s)\in\mathcal{A}^\star\times\mathcal{A}\times\mathcal{A}^\star \;:\; \sigma(j)=pis} [\mathbf{M}_\sigma^{-1}(\mathbf{x}+\mathbf{P}(s)), j]^\star.$$

We extend this definition to unions of faces: $\mathbf{E}_1^\star(\sigma)(P \cup Q) = \mathbf{E}_1^\star(\sigma)(P) \cup \mathbf{E}_1^\star(\sigma)(Q)$. Basic properties of dual substitutions are summarized in the proposition below. The first statement ensures that composition behaves well under products (note that the product is reversed). The second statement can be interpreted as a form of "linearity" of $\mathbf{E}_1^\star$, and allows us to specify a mapping $\mathbf{E}_1^\star$ simply by giving $\mathbf{M}_\sigma$ and the images of $[0, 1]^\star, [0, 2]^\star, [3, 1]^\star$. The last two statements establish fundamental links between discrete planes and dual substitutions. The proof of this proposition relies on scalar product inequalities using the definition of discrete planes and dual substitutions.

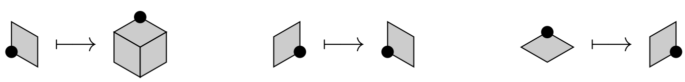
Proposition 1.2.4 ([AI01, Fer06]). *Let σ be a unimodular substitution. We have:*

(1) $\mathbf{E}_1^\star(\sigma \circ \sigma') = \mathbf{E}_1^\star(\sigma') \circ \mathbf{E}_1^\star(\sigma)$ *for every unimodular substitution* σ'*.*
(2) $\mathbf{E}_1^\star(\sigma)([\mathbf{x}, i]^\star) = \mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{E}_1^\star([\mathbf{0}, i]^\star)$ *for every face* $[\mathbf{x}, i]^\star$*.*
(3) $\mathbf{E}_1^\star(\sigma)(\Gamma_\mathbf{v}) = \Gamma_{{}^t\mathbf{M}_\sigma \mathbf{v}}$ *for every discrete plane* $\Gamma_\mathbf{v}$*.*
(4) If f and g are distinct faces in a common discrete plane $\Gamma_\mathbf{v}$, then $\mathbf{E}_1^\star(\sigma)(f) \cap \mathbf{E}_1^\star(\sigma)(g)$ does not contain any face.

Example 1.2.5. Let $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ be the Tribonacci substitution of Example 1.1.2. The action of $\mathbf{E}_1^\star(\sigma)$ on unit faces is given by

$$\begin{aligned}
\mathbf{E}_1^\star(\sigma)([\mathbf{x}, 1]^\star) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [(1, 0, -1), 1]^\star \cup [(0, 1, -1), 2]^\star \cup [\mathbf{0}, 3]^\star \\
\mathbf{E}_1^\star(\sigma)([\mathbf{x}, 2]^\star) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0}, 1]^\star \\
\mathbf{E}_1^\star(\sigma)([\mathbf{x}, 3]^\star) &= \mathbf{M}_\sigma^{-1}\mathbf{x} + [\mathbf{0}, 2]^\star,
\end{aligned}$$

which can also be represented as follows, where each black dot in the preimage stands for $\mathbf{x}$ and each black dot in the image stands for $\mathbf{M}_\sigma^{-1}\mathbf{x}$.



Example 1.2.6. Let $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ be the substitution defined in the previous example and let $\sigma : 1 \mapsto 12,\ 2 \mapsto 31,\ 3 \mapsto 1$. Iterating $\mathbf{E}_1^\star(\sigma)$ and $\mathbf{E}_1^\star(\tau)$ on the pattern $\mathcal{U}$ yields unions of faces of increasing size, which are shown in Figure 1.2. The fact that these patterns seem to be contained in discrete planes is confirmed by Proposition 1.2.4, (3) because the pattern $\mathcal{U}$ is contained in a discrete plane (see Remark 1.2.2), so all its forward images must also be.

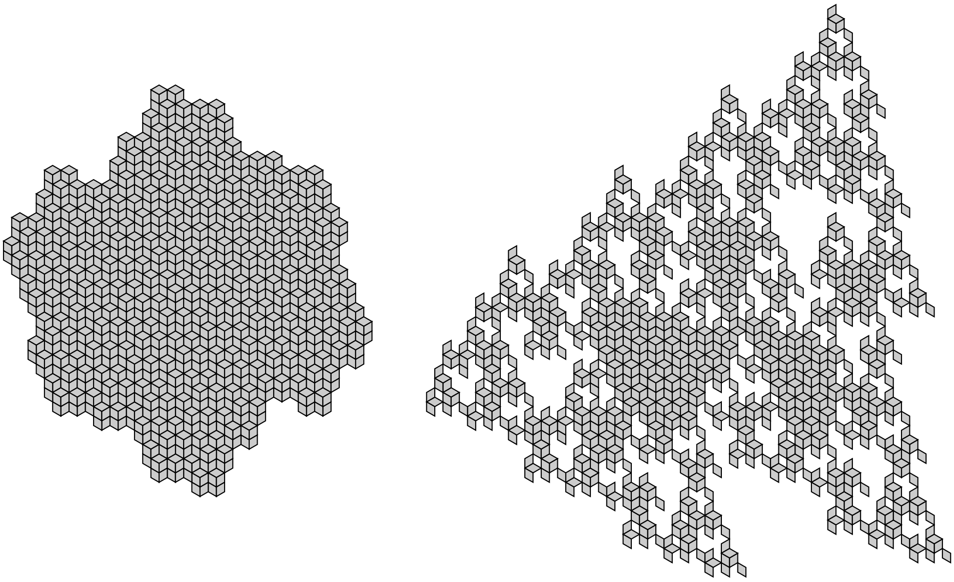


Figure 1.2: The patterns $\mathbf{E}_1^\star(\sigma)^{10}(\mathcal{U})$ and $\mathbf{E}_1^\star(\tau)^{10}(\mathcal{U})$, where σ and τ are the two substitutions defined in Example 1.2.6. These two patterns contains 1201 faces. (They have the same number of faces because the incidence matrices of σ and τ are equal.)

About the $\mathbf{E}_k^\star$ terminology The term "dual" comes from the fact that $\mathbf{E}_1^\star(\sigma)$ was originally introduced in [**AI01**] as the dual of a geometric realization of σ as a linear map in an n-dimensional vector space (where n is the size of the alphabet of σ). This is where the formula given in Definition 1.2.3 comes from. Stating and proving this rigorously can be done, but we will not do it here because using Definition 1.2.3 directly will be sufficient for our needs in Chapter 2.

An even more general setting is introduced in [**SAI01**], where the linear maps $\mathbf{E}_k$ and $\mathbf{E}_k^\star$ are introduced for every $k \in \{0, \dots, n\}$. Intuitively, $\mathbf{E}_k(\sigma)$ acts on k-dimensional objects, and $\mathbf{E}_k^\star(\sigma)$, which is the dual of $\mathbf{E}_k(\sigma)$, acts on $(n-k)$-dimensional objects. For instance $\mathbf{E}_0(\sigma)$ corresponds to the action of $\mathbf{M}_\sigma$ on $\mathbb{Z}$ (0-dimensional points), and $\mathbf{E}_1(\sigma)$ acts on unions on the 1-dimensional analogues of faces defined above. This is also consistent with our $\mathbf{E}_1^\star(\sigma)$ maps acting on faces of dimension $2 = 3 - 1$ for 3-letter substitutions.

Remark 1.2.7. We can observe that Definition 1.2.3 gives "suffix" version of dual substitutions, because $\mathbf{P}(s)$ is used to compute the position of the images, and the words s are the *suffixes* of the $\sigma(j)$. Several variants of this definition are found in the literature (sometimes a "prefix" version is used), but all of them are equivalent in the sense that using another variant essentially affects the resulting patterns up to an isometry only.

1.3 Rauzy fractals

Rauzy fractals were first introduced by Rauzy in [**Rau82**] to study a domain exchange map associated with the Tribonacci substitution. Similar objects were also considered in [**BT87, Thu89**] and a general definition for Pisot substitutions was given in [**AI01**] and [**CS01b**]. We will give two different definitions of Rauzy fractals:

- Definition 1.3.2, introduced in [**CS01b**], where the existence of the fractal follows from the fact that a well chosen projection of an infinite broken line stays within a bounded set, where the broken line is a geometrical version of a fixed point of the substitution. This can also be viewed as a "definition as an orbit closure": iterating σ to produce an infinite broken line can be viewed as iterating the dynamics of σ introduced in Section 1.4 (hence producing an infinite orbit).
- Definition 1.3.5, introduced in [**AI01**], where the existence of the fractal follows by Hausdorff convergence of a renormalized sequence of "polygonal approximations" of the fractal, which are obtained by iterating dual substitutions on a pattern.

The first definition will be mostly used in Chapter 4, where some Rauzy fractals of some reducible substitutions will be used. The second will be mostly used in Chapter 2, where we use we use combinatorial tools and dual substitutions to prove some properties of Rauzy fractals in Chapter 3.

Projection and renormalization

Before defining Rauzy fractals we must set up some notation for the various algebraic objects that will be used.

Projection Let σ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A} = \{1, \ldots, n\}$, such that the dominant eigenvalue β of $\mathbf{M}_\sigma$ is a real Pisot number of algebraic degree d. Denote by $\beta_1, \ldots, \beta_r$ the r real conjugates of β, and $\beta_{r+1}, \overline{\beta_{r+1}}, \ldots, \beta_{r+s}, \overline{\beta_{r+s}}$ the $2s$ complex conjugates of β (we have $r + 2s = d - 1$). Let $\mathbf{v}_\beta$ be a left β-eigenvector of $\mathbf{M}_\sigma$. The ***projection map*** we will use is given by

$$\begin{array}{rcccl} \pi_\sigma & : & \mathbb{R}^n & \to & \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^{d-1} \\ & & e_i & \mapsto & (\langle \mathbf{v}_{\beta_1}, e_i \rangle, \ldots, \langle \mathbf{v}_{\beta_{r+s}}, e_i \rangle) \end{array}$$

where each eigenvector $\mathbf{v}_{\beta_j}$ is obtained by replacing β by β_j in the coordinates of $\mathbf{v}_\beta$. Note that the conjugates $\overline{\beta_{r+1}}, \ldots, \overline{\beta_{r+s}}$ are not taken into account in the definition of π_σ. The projection π_σ will be used in Definitions 1.3.2 and 1.3.5 to define Rauzy fractals as the projections of some discrete objects in $\mathbb{R}^n$.

Renormalization The ***renormalization map*** $\mathbf{h}_\sigma : \mathbb{R}^r \times \mathbb{C}^s \to \mathbb{R}^r \times \mathbb{C}^s$ is defined by $\mathbf{h}_\sigma(x) = \mathbf{M}x$, where $\mathbf{M}$ is the diagonal matrix whose diagonal entries are $\beta_1, \ldots, \beta_{r+s}$. The mapping $\mathbf{h}_\sigma$ is contracting on $\mathbb{R}^r \times \mathbb{C}^s$ because $|\beta_i| < 1$ for $1 \leqslant i \leqslant r + s$. It corresponds to the action of $\mathbf{M}_\sigma$ before projecting by π_σ, *i.e.*, $\pi_\sigma \mathbf{M}_\sigma = \mathbf{h}_\sigma \pi_\sigma$.

This map will be in Definition 1.3.5 where Rauzy fractals will be defined as the Hausdorff limit of an infinite sequence of renormalized projections of polygonal sets. We will also use in Chapter 4 when dealing with the iterated function systems related with Rauzy fractals.

Remark 1.3.1. The definitions of Rauzy fractals we are going to give will use the projection $\pi_\sigma : \mathbb{R}^n \to \mathbb{R}^r \times \mathbb{C}^s$, so our Rauzy fractals will be subsets of $\mathbb{R}^r \times \mathbb{C}^s$. Many authors choose to define Rauzy fractals as a subset of the contracting plane of the matrix $\mathbf{M}_\sigma$, but this is equivalent to our definition because the action of $\mathbf{M}_\sigma$ on the contracting plane is equivalent to the action of $\mathbf{h}_\beta$ on $\mathbb{R}^r \times \mathbb{C}^s$ (see [**Sie00**] for more details). In particular, $\mathbb{R}^r \times \mathbb{C}^s$ and the contracting plane have the same dimension.

We have chosen the representation in $\mathbb{R}^r \times \mathbb{C}^s$ because it will be more convenient to use in the proofs of Chapter 4.

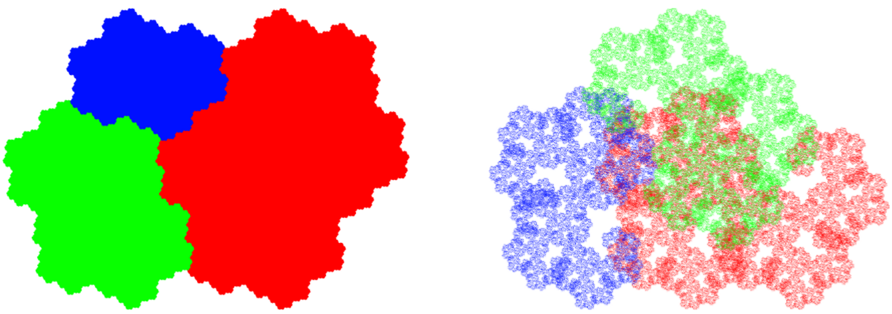
Definition as an orbit closure

Definition 1.3.2. Let σ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A}$ and let u be a periodic point of σ. The ***Rauzy fractal*** of σ (with respect to $\mathbf{v}_\beta$) is the set $\mathcal{T}_\sigma = \bigcup_{i\in\mathcal{A}} \mathcal{T}_\sigma(i)$, where for each $i \in \{1, \dots, n\}$, $\mathcal{T}_\sigma(i)$ is the ***subtile*** of type i given by

$$\mathcal{T}_\sigma(i) = \overline{\{\pi_\sigma \mathbf{P}(u_1 \dots u_n) : n \geqslant 1 \text{ and } u_{n+1} = i\}}.$$

Remark 1.3.3. The sets $\mathcal{T}_\sigma$ and $\mathcal{T}_\sigma(i)$ *do* depend on the choice of $\mathbf{v}_\beta$: the norm of $\mathbf{v}_\beta$ affects π_σ, so it affects the tiles up to an inflation factor. We insist on this point because specific choices of eigenvectors $\mathbf{v}_\beta$ will allow us to relate different Rauzy fractals in Propositions 4.3.2 and 4.4.1, without having to bother about which representation space is used. Also note that the choice of the periodic point u of σ does not affect the Rauzy fractal because σ is assumed to be primitive.

Example 1.3.4. Below are plotted the Rauzy fractals of $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ (left) and $\tau : 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 12$ (right).



These fractals look strikingly different, but both have nonempty-interior, both are connected, and both admit tilings of the plane. Such properties are reviewed later in this section.

Definition as a Hausdorff limit of polygons

We now give another definitions of Rauzy fractals, using dual substitutions. We will assume that the substitutions are irreducible, because it is not known how to give such a definition of Rauzy fractals in the general reducible case. Moreover, we will restrict to the 3-letter case because the below definitions will be only used for 3-letter substitutions. Note however that it is possible to define Rauzy fractals of irreducible Pisot substitutions in any dimension [**AI01**].

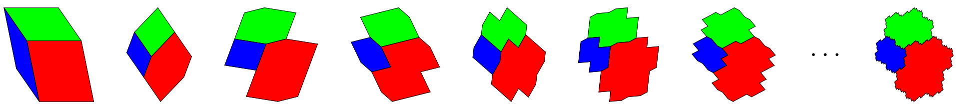
Let $\mathcal{U} = [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star$ be the "seed" from which we will start iterating $\mathbf{E}_1^\star(\sigma)$. Thanks to Proposition 1.2.4, the patterns we obtain are increasing patterns that are included in a discrete plane. We can then project and renormalize this sequence to get an infinite sequence of planar sets $P_1, P_2, \ldots$ defined by

$$P_n \ = \ \mathbf{h}_\sigma^n \pi_\sigma(\mathbf{E}_1^\star(\sigma)^n(\mathcal{U}))$$

for $n \geqslant 0$, where $\mathbf{h}_\sigma$ and π_σ are the renormalization and projection maps previously defined. It can be proved that $P_0, P_1, \ldots$ is a Hausdorff-convergent sequence in the metric space of compact subsets of the plane. See [**AI01**] for more details. This allows us to state the following definition.

Definition 1.3.5. Let σ be a unimodular Pisot irreducible substitution on alphabet $\{1, 2, 3\}$. The ***Rauzy fractal*** of σ is the Hausdorff limit of the sequence $(P_n)_{n \geqslant 0}$. The subtiles of $\mathcal{T}_\sigma$ can be defined in a similar way: the ***subtile*** $\mathcal{T}_\sigma(i)$ is the Hausdorff limit of the sequence of the sets $\mathbf{h}_\sigma^n \pi_\sigma(\mathbf{E}_1^\star(\sigma)^n([\mathbf{0}, i]^\star))$ as $n \to \infty$.

The convergence of $P_0, P_1, \ldots$ is illustrated below in the case of the Tribonacci substitution $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.



Properties of Rauzy fractals

Topological properties Rauzy fractals and their subtiles are always compact, locally connected and they are the closure of its interior. This was proved in the irreducible case in [**SW02**] and generalized to the reducible case in [**BBK06, EIR06**].

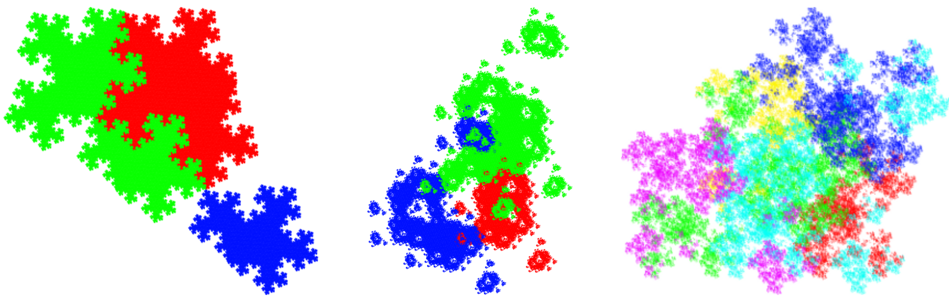
The tiles of a Rauzy fractals are also known to be the solution of graph-directed iterated function systems [**SW02, BS05**], which are classical objects used to represent fractal sets. More details are given in Section 4.2 for Rauzy fractals, and a general definition is given in Section 7.1, where we consider iterated functions in a more general context.

The box-counting and the Hausdorff dimension of the boundary of a Rauzy fractal always lie in the interval $]d-1, d[$, where d is the degree of the associated Pisot eigenvalue. An explicit formula for the box-counting dimension (in terms of spectral radii and eigenvalues) is given in [**Thu06**] in the case of irreducible substitutions. If, furthermore, all the eigenvalues of β have the same modulus, then the box-counting dimension and the Hausdorff dimension coincide. This latter assumption is an important one, which reflects the contrast between self-similar and self-affine fractals (which are notoriously more difficult to handle): if the eigenvalues have non-equal moduli, then the contraction factors are not uniform so the corresponding fractal is *not* self-similar (but self-affine). See Section 7.1 for more details about self-similar and self-affine fractals. The dimension formulas above have been generalized to the reducible case in [**ST09**]. See [**Fal03**] for some background on fractal dimensions.

Topological properties which are not always satisfied The zoology of Rauzy fractals is very rich from a topological point of view. Not all fractals are connected, some are homeomorphic to a disc, and some have nontrivial fundamental group (countable or not).

Also, the origin is not always an interior point of the fractal. These properties are extensively reviewed in [**ST09**], where many (semi-)algorithms are given. They are based on graph-theoretical constructions detecting boundary intersections, and tools from plane topology. See [**Hat02**] for background about the associated notions from algebraic topology such as the fundamental group.

The topological variety of Rauzy fractals is illustrated by the following three pictures, representing the respective Rauzy fractals of $\sigma_1 : 1 \mapsto 2132,\ 2 \mapsto 1,\ 3 \mapsto 2$, $\sigma_2 : 1 \mapsto 12233$, $2 \mapsto 123,\ 3 \mapsto 223$ and $\sigma_3 : 1 \mapsto 34,\ 2 \mapsto 543,\ 3 \mapsto 53,\ 4 \mapsto 64,\ 5 \mapsto 1,\ 6 \mapsto 2$.



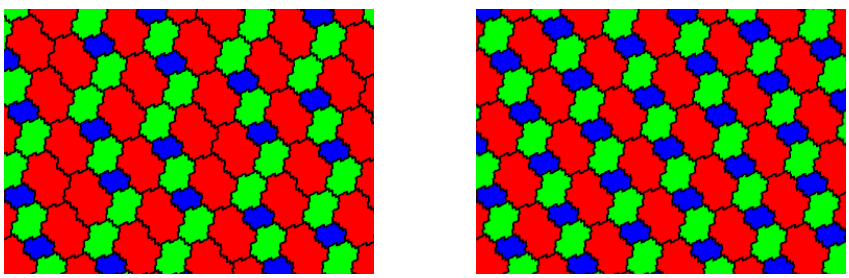
Tiling properties Rauzy fractals associated with a Pisot eigenvalue of degree d admit self-affine aperiodic tilings of the $(d-1)$-dimensional hyperplane, as stated in the next proposition, whose proof can be found in [**ST09**] or [**BR10**].

Proposition 1.3.6. *Let σ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A}$ (of size n), whose dominant eigenvalue has degree d. There exists a positive integer p such that the family*

$$\{\mathcal{T}_\sigma(i) + \pi_\sigma(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^n, 0 \leqslant \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}$$

*is a p-**multiple tiling** of a $(d-1)$-dimensional hyperplane, that is, almost every point belongs to exactly p subtiles in the tiling. If, moreover, σ satisfies a condition called the* super coincidence condition *(defined in Section 1.4), then the multiple tiling is a tiling.*

The self-affine aperiodic tiling is illustrated below left. When σ is irreducible, it is also possible to define a periodic tiling (as well as many other tilings), which is shown below right. Periodic tilings can also associated with some reducible substitutions in some case. See [**ST09**] for more details.



The tiling properties stated above have important consequences in dynamics, as will be stated in Section 1.4. Also, these properties are algorithmically decidable; see [**ST09**] and [**BR10**]. In Section 5.6 we give an explicit example of a reducible substitution where the tiles *do* overlap, and for which the multiple tiling of Proposition 1.3.6 is not a tiling.

Equivalence of the two definitions The two definitions of Rauzy fractals we have given (Definitions 1.3.2 and 1.3.5) are equivalent. This can be proved by showing that two fractals verify the same GIFS equation, which implies that they are equal because every GIFS has a unique solution. This uniqueness result is classical in fractal geometry; see [**Fal03, Fal97**] for a proof. GIFS will be discussed more in details in Sections 4.2 and 7.1.

1.4 Dynamics of substitutions

In this section we define the symbolic dynamical system (X_σ, S) of a substitution σ and we state the classical results about the various geometrical interpretations of (X_σ, S).

Subshifts

Let $\mathcal{A}$ be a finite alphabet. We endow the space $\mathcal{A}^{\mathbb{Z}}$ with the topology induced from the distance function $d(x, y) = 2^{-\min\{|n| : x_n \neq y_n\}}$. Two configurations are close if they agree on a large neighborhood of the origin. The ***shift map*** $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $S(x) = (x_{i+1})_{i \in \mathbb{Z}}$ for any $x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$.

A ***subshift*** is the symbolic dynamical system given by (X, S), where S is the shift map and $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is topologically closed and invariant by S. Equivalently, we can require that there exists a set of words $F \subseteq \mathcal{A}^\star$ such that X is equal to the set of sequences in which no word of F appears. Background notions on symbolic dynamics can be found in [**LM95**] or [**Kit98**]. In this section we will only deal with two-sided subshifts (sequences indexed by $\mathbb{Z}$) and not one-sided subshifts (indexed by $\mathbb{N}$). This removes the difficulty of having to deal with the fact that the shift map is not bijective in the one-sided case (where shifting a sequence deletes its first symbol).

Two dynamical systems (X, S) and (Y, T) are ***topologically conjugate*** if there exist a bijective continuous map $\varphi : X \to Y$ such that $\varphi \circ S = T \circ \varphi$. We say that (X, S) and (Y, T) are ***semi-conjugate*** if there exist two countable sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a continuous map $\varphi : X \to Y$ such that $\varphi \circ S = T \circ \varphi$, and the induced map $\varphi : X \setminus X_0 \to Y \setminus Y_0$ is bijective. If φ is only surjective, then (Y, T) is a ***factor*** of (X, S).

Substitutive dynamical systems

Let σ be a primitive substitution on alphabet $\mathcal{A}$, and let $x \in \mathcal{A}^{\mathbb{Z}}$ be a periodic point of σ (there exists $k \geqslant 0$ such that $\sigma^k(x) = x$). The ***substitutive subshift*** associated with σ is the subshift (X_σ, S) where X_σ is the topological closure in $\mathcal{A}^{\mathbb{Z}}$ of the orbit set $\{S^k(x) : k \in \mathbb{Z}\}$. Since σ is primitive, the definition of X_σ does not depend on the choice of x. Many facts about substitutive dynamical systems can be found in [**Que10**] or [**PF02**].

The Pisot case, coincidences conditions The case of Pisot substitutions has gained a lot of attention since the pioneering work of Rauzy [**Rau82**], who proved that (X_σ, S) is semi-conjugate to a toral translation in the case of the Tribonacci substitution. This has been generalized by [**AI01, CS01b**] to all Pisot substitutions, under the assumption of some combinatorial conditions known as *coincidence conditions*.

Such combinatorial conditions were first introduced by Dekking [**Dek78**], for the study of constant length substitutions. Later, a variant was introduced by Arnoux and Ito [**AI01**]: a substitution σ on alphabet $\mathcal{A}$ satisfies the ***strong coincidence condition*** if for every

$(j_1, j_2) \in \mathcal{A}^2$, there exists $k \in \mathbb{N}$ and $i \in \mathcal{A}$ such that $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$ or $\mathbf{P}(s_1) = \mathbf{P}(s_2)$. This condition ensures that (X_σ, S) is semi-conjugate to the dynamics induced by the domain exchange in the Rauzy fractal of σ (see [**AI01**] for more details).

A stronger condition, the ***super coincidence condition*** was then introduced independently in [**IR06**] and [**BK06**]. It ensures that (X_σ, S) is semi-conjugate to a toral translation. We do not define this condition in detail here. See [**BR10**, Chapter 5] for a survey about the various notions of coincidences. Note that super coincidence also implies that the (X_σ, S) has ***pure discrete spectrum***, which is a spectral reformulation of the fact that (X_σ, S) is semi-conjugate to a toral translation [**Wal82, Que10**].

The notion of coincidences that we will use for the dynamical applications in Chapter 3 is stated in the following proposition. It is expressed in terms of dual substitutions and was established in [**IR06**]; see also [**BR10**, Theorem 5.4.14].

Proposition 1.4.1 ([IR06]). *Let σ be a unimodular Pisot irreducible substitution. The system (X_σ, S) is semi-conjugate to a toral translation if and only if the patterns $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ cover balls of arbitrarily large radius when $n \to \infty$.*

The ***Pisot conjecture*** states that the above properties are always true, provided σ is unimodular, Pisot and irreducible. Counterexamples exist if the irreducibility assumption is dropped (see for example Section 5.6). The Pisot conjecture has been proved in the 2-letter case, by the combination of two results: (1) the strong coincidence condition alone is sufficient to ensure pure discrete spectrum in the two-letter case [**HS03**], (2) the strong coincidence condition holds in the two-letter case [**BD02**]. The conjecture is still open in the case of three letters or more. Also note that some similar conjectures have been formulated in the context of more geometrical substitution tiling spaces; see for example [**AL11, BG13**].

Substitutions, subshifts of finite type and Markov partitions

A nonempty subshift (X, S) is a ***subshift of finite type (SFT)*** if there exists a *finite* set of words $F \subseteq \mathcal{A}^\star$ such that X is equal to the set of sequences in which no word of F appears. Equivalently, we can require that X is the set of bi-infinite paths of edges in a finite graph (without labellings). SFTs are among the most studied symbolic dynamical systems (see for example [**LM95**]) and their first use in dynamics dates back at least to 1898 in the works of Hadamard on geodesics of surfaces with negative curvature [**Had98**]. SFTs have positive entropy and they always contain periodic points.

So far, we have seen that the dynamics of a Pisot substitution σ can be interpreted in three different ways: a symbolic shift (X_σ, S), a toral translation or a domain exchange on the Rauzy fractal $\mathcal{T}_\sigma$. A fourth equivalent interpretation can be given in terms of the action of an "adic transformation" on the infinite paths in a finite graph (called the prefix-suffix automaton of σ, see [**CS01a**]), or equivalently as an adic transformation on the infinite paths of a Bratteli diagram [**DHS99**] (the prefix-suffix automaton can be seen as a "folded up" Bratteli diagram in this case). Also, these four dynamics are very far from being SFTs: they are minimal and have entropy zero.

However, for each of the four dynamics above, there exists another corresponding dynamics on the same base set which is semi-conjugate to an SFT. In each of the four cases, this fundamentally different dynamics can be seen as another interpretation of the base set in question. For example, in the case of X_σ, the zero entropy dynamics corresponds to the action of the symbolic shift and the SFT dynamics corresponds to the action of σ itself on X_σ (more precisely, the action of "σ-desubstitution" on X_σ). This is summarized in the table below; more details can be found in [**Sie00, CS01a**].

Base set	Zero entropy dynamics	Positive entropy dynamics
Substitutive sequences X_σ	Shift	σ-desubstitution
Prefix-suffix sequences Σ	Adic transformation	Shift (SFT)
Rauzy fractal $\mathcal{T}_\sigma$	Domain exchange	Zoom in GIFS
Geometrical torus	Translation on $\mathbb{T}^{d-1}$	Automorphism $\mathbf{M}_\sigma$ of $\mathbb{T}^d$

Markov partitions for toral automorphisms A classical result in symbolic dynamics is the existence of Markov partitions for automorphisms of the torus (the action of $x \mapsto \mathbf{M}x$ of a $d \times d$ matrix $\mathbf{M}$ on the d-dimensional torus $\mathbb{T}^d$), when $\mathbf{M}$ is hyperbolic (no eigenvalues lie on the unit circle) [**Bow08**]. (See [**LM95**, Section 6.5] for a definition of Markov partitions.) In the case of 2×2 matrices, the Markov partitions can be explicitly constructed [**AW70, Adl98**], and the partition can be chosen to be very nice: two rectangles are enough. In dimension $d \geqslant 3$, we know that Markov partitions exist, but no general construction similar to the 2×2 has been given. A result of Bowen [**Bow78**] states that such partitions must have fractal boundary if $d \geqslant 3$, which reduces the hopes of finding nice explicit Markov partitions.

In the particular case where the hyperbolic matrix is the incidence matrix of a Pisot substitution, Rauzy fractals provide a way to construct explicit Markov partitions of the automorphism induced by $\mathbf{M}_\sigma$. Indeed, if such a Markov partition exists, then it induces a periodic tiling of the space. Intersecting the contracting space of the matrix with this tiling induces a self-similar tiling of the contracting plane. A first approach is then to try to construct a domain of the contracting space which is self-similar with respect to the action of $\mathbf{M}_\sigma$, and this is precisely what is achieved when good tiling properties of Rauzy fractals hold. Such an approach was initiated in [**IO93, KV98, Pra99**] and was proved to be successful in the Pisot case [**IR06, Sie00**], provided that the associated Rauzy fractal satisfies good tiling properties. An example is shown in Figure 1.3.

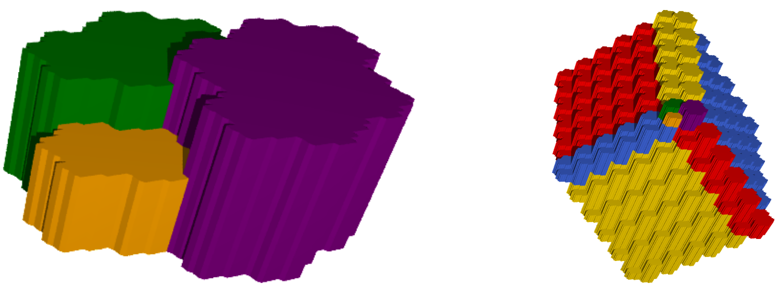


Figure 1.3: A fundamental domain (left), which admits a periodic tiling of the space (right). The corresponding partition of $\mathbb{T}^3$ by the three parts of the fundamental domain is a Markov partition for the toral automorphism $(\mathbb{T}^3, \mathbf{M}_\sigma)$, where $\mathbf{M}_\sigma$ is the incidence matrix associated with the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.

1.5 Substitutions and continued fraction algorithms

We now define the families of substitutions that will be studied in the next chapters. These families are naturally associated with the multidimensional continued fraction algorithms of the same name. We will give some specific properties of these algorithms for the Brun and Jacobi-Perron substitutions, which will be needed in Section 2.4.

Note that a fourth family of substitutions is considered in Section 3.4, which are associated with the so-called *fully subtractive algorithm*. We do not define it here because it will be only used as a tool in the concerned section. Links between the Arnoux-Rauzy substitutions and the fully subtractive algorithm, as well as some convergence results, are given in Section 3.3.

Arnoux-Rauzy substitutions

Arnoux-Rauzy sequences have been introduced in [**AR91**] in order to generalize the properties of Sturmian sequences to systems over a 3-letter alphabet. They are defined as the infinite sequences generated by iteration of the substitutions σ_1^{AR}, σ_2^{AR}, σ_3^{AR} defined below, where each σ_i^{AR} occurs infinitely often in the iteration. These sequences have minimal factor complexity $2n+1$, providing an analogue of the fact that Sturmian sequences have minimal complexity $n+1$ on 2-letter alphabets.

It was conjectured that Arnoux-Rauzy sequences correspond to natural codings of translations on the two-dimensional torus (as in the Sturmian case with rotations on the circle), but this conjecture has been disproved in [**CFZ00**]: counterexamples arise if the distribution of the σ_i^{AR} is too "unbalanced". More properties of Arnoux-Rauzy sequences can be found in [**BFZ05, CC06, CFM08, BCS13**].

The Arnoux-Rauzy substitutions can be naturally associated with a continued fraction algorithm. However, we will not need any specific properties of this continued fraction algorithm for the results of Chapters 2 and 3, because everything will be proved in a combinatorial way directly from the definition of σ_1^{AR}, σ_2^{AR} and σ_3^{AR}. Note, however, that the continued fraction algorithm associated with Arnoux-Rauzy substitutions has been proved convergent [**Sch00, AD13**], and that our results will allow us to deduce another proof of convergence in Section 3.3.

Let σ_1^{AR}, σ_2^{AR}, σ_3^{AR} be the ***Arnoux-Rauzy substitutions*** defined by

$$\sigma_1^{\mathsf{AR}} : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \qquad \sigma_2^{\mathsf{AR}} : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \qquad \sigma_3^{\mathsf{AR}} : \begin{cases} 1 \mapsto 13 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases}$$

and denote their associated dual substitutions by $\Sigma_i^{\mathsf{AR}} = \mathbf{E}_1^\star(\sigma_i^{\mathsf{AR}})$. These dual substitutions will be exploited in Chapter 2 (where we also describe them in more details). The following property ensures that admissible finite products of Arnoux-Rauzy substitutions are Pisot.

Proposition 1.5.1 ([AI01]). *Let σ be a product of σ_1^{AR}, σ_2^{AR} and σ_3^{AR} in which each substitution appears at least once. Then, σ is a Pisot irreducible substitution.*

We will restrict to the study of infinite sequences $(\sigma_{i_n}^{\mathsf{AR}})_{n\in\mathbb{N}}$ in which each σ_i^{AR} occurs infinitely often. Such sequences will be called ***Arnoux-Rauzy admissible sequences***.

Brun substitutions

Let $\mathbf{v} \in \mathbb{R}^3_{\geqslant 0}$ such that $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$. The ***Brun algorithm*** [**Bru58**] is one of the possible natural generalizations of Euclid's algorithm: subtract the second largest component of $\mathbf{v}$ to the largest and reorder the result:

$$\mathbf{v} \;\mapsto\; \begin{cases} (\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 - \mathbf{v}_2) & \text{if } \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3 - \mathbf{v}_2 \\ (\mathbf{v}_1,\ \mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_2) & \text{if } \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_2 < \mathbf{v}_2 \\ (\mathbf{v}_3 - \mathbf{v}_2,\ \mathbf{v}_1,\ \mathbf{v}_2) & \text{if } \mathbf{v}_3 - \mathbf{v}_2 < \mathbf{v}_1 \leqslant \mathbf{v}_2. \end{cases}$$

Iterating this map starting from $\mathbf{v}^{(0)} = \mathbf{v}$ yields an infinite sequence of vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots$ and the algorithm can be rewritten in matrix form: $\mathbf{v}^{(n)} = \mathbf{M}_{i_n}^{-1}\mathbf{v}^{(n-1)}$, where

$$\mathbf{M}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad \mathbf{M}_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{M}_3^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The ***Brun expansion*** of a vector $\mathbf{v} \in \mathbb{R}^3_{\geqslant 0}$ is the infinite sequence $i_1, i_2, \ldots$ obtained above. In this document, the overscript notation $\mathbf{v}^{(n)}$ will be only used to denote sequences of vectors produced by a continued fraction algorithm.

Example 1.5.2.

- If $\mathbf{v} = (1, 2, 5)$ then $(i_n) = 12132111\ldots$ is an eventually constant sequence containing infinitely many 1's but finitely many 2's and 3's.
- If $\mathbf{v} = (1, 2, \sqrt{5})$ then $(i_n) = 313\ 1112\ 1112\ldots$ is an eventually periodic sequence containing infinitely many 1's and 2's but finitely many 3's.
- If $\mathbf{v} = (1, 2, e)$ then $(i_n) = 313212221112\ldots$ is an aperiodic sequence containing infinitely many 1's and 2's but finitely many 3's.
- If $\mathbf{v} = (1, \beta, \beta^2)$ then $(i_n) = 1131\ 132\ 132\ \ldots$ is an eventually periodic sequence containing infinitely many 1's, 2's and 3's, where $\beta \approx 3.21$ is the dominant Pisot cubic root of $x^3 - 3x^2 - x + 1$.
- If $\mathbf{v} = (1, e, \pi)$ then $(i_n) = 31232331211113\ldots$ contains infinitely many 1's, 2's and 3's (provided e and π are linearly independent).

As suggested by Example 1.5.2, the Brun algorithm is able to "detect" totally irrational vectors by producing an expansion with infinitely many 3's. Moreover, it can be proved that the algorithm is convergent, *i.e.*, that such an expansion determines a unique vector.

Proposition 1.5.3 ([Bru58]). *The Brun expansion of $\mathbf{v} \in \mathbb{R}^3_{\geqslant 0}$ contains infinitely many 3's if and only if $\mathbf{v}$ is totally irrational. Moreover, the Brun algorithm is* ***convergent****: for every such expansion $(i_n)_{n\in\mathbb{N}}$, there is a unique vector $\mathbf{v}$ whose expansion is $(i_n)_{n\in\mathbb{N}}$.*

We now define the ***Brun substitutions*** by

$$\sigma_1^{\mathsf{Brun}} : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \qquad \sigma_2^{\mathsf{Brun}} : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \sigma_3^{\mathsf{Brun}} : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 13 \end{cases}$$

and we denote their associated dual substitutions by $\Sigma_i^{\mathsf{Brun}} = \mathbf{E}_1^\star(\sigma_i^{\mathsf{Brun}})$ for $i \in \{1, 2, 3\}$.

These substitutions are chosen in such a way that $\mathbf{M}_i = {}^t\mathbf{M}_{\sigma_i^{\mathsf{Brun}}}$ for $i \in \{1,2,3\}$, where the $\mathbf{M}_i$ are the matrices associated with the Brun algorithm above. Hence, thanks to Proposition 1.2.4, we have $\Sigma_i^{\mathsf{Brun}}(\Gamma_{\mathbf{v}}) = \Gamma_{\mathbf{M}_i\mathbf{v}}$, so if $(i_n)_{n\geqslant 1}$ is the Brun-expansion of $\mathbf{v}$, we have

$$\Sigma_{i_n}^{\mathsf{Brun}}(\Gamma_{\mathbf{v}^{(n)}}) = \Gamma_{\mathbf{M}_{i_n}\mathbf{v}^{(n)}} = \Gamma_{\mathbf{v}^{(n-1)}},$$

for all $n \geqslant 1$ and in particular $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\Gamma_{\mathbf{v}^{(n)}}) = \Gamma_{\mathbf{v}}$. This fact, together with the convergence of the algorithm, will be used in Section 2.4. The following proposition guarantees that products of Brun substitutions enjoy nice algebraic properties.

Proposition 1.5.4 ([AD13]). *For every $(i_1,\ldots,i_n) \in \{1,2,3\}^n$, the product $\sigma_{i_1}^{\mathsf{Brun}} \cdots \sigma_{i_n}^{\mathsf{Brun}}$ is irreducible Pisot if and only if $i_k = 3$ at least once.*

Since we are interested in the Pisot case only (for example to be able to define Rauzy fractals), we will restrict to the sequences characterized by Proposition 1.5.4: a sequence $(i_n)_n \in \{1,2,3\}^{\mathbb{N}}$ is ***Brun-admissible*** if it contains infinitely many 3's.

Jacobi-Perron substitutions

Let $\mathbf{v} \in \mathbb{R}^3_{\geqslant 0}$ be such that $\mathbf{v}_1 \leqslant \mathbf{v}_3$ and $\mathbf{v}_2 \leqslant \mathbf{v}_3$. The ***Jacobi-Perron algorithm*** [**Jac68, Per07, Sch00**] consists in iterating the map $\mathbf{v} \mapsto (\mathbf{v}_2 - a\mathbf{v}_1,\ \mathbf{v}_3 - b\mathbf{v}_1,\ \mathbf{v}_1)$, where $a = \lfloor \mathbf{v}_2/\mathbf{v}_1 \rfloor$ and $b = \lfloor \mathbf{v}_3/\mathbf{v}_1 \rfloor$. Like with the Brun algorithm, we obtain an infinite sequence of vectors $\mathbf{v}^{(0)} = \mathbf{v}, \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots$ such that $\mathbf{v}^{(n)} = \mathbf{M}_{a_n,b_n}^{-1}\mathbf{v}^{(n-1)}$ where $\mathbf{M}_{a,b}$ is a matrix with non-negative integer entries, and we can define the ***Jacobi-Perron expansion*** of $\mathbf{v}$ as the infinite sequence $(a_n,b_n)_{n\geqslant 1}$. It can be proved that $(a_n,b_n)_{n\geqslant 1}$ is the Jacobi-Perron expansion of some vector $\mathbf{v}$ if and only if for every $n \geqslant 1$ we have $0 \leqslant a_n \leqslant b_n$, $b_n \neq 0$, and $a_n = b_n$ implies $a_{n+1} \neq 0$ (see [**Sch73, Sch00, Bre81**]). We refer to such a sequence $(a_n,b_n)_{n\geqslant 1}$ as a ***Jacobi-Perron-admissible*** sequence.

Proposition 1.5.5 ([Per07]). *A sequence $(a_n,b_n)_{n\geqslant 1}$ is a Jacobi-Perron-admissible sequence if and only if it is the Jacobi-Perron expansion of totally irrational vector $\mathbf{v} \in \mathbb{R}^3_{\geqslant 0}$. Moreover, the Jacobi-Perron algorithm is* ***convergent****: for every such expansion $(a_n,b_n)_{n\in\mathbb{N}}$, there is a unique vector $\mathbf{v}$ whose expansion is $(a_n,b_n)_{n\in\mathbb{N}}$.*

Let the ***Jacobi-Perron substitutions*** be defined by

$$\sigma_{a,b}^{\mathsf{JP}} : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 13^a \\ 3 \mapsto 23^b \end{cases}$$

for all $a,b \geqslant 0$, and set $\Sigma_{a,b}^{\mathsf{JP}} = \mathbf{E}_1^\star(\sigma_{a,b}^{\mathsf{JP}})$. These substitutions are chosen in such a way that $\mathbf{M}_{a,b} = {}^t\mathbf{M}_{\sigma_{a,b}^{\mathsf{JP}}}$ for every $a,b \geqslant 0$, so the correspondence with dual substitutions and discrete planes is analogous as described above for the Brun substitutions. We also have:

Proposition 1.5.6 ([DFP04]). *Every product $\sigma_{a_1,b_1}^{\mathsf{JP}} \cdots \sigma_{a_n,b_n}^{\mathsf{JP}}$ is irreducible Pisot if $0 \leqslant a_n \leqslant b_n$ and $b_n \neq 0$ for all $n \geqslant 1$.*

Note that, unlike in the Arnoux-Rauzy and Brun cases, Jacobi-Perron substitutions do not consist of a finite number of substitutions, but of infinitely many substitutions parametrized by $a,b \in \mathbb{N}$. In essence this reflects the fact that the associated continued fraction algorithm is *multiplicative*, in opposition with the *additive* algorithms of Arnoux-Rauzy or Brun. Such notions are defined formally in [**Sch00**].

Additive version of Jacobi-Perron substitutions

When dealing with Jacobi-Perron substitutions in Chapter 2, we will work directly on the substitutions $\Sigma^{\mathsf{JP}}_{a,b}$, except in Section 2.4 when we will construct generation graphs, because our construction only applies for *finitely many* substitutions (and there are infinitely many substitutions $\sigma^{\mathsf{JP}}_{a,b}$).

Hence we will need to decompose $\sigma^{\mathsf{JP}}_{a,b}$ in products using finitely many different substitutions. A first natural additive substitutive realization of Jacobi-Perron algorithm is given by

$$\tau_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 3 \end{cases} \qquad \tau_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 31 \end{cases} \qquad \tau_3 : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 2 \end{cases}$$

with $\sigma^{\mathsf{JP}}_{a,b} = \tau_3\tau_1^a\tau_2^b$. (Note that $\tau_1\tau_2 = \tau_2\tau_1$.) The problem with this decomposition is that it is not restrictive enough, which makes the generation graphs associated with this family too difficult to handle.

So we choose another additive decomposition, which enforces the constraint $0 \leqslant a \leqslant b$ and $b \geqslant 1$ to be satisfied in the additive products. Let

$$\theta_1 = \tau_2 \qquad \theta_2 = \tau_1\tau_2 \qquad \theta_3 = \tau_3\tau_2 \qquad \theta_4 = \tau_3\tau_1\tau_2.$$

We have

$$\sigma^{\mathsf{JP}}_{a,b} = \begin{cases} \theta_3\theta_1^{b-1} & \text{if } a = 0 \\ \theta_3\theta_1^{b-a-1}\theta_2^a \;=\; \theta_4\theta_1^{b-a}\theta_2^{a-1} & \text{if } 0 < a < b \\ \theta_4\theta_2^{a-1} & \text{if } a = b. \end{cases}$$

Finally, we define $\Theta_i = \mathbf{E}_1^\star(\theta_i)$ for $i \in \{1,2,3,4\}$, the substitutions that we will use in Section 2.4 to construct generation graphs for the Jacobi-Perron substitutions.

Note that the "rhythm" is provided by θ_3 and θ_4, in the sense that every product $\theta_{i_1}\cdots\theta_{i_n}$ starting with θ_3 or θ_4 can be uniquely decomposed into a product of $\sigma^{\mathsf{JP}}_{a_k,b_k}$, thanks to the fact that in the above product decomposition, each possibility contains exactly one θ_3 or θ_4.

Chapter 2

Generating discrete planes with substitutions

In this chapter we prove some results about discrete plane generation with the dual substitutions associated with the Arnoux-Rauzy, Brun and Jacobi-Perron algorithms. We will exploit these results in Chapter 3 to deduce some other results about these families of substitutions. This is joint work with Valérie Berthé, Jérémie Bourdon and Anne Siegel [**BJS13, BJS12, BBJS13**]. The content of Section 2.6.8 is joint work with Valérie Berthé, Damien Jamet and Xavier Provençal [**BJJP13**].

Contents

2.1 General strategy

Our aim in this chapter is to establish the following discrete plane generation results for the dual substitutions associated with the Arnoux-Rauzy, Brun and Jacobi-Perron algorithms (which are defined in Section 1.5).

- There exist finite "seeds" $\mathcal{V}$ (patterns) such that iterating any sequence of substitutions from $\mathcal{V}$ yields patterns covering arbitrarily large discs centered at the origin. This is proved in Theorem 2.5.1 for our three families of substitutions. (Such seeds were known to exist only in the case of a single substitution [**BR10**].)
- An effective characterization of the sequences for which the seed $\mathcal{U}$ satisfies the above property is given in Theorem 2.5.2 for the Brun substitutions and in Theorem 2.5.4 for the Jacobi-Perron substitutions, where $\mathcal{U}$ is the seed $= [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star$.
- Iterating any sequence of substitutions from $\mathcal{U}$ yields patterns covering arbitrarily large discs, which are not necessarily centered at the origin. This is proved in Theorem 2.5.5 for our three families of substitutions.

This is illustrated by the following example. Let

$$(i_n) = 232\ 232\ 232\ \cdots \quad \text{and} \quad (j_n) = 2311\ 2311\ 2311\ \cdots$$

be two infinite (periodic) sequences. Figure 2.1 (left) suggests that $\Sigma^{\mathsf{Brun}}_{i_1} \cdots \Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U})$ covers arbitrarily large discs centered at the origin as $n \to \infty$, but Figure 2.1 (right) suggests that $\Sigma^{\mathsf{Brun}}_{j_1} \cdots \Sigma^{\mathsf{Brun}}_{j_n}(\mathcal{U})$ does **not** cover arbitrarily large discs centered at the origin as $n \to \infty$. This can be proved rigorously thanks to Theorem 2.5.4.

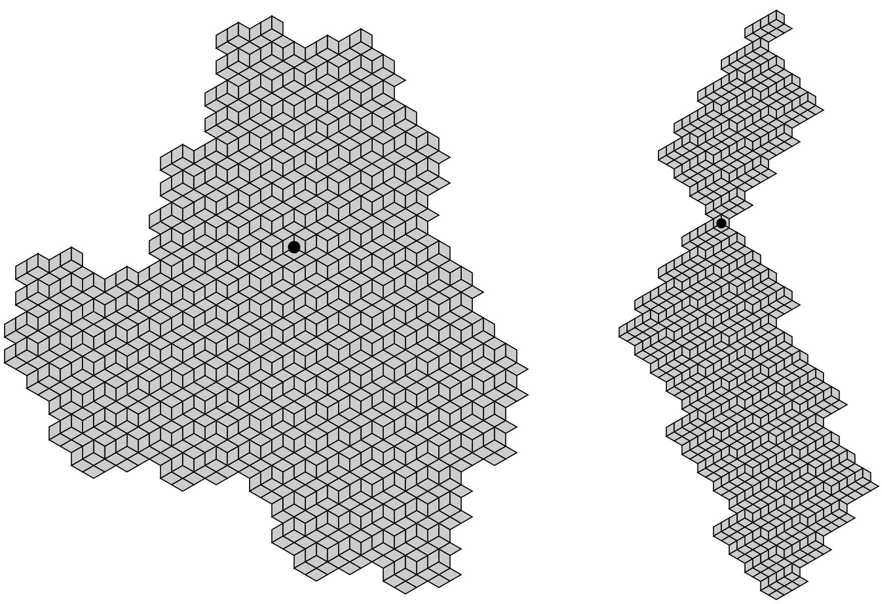


Figure 2.1: On the left, the pattern $(\Sigma^{\mathsf{Brun}}_2 \Sigma^{\mathsf{Brun}}_3 \Sigma^{\mathsf{Brun}}_2)^5(\mathcal{U})$. On the right, the pattern $(\Sigma^{\mathsf{Brun}}_2 \Sigma^{\mathsf{Brun}}_3 \Sigma^{\mathsf{Brun}}_1 \Sigma^{\mathsf{Brun}}_1)^4(\mathcal{U})$. The origin is marked by a black dot in both pictures. The Rauzy fractals associated with these two infinite periodic products are plotted in Figure 3.1.

The annulus property The patterns generated by iterating dual substitutions can have complicated shapes, so there is no obvious way of proving that such a sequence covers arbitrarily large discs. (In contrast, such a property is obvious for square 2×2 substitutions of the plane, because the generated patterns are simply squares of size 2^n.)

One approach for proving this property in the context of dual substitutions has been initiated by Ito and Ohtsuki [**IO94**]. The idea is to make sure that the generated patterns contain an increasing number of annuli of positive width, and hence cover arbitrarily large discs. This requires that the *annulus property* holds, *i.e.*, that the image of an annulus by a dual substitution remains an annulus. However, the annulus property is wrong if no additional assumptions are made on the annuli, as shown in Figure 2.2. This is why we must introduce *covering properties* in Section 2.2, which are some combinatorial restrictions on the annuli. These restrictions allow us to prove the annulus property in Section 2.3. The proofs require many verifications which are specific to the set of substitutions under study. Hence the results given in Sections 2.2 and 2.3 are stated in a generic way (they do not depend on the choice of substitutions), and proving that they hold for our families of substitutions is reduced to technical verifications, which are carried out in Section 2.6.

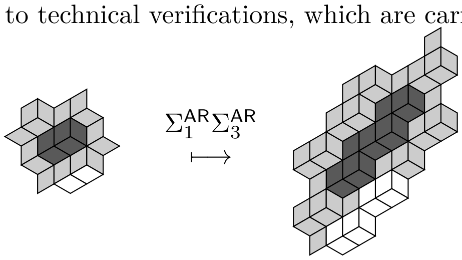


Figure 2.2: On the left, the faces in light gray do not form a suitable annulus around the faces in dark gray, because the annulus is broken in the image by $\Sigma_1^{\mathsf{AR}}\Sigma_3^{\mathsf{AR}}$. This problem is fixed by adding the faces in white to the annulus. The covering conditions introduced in Section 2.2 ensure that such problems do not happen. Similar examples with the Brun and Jacobi-Perron substitutions are given in Example 2.3.3.

Generation graphs For our strategy to work, establishing the annulus property is necessary but not sufficient: we must also make sure that, starting from a seed, at least one annulus is eventually generated by iterating sufficiently many substitutions.

In the case of Arnoux-Rauzy substitutions, such a verification can be made by explicitly tracking all the possible patterns "growing" around the seed $\mathcal{U}$, and checking that an annulus is eventually generated in all cases. This is done thanks to graph shown in Figure 2.4 used in the proof of Lemma 2.6.13, and this proves that the seed $\mathcal{U}$ is always enough for Arnoux-Rauzy substitutions.

However, the situation is quite different with the Brun or the Jacobi-Perron substitutions, because there are some sequences of substitutions which do not generate annuli around the seed, as we have seen in the example of Figure 2.1. Therefore, the reasoning described above for the Arnoux-Rauzy substitutions becomes unsuitable. Trying to build a graph similar to the one used for the Arnoux-Rauzy substitutions in Lemma 2.6.13 produces graphs which are too complicated to handle. Moreover, trying to use this approach to prove that annuli are always generated when starting from a seed larger than $\mathcal{U}$ considerably increases the number of cases to check.

This leads us to introduce another kind of generation graphs in Section 2.4 where we do not track all the possible patterns around a seed, but rather all the possible sequences of preimages of each *face* of the annuli we want to generate. This yields much smaller graphs, whose vertices are faces (and not patterns), and which can be constructed algorithmically. Again, these constructions (stated in Section 2.4) are generic, and the specific constructions are carried out in Sections 2.6.5 and 2.6.6 for the Brun and Jacobi-Perron substitutions.

2.2 Covering properties

We now introduce $\mathcal{L}$-coverings and strong $\mathcal{L}$-coverings, which are the combinatorial tools that will be used in order to prove the annulus property in Section 2.3.

We call a ***pattern*** any finite union of faces. A pattern is said to be ***edge-connected*** if any two faces are connected by a path of faces $f_1, \dots, f_n$ such that f_k and f_{k+1} share an edge, for all $k \in \{1, \dots, n-1\}$. In the definitions below, $\mathcal{L}$ will always denote a set of patterns which is closed by translation of $\mathbb{Z}^3$, so we will define such sets by giving only one element of each translation class.

Definition 2.2.1. Let $\mathcal{L}$ be a set of patterns. A pattern P is ***$\mathcal{L}$-covered*** if for all faces $e, f \in P$, there exist patterns $Q_1, \dots, Q_n \in \mathcal{L}$ such that

(1) $e \in Q_1$ and $f \in Q_n$;
(2) $Q_k \cap Q_{k+1}$ contains at least one face, for all $k \in \{1, \dots, n-1\}$;
(3) $Q_k \subseteq P$ for all $k \in \{1, \dots, n\}$.

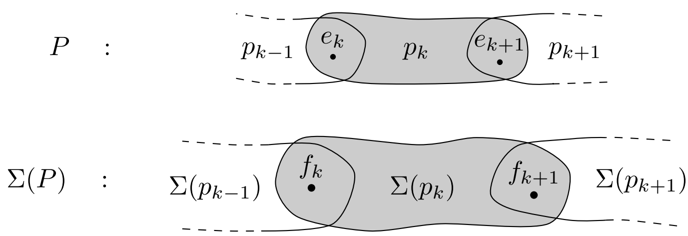
The next proposition, due to [**IO94**], gives a sufficient combinatorial criterion to prove that a dual substitution preserves $\mathcal{L}$-covering.

Proposition 2.2.2. *Let $\mathcal{L}$ be a set of patterns, P be an $\mathcal{L}$-covered pattern and Σ be a dual substitution. If $\Sigma(Q)$ is $\mathcal{L}$-covered for every $Q \in \mathcal{L}$, then $\Sigma(P)$ is $\mathcal{L}$-covered.*

Proof. Let us call an ***$\mathcal{L}$-chain*** from a face e to a face f a sequence of patterns $Q_1, \dots, Q_n \in \mathcal{L}$ that satisfies Properties (1), (2) and (3) of Definition 2.2.1.

Let f and f' be two faces of $\Sigma(P)$. To prove that $\Sigma(P)$ is $\mathcal{L}$-covered, we need to construct an $\mathcal{L}$-chain from f to f'. Let e and e' be two faces of P such that $f \in \Sigma(e)$ and $f' \in \Sigma(e')$. Since *by*P is $\mathcal{L}$-covered, there exists an $\mathcal{L}$-chain $(p_1, \dots, p_n)$ from e to e'. For all $k \in \{2, \dots, n-1\}$, let f_k be a face of $\Sigma(p_k \cap p_{k+1})$, and let $f_1 = f$, $f_n = f'$. Such a face f_k exists because $(p_1, \dots, p_n)$ is an $\mathcal{L}$-chain.

For all $k \in \{1, \dots, n-1\}$, there exists an $\mathcal{L}$-chain from f_k to f_{k+1}, because f_k and f_{k+1} are in $\Sigma(p_{k+1})$ and $\mathcal{L}$ is stable under Σ; see below.

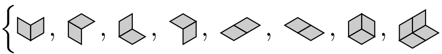


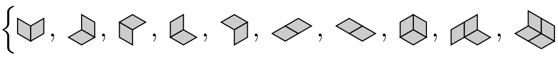
The concatenation of the $\mathcal{L}$-chains from f_k to f_{k+1} yields an $\mathcal{L}$-chain from f to f', so the proposition is proved. ⬠

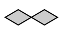
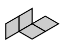
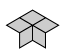
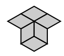
We will see in Example 2.3.3 that $\mathcal{L}$-coverings are not sufficient for our purposes, so we introduce *strong* $\mathcal{L}$-coverings below.

Definition 2.2.3. A pattern P is ***strongly*** $\boldsymbol{\mathcal{L}}$***-covered*** if P is $\mathcal{L}$-covered and if for every pattern $X \subseteq P$ that is edge-connected and consists of two faces, there exists a pattern $Y \in \mathcal{L}$ such that $X \subseteq Y \subseteq P$.

Example 2.2.4. Consider the following two sets of patterns (which will later be used with the Brun and Jacobi-Perron substitutions in Section 2.6):

$\mathcal{L}^{\mathsf{Brun}} = \{$  $\}$,

$\mathcal{L}^{\mathsf{JP}} = \{$  $\}$,

and let $P_1 =$ , $P_2 =$ , $P_3 =$ , $P_4 =$ . We have:

- P_1 is neither $\mathcal{L}^{\mathsf{Brun}}$- nor $\mathcal{L}^{\mathsf{JP}}$-covered;
- P_2 is not $\mathcal{L}^{\mathsf{Brun}}$-covered, but it is strongly $\mathcal{L}^{\mathsf{JP}}$-covered;
- P_3 is $\mathcal{L}^{\mathsf{Brun}}$- and $\mathcal{L}^{\mathsf{JP}}$-covered, but not strongly covered.;
- P_4 is strongly $\mathcal{L}^{\mathsf{Brun}}$- and $\mathcal{L}^{\mathsf{JP}}$-covered.

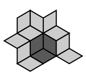
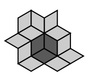
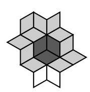
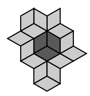
2.3 The annulus property

In this section we define $\mathcal{L}$-annuli and we introduce Property A, which is a sufficient condition under which we can prove that a dual substitution Σ verifies the ***annulus property***: if A is an $\mathcal{L}$-annulus of P, then $\Sigma(A)$ is an $\mathcal{L}$-annulus of $\Sigma(P)$ (Proposition 2.3.5). The ***boundary*** ∂P of a pattern P is the union of the edges e of the faces of f such that e is contained in one face only. We recall that $\mathcal{U} =$  $= [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star$.

Definition 2.3.1. An $\boldsymbol{\mathcal{L}}$***-annulus*** of a simply connected pattern P is a pattern A such that A is strongly $\mathcal{L}$-covered, A and P have no face in common, and $P \cap \partial(P \cup A) = \varnothing$.

Note that the condition $P \cap \partial(P \cup A) = \varnothing$ in the above definition is a concise way to express the fact that the $\mathcal{L}$-covered set$A$ is a "good surrounding" of $\mathcal{U}$.

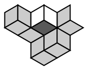
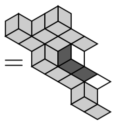
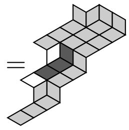
Example 2.3.2. Let A_1, A_2, A_3 and A_4 be defined by

$A_1 \cup \mathcal{U} =$  $A_2 \cup \mathcal{U} =$  $A_3 \cup \mathcal{U} =$  $A_4 \cup \mathcal{U} =$ ,

where $\mathcal{U}$ is depicted in dark gray. We have:

- A_1 is not an annulus of $\mathcal{U}$ because $\mathcal{U} \cap \partial(\mathcal{U} \cup A_1)$ is nonempty (it contains an edge).
- A_2 is not an annulus of $\mathcal{U}$ because $\mathcal{U} \cap \partial(\mathcal{U} \cup A_2)$ is nonempty (it contains a point).
- A_3 is not an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{U}$ because it is not strongly $\mathcal{L}^{\mathsf{Brun}}$-covered. Indeed, if $X =$  $\subseteq A_3$ is the pattern depicted in white in the picture, there does not exist a pattern $Y \in \mathcal{L}^{\mathsf{Brun}}$ such that $X \subseteq Y \subseteq A_3$.
- A_4 is an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{U}$.

Example 2.3.3. Let P be a pattern that consists of the single face $[\mathbf{0}, 3]^\star$ (shown in dark gray below) and let A be the set of faces surrounding P defined in the picture below.

$A \cup P =$  $\Sigma_1^{\mathsf{Brun}}\Sigma_3^{\mathsf{Brun}}(A \cup P) =$  $\Sigma_{0,2}^{\mathsf{JP}}(A \cup P) =$ 

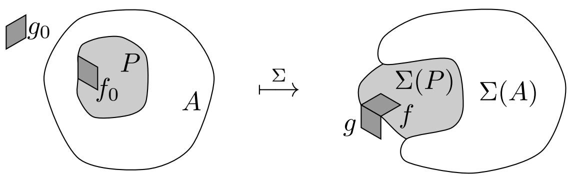
The pattern A is both $\mathcal{L}^{\mathsf{Brun}}$- and $\mathcal{L}^{\mathsf{JP}}$-covered, but *not* strongly (so it is not a valid annulus). Its images by Brun or Jacobi-Perron substitutions fail to be topological annuli (as shown above), which illustrates the necessity of *strong* coverings if we want the image of an annulus to remain an annulus. The faulty two-face pattern and its images are shown in white.

Definition 2.3.4. Let Σ be a dual substitution, and let $\mathcal{L}$ a set of edge-connected patterns. We say that ***Property A*** holds for Σ with respect to $\mathcal{L}$ if for every connected two-face pattern $f \cup g$ and for every disconnected two-face pattern $f_0 \cup g_0$ such that $f \in \Sigma(f_0)$ and $g \in \Sigma(g_0)$, the following holds: there do not exist a pattern P and an $\mathcal{L}$-annulus A of P both included in a common discrete plane Γ such that $f_0 \in P$ and $g_0 \in \Gamma \setminus (A \cup P)$.

Thanks to the following proposition, Property A (together with strong $\mathcal{L}$-covering assumptions) implies the annulus property. The main interest of Property A is that it can be checked by treating finitely many cases, by enumerating all the two-face connected patterns $f \cup g$ that admit a disconnected preimage. This is done for the Arnoux-Rauzy, Brun and Jacobi-Perron substitutions in Propositions 2.6.5, 2.6.7 and 2.6.8.

Proposition 2.3.5. *Let Σ be a dual substitution and $\mathcal{L}$ be a set of edge-connected patterns such that Property A holds for Σ with respect to $\mathcal{L}$. Assume that the image by Σ of every strongly $\mathcal{L}$-covered pattern is strongly $\mathcal{L}$-covered. Let P be a pattern and A be an $\mathcal{L}$-annulus of P, both included in a common discrete plane. Then $\Sigma(A)$ is an $\mathcal{L}$-annulus of $\Sigma(P)$.*

Proof. The pattern A is strongly $\mathcal{L}$-covered because it is an $\mathcal{L}$-annulus, so $\Sigma(A)$ is also strongly $\mathcal{L}$-covered, by assumption. It remains to show that $\Sigma(P) \cap \partial(\Sigma(P) \cup \Sigma(A)) = \varnothing$. Suppose the contrary. This means that there exist faces f, g, f_0, g_0 such that $f \in \Sigma(f_0)$, $g \in \Sigma(g_0)$, $f \cup g$ is connected, and $f_0 \cup g_0$ is disconnected as shown below.



These are precisely the conditions stated in Property A, so such a situation cannot occur and the proposition holds. ⬠

The annulus property will be crucially used in the proof of Theorem 2.5.1: it ensures that a quantity called the *minimal combinatorial radius* (defined in Section 2.5) is strictly increasing under iteration of dual substitutions.

2.4 Generation graphs

Let $\Sigma_1, \ldots, \Sigma_\ell$ be dual substitutions and let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ be seeds from which we want to iterate the Σ_i. The aim of this section is to characterize the sequences $(i_n)_{n \in \mathbb{N}} \in \{1, \ldots, \ell\}^{\mathbb{N}}$ such that for n large enough and for every $\mathcal{V}_i$, the patterns $\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{V}_i)$ eventually contain an $\mathcal{L}$-covered annulus of some $\mathcal{V}_j$ (where $\mathcal{L}$ is a set of patterns associated with the substitutions Σ_i). In practice, the annuli $\mathcal{V}_i$ will often consist of an annulus of $\mathcal{U} =$  or of a slightly larger pattern.

This characterization is achieved thanks to the algorithmic construction of *generation graphs* below. We start with a set $\mathcal{X}$ that contains the faces of all the possible $\mathcal{L}$-annuli of the seeds $\mathcal{V}_i$, and we recursively backtrack their preimages, encoding all the information in the graph. Intuitively, if $\mathcal{V}_i$ is such that every "backward" path of preimages from $\mathcal{X}$ eventually comes back to $\mathcal{V}_i$, then the seeds $\mathcal{V}_i$ always generates an $\mathcal{L}$-annulus of another $\mathcal{V}_j$. This observation is formalized in Proposition 2.4.4 below, which provides us with the desired characterizations of sequences $(i_n)_{n \in \mathbb{N}} \in \{1, \ldots, \ell\}^{\mathbb{N}}$. We need two technical assumptions for the above method to work in practice.

- We use a ***filter set*** $\mathcal{F}$, by restricting the allowed preimages to $\mathcal{F}$ only. This makes generation graphs more manageable by eliminating some useless faces. For example, the set $\mathcal{F}^{\mathsf{Brun}}$ that will be used with the Brun substitutions is the set of all the faces that belong to a discrete plane $\Gamma_{\mathbf{v}}$ with $0 < \mathbf{v}_1 < \mathbf{v}_2 < \mathbf{v}_3$.
- Moreover, we assume that $\Sigma_1, \ldots, \Sigma_\ell$ are substitutions associated with a *convergent* continued fraction algorithm, in the same sense as described for the Brun and Jacobi-Perron substitutions in Propositions 1.5.3 and 1.5.5, namely:
  - To every admissible sequence $(i_n)_{n \in \mathbb{N}} \in \{1, \ldots, \ell\}^{\mathbb{N}}$ corresponds a unique vector $\mathbf{v}$ with positive coordinates.
  - To this vector are associated the vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots$, which are such that $\Gamma_{\mathbf{v}} = \Sigma_{i_1} \cdots \Sigma_{i_n}(\Gamma_{\mathbf{v}^{(n)}})$ for all $n \geqslant 1$.

Definition 2.4.1. Let $\Sigma_1, \ldots, \Sigma_\ell$ be dual substitutions, $\mathcal{F}$ be a family of faces (the "filter set") and $\mathcal{X}$ be a finite set of faces (the "initial set"). The ***generation graph*** associated with $\Sigma_1, \ldots, \Sigma_\ell$, $\mathcal{F}$ and $\mathcal{X}$ is defined as the graph $\mathcal{G} = \bigcup_{n \geqslant 0} \mathcal{G}_n$ where $(\mathcal{G}_n)_{n \geqslant 0}$ is the sequence of directed graphs (whose vertices are faces), defined by induction as follows.

(1) *Initialization.* $\mathcal{G}_0$ has no edges and its set of vertices is $\mathcal{X}$.
(2) *Iteration.* Suppose that $\mathcal{G}_n$ is constructed for some $n \geqslant 0$. Start with $\mathcal{G}_{n+1}$ equal to $\mathcal{G}_n$. Then, for each vertex f of $\mathcal{G}_n$, for each $i \in \{1, \ldots, \ell\}$, and for each $g \in \mathcal{F}$ such that $f \in \Sigma_i(g)$, add the vertex g and the edge $g \xrightarrow{i} f$ to $\mathcal{G}_{n+1}$.

Remark 2.4.2. The non-decreasing union $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$ considered in Definition 2.4.1 is not necessarily finite. In fact, $\mathcal{G}$ is finite if and only if $\mathcal{G}_n = \mathcal{G}_{n+1}$ for some $n \in \mathbb{N}$. This is the case for the set of Brun substitutions, as we will see in Section 2.6.5. However, even if $\mathcal{G}_n = \mathcal{G}_{n+1}$ never occurs, the infinite graph $\mathcal{G}$ can still be successfully exploited, as will be done in Section 2.6.6 for the Jacobi-Perron substitutions.

Remark 2.4.3. The orientation of edges in generation graphs has been chosen in such a way that it agrees with the usual arrow notation for composition of functions. In particular, for every path $f_n \xrightarrow{i_n} \cdots \xrightarrow{i_2} f_1 \xrightarrow{i_1} f_0$, we have $f_0 \in \Sigma_{i_1} \cdots \Sigma_{i_n}(f_n)$.

Proposition 2.4.4. *Let*

- $\Sigma_1, \ldots, \Sigma_\ell$ *be substitutions associated with a convergent continued fraction algorithm;*
- $\mathcal{F}$ *be a set of faces such that* $\mathcal{F} \subset \bigcup_{\mathbf{v}>0} \Gamma_\mathbf{v}$ *and such that if* $\Gamma_\mathbf{v} \subseteq \mathcal{F}$*, then* $\Gamma_{\mathbf{v}^{(n)}} \subseteq \mathcal{F}$ *for all* $n \geqslant 1$*;*
- $\mathcal{X} \subseteq \mathcal{F}$ *be a finite set of faces;*
- $\mathcal{G}$ *be the generation graph constructed with substitutions* $\Sigma_1, \ldots, \Sigma_\ell$*, filter set* $\mathcal{F}$ *and initial set* $\mathcal{X}$*.*

For every admissible sequence $(i_n)_{n\in\mathbb{N}} \in \{1, \ldots, \ell\}^\mathbb{N}$ *(which determines a unique vector* $\mathbf{v}$*), for every finite pattern* $\mathcal{V}$*, and for every* $f_0 \in \mathcal{X} \cap \Gamma_\mathbf{v}$*, we have:*

(1) If $f_0 \notin \Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{V})$*, then there exists a path* $f_n \xrightarrow{i_n} \cdots f_1 \xrightarrow{i_1} f_0$ *in* $\mathcal{G}$ *with* $f_n \notin \mathcal{V}$*.*

(2) For every path $f_n \xrightarrow{i_n} \cdots f_1 \xrightarrow{i_1} f_0$ *in* $\mathcal{G}$ *such that* $f_n \notin \mathcal{U}$*, we have* $f_0 \notin \Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{U})$*.*

Proof. (1). Let n be such that $f_0 \notin \Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{V})$. We have $\Gamma_\mathbf{v} = \Sigma_{i_1} \cdots \Sigma_{i_n}(\Gamma_{\mathbf{v}^{(n)}})$ for all $n \geqslant 1$, so Statement (3) of Proposition 1.2.4 implies that there exist faces $f_1, \ldots, f_n$ such that $f_k \in \Gamma_{\mathbf{v}^{(k)}}$ and $f_{k-1} \in \Sigma_{i_k}(f_k)$ for every $k \in \{1, \ldots, n\}$. By assumption on $\mathcal{F}$ we have $\Gamma_{\mathbf{v}^{(k)}} \subseteq \mathcal{F}$ for all $k \geqslant 0$, so $f_1, \ldots, f_n$ are all in the vertices of $\mathcal{G}$. By assumption on f_0 we cannot have $f_n \in \mathcal{V}$, so the faces $f_0, f_1, \ldots, f_n$ yield the required path in $\mathcal{G}$.

(2). Let $f_n \xrightarrow{i_n} \cdots f_1 \xrightarrow{i_1} f_0$ be a path in $\mathcal{G}$ such that $f_n \notin \mathcal{U}$. The patterns $\mathcal{U}$ and f_n are disjoint patterns included in a common discrete plane (because $\mathcal{U}$ is included in every discrete plane). Hence, Statement (4) of Proposition 1.2.4 implies that $\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{U})$ and $\Sigma_{i_1} \cdots \Sigma_{i_n}(f_n)$ do not have any face in common for all $n \geqslant 1$, which implies that $f_0 \notin \Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{U})$ because $f_0 \in \Sigma_{i_1} \cdots \Sigma_{i_n}(f_n)$. ⬠

Remark 2.4.5. Statement (1) of Proposition 2.4.4 will be used to obtain "positive" results, such as the fact that a given seed $\mathcal{V}$ always generates a full discrete plane (see Theorem 2.5.1). Conversely, Statement (2) will be used in Theorems 2.5.2 and 2.5.4 to characterize the sequences $(i_n)_{n\in\mathbb{N}}$ for which iterating the corresponding sequence of dual substitutions fails to generate an entire discrete plane when starting from $\mathcal{U}$.

2.5 Main results

We now state our main results relative to discrete plane generation. In order to express the fact that some patterns grow by iterating substitutions, we need to define the ***minimal combinatorial radius*** $\mathrm{rad}(P)$ of a pattern P containing the seed $\mathcal{U} =$ . It is equal to the length of the shortest sequence of faces $f_1, \ldots, f_n$ in P such that $f_1 \in \mathcal{U}$, f_i and f_{i+1} share an edge, and f_n shares an edge with the boundary of P. Intuitively, $\mathrm{rad}(P)$ measures the minimal distance between $\mathbf{0}$ and the boundary of P.

The proofs of the results of this section depend on some "technical" results, which are proved separately in Section 2.6 in order to improve the readability of the proofs. These technical results depend on some enumerations performed with the Sage software [**Sag**], see p. 20.

Existence of finite seeds

We start by stating the existence of finite seeds such that iterating *any* admissible sequence of substitutions generates patterns with arbitrarily large minimal combinatorial radius. In the Arnoux-Rauzy case, the seed $\mathcal{U}$ is enough, but in the Brun and Jacobi-Perron cases, we need the (slightly larger) seeds $\mathcal{V}_1^{\mathsf{Brun}}, \mathcal{V}_2^{\mathsf{Brun}}$ and $\mathcal{V}_1^{\mathsf{JP}}, \mathcal{V}_2^{\mathsf{JP}}, \mathcal{V}_3^{\mathsf{JP}}, \mathcal{V}_4^{\mathsf{JP}}$. These finite seeds are defined in Proposition 2.6.3 and 2.6.4.

Theorem 2.5.1. *For every $R \geqslant 0$ there exists $N \geqslant 0$ such that*

(1) $\operatorname{rad}(\Sigma_{i_1}^{\mathsf{AR}} \cdots \Sigma_{i_n}^{\mathsf{AR}}(\mathcal{U})) \geqslant R$ *for every* $(i_1 \cdots i_n) \in \{1,2,3\}^n$ *that contains more than* N *occurrences of* 1, 2 *and* 3*;*
(2) $\operatorname{rad}(\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{V})) \geqslant R$ *for every* $(i_1 \cdots i_n) \in \{1,2,3\}^n$ *that contains more than* N *occurrences of* 3*, and for every* $\mathcal{V} \in \{\mathcal{V}_1^{\mathsf{Brun}}, \mathcal{V}_2^{\mathsf{Brun}}\}$*;*
(3) $\operatorname{rad}(\Sigma_{a_1,b_1}^{\mathsf{JP}} \cdots \Sigma_{a_n,b_n}^{\mathsf{JP}}(\mathcal{V})) \geqslant R$ *for every admissible Jacobi-Perron expansion* (a_1, b_1)*, ...,* (a_n, b_n) *such that* $n \geqslant N$*, and for every* $\mathcal{V} \in \{\mathcal{V}_1^{\mathsf{JP}}, \mathcal{V}_2^{\mathsf{JP}}, \mathcal{V}_3^{\mathsf{JP}}, \mathcal{V}_4^{\mathsf{JP}}\}$*.*

Proof. The intuitive idea behind this proof is simple: if we are given n disjoint concentric annuli around a seed $\mathcal{V}$, then applying sufficiently many substitutions will:

- preserve the n concentric annuli (thanks to the annulus property),
- generate a new annulus around the seed $\mathcal{V}$ (thanks to generation graphs),

so we have $n+1$ disjoint concentric annuli, and repeating this reasoning yields the result. Note that by "disjoint", we mean "which do not have any faces in common". We now formalize the above reasoning to prove (2). By virtue of Lemma 2.6.15, there exists $M \geqslant 1$ such that for every $(i_1, \dots, i_n) \in \{1,2,3\}^n$ containing more than M occurrences of 3, the pattern $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{V})$ contains an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$.

Let $R \geqslant 0$. We prove that taking $N = R \times M$ satisfies (2). Let $(i_1, \dots, i_n) \in \{1,2,3\}^n$ containing more than N occurrences of 3. We write $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}} = \Sigma_1 \cdots \Sigma_R$, where each Σ_k is a product of Σ_i^{Brun} containing Σ_3^{Brun} at least M times. To prove that that $\operatorname{rad}(\Sigma_1 \cdots \Sigma_R(\mathcal{V})) \geqslant R$, we prove that $\Sigma_1 \cdots \Sigma_R(\mathcal{V})$ contains R disjoint concentric annuli $A_1, \dots, A_R$ such that A_1 is an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{V}_j^{\mathsf{Brun}}$, and for all $k \in \{1, \dots, R-1\}$, A_k is an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $A_{k-1} \cup \cdots \cup A_1 \cup \mathcal{V}_j^{\mathsf{Brun}}$ for some $j \in \{1,2\}$.

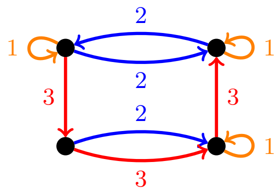
The reasoning goes by induction. First, $\Sigma_1^{\mathsf{Brun}}(\mathcal{V})$ contains an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{V}_j^{\mathsf{Brun}}$ for some $j \in \{1,2\}$, because Σ_R contains at least M occurrences Σ_3^{Brun}. Now suppose that for some $k \in \{1, \dots, R-1\}$, the pattern $\Sigma_{R-k+1} \cdots \Sigma_R(\mathcal{V})$ contains k disjoint annuli $A_1, \dots, A_k$ around $\mathcal{V}_j^{\mathsf{Brun}}$, like above (with $j \in \{1,2\}$). We use the annulus property for Brun substitutions: thanks to Proposition 2.3.5 (the annulus property), Proposition 2.6.7 (strong covering conditions) and Proposition 2.6.10 (Property A), the patterns $\Sigma_{R-k}(A_1), \dots, \Sigma_{R-k}(A_k)$ are k concentric $\mathcal{L}^{\mathsf{Brun}}$ annuli. Moreover, $\Sigma_{R-k}(\mathcal{V}_j^{\mathsf{Brun}})$ contains an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$. In total this makes $k+1$ annuli contained in $\Sigma_{R-k}\Sigma_{R-k+1} \cdots \Sigma_R(\mathcal{V})$. They are all disjoint thanks to Proposition 1.2.4, so the induction step holds and (2) is proved.

Statements (1) and (3) can be proved by the same reasoning. For Arnoux-Rauzy substitutions, we use Lemma 2.6.13 (to generate an annulus around the seed), Proposition 2.3.5 (the annulus property), Proposition 2.6.5 (strong covering conditions), and Proposition 2.6.9 with Lemma 2.6.6 (Property A). For Jacobi-Perron substitutions we use Lemma 2.6.15 (to generate an annulus around the seed), Proposition 2.3.5 (the annulus property), Proposition 2.6.8 (strong covering conditions), and Proposition 2.6.12 (Property A). Note that in the proofs of (2) and (3), we have implicitly used the fact that if a pattern contains $\mathcal{U}$ and an $\mathcal{L}^{\mathsf{Brun}}$- or $\mathcal{L}^{\mathsf{JP}}$-annulus of $\mathcal{U}$, then it contains one of the seeds $\mathcal{V}_i^{\mathsf{Brun}}$ or $\mathcal{V}_i^{\mathsf{JP}}$. This is proved in Proposition 2.6.3 for Brun substitutions and in Proposition 2.6.4 for Jacobi-Perron substitutions. ⬠

Characterizing sequences for which $\mathcal{U}$ is not a seed

We now give characterizations of the "bad" sequences of substitutions, that is, the sequences of substitutions such that the seed $\mathcal{U}$ does *not* generate patterns with arbitrarily large combinatorial radius. These results are only stated for Brun and Jacobi-Perron substitutions because Theorem 2.5.1 implies that there are no such sequences for Arnoux-Rauzy substitutions.

Theorem 2.5.2. *Let $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$ be such that $i_n = 3$ infinitely often, and let $\mathbf{v}$ be the vector whose Brun expansion is $(i_n)_{n\in\mathbb{N}}$. Then $\bigcup_{n\geqslant 0} \Sigma^{\mathsf{Brun}}_{i_1}\cdots\Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U})$ is a strict subset of $\Gamma_{\mathbf{v}}$ if and only if there exists N such that $(i_n)_{n\geqslant N}$ is the labelling of an infinite backward path $\cdots \xrightarrow{i_{N+2}} \bullet \xrightarrow{i_{N+1}} \bullet \xrightarrow{i_N} \bullet$ in the following graph.*



Proof. First, note that the 4-vertex graph given in the statement of Theorem 2.5.2 and the graph $\mathcal{G}^{\mathsf{Brun}}$ given in Section 2.6.5, admit the same sets of infinite path labellings containing infinitely many 3's. (The 4-vertex graph above is a "reduced" version of $\mathcal{G}^{\mathsf{Brun}}$.) Also note that the inclusion $\Sigma^{\mathsf{Brun}}_{i_1}\cdots\Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U}) \subseteq \Gamma_{\mathbf{v}}$ is true for all $n \geqslant 1$, because $(i_n)_{n\in\mathbb{N}}$ is the Brun-expansion of $\mathbf{v}$ and because $\mathcal{U}$ is included in every discrete plane.

Now, if $\bigcup_{n\geqslant 1} \Sigma^{\mathsf{Brun}}_{i_1}\cdots\Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U})$ is a strict subset of $\Gamma_{\mathbf{v}}$, then, thanks to Proposition 2.6.3, there must exist $f_0 \in (\Gamma_{\mathbf{v}} \cap (\mathcal{V}^{\mathsf{Brun}}_1 \cup \mathcal{V}^{\mathsf{Brun}}_2))$ such that $f_0 \notin \bigcup_{n\geqslant 1} \Sigma^{\mathsf{Brun}}_{i_1}\cdots\Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U})$ is strictly contained in $\Gamma_{\mathbf{v}}$ (otherwise Theorem 2.5.1 would imply equality). Hence, Proposition 2.4.4 (1) applied to the graph $\mathcal{G}^{\mathsf{Brun}}$ (computed in Section 2.6.5) proves that such an infinite path must appear in $\mathcal{G}^{\mathsf{Brun}}$. Conversely, the existence of such a path proves that the corresponding face $f_0 \in (\Gamma_{\mathbf{v}} \cap (\mathcal{V}^{\mathsf{Brun}}_1 \cup \mathcal{V}^{\mathsf{Brun}}_2))$ will be missing from $\Sigma^{\mathsf{Brun}}_{i_1}\cdots\Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U})$ for all $n \geqslant 1$, thanks to Proposition 2.4.4 (2). ⬠

Example 2.5.3. For a *finite* product of Brun substitutions $\sigma = \sigma^{\mathsf{Brun}}_{i_1}\cdots\sigma^{\mathsf{Brun}}_{i_m}$, Theorem 2.5.2 implies that the patterns $\mathbf{E}^\star_1(\sigma)^n(\mathcal{U})$ generate a full discrete plane if and only if there is a loop labelled by a power of the word $i_m \ldots i_1$ in the graph. This is the case for example with $i_4 i_3 i_2 i_1 = 3332$, but not with $i_3 i_2 i_1 = 332$ (it is easy to check that there is no loop labelled by a power of 332). Note that in some cases we must take a power of $i_m \ldots i_1$ for such a loop to exist, which is the case for example with $i_2 i_1 = 32$, because there is a loop labelled by 3232 but no loop labelled by 32.

It is also interesting to note that many "language-theoretical" corollaries can be derived from Theorem 2.5.2. For example, if $\sigma = \sigma^{\mathsf{Brun}}_{i_1}\cdots\sigma^{\mathsf{Brun}}_{i_m}$ and $i_1\cdots i_m = 3332^k$, then $\bigcup_{n\geqslant 1}\mathbf{E}^\star_1(\sigma)^n(\mathcal{U})$ is a strict subset of $\Gamma_{\mathbf{v}}$ if and only if k is odd.

Theorem 2.5.4. *Let $(a_n, b_n)_{n\in\mathbb{N}}$ be an admissible Jacobi-Perron expansion and let $\mathbf{v}$ be a vector whose Jacobi-Perron expansion is $(a_n, b_n)_{n\in\mathbb{N}}$. Then $\bigcup_{n\geqslant 1}\Sigma^{\mathsf{JP}}_{a_1,b_1}\cdots\Sigma^{\mathsf{JP}}_{a_n,b_n}(\mathcal{U}))$ is a strict subset of $\Gamma_{\mathbf{v}}$ if and only if there exists $\ell \geqslant 1$ such that for every $k \geqslant 0$, we have*

$$a_{\ell+3k} = 0, \qquad a_{\ell+3k+1} = b_{\ell+3k+1}, \qquad 0 < a_{\ell+3k+2} < b_{\ell+3k+2}.$$

Proof. Let $(i_n)_{n\in\mathbb{N}}$ be the additive Jacobi-Perron expansion corresponding to $(a_n, b_n)_{n\in\mathbb{N}}$. Because this additive expansion is admissible, it must contain infinitely many 3's or 4's. Like in the proof of Theorem 2.5.4, the infinite labelling sequences of the graph $\mathcal{G}^{\mathsf{JP}}$ defined in Section 2.6.6 give us the desired characterization, thanks to Proposition 2.4.4.

Indeed, in the graph $\mathcal{G}^{\mathsf{JP}}$, following an infinite path containing infinitely many 3's or 4's forces us to turn clockwise, visiting f_c, f_e and f_d cyclically and in this order. The result then follows from the definition of the additive decomposition of $\Sigma^{\mathsf{JP}}_{a,b}$ by $\Theta_1, \Theta_2, \Theta_3, \Theta_4$, and by the following facts, which can easily be checked:

- any path from f_c to f_e in $\mathcal{G}^{\mathsf{JP}}$ corresponds to $a_n = 0$,
- any path from f_d to f_c in $\mathcal{G}^{\mathsf{JP}}$ corresponds to $a_n = b_n$,
- any path from f_e to f_d in $\mathcal{G}^{\mathsf{JP}}$ corresponds to $0 < a_n < b_n$. ⬠

We have recovered in Theorem 2.5.4 the same characterization as the one obtained in [**IO94**], by using a different notion of generation graphs, which does not require the same succession of lemmas as found in [**IO94**].

Translates of seeds always occur

The next theorem states that the seed $\mathcal{U}$ is always sufficient to generate *translates* of patterns with arbitrarily large radius (even though $\mathcal{U}$ is not always sufficient to generate patterns of arbitrarily large radius centered at the origin). As mentioned previously, we do not state this result for Arnoux-Rauzy substitutions because it is directly implied by Theorem 2.5.1.

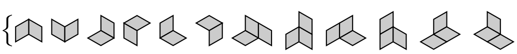
Theorem 2.5.5. *For every $R \geqslant 0$ there exists $N \geqslant 0$ such that*

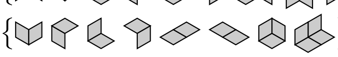
(1) $\Sigma^{\mathsf{Brun}}_{i_1} \cdots \Sigma^{\mathsf{Brun}}_{i_n}(\mathcal{U})$ *contains a translate of a pattern P with* $\mathrm{rad}(P) \geqslant R$ *for every* $(i_1 \cdots i_n) \in \{1,2,3\}^n$ *that contains more than N occurrences of* 3*;*
(2) $\Sigma^{\mathsf{JP}}_{a_1,b_1} \cdots \Sigma^{\mathsf{JP}}_{a_n,b_n}(\mathcal{U})$ *contains a translate of a pattern P with* $\mathrm{rad}(P) \geqslant R$ *for every admissible Jacobi-Perron expansion* $(a_1, b_1), \ldots, (a_n, b_n)$ *such that* $n \geqslant N$*.*

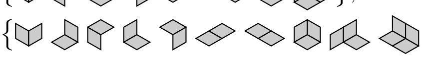
Proof. Since we are dealing with *translates* of seeds, we cannot use generation graphs like in the previous theorems. We must track "by hand" all the possible sequences of iterations starting from $\mathcal{U}$, and explicitly check that a translate of one of the seeds of Theorem 2.5.1 is eventually generated. This is done in details in Lemma 2.6.18 for (1) and in Lemma 2.6.19 for (2). Then, the result follows directly from Theorem 2.5.1: once a translate of a seed occurs, its images will be translates of patterns with arbitrarily large combinatorial radius. (Note that by "linearity" of $\mathbf{E}_1^\star$ substitutions, the image of a translate of a pattern is always a translate of its image.) ⬠

2.6 Technical proofs

The following sets of patterns will be used throughout this section:

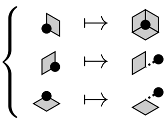
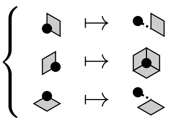
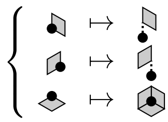
$\mathcal{L}^{\mathsf{AR}} = \{$  $\}$,

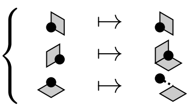
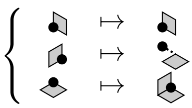
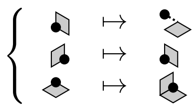
$\mathcal{L}^{\mathsf{Brun}} = \{$  $\}$,

$\mathcal{L}^{\mathsf{JP}} = \{$  $\}$.

Remark 2.6.1. We will often use the arithmetic restrictions of the definition of discrete planes (Definition 1.2.1) in order to simplify the combinatorics of the patterns that occur in a discrete plane. For example, if $\mathbf{v}_1 \leqslant \mathbf{v}_3$ and $\mathbf{v}_2 \leqslant \mathbf{v}_3$, then $\Gamma_{\mathbf{v}}$ cannot contain any translate of the two-face pattern $[\mathbf{0},1]^\star \cup [(0,1,0),1]^\star =$  or $[\mathbf{0},2]^\star \cup [(0,0,1),2]^\star =$ . If moreover $\mathbf{v}_1 \leqslant \mathbf{v}_2$, then the pattern $[\mathbf{0},1]^\star \cup [(0,1,0),1]^\star =$  also never occurs.

We give below a visual representation of the dual substitutions that will be manipulated in this section. The black dot stands for $\mathbf{x}$ on the left hand sides and for $\mathbf{M}_\sigma^{-1}\mathbf{x}$ on the right-hand sides.

Σ_1^{AR} :  Σ_2^{AR} :  Σ_3^{AR} : 

Σ_1^{Brun} :  Σ_2^{Brun} :  Σ_3^{Brun} : 

$$\Sigma_{a,b}^{\mathsf{JP}} : \begin{cases} [\mathbf{0},1]^\star & \mapsto & [(a,0,0),2]^\star \\ [\mathbf{0},2]^\star & \mapsto & [(b,0,0),3]^\star \\ [\mathbf{0},3]^\star & \mapsto & [(0,0,0),1]^\star \;\cup\; \bigcup_{k=0}^{a-1}[(k,0,0),2]^\star \;\cup\; \bigcup_{k=0}^{b-1}[(k,0,0),3]^\star. \end{cases}$$

Preimages by dual substitutions In the proofs of this chapter, we will often have to compute preimages of faces by dual substitutions. By abuse of notation we will write $\Sigma_k^{-1}([\mathbf{x},i]^\star)$ for the union of faces $[\mathbf{y},j]^\star$ such that $\Sigma_k([\mathbf{y},j]^\star)$ contains the face $[\mathbf{x},i]^\star$. This notation is also extended to finite unions of faces.

Lemma 2.6.2 describes the preimages by Arnoux-Rauzy substitutions. It can be proved by an easy enumeration of cases. We will not give such lemmas for the Brun and Jacobi-Perron substitutions, even though they will be implicitly used in the proofs of this section when enumerating preimages.

Lemma 2.6.2. *Let* $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) \in \mathbb{Z}^3$. *We have*

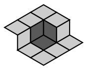
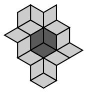
$$(\Sigma_1^{\mathsf{AR}})^{-1}([\mathbf{x},i]^\star) = \begin{cases} \left[\left(\begin{smallmatrix} x+y+z \\ y \\ z \end{smallmatrix}\right),1\right]^\star & \text{if } i=1 \\ \left[\left(\begin{smallmatrix} x+y+z \\ y \\ z \end{smallmatrix}\right),1\right]^\star \cup \left[\left(\begin{smallmatrix} x+y+z-1 \\ y \\ z \end{smallmatrix}\right),i\right]^\star & \text{if } i=2,3 \end{cases},$$

$$(\Sigma_2^{\mathsf{AR}})^{-1}([\mathbf{x},i]^\star) = \begin{cases} \left[\left(\begin{smallmatrix} x \\ x+y+z \\ z \end{smallmatrix}\right),2\right]^\star & \text{if } i=2 \\ \left[\left(\begin{smallmatrix} x \\ x+y+z \\ z \end{smallmatrix}\right),2\right]^\star \cup \left[\left(\begin{smallmatrix} x \\ x+y+z-1 \\ z \end{smallmatrix}\right),i\right]^\star & \text{if } i=1,3 \end{cases},$$

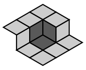
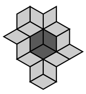
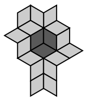
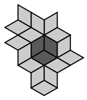
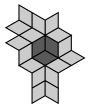
$$(\Sigma_3^{\mathsf{AR}})^{-1}([\mathbf{x},i]^\star) = \begin{cases} \left[\left(\begin{smallmatrix} x \\ y \\ x+y+z \end{smallmatrix}\right),3\right]^\star & \text{if } i=3 \\ \left[\left(\begin{smallmatrix} x \\ y \\ x+y+z \end{smallmatrix}\right),3\right]^\star \cup \left[\left(\begin{smallmatrix} x \\ y \\ x+y+z-1 \end{smallmatrix}\right),i\right]^\star & \text{if } i=2,3 \end{cases}.$$

2.6.1 Minimal annuli and seeds

Proposition 2.6.3 (Brun minimal annuli). *Let P be a pattern contained in a discrete $\Gamma_{\mathbf{v}}$ with $0 < \mathbf{v}_1 < \mathbf{v}_2 < \mathbf{v}_3$. If P contains $\mathcal{U}$ and an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{U}$, then P must contain one of the following two patterns, each of which contains an $\mathcal{L}^{\mathsf{Brun}}$-annuli (shown in light gray) of $\mathcal{U}$ (shown in dark gray):*

$\mathcal{V}_1^{\mathsf{Brun}} =$  $\mathcal{V}_2^{\mathsf{Brun}} =$  .

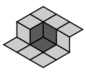
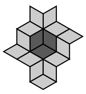
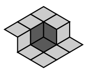
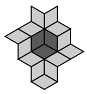
Proof. According to Remark 2.6.1, the algebraic restriction $0 < \mathbf{v}_1 < \mathbf{v}_2 < \mathbf{v}_3$ implies that $\Gamma_{\mathbf{v}}$ must contain one of the following patterns surrounding $\mathcal{U}$:

 ,  ,  ,  or  .

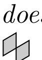
The first two possibilities correspond to the $\mathcal{L}^{\mathsf{Brun}}$-annuli given in the statement of the proposition. The three other patterns can be ruled out since they contain ,  or , which is forbidden by Remark 2.6.1 because $\mathbf{v}_1 < \mathbf{v}_2 < \mathbf{v}_3$. Lastly, we check that these two patterns do contain $\mathcal{L}^{\mathsf{Brun}}$-annuli of $\mathcal{U}$. ⬠

Similarly we can prove the following.

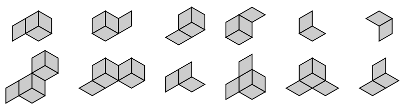
Proposition 2.6.4 (Jacobi-Perron minimal annuli). *Let P be a pattern contained in a discrete plane $\Gamma_{\mathbf{v}}$ with $0 < \mathbf{v}_1 < \mathbf{v}_3$ and $0 < \mathbf{v}_2 < \mathbf{v}_3$. If P contains $\mathcal{U}$ and an $\mathcal{L}^{\mathsf{JP}}$-annulus of $\mathcal{U}$, then P must contain one of the following four patterns, each of which contains and $\mathcal{L}^{\mathsf{JP}}$-annuli (shown in light gray) of $\mathcal{U}$ (shown in dark gray):*

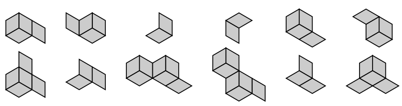
$\mathcal{V}_1^{\mathsf{JP}} =$  $\mathcal{V}_2^{\mathsf{JP}} =$  $\mathcal{V}_3^{\mathsf{JP}} =$  $\mathcal{V}_4^{\mathsf{JP}} =$  .

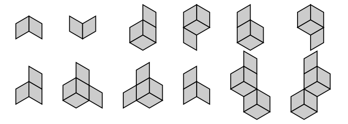
2.6.2 Covering properties

Proposition 2.6.5 (Strong $\mathcal{L}^{\mathsf{AR}}$-covering). *Let P be a strongly $\mathcal{L}^{\mathsf{AR}}$-covered pattern that does not contain any translate of one of the three patterns*  $= [\mathbf{0}, 1]^\star \cup [(0, 1, 1), 1]^\star$,  $= [\mathbf{0}, 2]^\star \cup [(1, 0, 1), 2]^\star$, *or*  $= [\mathbf{0}, 3]^\star \cup [(1, 1, 0), 3]^\star$. *Then $\Sigma_i^{\mathsf{AR}}(P)$ is strongly $\mathcal{L}^{\mathsf{AR}}$-covered for each $i \in \{1, 2, 3\}$.*

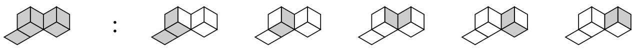
Proof. First, $\Sigma_i^{\mathsf{AR}}(P)$ is $\mathcal{L}^{\mathsf{AR}}$-covered thanks to Proposition 2.2.2, because $\Sigma_i^{\mathsf{AR}}(Q)$ is $\mathcal{L}^{\mathsf{AR}}$-covered for every $Q \in \mathcal{L}^{\mathsf{AR}}$. This makes a total of 36 patterns to check:

$\Sigma_1^{\mathsf{AR}}(\mathcal{L}^{\mathsf{AR}}) = \Bigg\{$  $\Bigg\}$;

$\Sigma_2^{\mathsf{AR}}(\mathcal{L}^{\mathsf{AR}}) = \Bigg\{$  $\Bigg\}$;

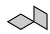
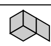
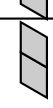
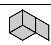
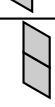
$\Sigma_3^{\mathsf{AR}}(\mathcal{L}^{\mathsf{AR}}) = \Big\{$  $\Big\}$.

This can easily be checked for each of these patterns, as for the following pattern:



Now we prove the *strong* covering part. Let $\Sigma = \Sigma_i^{\mathsf{AR}}$ for some $i \in \{1,2,3\}$. Let $X \subseteq \Sigma(P)$ be a two-face connected pattern. If X is a translate of , , , , or , then the strong covering condition is trivially verified because X is itself in $\mathcal{L}^{\mathsf{AR}}$.

It remains to treat the case of the six remaining possibilities for X up to translation. We have $X \subseteq \Sigma(P)$, so it is sufficient to check that for every pattern $X_0 \subseteq P$ such that $X \subseteq \Sigma(X_0)$, there exists $Y \in \mathcal{L}^{\mathsf{AR}}$ such that $X \subseteq Y \subseteq \Sigma(X_0)$. Moreover, since X is a two-face pattern, we can restrict to the case where X_0 consists of two faces only, which leaves a finite number of cases to check for X_0. Suppose that $X =$. The following table displays all the possibilities for X_0.

i	2	2	3	3
X_0 such that $X \subseteq \Sigma_i^{\mathsf{AR}}(X_0)$				
$\Sigma_i^{\mathsf{AR}}(X_0)$				

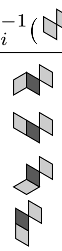
In the first and third columns, there is indeed a pattern $Y \in \mathcal{L}^{\mathsf{AR}}$ such that $X \subseteq Y \subseteq \Sigma(X_0)$, namely $Y =$, which settles the case of these two columns. The case of the fourth column is settled directly because we have assumed that P does not contain the pattern . For the second column, since P is itself strongly $\mathcal{L}^{\mathsf{AR}}$-covered, X_0 is contained in some $Y_0 \in \mathcal{L}^{\mathsf{AR}}$ (here,). Hence, $\Sigma_2^{\mathsf{AR}}(Y_0) \subseteq \Sigma_2^{\mathsf{AR}}(P)$, with $\Sigma_2^{\mathsf{AR}}(Y_0) \in \mathcal{L}^{\mathsf{AR}}$, so the property holds.

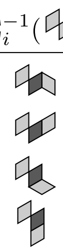
The five remaining cases for X can be dealt with in exactly the same way as above: in each of the five cases, there are four preimages to check, which are symmetrical copies of the ones in the table above. ⬠

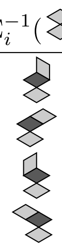
The following lemma will be needed for Arnoux-Rauzy substitutions, to handle the three patterns forbidden in Proposition 2.6.5.

Lemma 2.6.6. *Let Σ be an arbitrary product of Σ_1^{AR}, Σ_2^{AR} and Σ_3^{AR}. The discrete plane $\Sigma(\Gamma_{(1,1,1)})$ contains no translate of any of the three patterns , , .*

Proof. We prove the result by induction on the size of the product Σ. First, thanks to Remark 2.6.1, $\Gamma_{(1,1,1)}$ does not contain any of the three patterns in question. Now, in the table below, we have listed all the preimages of the three patterns (in light gray), together with their only possible "completion" within a discrete plane (in dark gray). This completion is deduced from the arithmetic description of discrete planes provided by Remark 2.6.1.

i	$\Sigma_i^{-1}(\cdot)$
2	
2	
3	
3	

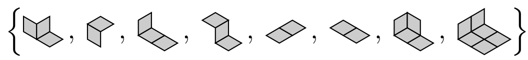
i	$\Sigma_i^{-1}(\cdot)$
1	
1	
3	
3	

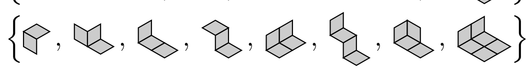
i	$\Sigma_i^{-1}(\cdot)$
1	
1	
2	
2	

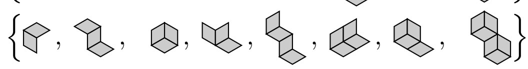
Let us work by contradiction and assume that a translate of one of the three patterns occurs in $\Sigma(\Gamma_{(1,1,1)})$ but not in $\Sigma'(\Gamma_{(1,1,1)})$, where $\Sigma = \Sigma_i^{\mathsf{AR}} \circ \Sigma'$, for some $i \in \{1,2,3\}$. The above table, together with the injectivity of Σ_i^{AR} (see Proposition 1.2.4), implies that a translate of one of the three patterns must occur in $\Sigma'(\Gamma_{(1,1,1)})$, which yields the desired contradiction. ⬠

Proposition 2.6.7 (Strong $\mathcal{L}^{\mathsf{Brun}}$-covering). *Let P be an $\mathcal{L}^{\mathsf{Brun}}$-covered pattern such that the patterns , and do not occur in P. Then $\Sigma_i^{\mathsf{Brun}}(P)$ is strongly $\mathcal{L}^{\mathsf{Brun}}$-covered for each $i \in \{1,2,3\}$.*

Proof. First, $\Sigma_i^{\mathsf{Brun}}(P)$ is $\mathcal{L}^{\mathsf{Brun}}$-covered thanks to Proposition 2.2.2, because $\Sigma_i^{\mathsf{Brun}}(Q)$ is $\mathcal{L}^{\mathsf{Brun}}$-covered for every $Q \in \mathcal{L}^{\mathsf{Brun}}$, as can be verified from the equalities below:

$\Sigma_1^{\mathsf{Brun}}(\mathcal{L}^{\mathsf{Brun}}) =$ 

$\Sigma_2^{\mathsf{Brun}}(\mathcal{L}^{\mathsf{Brun}}) =$ 

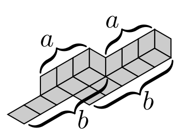
$\Sigma_3^{\mathsf{Brun}}(\mathcal{L}^{\mathsf{Brun}}) =$ .

It remains to prove that $\Sigma_i^{\mathsf{Brun}}(P)$ is *strongly* $\mathcal{L}^{\mathsf{Brun}}$-covered. Let $X \subseteq \Sigma_i^{\mathsf{Brun}}(P)$ be an edge-connected two-face pattern. If X is a translate of one of the first 6 patterns of $\mathcal{L}^{\mathsf{Brun}}$, then the result trivially holds (take $Y = X$ in Definition 2.2.3).

If X is a translate of $f \cup g =$, then there must exist $f_0, g_0 \in P$ with $f_0 \neq g_0$ such that $f \in \Sigma_i^{\mathsf{Brun}}(f_0)$ and $g \in \Sigma_i^{\mathsf{Brun}}(g_0)$. Indeed, we can check that X cannot be contained in the image of single face, by definition of the Σ_i^{Brun} (see their definition in Section 1.5). Enumerating all the possible such preimages f_0, g_0 gives $f_0 \cup g_0 =$ (for Σ_1^{Brun}), $f_0 \cup g_0 =$ (for Σ_2^{Brun}), or $f_0 \cup g_0 =$ (for Σ_3^{Brun}). For $i \in \{1,2,3\}$ we have $\Sigma_i^{\mathsf{Brun}}(f_0 \cup g_0) =$ so $X \subseteq$ $\subseteq \Sigma_i^{\mathsf{Brun}}(P)$, which enables us to take $Y =$ $\in \mathcal{L}^{\mathsf{Brun}}$. The cases $X =$ or can be settled exactly in the same way. We have treated 9 cases over 12 possible cases for X, so the proof is complete because we have ruled out the 3 remaining patterns in the statement of the proposition. ⬠

Proposition 2.6.8 (Strong $\mathcal{L}^{\mathsf{JP}}$-covering). *Let P be an $\mathcal{L}^{\mathsf{JP}}$-covered pattern avoiding the patterns and . Then $\Sigma_{a,b}^{\mathsf{JP}}(P)$ is strongly $\mathcal{L}^{\mathsf{JP}}$-covered for all $0 \leqslant a \leqslant b$, with $b \neq 0$.*

Proof. First we must prove that $\Sigma_{a,b}^{\mathsf{JP}}(P)$ is $\mathcal{L}^{\mathsf{JP}}$-covered. Thanks to Proposition 2.2.2 it is enough to prove that $\Sigma_{a,b}^{\mathsf{JP}}(Q)$ is $\mathcal{L}^{\mathsf{JP}}$-covered for every $Q \in \mathcal{L}^{\mathsf{JP}}$. Suppose that $Q =$. The pattern $\Sigma_{a,b}^{\mathsf{JP}}(Q)$ is of the form



(in this example $a = 3$ and $b = 5$). If $a = 0$, $\Sigma^{\mathsf{JP}}_{a,b}(Q)$ can be $\mathcal{L}^{\mathsf{JP}}$-covered using the patterns , and . If $a \neq 0$, $\Sigma^{\mathsf{JP}}_{a,b}(Q)$ can be $\mathcal{L}^{\mathsf{JP}}$-covered using the patterns , , , and . A similar reasoning can be carried out for each of the other 9 patterns $Q \in \mathcal{L}^{\mathsf{JP}}$, which proves that P is $\mathcal{L}^{\mathsf{JP}}$-covered.

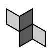
It remains to prove the *strong* $\mathcal{L}^{\mathsf{JP}}$-covering. Let $X \subseteq \Sigma^{\mathsf{JP}}_{a,b}(P)$ be an edge-connected two-face pattern. If X is a translate of one of the first 7 patterns of $\mathcal{L}^{\mathsf{JP}}$, then the result is obvious, because X is itself in $\mathcal{L}^{\mathsf{JP}}$. If X is a translate of , then we must have $X = \subseteq \subseteq \Sigma^{\mathsf{JP}}_{a,b}(P)$, because in an image by $\Sigma^{\mathsf{JP}}_{a,b}$, a face of type 1 must come from a face of type 3, so there must also be a face of type 3 at the same position (because $b \neq 0$). If X is a translate of , a similar reasoning yields $X = \subseteq \subseteq \Sigma^{\mathsf{JP}}_{a,b}(P)$. If X is a translate of , then X is in the image of a face of type 3, or has one face in the image of a face of type 2, and the other face in the image of a face of type 3. In both cases, we must have $X = \subseteq \subseteq \Sigma^{\mathsf{JP}}_{a,b}(P)$ (since $b \geqslant a$).

We have treated 10 cases over 12 possible cases for X, and the two remaining patterns have been ruled out in the statement of the proposition. ⬠

2.6.3 Property A

Proposition 2.6.9 (Property A for Arnoux-Rauzy). *Property A holds for Arnoux-Rauzy substitutions with $\mathcal{L}^{\mathsf{AR}}$, when restricted to planes $\Gamma_{\mathbf{v}}$ avoiding , , or .*

Proof. We enumerate all the faces f, g, f_0, g_0 such that $f \in \Sigma^{\mathsf{AR}}_i(f_0)$, $g \in \Sigma^{\mathsf{AR}}_i(g_0)$, $f \cup g$ is connected and $f_0 \cup g_0$ is disconnected, for some $i \in \{1, 2, 3\}$. All the possibilities are given in Figure 2.3, where the faces plotted in dark gray correspond to the only possible of $f_0 \cup g_0$ within an admissible discrete plane, with respect to Remark 2.6.1. The first such case is the following:

$$f \cup g \cup X \ =$$ ,

where $X = [\mathbf{0}, 2]^\star \cup [(0,0,1), 2]^\star$ is shown in dark gray. We must have $X \subseteq A$ because otherwise A would not be an annulus of P. However, this situation is impossible: there does not exist any pattern $Y \in \mathcal{L}^{\mathsf{AR}}$ such that $X \subseteq Y \subseteq A$ because Y would then overlap with f_0 or g_0, which both are not in A by assumption. This implies that A is not strongly $\mathcal{L}^{\mathsf{AR}}$-covered, a contradiction.

The same reasoning applies to all the other possible cases for $f_0 \cup g_0$, and by assumption we have not considered the (problematic) patterns , , . ⬠

Proposition 2.6.10 (Property A for Brun). *Property A holds for Brun substitutions with $\mathcal{L}^{\mathsf{Brun}}$, when restricted to planes $\Gamma_{\mathbf{v}}$ with $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$.*

Proof. There are finitely many two-face connected patterns $f \cup g$, so we can enumerate all the faces f, g, f_0, g_0 that satisfy $f \in \Sigma(f_0)$, $g \in \Sigma(g_0)$, $f \cup g$ is connected and $f_0 \cup g_0$ is disconnected (see Definition 2.3.4), for $\Sigma = \Sigma^{\mathsf{Brun}}_1$, Σ^{Brun}_2 and Σ^{Brun}_3. It turns out that there are 9 such possibilities, where the corresponding values for $f_0 \cup g_0$ are shown in the table below.

Σ^{Brun}_1	Σ^{Brun}_2	Σ^{Brun}_3
$[\mathbf{0}, 2]^\star \cup [(0,1,0), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(1,0,-1), 3]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,1,-1), 3]^\star$
$[\mathbf{0}, 2]^\star \cup [(1,-1,0), 2]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,1,1), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,0,1), 2]^\star$
$[\mathbf{0}, 2]^\star \cup [(0,1,1), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,0,1), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(1,0,1), 2]^\star$

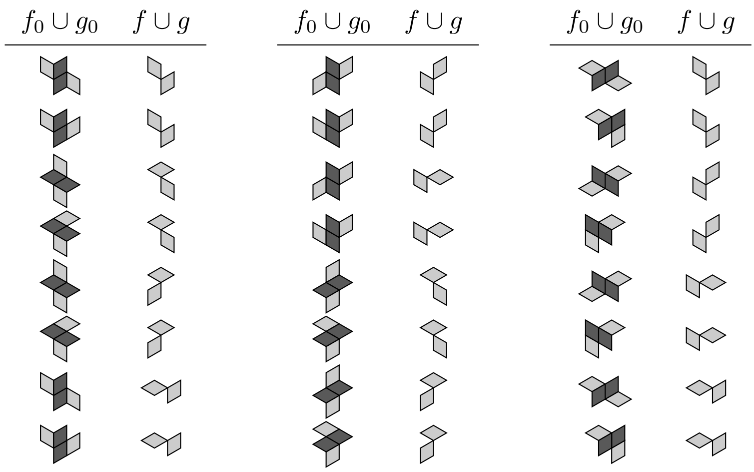
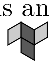


Figure 2.3: Disconnected preimages of connected two-face patterns by Σ_1^{AR} (left), Σ_2^{AR} (middle) and Σ_3^{AR} (right). The only possible completion of $f_0 \cup g_0$ within an admissible discrete plane are shown in dark gray.

Let us treat the case $f_0 \cup g_0 = [\mathbf{0}, 2]^\star \cup [(1,-1,0), 2]^\star$. Suppose that there exists a pattern P and an $\mathcal{L}^{\mathsf{Brun}}$-annulus A of P that is included in a discrete plane such that $f_0 \in P$ and $g_0 \notin A$. Because A is an annulus of P, every extension of $f_0 \cup g_0$ within a discrete plane must be of the form  or  , where $f_0 \cup g_0$ is shown in light gray and the dark gray faces are in A.

The first case cannot happen because it contains an occurrence of , which is forbidden since we are restricted to discrete planes $\Gamma_{\mathbf{v}}$ with $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$ (see Remark 2.6.1). The second case also cannot happen, because A is strongly $\mathcal{L}^{\mathsf{Brun}}$-covered. Indeed,  $\subseteq A$, so there must exist a translate of a pattern of $\mathcal{L}^{\mathsf{Brun}}$ that is included in A and that contains . The only such pattern in $\mathcal{L}^{\mathsf{Brun}}$ is  (note that  $\notin \mathcal{L}^{\mathsf{Brun}}$). This is impossible because then  and $f_0 \cup g_0$ overlap, which is a contradiction because $f_0, g_0 \notin A$ and  $\subseteq A$. The same reasoning applies to the eight other cases. ⬠

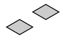
In order to prove Property A for Jacobi-Perron substitutions (Proposition 2.6.12 below), we first need to prove Lemma 2.6.11 below, which describes all the possible disconnected preimages by $\Sigma_{a,b}^{\mathsf{JP}}$ of a two-face connected pattern. A striking fact is that there are only three possible such preimages, despite the fact that a and b can take infinitely many values.

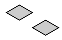
Lemma 2.6.11. *Let f and g be two faces such that*

(1) $f \cup g$ is included in a discrete plane $\Gamma_{\mathbf{v}}$ with $0 < \mathbf{v}_1 \leqslant \mathbf{v}_3$ and $0 < \mathbf{v}_2 \leqslant \mathbf{v}_3$;
(2) $f \cup g$ is disconnected;
(3) $\Sigma_{a,b}^{\mathsf{JP}}(f \cup g)$ is connected for some $0 \leqslant a \leqslant b, b \neq 0$.

Then $f \cup g$ is a translate of one of the three patterns

P_1 =  = $[\mathbf{0},3]^\star \cup [(1,0,-1),3]^\star$,

P_2 =  = $[\mathbf{0},3]^\star \cup [(1,-1,-1),3]^\star$,

P_3 =  = $[\mathbf{0},3]^\star \cup [(-1,1,-1),3]^\star$.

Proof. We first need the following easy preliminary fact (Equation 2.1 below), which can be checked by inspection of finitely many cases. A two-face pattern $P = [\mathbf{x}, i]^\star \cup [\mathbf{y}, j]^\star$ is edge-connected if and only $\mathbf{y} - \mathbf{x} \in V_{ij}$, where

$$\begin{aligned}
V_{11} &= \{\pm(0,1,0), \pm(0,0,1), \pm(0,1,-1), \pm(0,1,1)\},\\
V_{22} &= \{\pm(1,0,0), \pm(0,0,1), \pm(1,0,-1), \pm(1,0,1)\},\\
V_{33} &= \{\pm(1,0,0), \pm(0,1,0), \pm(1,-1,0), \pm(1,1,0)\},\\
V_{12} &= \{(0,0,0), (-1,1,0), \pm(0,0,1), (0,1,-1), (-1,1,-1), (-1,1,1), (-1,0,1)\},\\
V_{13} &= \{(0,0,0), (-1,0,1), \pm(0,1,0), (-1,1,0), (-1,1,1), (-1,-1,1), (0,-1,1)\},\\
V_{23} &= \{(0,0,0), (0,-1,1), \pm(1,0,0), (-1,0,1), (-1,-1,1), (1,-1,1), (1,-1,0)\},\\
V_{21} &= -V_{12}, \quad V_{31} = -V_{13}, \quad V_{32} = -V_{22}.
\end{aligned} \tag{2.1}$$

In contrast with the Brun substitutions, we cannot prove Lemma 2.6.11 by enumerating all the possibilities because a and b can take infinitely many values. However, the problem can be reduced to solving a simple family of linear equations, which then allows a systematic study of the disconnected preimages of $f \cup g$.

We will show that if Condition (3) holds, then Condition (1) or (2) fails, unless $f \cup g$ is a translate of P_1, P_2 or P_3. We will only treat the cases $(i,j) = (1,1)$ and $(i,j) = (3,3)$, where i is the type of f and j is the type of g. These two cases correspond to the simplest and the most complicated cases, respectively. The other cases can be treated similarly. Let $\mathbf{M}_{a,b} = \mathbf{M}_{\sigma^{\mathsf{JP}}_{a,b}}$.

Case $(i,j) = (1,1)$ Let $f = [\mathbf{x}, 1]^\star$ and $g = [\mathbf{y}, 1]^\star$ be two faces contained in a same discrete plane, and assume that Condition (3) holds. There exist $0 \leqslant a \leqslant b$ with $b \neq 0$ such that the pattern

$$\Sigma^{\mathsf{JP}}_{a,b}(f \cup g) \;=\; [\mathbf{M}_{a,b}^{-1}\mathbf{x} + (a,0,0), 2]^\star \cup [\mathbf{M}_{a,b}^{-1}\mathbf{y} + (a,0,0), 2]^\star$$

is connected. Consequently, we have $\mathbf{M}_{a,b}^{-1}(\mathbf{y} - \mathbf{x}) \in V_{22}$ thanks to Equation (2.1). It follows that

$$\mathbf{y} - \mathbf{x} \;\in\; \{\mathbf{M}_{a,b}\mathbf{v} : \mathbf{v} \in V_{22}\} \;=\; \left\{\pm\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}0\\1\\b-1\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}0\\1\\b\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}0\\1\\b+1\end{smallmatrix}\right)\right\}.$$

Since a and b can take all possible values with $0 \leqslant a \leqslant b$ and $b \neq 0$, this gives

$$\mathbf{y} - \mathbf{x} \;\in\; \left\{\pm\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}0\\1\\t\end{smallmatrix}\right) : t \geqslant 0\right\}.$$

Now, if $\mathbf{y} - \mathbf{x} = \pm(0,0,1)$ or $\pm(0,1,0)$, then $\mathbf{y} - \mathbf{x} \in V_{11}$, so $f \cup g$ is connected (by Equation (2.1)), which contradicts Condition (2). If $\mathbf{y} - \mathbf{x} = (0,1,t)$ with $t \geqslant 1$ then Condition (1) is violated by the fact that there exists no discrete plane $\Gamma_{\mathbf{v}}$ with $0 < \mathbf{v}_1 < \mathbf{v}_3$ and $0 < \mathbf{v}_2 < \mathbf{v}_3$ which contains both f and g. Indeed, if it were the case, Definition 1.2.1 would imply that $-\mathbf{v}_1 < \mathbf{v}_2 + t\mathbf{v}_3 < \mathbf{v}_1$, which is impossible for $0 < \mathbf{v}_1 < \mathbf{v}_3$ and $0 < \mathbf{v}_2 < \mathbf{v}_3$. The same reasoning applies to the remaining case $\mathbf{y} - \mathbf{x} = (0, -1, -t)$, with $t \geqslant 1$.

Case $(i, j) = (3, 3)$ Let $f = [\mathbf{x}, 3]^\star$ and $g = [\mathbf{y}, 3]^\star$ be two faces contained in a same discrete plane, and assume that Condition (3) holds. According to Equation (2.1), the fact that $\Sigma^{\mathsf{JP}}_{a,b}(f \cup g)$ is connected implies that one of the following six relations:

$$\begin{array}{lclcll}
\mathbf{M}_{a,b}^{-1}(\mathbf{y}-\mathbf{x}) & & & \in & V_{11}, & \\
\mathbf{M}_{a,b}^{-1}(\mathbf{y}-\mathbf{x}) & + & (k,0,0) & \in & V_{12} & (0 \leqslant k \leqslant a-1), \\
\mathbf{M}_{a,b}^{-1}(\mathbf{y}-\mathbf{x}) & + & (\ell,0,0) & \in & V_{13} & (0 \leqslant \ell \leqslant b-1), \\
\mathbf{M}_{a,b}^{-1}(\mathbf{y}-\mathbf{x}) & + & (k-k',0,0) & \in & V_{22} & (0 \leqslant k,k' \leqslant a-1), \\
\mathbf{M}_{a,b}^{-1}(\mathbf{y}-\mathbf{x}) & + & (\ell-k,0,0) & \in & V_{23} & (0 \leqslant k \leqslant a-1,\ 0 \leqslant \ell \leqslant b-1), \\
\mathbf{M}_{a,b}^{-1}(\mathbf{y}-\mathbf{x}) & + & (\ell-\ell',0,0) & \in & V_{33} & (0 \leqslant \ell,\ell' \leqslant b-1),
\end{array}$$

for some $0 \leqslant a \leqslant b$ with $b \neq 0$. This is equivalent to

$$\mathbf{y} - \mathbf{x} \in \left\{ \left(\begin{smallmatrix}0\\0\\s\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}1\\0\\s\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\1\\s\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}0\\1\\t\end{smallmatrix}\right), \pm\left(\begin{smallmatrix}1\\1\\t\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\-1\\-t\end{smallmatrix}\right) : s \in \mathbb{Z}, t \geqslant 0 \right\}.$$

This either contradicts Condition (1) or (2), or else implies that $f \cup g$ is equal to P_1, P_2 or P_3. Indeed, let $\Gamma_{\mathbf{v}}$ be a discrete plane that contains both f and g. Thanks to Definition 1.2.1, we can restrict the possible values of $\mathbf{x} - \mathbf{y}$ even further. The remaining valid solutions are shown in the following table, where $s \in \mathbb{Z}$, $t \geqslant 0$, $0 < \mathbf{v}_1 < \mathbf{v}_3$ and $0 < \mathbf{v}_2 < \mathbf{v}_3$. We then observe that either $\mathbf{y} - \mathbf{x} \in V_{33}$ (so Condition (2) is violated because $f \cup g$ is connected thanks to Equation (2.1)), or that $f \cup g$ is equal to P_1, P_2 or P_3.

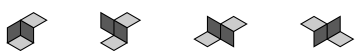
Inequalities	Solutions
$-\mathbf{v}_3 < s\mathbf{v}_3 < \mathbf{v}_3$	$s = 0$: $\mathbf{y} - \mathbf{x} \in V_{33}$
$-\mathbf{v}_3 < \mathbf{v}_1 + s\mathbf{v}_3 < \mathbf{v}_3$	$s = 0$: $\mathbf{y} - \mathbf{x} \in V_{33}$; $s = 1$: P_1
$-\mathbf{v}_3 < -\mathbf{v}_1 + \mathbf{v}_2 + s\mathbf{v}_3 < \mathbf{v}_3$	$s = 0$: $\mathbf{y} - \mathbf{x} \in V_{33}$; $s = -1$: P_3; $s = 1$: P_2
$-\mathbf{v}_3 < \mathbf{v}_2 + t\mathbf{v}_3 < \mathbf{v}_3$	$t = 0$: $\mathbf{y} - \mathbf{x} \in V_{33}$
$-\mathbf{v}_3 < \mathbf{v}_1 + \mathbf{v}_2 + t\mathbf{v}_3 < \mathbf{v}_3$	$t = 0$: $\mathbf{y} - \mathbf{x} \in V_{33}$
$-\mathbf{v}_3 < \mathbf{v}_1 - \mathbf{v}_2 - t\mathbf{v}_3 < \mathbf{v}_3$	$t = 0$: $\mathbf{y} - \mathbf{x} \in V_{33}$; $t = 1$: P_3

⬠

Proposition 2.6.12 (Property A for Jacobi-Perron). *Property A holds for Jacobi-Perron substitutions with $\mathcal{L}^{\mathsf{JP}}$, when restricted to discrete planes $\Gamma_{\mathbf{v}}$ with $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_3$ and $0 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$.*

Proof. Thanks to Lemma 2.6.11, if f, g, f_0, g_0 satisfy the conditions required in Definition 2.3.4 for some Jacobi-Perron substitution, then $f_0 \cup g_0$ must be equal to a translate of one of the three patterns P_1, P_2, P_3 of the statement of Lemma 2.6.11.

Suppose that there exists a pattern P and an $\mathcal{L}^{\mathsf{JP}}$-annulus A of P that is included in a discrete plane such that $f_0 \in P$ and $g_0 \notin A \cup P$. Because A is an annulus of P, every extension of $f_0 \cup g_0$ within a discrete plane must be of the form

,

where $f_0 \cup g_0$ is shown in light gray and the dark gray faces are in A. Now, similarly as in the proof of Proposition 2.6.10, a contradiction must occur in each case, thanks to the strong $\mathcal{L}^{\mathsf{JP}}$-covering of A and the precise choice of patterns in $\mathcal{L}^{\mathsf{JP}}$. ⬠

2.6.4 Generation graph for the Arnoux-Rauzy substitutions

We now give a direct proof of the fact that iterating an admissible Arnoux-Rauzy sequence of substitutions starting from $\mathcal{U}$ always yields an $\mathcal{L}^{\mathsf{AR}}$-annulus of $\mathcal{U}$. This will be done thanks to the graph used in the proof of Lemma 2.6.13 which tracks all the possible forward images of $\mathcal{U}$ by Arnoux-Rauzy substitutions.

Let us note that the existence of a finite graph such as the one in Figure 2.4 which provides the description of the possible annuli surrounding the unit cube is the key ingredient of the proof. There is no reason *a priori* for such a finite graph to exist for a given set of substitutions.

Lemma 2.6.13. *Let Σ and Σ' be two products of Σ_1^{AR}, Σ_2^{AR} and Σ_3^{AR} in which each substitution appears at least once. Then, $\Sigma\Sigma'(\mathcal{U}) \setminus \mathcal{U}$ contains an $\mathcal{L}^{\mathsf{AR}}$-annulus of $\mathcal{U}$.*

Proof. Let $w, w' \in \{1,2,3\}^n$ be such that $w = i_1 \cdots i_n$, $w' = i'_1 \cdots i'_m$, $\Sigma = \Sigma_{i_1}^{\mathsf{AR}} \cdots \Sigma_{i_n}^{\mathsf{AR}}$, and $\Sigma = \Sigma_{i'_1}^{\mathsf{AR}} \cdots \Sigma_{i'_m}^{\mathsf{AR}}$. This proof is a case study, which we formalize as the study of the directed graph in Figure 2.4. The vertices of this finite graph are patterns. For every edge $P \xrightarrow{i} Q$ we have $Q \subseteq \Sigma_i^{\mathsf{AR}}(P)$, and every vertex has three outgoing edges with distinct labels 1, 2 and 3 (for the sake of clarity, loops are not drawn in Figure 2.4). The six extremal patterns are valid $\mathcal{L}^{\mathsf{AR}}$-annuli.

It can be checked that one of the six extremal vertices is always reached when following the path of labelled edges given by ww' starting at the top vertex $\mathcal{U}$. Indeed, we can assume without loss of generality (by symmetry) that $w = 1^m 2^n 3x$, where $m, n \geqslant 1$ and $x \in \{1,2,3\}^\star$. If we follow the path described by w starting from $\mathcal{U}$ in the graph, then just after having read the first 3 in w, we can only be at pattern P_{123} or P_{213}, where $P_{j_1 \cdots j_k}$ denotes the pattern reached by following $j_1 \cdots j_k$ in the graph. Hence after having read the first w we are either in P_{123} or P_{213}, or we have reach an $\mathcal{L}^{\mathsf{AR}}$-annulus. So, after reading ww', an $\mathcal{L}^{\mathsf{AR}}$-annulus is necessarily reached because w' contains at least a 1. ⬠

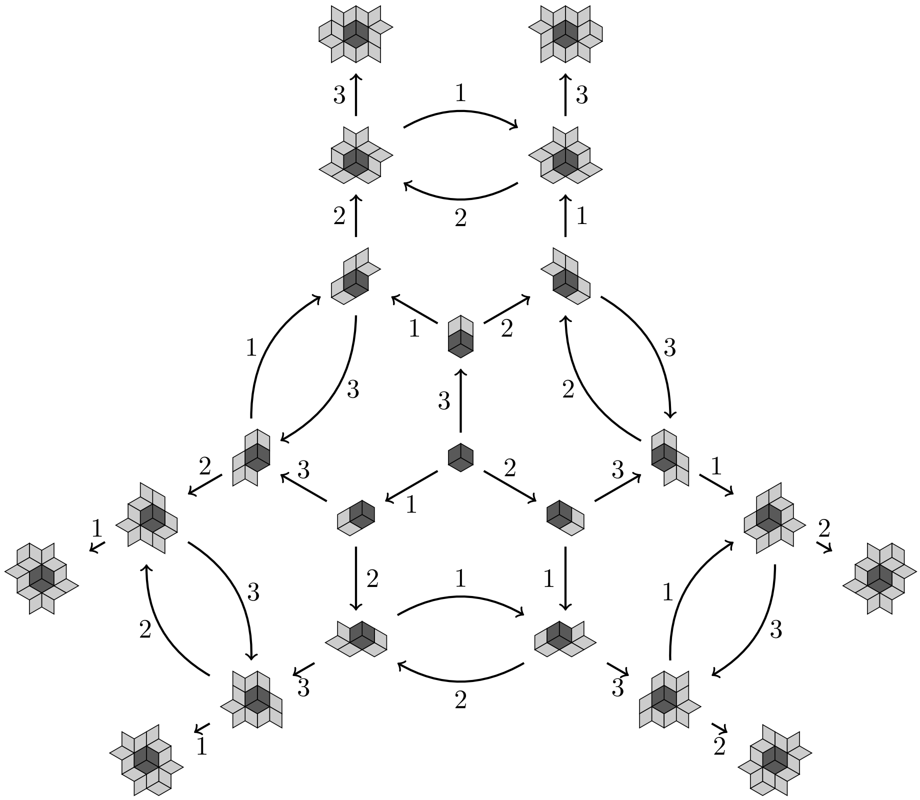


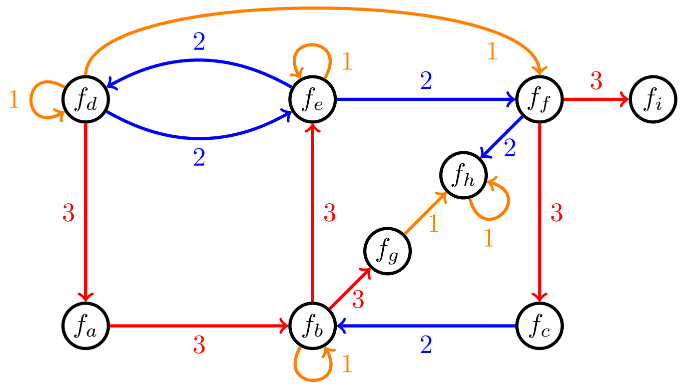
Figure 2.4: The graph used in the proof of Lemma 2.6.13 for Arnoux-Rauzy substitutions. Each vertex has three outgoing edges with distinct labels 1, 2 or 3. (The 9 loops and the 18 outgoing edges of the extremal vertices are not drawn.) Note that this not a *generation graph* in the sense of Definition 2.4.1, it was not constructed like a generation graph, and the vertices are not faces but patterns.

2.6.5 Generation graphs for the Brun substitutions

Graph $\mathcal{G}^{\mathsf{Brun}}$ for the seed $\mathcal{U}$ We now construct generation graphs for the substitutions $\Sigma_1^{\mathsf{Brun}}, \Sigma_2^{\mathsf{Brun}}, \Sigma_3^{\mathsf{Brun}}$. To construct the graphs below we use the filter $\mathcal{F}^{\mathsf{Brun}}$ equal to the set of all the faces f that belong to a discrete plane $\Gamma_{\mathbf{v}}$ with $0 < \mathbf{v}_1 < \mathbf{v}_2 < \mathbf{v}_3$.

Let $\mathcal{G}^{\mathsf{Brun}}$ be the generation graph (see Definition 2.4.1) associated with $\Sigma_1^{\mathsf{Brun}}, \Sigma_2^{\mathsf{Brun}}, \Sigma_3^{\mathsf{Brun}}$, the filter $\mathcal{F}^{\mathsf{Brun}}$ and the initial set of faces $\mathcal{X} = \mathcal{V}_1^{\mathsf{Brun}} \cup \mathcal{V}_2^{\mathsf{Brun}} \setminus \mathcal{U}$ (the union of the minimal annuli given in Proposition 2.6.3). Its computation stops after two iterations of the algorithm, that is, $G = G_2$. It has 19 vertices and 47 edges.

Since we are interested only in Brun expansions containing infinitely many 3's, we remove from $\mathcal{G}^{\mathsf{Brun}}$ all the vertices which are not contained in an infinite path containing infinitely many 3's. Doing so yields the following graph, whose properties we will exploit in the proof of Theorem 2.5.2.



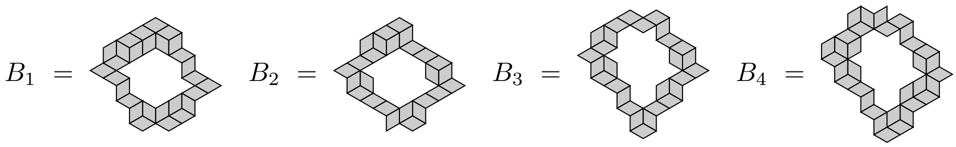
The faces corresponding to the vertices of the graph are

$$
\begin{aligned}
f_a &= [(1,1,-1),1]^\star & f_d &= [(-1,1,0),2]^\star & f_g &= [(-1,0,1),2]^\star \\
f_b &= [(1,-1,1),3]^\star & f_e &= [(-1,0,1),3]^\star & f_h &= [(-1,-1,1),3]^\star \\
f_c &= [(1,1,-1),2]^\star & f_f &= [(-1,1,0),3]^\star & f_i &= [(1,1,-1),3]^\star.
\end{aligned}
$$

Example 2.6.14. The vertex f_e is at the end of the following infinite backward path consisting of the following cycle: $\cdots f_e \xrightarrow{2} f_d \xrightarrow{3} f_a \xrightarrow{3} f_b \xrightarrow{3} f_e$. So Proposition 2.4.4 (2) implies that $f_e \notin (\Sigma_3^{\mathsf{Brun}}\Sigma_3^{\mathsf{Brun}}\Sigma_3^{\mathsf{Brun}}\Sigma_2^{\mathsf{Brun}})^n(\mathcal{U})$ for every $n \geqslant 0$.

On the other hand we also know that for every sequence $(i_n)_{n\in\mathbb{N}}$ such that iterating the $\Sigma_{i_n}^{\mathsf{Brun}}$ from $\mathcal{U}$ fails to generate an $\mathcal{L}^{\mathsf{Brun}}$-annulus $\mathcal{U}$, then the faulty sequence $(i_n)_{n\in\mathbb{N}}$ must appear in $\mathcal{G}^{\mathsf{Brun}}$ as the label of an infinite backward path avoiding $\mathcal{U}$, thanks to Proposition 2.4.4 (1). This allows us to detect such faulty sequences.

Graph $\mathcal{H}^{\mathsf{Brun}}$ for the seeds $\mathcal{V}_i^{\mathsf{Brun}}$ We now construct another generation graph for the Brun substitutions, with the seeds $\mathcal{V}_1^{\mathsf{Brun}}$ and $\mathcal{V}_2^{\mathsf{Brun}}$ instead of $\mathcal{U}$. This time, *every* Brun-admissible sequence allows the generation of some $\mathcal{L}^{\mathsf{Brun}}$-annuli around $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$. Before we compute $\mathcal{H}^{\mathsf{Brun}}$ we must enumerate all the possible $\mathcal{L}^{\mathsf{Brun}}$-annuli of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$. This can be done similarly as in the proof of Proposition 2.6.3, and yields the following four possible annuli.



Let $\mathcal{H}^{\mathsf{Brun}}$ be the generation graph (according to Definition 2.4.1) constructed with substitutions $\Sigma_1^{\mathsf{Brun}}, \Sigma_2^{\mathsf{Brun}}, \Sigma_3^{\mathsf{Brun}}$, filter $\mathcal{F}^{\mathsf{Brun}}$ and initial set $\mathcal{X} = B_1 \cup B_2 \cup B_3 \cup B_4$ (a total of 60 faces). Its computation stops after six iterations of the algorithm ($G = G_6$). It has 101 vertices and 240 edges. We use the structure of $\mathcal{H}^{\mathsf{Brun}}$ to prove the following lemma.

Lemma 2.6.15. *Let $\mathcal{V} \in \{\mathcal{V}_1^{\mathsf{Brun}}, \mathcal{V}_2^{\mathsf{Brun}}\}$. There exists N such that for every $(i_1, \ldots, i_n) \in \{1,2,3\}^n$ containing more than N occurrences of 3, the pattern $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{V})$ contains an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$.*

Proof. Suppose that such an N does not exist. Then, by a compactness argument, we can prove that there exists an infinite sequence $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$ containing infinitely many 3's such that for every $n \geqslant 0$, the pattern $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{V})$ does not contain an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$.

Because B_1, B_2, B_3, B_4 are all the possible $\mathcal{L}^{\mathsf{Brun}}$-annuli of $\mathcal{V}_1^{\mathsf{Brun}}$ and $\mathcal{V}_2^{\mathsf{Brun}}$, this means that there exists a face $f_0 \in \Gamma_{\mathbf{v}} \cap (B_1 \cup B_2 \cup B_3 \cup B_4)$ such that $f_0 \notin \Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{V})$ for all $n \geqslant 0$, where $\mathbf{v}$ is the unique vector determined by the Brun-admissible sequence (i_n).

Since the generation graph $\mathcal{H}^{\mathsf{Brun}}$ constructed above is finite, we can check that there exists $M \geqslant 1$ such that if $f_n \xrightarrow{j_n} \cdots f_1 \xrightarrow{j_1} f_0$ is a directed path in $\mathcal{H}^{\mathsf{Brun}}$ with $j_k = 3$ more than M times, then we have $f_n \in \mathcal{V}$. One possible way to check this property is to verify that every cycle of $\mathcal{H}^{\mathsf{Brun}}$ that contains an edges labelled by 3 consists only of faces in $\mathcal{V}$.

However, Statement (1) of Proposition 2.4.4 implies that for every $n \geqslant 0$ there exists a path $f_n \xrightarrow{i_n} \cdots f_1 \xrightarrow{i_1} f_0$ with $f_n \notin \mathcal{V}$ in $\mathcal{H}^{\mathsf{Brun}}$. This is a contradiction because the sequence (i_n) contains infinitely manys 3's, so the proposition is proved. ⬠

2.6.6 Generation graphs for the Jacobi-Perron substitutions

In this section we take the filter $\mathcal{F}^{\mathsf{JP}}$ to be the set of faces f that belong to a discrete plane $\Gamma_{\mathbf{v}}$ such that and $0 < \mathbf{v}_1 < \mathbf{v}_3$ and $0 < \mathbf{v}_2 < \mathbf{v}_3$.

The graph $\mathcal{G}^{\mathsf{JP}}$ Let $\mathcal{G}^{\mathsf{JP}}$ be the generation graph (see Definition 2.4.1) associated with the additive Jacobi-Perron substitutions $\Theta_1, \Theta_2, \Theta_3, \Theta_4$, and the Jacobi-Perron minimal annuli $\mathcal{X} = \mathcal{V}_1^{\mathsf{JP}} \cup \mathcal{V}_2^{\mathsf{JP}} \cup \mathcal{V}_3^{\mathsf{JP}} \cup \mathcal{V}_4^{\mathsf{JP}}$ (defined in Proposition 2.6.4). Unlike the graph $\mathcal{G}^{\mathsf{Brun}}$, the graph $\mathcal{G}^{\mathsf{JP}}$ is infinite (we never have $\mathcal{G}_n^{\mathsf{JP}} = \mathcal{G}_{n+1}^{\mathsf{JP}}$). However, the structure of $\mathcal{G}^{\mathsf{JP}}$ is simple enough, so we can describe it in detail. In order to do so we need the following lemma, which can be proved easily by studying the preimages of the additive Jacobi-Perron substitutions $\Theta_1, \Theta_2, \Theta_3$ and Θ_4.

Lemma 2.6.16. *For $n \geqslant 0$, let*

$$\begin{aligned} e_n &= [(-n,1,0),3]^\star & g_n &= [(-n,-1,1),3]^\star \\ f_n &= [(-n,1,0),1]^\star & h_n &= [(-n,-1,1),1]^\star. \end{aligned}$$

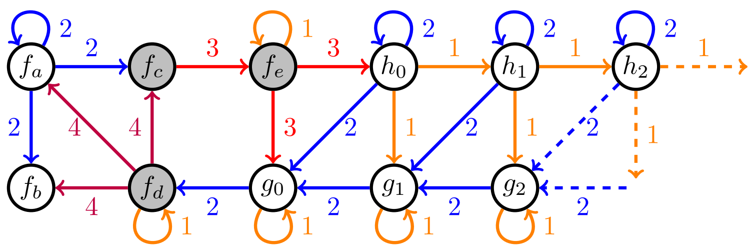
For every integer $n \geqslant 1$, the only preimages of e_n, f_n, g_n, h_n by one of the additive Jacobi-Perron substitutions $\Theta_1, \Theta_2, \Theta_3$ or Θ_4 are given by

$$\begin{array}{llll} e_n \in \Theta_1(e_{n+1}) & f_n \in \Theta_1(f_n) & g_n \in \Theta_1(g_n) & h_n \in \Theta_1(h_{n-1}) \\ e_n \in \Theta_1(f_n) & f_n \in \Theta_2(h_{n-1}) & g_n \in \Theta_1(h_{n-1}) & h_n \in \Theta_2(h_n). \\ e_n \in \Theta_2(e_n) & & g_n \in \Theta_2(h_n) & \\ e_n \in \Theta_2(f_{n-1}) & & g_n \in \Theta_2(h_{n+1}) & \end{array}$$

We can now describe the infinite graph $\mathcal{G}^{\mathsf{JP}}$ defined above. Let $\mathcal{G}_3^{\mathsf{JP}}$ be the graph obtained at the third step in the computation of the generation graph in Definition 2.4.1. This graph has 33 vertices and 93 edges. The only faces of $\mathcal{G}_3^{\mathsf{JP}}$ that admit preimages allowed by the filter set $\mathcal{F}^{\mathsf{JP}}$ which are not vertices of the graph $\mathcal{G}_3^{\mathsf{JP}}$ are e_4, f_3, g_4 and h_3.

From Lemma 2.6.16 we deduce that $\mathcal{G}^{\mathsf{JP}}$ has two infinite branches, the first consisting of vertices e_n, f_n and the second consisting of vertices g_n, h_n, and the only edge labels appearing in these infinite branches are 1 and 2.

Since we are interested only in the additive Jacobi-Perron-admissible sequences $(i_n)_{n\in\mathbb{N}}$ (that is, containing infinitely many edge labelled by 3 or 4), we remove from $\mathcal{G}^{\mathsf{JP}}$ all the vertices which are not at the beginning of a backward infinite path labelled by such a sequence $(i_n)_{n\in\mathbb{N}}$. This yields the following subgraph of $\mathcal{G}^{\mathsf{JP}}$ (which now contains only one infinite branch):



where the faces are given by

$$\begin{array}{ll} f_a = [(1,1,-1),1]^\star & f_e = [(-1,1,1),3]^\star \\ f_b = [(1,1,-1),3]^\star & g_n = [(-n,-1,1),3]^\star \text{ for } n \geqslant 0 \\ f_c = [(1,1,-1),2]^\star & h_n = [(-n,-1,1),1]^\star \text{ for } n \geqslant 0 \\ f_d = [(1,-1,1),3]^\star & \end{array}$$

and the edges are the ones shown above, plus, for every $n \geqslant 0$,

$$g_n \xrightarrow{1} g_n, \quad h_n \xrightarrow{2} h_n, \quad h_n \xrightarrow{2} g_n, \quad h_n \xrightarrow{1} g_{n+1}, \quad h_n \xrightarrow{1} h_{n+1}, \quad g_{n+1} \xrightarrow{2} g_n.$$

Moreover we have deleted the edge $f_e \xrightarrow{4} f_d$ in the above graph, because every infinite path containing it is incompatible with the condition $a_n = b_n \Rightarrow a_{n+1} \neq 0$ on Jacobi-Perron expansions $(a_n, b_n)_{n\geqslant 1}$ (see Section 1.5). Indeed, if the edge $f_e \xrightarrow{4} f_d$ is allowed, then the forbidden product $\Theta_4\Theta_1^k\Theta 3 = \Sigma_{1,1}^{\mathsf{JP}}\Sigma_{0,k+1}^{\mathsf{JP}}$ is allowed for some $k \geqslant 0$.

The graph $\mathcal{H}^{\mathsf{JP}}$ We now construct a generation graph $\mathcal{H}^{\mathsf{JP}}$ for the seeds $\mathcal{V}_1^{\mathsf{JP}}$, $\mathcal{V}_2^{\mathsf{JP}}$, $\mathcal{V}_3^{\mathsf{JP}}$ and $\mathcal{V}_4^{\mathsf{JP}}$ instead of $\mathcal{U}$. Before computing $\mathcal{H}^{\mathsf{JP}}$ we must enumerate all the possible $\mathcal{L}^{\mathsf{JP}}$-annuli of the $\mathcal{V}_i^{\mathsf{JP}}$. This can be done similarly as in the proof of Proposition 2.6.3, and yields eight possible annuli $B_1, \dots, B_8$, which are similar to the annuli $B_1, \dots, B_4$ computed for the graph $\mathcal{H}^{\mathsf{Brun}}$.

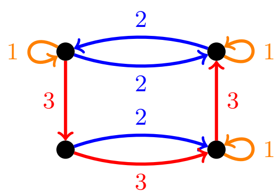
Let $\mathcal{H}^{\mathsf{JP}}$ be the generation graph (according to Definition 2.4.1) associated with the additive Jacobi-Perron substitutions $\Theta_1, \Theta_2, \Theta_3, \Theta_4$, the initial set $\mathcal{X} = B_1 \cup \cdots \cup B_8$ and the filter $\mathcal{F}^{\mathsf{JP}}$. Thanks to the structure of $\mathcal{H}^{\mathsf{JP}}$ we have the following lemma, which can be proved in exactly the same way as the corresponding Lemma 2.6.15 for the Brun substitutions.

Lemma 2.6.17. *Let $\mathcal{V} \in \{\mathcal{V}_1^{\mathsf{JP}}, \mathcal{V}_2^{\mathsf{JP}}, \mathcal{V}_3^{\mathsf{JP}}, \mathcal{V}_4^{\mathsf{JP}}\}$. There exists N such that for every $(i_1, \dots, i_n) \in \{1,2,3,4\}^n$ containing more than N occurrences of 3 or 4, the pattern $\Theta_{i_1} \cdots \Theta_{i_n}(\mathcal{V})$ contains an $\mathcal{L}^{\mathsf{JP}}$-annulus of $\mathcal{V}_1^{\mathsf{JP}}$, $\mathcal{V}_2^{\mathsf{JP}}$, $\mathcal{V}_3^{\mathsf{JP}}$ or $\mathcal{V}_4^{\mathsf{JP}}$.*

2.6.7 Generating translates of seeds

Lemma 2.6.18. *There exists $N \geqslant 0$ such that if $(i_1, \cdots, i_n) \in \{1,2,3\}^n$ contains more than N occurrences of 3, then $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{U})$ contains a translate of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$.*

Proof. Suppose that such an N does not exist. Then, by a compactness argument, there exists an infinite sequence $(i_n)_{n\in\mathbb{N}}$ such that $\bigcup_{n\geqslant 0} \Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{U})$ does not contain any translate of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$. By Theorem 2.5.2, there exists M such that $(i_n)_{n\in\mathbb{N}}$ must be the labelling of an infinite backward path $\cdots \overset{i_{M+1}}{\to} \bullet \overset{i_M}{\to} \bullet$ in the following graph.



Let $k \geqslant 0$ be such that every path $\bullet \overset{i_{M+k}}{\to} \cdots \overset{i_M}{\to} \bullet$ in the graph is such that the word $i_{M+k} \cdots i_M$ starts with a word contained in one of the following rational languages:

- $L21^\star321^\star31^\star2$,
- $L21^\star331^\star31^\star21^\star2$,
- $L21^\star331^\star31^\star21^\star33$,
- $L21^\star331^\star31^\star21^\star321^\star31^\star2(1 \cup 2 \cup 3)$,

where $L = (2\cup 1)^\star 3(2\cup 1)^\star 3(2\cup 1)^\star 3(2\cup 1)^\star$. It can be checked by inspection that such a k exists: by following the edges we must turn in counter-clockwise direction along the four vertices, because $i_n = 3$ infinitely often. Hence, L is necessarily obtained because it only requires three 3's, and the rest of each language is obtained by starting at the top-right vertex of the graph, and enumerating all the possible path labellings encountered.

We now prove that $\Sigma_{i_M}^{\mathsf{Brun}} \cdots \Sigma_{i_{M+k}}^{\mathsf{Brun}}(\mathcal{U})$ contains a translate of $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$. This is done thanks to the graph shown in Figure 2.5: following any corresponding path labelled by $i_M, \dots, i_{M+k}$ starting from $\mathcal{U}$ in the graph eventually yields to $\mathcal{V}_1^{\mathsf{Brun}}$ or $\mathcal{V}_2^{\mathsf{Brun}}$, which proves our claim because for every edge $P \overset{i}{\longrightarrow} Q$ in the graph, $\Sigma_i^{\mathsf{Brun}}(P)$ contains a translated copy of Q. It follows that the lemma is proved, by contradiction of the assumption made on $(i_n)_{n\in\mathbb{N}}$, because $\Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_{M+k}}^{\mathsf{Brun}}(\mathcal{U})$ must contain a translate of one of the $\mathcal{V}_i^{\mathsf{Brun}}$. ⬠

Note that unlike the generation graphs constructed in Sections 2.6.5 and 2.6.6, the graph of Figure 2.5 used in the proof of Lemma 2.6.18 has been constructed "by hand" and not algorithmically. *Constructing* such a graph is tedious, but checking its validity is easy (for example using computer algebra).

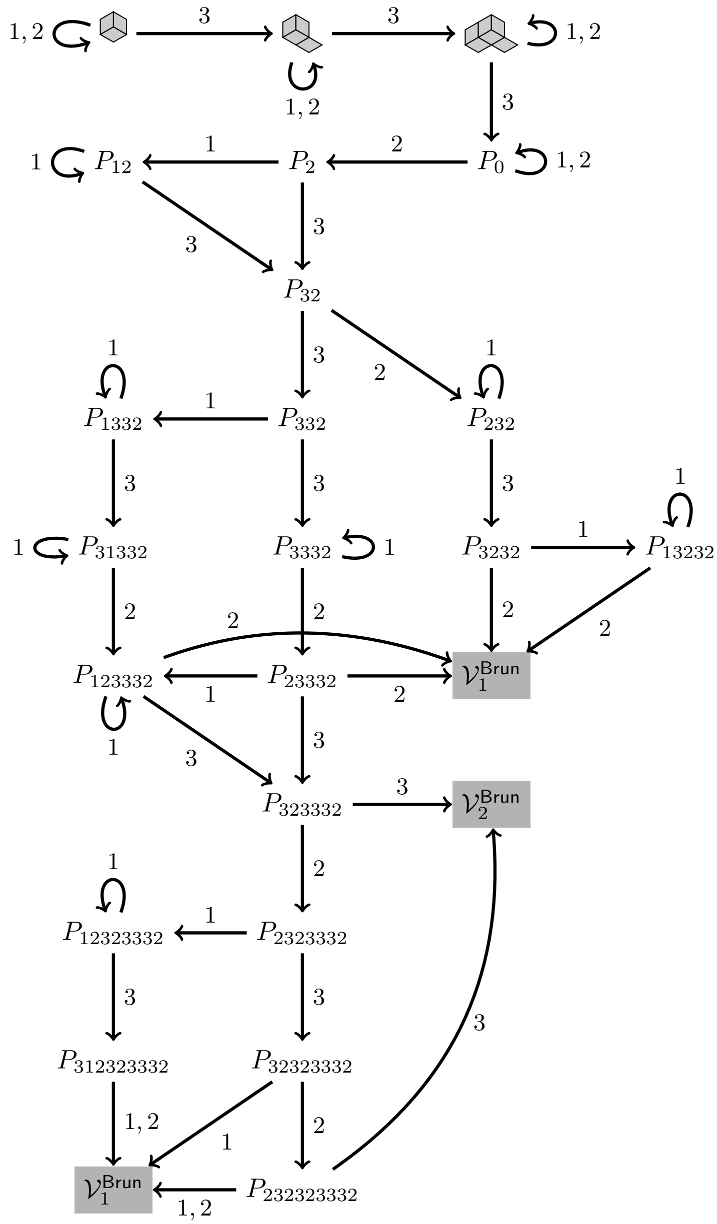
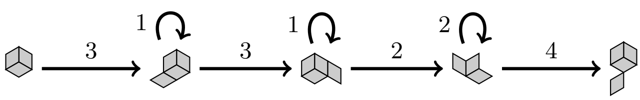


Figure 2.5: The graph used in the proof of Lemma 2.6.18. Each vertex is a pattern, and there is an edge $P \xrightarrow{i} Q$ if $\Sigma_i^{\mathsf{Brun}}(P)$ contains a *translated copy* of Q.

We now prove an analogue of the previous lemma for Jacobi-Perron substitutions. The proof is easier (we need a smaller graph) because the set of "bad" Jacobi-Perron is more constrained by Theorem 2.5.4.

Lemma 2.6.19. *There exists $N \geqslant 0$ such that if $n \geqslant N$ and $(a_1, b_1), \ldots, (a_n, b_n)$ is an admissible Jacobi-Perron expansion, then $\Sigma^{\mathsf{JP}}_{a_1,b_1} \cdots \Sigma^{\mathsf{JP}}_{a_n,b_n}(\mathcal{U})$ contains a translate of $\mathcal{V}^{\mathsf{JP}}_1$, $\mathcal{V}^{\mathsf{JP}}_2$, $\mathcal{V}^{\mathsf{JP}}_3$ or $\mathcal{V}^{\mathsf{JP}}_4$.*

Proof. Similarly as in the proof of Lemma 2.6.18, we can apply Theorem 2.5.4, so that the result only needs to be proved for products of the form $\Theta_{i_1} \cdots \Theta_{i_n}$ with $i_1 \cdots i_n \in 31^\star 31^\star 22^\star 4$, which is a translation of the condition given in Theorem 2.5.4 in terms of additive Jacobi-Perron substitutions. This is settled by the following graph, in which every edge $P \xrightarrow{i} Q$ means that $\Sigma^{\mathsf{Brun}}_i(P)$ contains a translated copy of Q.



Finally, it can be checked that the last pattern is sufficient to generate a seed $\mathcal{V}^{\mathsf{JP}}_i$, thanks to Theorem 2.5.4. ⬠

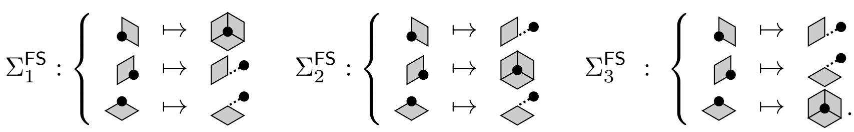
2.6.8 The case of the fully subtractive algorithm

In Section 3.4 we will need some discrete plane generation results for the substitutions associated with the *fully subtractive* continued fraction algorithm (defined in Section 3.4). We deal with this family of substitutions separately from the Arnoux-Rauzy, Brun and Jacobi-Perron because it will be used only in the discrete-geometrical applications of Section 3.4. These substitutions behave in a way similar to the Arnoux-Rauzy ones, in the sense that good annulus is always generated from $\mathcal{U}$.

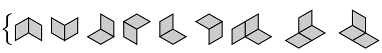
We then define

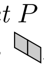
$$\sigma^{\mathsf{FS}}_1 = \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 21 \\ 3 \mapsto 31 \end{cases} \qquad \sigma^{\mathsf{FS}}_2 = \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 12 \\ 3 \mapsto 32 \end{cases} \qquad \sigma^{\mathsf{FS}}_3 = \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 13 \\ 3 \mapsto 23, \end{cases}$$

and their associated dual substitutions given by

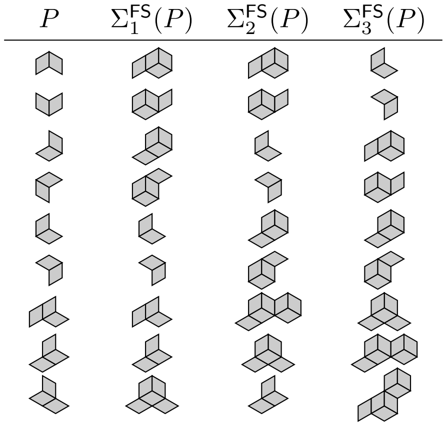


The following set of patterns will used to establish good covering properties for the subsitutions Σ^{FS}_i:

$\mathcal{L}^{\mathsf{FS}} = \{$  $\}$.

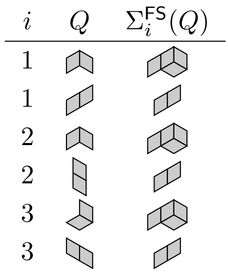
Proposition 2.6.20 (Strong $\mathcal{L}^{\mathsf{FS}}$-covering). *Let P be a strongly $\mathcal{L}^{\mathsf{FS}}$-covered pattern which is contained in a stepped plane that avoids* , ,  *and* . *Then $\Sigma^{\mathsf{FS}}_i(P)$ is strongly $\mathcal{L}^{\mathsf{FS}}$-covered for every $i \in \{1, 2, 3\}$.*

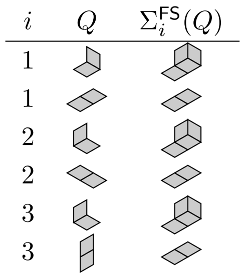
Proof. Let $i \in \{1,2,3\}$. We must first prove that $\Sigma_i^{\mathsf{FS}}(P)$ is $\mathcal{L}^{\mathsf{FS}}$-covered. The proof relies on Proposition 2.2.2 by checking that $\Sigma_i^{\mathsf{FS}}(Q)$ is $\mathcal{L}^{\mathsf{FS}}$-covered for all $Q \in \mathcal{L}^{\mathsf{FS}}$. This makes a total of twenty-seven patterns to check.

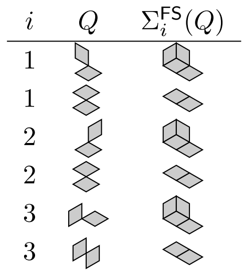


Now, let $X \subseteq \Sigma_i^{\mathsf{FS}}(P)$ be a two-face connected pattern. We must prove that there exists $Y \in \mathcal{L}^{\mathsf{FS}}$ such that $X \subseteq Y \subseteq \Sigma(P)$. If X is a translation of one of the six patterns , , , , , , then the strong covering condition is trivially verified by taking $Y = X$. By assumption and by Lemma 2.6.21, $\Sigma(P)$ does not contain any translate of , or , so we ignore these cases for X.

It remains to treat the cases $X =$, or . We have $X \subseteq \Sigma(P)$, so it is sufficient to check that for every pattern $Q \subseteq P$ such that $X \subseteq \Sigma(Q)$, there exists $Y \in \mathcal{L}^{\mathsf{FS}}$ such that $X \subseteq Y \subseteq \Sigma(Q)$. Moreover, since X is a two-face pattern, we can restrict to the case where Q consists of two faces only, which leaves a finite number of cases to check for Q:







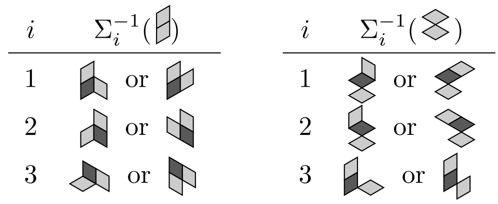
The case of the first, third and fifth rows of each table is settled, by taking $Y =$ $\in \mathcal{L}^{\mathsf{FS}}$, $Y =$ $\in \mathcal{L}^{\mathsf{FS}}$, or $Y =$ $\in \mathcal{L}^{\mathsf{FS}}$. For the second row of the first table with $Q =$, P is strongly $\mathcal{L}^{\mathsf{FS}}$-covered so we have $Q \subseteq Y_0 \subseteq P$ with $Y_0 =$ $\in \mathcal{L}^{\mathsf{FS}}$. It follows that $X \subseteq \Sigma_1^{\mathsf{FS}}(Y_0) =$, so taking $Y = \Sigma_1^{\mathsf{FS}}(Y_0) \in \mathcal{L}^{\mathsf{FS}}$ works. The cases of the second and fourth rows of the second table can be dealt with similarly.

In the last row of the third table with we have $Q =$, so must appear in Γ, which is forbidden by assumption. This can be seen by using Remark 2.6.1 to compute the only possible "completion" of Q within Γ (shown in dark gray): . In all the remaining cases, Q is a pattern which is not allowed in Γ by assumption, so they can be ignored. ⬠

The following lemma will be needed to to handle the three patterns forbidden in Proposition 2.6.20. It is analogous to Lemma 2.6.6 that is used with the Arnoux-Rauzy substitutions.

Lemma 2.6.21. *Let Γ be a stepped plane that does not contain any translate of one of the patterns , , and . Then no translate of any of these four patterns appears in $\Sigma_1^{\mathsf{FS}}(\Gamma)$ for all $i \in \{1,2,3\}$.*

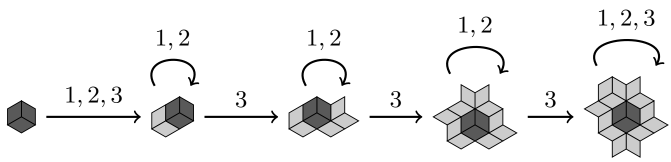
Proof. The patterns and admit no preimage that is included in a discrete plane, as can be checked using Remark 2.6.1. For the two other cases, below are listed their possible preimages (in light gray), together with their only possible "completion" within a discrete plane (in dark gray) obtained thanks to Remark 2.6.1.



These two tables show that if one of these two patterns appears in $\Sigma(\Gamma)$, then one of the four patterns must appear in Γ, which concludes the proof. ⬠

Lemma 2.6.22. *Let Σ be a product of Σ_1^{FS}, Σ_2^{FS} and Σ_3^{FS} in which Σ_3^{FS} appears at least four times. Then $\Sigma(\mathcal{U})$ contains an $\mathcal{L}^{\mathsf{FS}}$-annulus of $\mathcal{U}$.*

Proof. In the below graph, we have $Q \subseteq \Sigma_i^{\mathsf{FS}}(P)$ for every edge $P \xrightarrow{i} Q$, so the result follows.

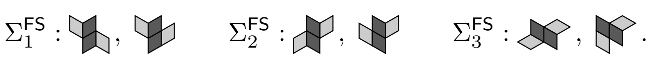


⬠

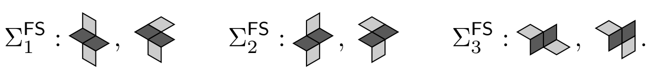
Proposition 2.6.23 (Property A for fully subtractive). *Property A holds for $\Sigma_1^{\mathsf{FS}}, \Sigma_2^{\mathsf{FS}}, \Sigma_3^{\mathsf{FS}}$ with $\mathcal{L}^{\mathsf{FS}}$, when restricted to discrete planes avoiding , , and .*

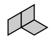
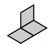
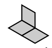
Proof. We enumerate all the faces f, g, f_0, g_0 such that $f \in \Sigma_i^{\mathsf{FS}}(f_0)$, $g \in \Sigma_i^{\mathsf{AR}}(g_0)$, $f \cup g$ is connected and $f_0 \cup g_0$ is disconnected, for some $i \in \{1,2,3\}$. All the possibilities are given in Figure 2.3, where the faces plotted in dark gray correspond to the only possible of $f_0 \cup g_0$ within an admissible discrete plane, with respect to Remark 2.6.1.

The first such possibilities are $f \cup g =$, or , but cases can be ignored thanks to Lemma 2.6.21. Another possibility is $f \cup g =$, which admits six disconnected preimages $f_0 \cup g_0$ (two for each Σ_i^{FS}). These preimages are shown below (in light gray), together with their only possible completion within a stepped plane (in dark gray):



The patterns that appear in dark gray are forbidden by Lemma 2.6.21, so this case is settled. The last two possibilities are $f \cup g =$ or . Below (in light gray) are all the possible preimages $f_0 \cup g_0$ (which are the same for the two possibilities), and in dark gray is shown their only possible completion X within a stepped plane:



Now, we have $X \subseteq A$ because $f_0 \cap g_0$ is disconnected, but this contradicts the strong $\mathcal{L}^{\mathsf{FS}}$-covering of A. Indeed, X is a two-face connected pattern, but there cannot exist a pattern $Y \in \mathcal{L}^{\mathsf{FS}}$ such that $X \subseteq Y \subseteq A$ because then we must have $Y =$ ,  or , so Y must overlap with f_0 or g_0, which is impossible because f_0, g_0 are not in A. ⬠

Chapter 3

Applications

We give some applications of the discrete plane generation results established in the previous chapter. We start with some dynamical properties of finite products of Arnoux-Rauzy, Brun and Jacobi-Perron substitutions in Section 3.1, and some topological properties of their Rauzy fractals in Section 3.2. Some number theoretical properties of the associated number systems are given in 3.3. Finally, a discrete geometrical result is obtained in Section 3.4, using the methods of Chapter 2. The first three sections are joint work with Valérie Berthé and Anne Siegel [**BJS13, BJS12, BBJS13**], and Section 3.4 is joint work with Valérie Berthé, Damien Jamet and Xavier Provençal [**BJJP13**].

Contents

Before stating our results we recall the following:

- the product $\sigma^{\mathsf{AR}}_{i_1} \cdots \sigma^{\mathsf{AR}}_{i_n}$ is ***Arnoux-Rauzy-admissible*** if $(i_1, \ldots, i_n)$ contains each symbol 1, 2 and 3 at least once;
- the product $\sigma^{\mathsf{Brun}}_{i_1} \cdots \sigma^{\mathsf{Brun}}_{i_n}$ is ***Brun-admissible*** if $i_k = 3$ for some $n \in \{1, \ldots, n\}$;
- the product $\sigma^{\mathsf{JP}}_{a_1,b_1} \cdots \sigma^{\mathsf{JP}}_{a_n,b_n}$ is ***Jacobi-Perron-admissible*** if for every $1 \leqslant k \leqslant n$, we have $0 \leqslant a_k \leqslant b_k$, $b_k \neq 0$, and $a_k = b_k$ implies $a_{k+1} \neq 0$.

3.1 Dynamical properties of products of substitutions

Apart from the case of irreducible unimodular Pisot substitutions on two letters, for which the Pisot conjecture is true [**BD02, HS03**], there are few results about the dynamics of infinite families of substitutions. One such result states that some families of substitutions arising from β-numeration are semi-conjugate to toral translations; see [**Aki00, BBK06**].

The tools developed in Chapter 2 enable us to establish dynamical properties of the infinite families consisting of arbitrary finite products of Arnoux-Rauzy, Brun or Jacobi-Perron substitutions in Theorem 3.1.1 below.

Even though our tools are not fully algorithmic, they can be seen as an extension of the techniques known for the study of a *single* substitution, where many algorithms have been developed.

Theorem 3.1.1. *For every admissible finite product σ of Arnoux-Rauzy, Brun or Jacobi-Perron substitutions, the system (X_σ, S) is semi-conjugate to a translation on the two-dimensional torus.*

Proof. This theorem follows directly from Proposition 1.4.1 (due to [**IR06**]), which states that (X_σ, S) is semi-conjugate to a toral translation if and only if $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ covers translates of patterns with arbitrarily large combinatorial radius. Indeed, this property is proved in Theorem 2.5.1 for Arnoux-Rauzy substitutions, and in Theorem 2.5.5 for Brun and Jacobi-Perron substitutions. ⬠

Note that another proof of Theorem 3.1.1 for Arnoux-Rauzy substitutions has been obtained in [**BŠW13**] by different methods (homological tools).

Markov partition for toral automorphisms Another dynamical application of Theorems 2.5.1 and 2.5.5 is that for every matrix $\mathbf{M}$ which is a product of incidences matrices of Arnoux-Rauzy, Brun or Jacobi-Perron substitutions, an explicit Markov partition can be constructed for the toral automorphisms $(\mathbb{T}^3, \mathbf{M})$, as explained in Section 1.4 and in Figure 1.3 page 32. Moreover, the domains of the Markov partition in question are connected, thanks to Theorem 3.2.1.

3.2 Topological properties of Rauzy fractals of products of substitutions

Before stating the results of this section, we mention that there are many connections between some topological properties of Rauzy fractals (such as being connected or having zero interior point) and some number theoretical properties of the associated dominant Pisot eigenvalue. This is described more in detail in Section 3.3.

Connectedness

The following result states that the Rauzy fractals of the substitutions in the families we consider are all connected. Contrary to the other results of this section, this can be proved by using $\mathcal{L}$-coverings only (there is no need for *strong* coverings or the annulus property).

Theorem 3.2.1. *The Rauzy fractal of every admissible finite product of Arnoux-Rauzy, Brun or Jacobi-Perron substitutions is connected.*

Proof. Let σ be such an admissible product. Thanks to Propositions 2.6.5, 2.6.7 and 2.6.8, these patterns $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ are all $\mathcal{L}$-connected, for the case of Arnoux-Rauzy, Brun and Jacobi-Perron substitutions. (Where $\mathcal{L}$ is the set of patterns associated with the substitutions.)

Since $\mathcal{L}$ consists of connected patterns only, the definition of $\mathcal{L}$-coverings (Definition 2.2.1) implies that every $\mathcal{L}$-covered pattern is path-connected. It follows that the patterns $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ are connected for every $n \geqslant 0$, so the Rauzy fractal of σ is a Hausdorff limit of connected sets, according to Definition 1.3.5. The result then follows from the fact that connectedness is preserved by the Hausdorff limit. ⬠

There are examples of such connected fractals which are not simply connected. This raises the question of characterizing the products that yield a simply connected fractal, which is an interesting and difficult question. (See Example 5.3 for a non-simply connected Arnoux-Rauzy fractal.)

Remark 3.2.2. It is interesting to note that, even though some of these fractals are not simply connected, all of them are the Hausdorff limit of simply connected polygons. The simply connected polygons in question are the polygons $P_n = \mathbf{h}_\sigma^n \pi_\sigma(\mathbf{E}_1^\star(\sigma)^n(\mathcal{U}))$ given in Definition 1.3.5. One way to prove that each P_n is simply connected is to prove that P_n admits a tiling of the plane and does not have any cut point. We will not prove this formally here, but this tiling property can be deduced from the fact that $\mathcal{U} =$ tiles the discrete plane $\Gamma_{(1,1,1)}$ (periodically), so $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ tiles $\mathbf{E}_1^\star(\sigma)^n(\Gamma_{(1,1,1)})$ for all $n \geqslant 0$ thanks to Proposition 1.2.4. (The fact that $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ does not have cut points follows from the $\mathcal{L}$-covering properties, because the patterns in $\mathcal{L}$ do not have cut points as well.)

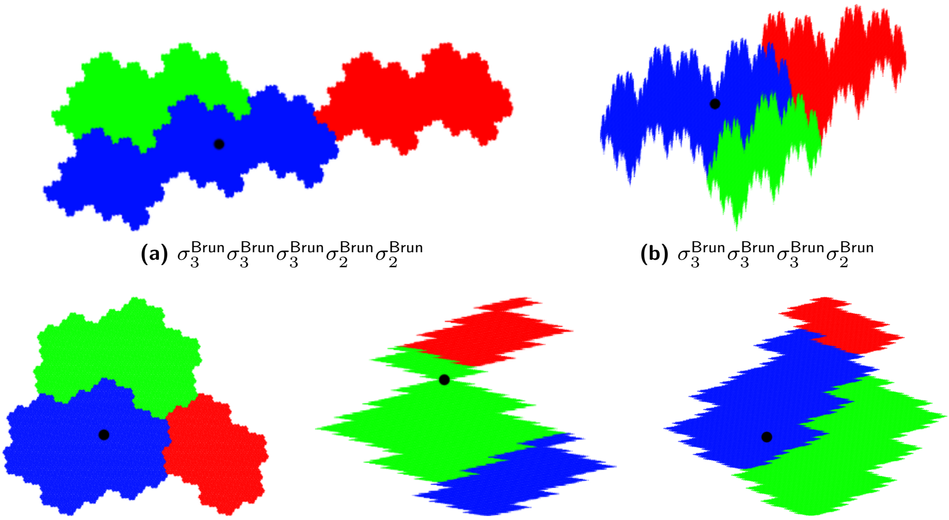
Zero interior point

Discrete plane generation properties can be interpreted in terms of Rauzy fractals. More precisely, the origin is as an interior point of the Rauzy fractal of σ if and only if the patterns $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ generate patterns with arbitrarily large combinatorial radius centered at the origin (see [**BS05**] or [**ST09**]). This has the following immediate consequences for Arnoux-Rauzy substitutions (by Theorem 2.5.1) and for Brun and Jacobi-Perron substitutions (by Theorems 2.5.2 and 2.5.4).

Theorem 3.2.3. *We have:*

- *The origin is an interior point of the Rauzy fractal of every admissible product of Arnoux-Rauzy substitutions.*
- *The origin is **not** an interior point of the Rauzy fractal of an admissible product $\sigma_{i_1}^{\mathsf{Brun}} \cdots \sigma_{i_n}^{\mathsf{Brun}}$ if and only if there exists a cycle $\bullet \xrightarrow{i_1} \bullet \xrightarrow{i_2} \cdots \xrightarrow{i_n} \bullet$ in the directed graph given in Theorem 2.5.2.*
- *The origin is **not** an interior point of the Rauzy fractal of an admissible product $\sigma_{a_1,b_1}^{\mathsf{JP}} \cdots \sigma_{a_n,b_n}^{\mathsf{JP}}$ if and only if the infinite sequence $((a_1, b_1), \ldots, (a_n, b_n))^\infty$ satisfies the condition given in Theorem 2.5.4.*

This theorem is illustrated on some finite products of Brun substitutions in Figure 3.1. The comments made in Example 2.5.3 also apply to the finite products of substitutions considered in Theorem 3.2.3. For example, the origin is an interior point of the Rauzy fractal of $\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}(\sigma_2^{\mathsf{Brun}})^k$ if and only if k is even; see Figure 3.1, (a) and (b) for the associated Rauzy fractals when $k = 1$ or 2.



(a) $\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_2^{\mathsf{Brun}}\sigma_2^{\mathsf{Brun}}$ **(b)** $\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_2^{\mathsf{Brun}}$

(c) $\sigma_2^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_2^{\mathsf{Brun}}$ **(d)** $\sigma_1^{\mathsf{Brun}}\sigma_1^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_2^{\mathsf{Brun}}$ **(e)** $\sigma_2^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_1^{\mathsf{Brun}}\sigma_1^{\mathsf{Brun}}$

Figure 3.1: Some Rauzy fractals of products of Brun substitutions. Only (b) and (d) do not have the origin as an interior point. An example of a product whose Rauzy fractal has zero interior point but whose reverse product does not is provided by (d) and (e). The fractals of (c) and (d) correspond to the two examples of substitutions shown in Figure 2.1 of Section 2.1, where (c) generates a full discrete plane but (d) does not.

3.3 Number-theoretical implications

Links with topology of Rauzy fractals

The *(F) property* (introduced in [**FS92**]), which expresses some finiteness properties of digital expansions in non-integer base β, can be reformulated in topological terms: it is equivalent to the fact that the origin is an interior point of the central tile associated with β [**Aki99, Aki02**].

Several variants of the (F) property have been proposed, one of them being the *extended (F) property*, introduced in [**BS05, FT06**] to extend the classical (F) property to the numeration system associated with a substitution σ. The extended (F) property can also be stated in topological terms: it is true if and only if the origin is an interior point of the Rauzy fractal associated with σ.

Consequently, Theorem 3.2.3, which characterizes the products of Arnoux-Rauzy, Brun and Jacobi-Perron substitutions for which the origin is an interior point of the Rauzy fractal, also provides a characterization of when such a product of substitutions satisfies the extended (F) property.

The connectedness of the fractal is conjectured to guarantee explicit relations between the norm of β and the β-expansion of 1 [**AG05**]. The properties of rational numbers with purely periodic β-expansions are closely related to the shape of the boundary of the Rauzy fractal [**ABBS08, AFSS10**].

In Diophantine approximation, explicit computation of the size of the largest ball contained in the fractal provides the best possible simultaneous approximations of some two-dimensional vectors with respect to a specific norm [**HM06**].

Fractal tiles for cubic number fields

Thanks to a result by Paysant-Le-Roux and Dubois [**PD84**] we are able to associate Rauzy fractal dynamics with every cubic real field in the theorem below.

Theorem 3.3.1. *For every cubic real extension $\mathbb{K}$ of $\mathbb{Q}$, there exist $\alpha, \beta \in \mathbb{K}$ and a 3-letter unimodular Pisot irreducible substitution σ such that $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$ and such that the toral translation $(\mathbb{T}^2, x \mapsto x + \binom{\alpha}{\beta})$ is semi-conjugate to (X_σ, S).*

Proof. According to [**PD84**, Proposition IV], if $\mathbb{K}$ is a totally real cubic field, then there exists a Pisot number β that generates $\mathbb{K}$, with minimal polynomial $P(X) = X^3 - c_2X^2 + c_1X - 1 = 0$, where $c_2 \geqslant 2c_1 - 2$ and $c_1 \geqslant 3$. If we set

$$\sigma = \sigma^{\mathsf{JP}}_{0,1}\sigma^{\mathsf{JP}}_{0,1}\sigma^{\mathsf{JP}}_{c_1-3,c_2-c_1}$$

(an admissible product because $c_2 - c_1 > c_1 - 2$), then $\mathbf{M}_\sigma$ has characteristic polynomial $P(X)$. Since $\mathbf{M}_\sigma$ is primitive, it admits a positive eigenvector $\mathbf{v}$ with $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3 = 1$ and we have $\mathbb{Q}(\mathbf{v}_2, \mathbf{v}_3) = \mathbb{K}$. The Jacobi-Perron expansion of $\mathbf{v}$ has period $(c_1-3, c_2-c_1)(0, 1), (0, 1)$ (this is an admissible expansion). Now consider the projection $\pi_{\mathbf{v},\mathbf{1}^\perp}$ along $\mathbf{v}$ onto the plane with normal vector $(1, 1, 1)$. This projection expresses as

$$\pi_{\mathbf{v},\mathbf{1}^\perp}(x, y, x) = (\mathbf{v}_2(x + y + z) - x)(\mathbf{e}_3 - \mathbf{e}_1) + (\mathbf{v}_3(x + y + z) - y)(\mathbf{e}_3 - \mathbf{e}_2).$$

The vectors $\pi_{\mathbf{v},\mathbf{1}^\perp}(\mathbf{e}_i)$ expressed in the basis $(\mathbf{e}_3 - \mathbf{e}_1, \mathbf{e}_3 - \mathbf{e}_2)$ are congruent to the translation by $(\mathbf{v}_2, \mathbf{v}_3)$ modulo $\mathbb{Z}^2$. The projection $\pi_{\mathbf{v},\mathbf{1}^\perp}$ can also be used to define the Rauzy fractal of σ (instead of the projection π_σ defined in Section 1.3), so the results follows by taking $\alpha = \mathbf{v}_2, \beta = \mathbf{v}_3$ thanks to Proposition 1.4.1 and Theorem 2.5.5. ⬠

Convergence of multidimensional continued fraction algorithms

The Arnoux-Rauzy substitutions $\Sigma^{\mathsf{AR}}_1, \Sigma^{\mathsf{AR}}_2, \Sigma^{\mathsf{AR}}_3$ (defined in section 1.5) and the substitutions $\Sigma^{\mathsf{FS}}_1, \Sigma^{\mathsf{FS}}_2, \Sigma^{\mathsf{FS}}_3$ (defined in Section 3.4) have natural interpretations in terms of multidimensional continued fraction algorithms, similarly as what we have described in Section 1.5 for the Brun and Jacobi-Perron substitutions. To avoid confusion, in this section we will denote by:

- $\mathbf{F}^{\mathsf{AR}}$ the continued fraction algorithm associated with $\Sigma^{\mathsf{AR}}_1, \Sigma^{\mathsf{AR}}_2, \Sigma^{\mathsf{AR}}_3$,
- $\mathbf{F}^{\mathsf{FS}}$ the continued fraction algorithm associated with $\Sigma^{\mathsf{FS}}_1, \Sigma^{\mathsf{FS}}_2, \Sigma^{\mathsf{FS}}_3$.

The associated continued fraction algorithms can be obtained via the incidence matrices of these substitutions, where each step of the algorithm is given by $\mathbf{v}^{(n-1)} = {}^{\mathrm{t}}\mathbf{M}_{\Sigma^{\mathsf{AR}}_{i_n}}\mathbf{v}^{(n)}$

or $\mathbf{v}^{(n-1)} = {}^{t}\mathbf{M}_{\Sigma^{\mathsf{FS}}_{i_n}} \mathbf{v}^{(n)}$ as described in Section 1.5 for the Brun and Jacobi-Perron substitutions. This yields the following two maps.

$$\mathbf{F}^{\mathsf{AR}} : \mathbf{v} \mapsto \begin{cases} (\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } 0 < \mathbf{v}_1 \leqslant \mathbf{v}_2 \text{ and } 0 < \mathbf{v}_1 \leqslant \mathbf{v}_3 \\ (\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2) & \text{if } 0 < \mathbf{v}_2 \leqslant \mathbf{v}_1 \text{ and } 0 < \mathbf{v}_2 \leqslant \mathbf{v}_3 \\ (\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3) & \text{if } 0 < \mathbf{v}_3 \leqslant \mathbf{v}_1 \text{ and } 0 < \mathbf{v}_3 \leqslant \mathbf{v}_2, \end{cases}$$

$$\mathbf{F}^{\mathsf{FS}} : \mathbf{v} \mapsto \begin{cases} (\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } 0 < \mathbf{v}_1 \leqslant \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \\ (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } 0 < \mathbf{v}_2 - \mathbf{v}_1 < \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \\ (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_1) & \text{if } 0 < \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 < \mathbf{v}_1. \end{cases}$$

These two algorithms are very similar: in both of them, the smallest coordinate of $\mathbf{v}$ is subtracted to the two others. For this reason $\mathbf{F}^{\mathsf{AR}}$ and $\mathbf{F}^{\mathsf{FS}}$ are called *fully subtractive* algorithms. The only difference is that $\mathbf{F}^{\mathsf{FS}}$ reorders coordinates at each step.

The discrete plane generation properties established in Chapter 2 allow us to recover some convergence properties of these two algorithms. We can define the $\mathbf{F}^{\mathsf{AR}}$- and $\mathbf{F}^{\mathsf{FS}}$-expansions of a vector $\mathbf{v}$ in a similar way as in Section 1.5.

Theorem 3.3.2. *The continued fraction algorithms associated with $\mathbf{F}^{\mathsf{AR}}$ and $\mathbf{F}^{\mathsf{FS}}$ are convergent. More precisely:*

- *For every sequence $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$, in which 1, 2 and 3 each occur infinitely often, there exists a unique vector $\mathbf{v} \in \mathbb{R}^3_{>0}$ whose $\mathbf{F}^{\mathsf{AR}}$-expansion is $(i_n)_{n\in\mathbb{N}}$.*
- *For every sequence $(i_n)_{n\in\mathbb{N}} \in \{1,2,3\}^{\mathbb{N}}$, in which 3 occurs infinitely often, there exists a unique vector $\mathbf{v} \in \mathbb{R}^3_{>0}$ whose $\mathbf{F}^{\mathsf{FS}}$-expansion is $(i_n)_{n\in\mathbb{N}}$.*

Proof. The statement for the Arnoux-Rauzy case is a direct consequence of the fact that the sequence of vectors

$$\mathbf{w}^{(n)} \;:=\; {}^{t}\mathbf{M}_{\sigma^{\mathsf{AR}}_{i_1}} \cdots {}^{t}\mathbf{M}_{\sigma^{\mathsf{AR}}_{i_n}} \mathbf{w}$$

is convergent for every admissible infinite sequence $(i_n)_{n\in\mathbb{N}}$ and for every vector $\mathbf{w} \in \mathbb{R}^3_{>0}$. Let us then prove this fact. Let $(i_n)_{n\in\mathbb{N}}$ be an admissible expansion and let $\mathbf{w} \in \mathbb{R}^3_{>0}$ be arbitrary. We have $\mathcal{U} \subseteq \Gamma_{\mathbf{w}}$, so we can prove by induction, using Proposition 1.2.4, that

$$\Sigma^{\mathsf{AR}}_{i_1} \cdots \Sigma^{\mathsf{AR}}_{i_n}(\mathcal{U}) \;\subseteq\; \Sigma^{\mathsf{AR}}_{i_1} \cdots \Sigma^{\mathsf{AR}}_{i_n}(\Gamma_{\mathbf{w}^{(n)}}).$$

for all $n \geqslant 1$. The result now follows because thanks to Theorem 2.5.1, the patterns $\Sigma^{\mathsf{AR}}_{i_1} \cdots \Sigma^{\mathsf{AR}}_{i_n}(\mathcal{U})$ have arbitrarily large combinatorial radius as $n \to \infty$, so the vectors $\mathbf{w}^{(n)}$ are constrained to a unique direction in the limit. The situation is identical with the substituions Σ^{FS}_{i}. ⬠

Note that we cannot directly deduce the same results for the Brun and Jacobi-Perron algorithms because we have used their convergence properties to establish their discrete plane generation properties in Section 2.4. Stronger (almost-everywhere) convergence properties of such algorithms can be obtained by other methods [**AD13**]; see also [**Sch00**].

Remark 3.3.3. Unlike the Brun and Jacobi-Perron algorithms, the set of vectors that admit an admissible $\mathbf{F}^{\mathsf{AR}}$- or $\mathbf{F}^{\mathsf{FS}}$-expansion is not full. The set of convergence of the two algorithms is in fact a rather complicated set (plotted below). In the case of $\mathbf{F}^{\mathsf{AR}}$ it is called the “Rauzy gasket” and has been proved to be homeomorphic to the Sierpiński triangle [**AS13a**]. Many properties of this set are currently unknown (its Lebesgue measure, its fractal dimension, or whether it can be decided if a given vector belongs to it). See Figure 3.2.

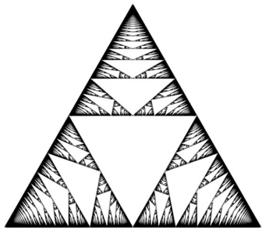


Figure 3.2: The Rauzy gasket. Picture from [**AS13a**].

3.4 Critical connectedness of arithmetical discrete planes

This section is devoted to an application in discrete geometry, concerning discrete planes defined by taking the vertices in $\mathbb{Z}^3$ which are contained in an ω-neighborhood of an actual two-dimensional plane in $\mathbb{R}^3$ (Definition 3.4.1). The number $\omega > 0$ is referred to as the ***thickness*** of the discrete plane, and modifying it naturally has an impact on the features of the discrete plane.

The aim of this section is to study some properties of the infimum thickness $\Omega(\mathbf{v})$ for which the discrete plane of normal vector $\mathbf{v}$ is 2-connected (*i.e.*, is "path-connected" by paths of adjacent vertices, see Definition 3.4.2).

In [**JT09**] an algorithm was given to compute this infimum $\Omega(\mathbf{v})$. However, it is not known if a discrete plane is 2-connected when its thickness is precisely this infimum. We answer this question in Theorem 3.4.8 by giving a characterization of which discrete planes are 2-connected at thickness $\Omega(\mathbf{v})$. Again, we will use some discrete plane generation methods developed in Chapter 2.

Arithmetical discrete planes and connecting thickness

In Chapters 1 and 2 we have been working with *discrete planes* (Definition 1.2.1), which consist of an infinite union of faces $[\mathbf{x}, i]^\star$ for $x \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$. In this section we will consider a more basic notion of discrete planes, *arithmetical discrete planes*, consisting only of elements of $\mathbb{Z}^3$, which we will interpret as "voxels" (three-dimensional pixels). Their definition below has been introduced in [**Rev91, And03**]. Denote by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ the canonical basis of $\mathbb{R}^3$, and by $\langle \cdot, \cdot \rangle$ the usual scalar product.

Definition 3.4.1. Let $\mathbf{v} \in \mathbb{R}^3_{\geq 0}$ and $\omega \in \mathbb{R}_+$. The ***arithmetical discrete plane*** with normal vector $\mathbf{v}$, and thickness ω is the set $\mathfrak{P}(\mathbf{v}, \omega) \subseteq \mathbb{Z}^3$ defined by

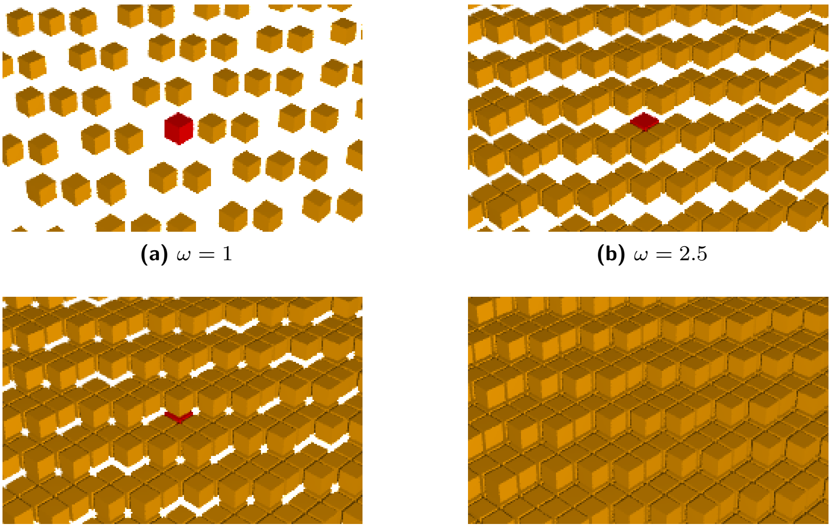
$$\mathfrak{P}(\mathbf{v}, \omega) = \left\{ \mathbf{x} \in \mathbb{Z}^3 : 0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \omega \right\}.$$

Definition 3.4.2. Two distinct $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^3$ are 2-***adjacent*** if $\|\mathbf{x} - \mathbf{y}\|_1 = 1$. A subset $X \subseteq \mathbb{Z}^3$ is 2-***connected*** if for every $\mathbf{x}, \mathbf{y} \in X$, there exist $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \in X$ such that

$\mathbf{x}^{(i)}$ and $\mathbf{x}^{(i+1)}$ are 2-adjacent for all $i \in \{1, \ldots, n-1\}$, with $\mathbf{x}^{(1)} = \mathbf{x}$ and $\mathbf{x}^{(n)} = \mathbf{y}$. The ***connecting thickness*** $\Omega(\mathbf{v})$ of $\mathbf{v} \in \mathbb{R}_+^3$ is defined by:

$$\Omega(\mathbf{v}) = \inf \{\omega \in \mathbb{R}_+ : \mathfrak{P}(\mathbf{v}, \omega) \text{ is 2-connected}\}.$$

The two above definitions are illustrated in Figure 3.3. Note that these definitions focus on 2-connectedness, but similar definitions are also possible for k-connectedness with $k = 0$ or 1. However, the value of $\Omega(\mathbf{v})$ for such alternative definitions can be directly deduced from the value of $\Omega(\mathbf{v})$ for 2-connectedness (see [**JT06**]), so it is natural to restrict to 2-connectedness.



(a) $\omega = 1$ **(b)** $\omega = 2.5$ **(c)** $\omega = 4$ **(d)** $\omega = 6$

Figure 3.3: The arithmetical discrete plane $\mathfrak{P}((1, \sqrt{2}, \pi), \omega)$, with varying thickness ω. In (a) and (b), $\mathfrak{P}(\mathbf{v}, 1)$ and $\mathfrak{P}(\mathbf{v}, 2.5)$ are not 2-connected, but in (c) and (d), $\mathfrak{P}(\mathbf{v}, 4)$ and $\mathfrak{P}(\mathbf{v}, 6)$ are 2-connected. It follows that $1 < \Omega(\mathbf{v})$, $2.5 \leqslant \Omega(\mathbf{v})$, $4 \geqslant \Omega(\mathbf{v})$ and $6 > \Omega(\mathbf{v})$.

Computing connecting thickness: the fully subtractive algorithm

In order to compute $\Omega(\mathbf{v})$, we can assume without loss of generality that $0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3$. We thus restrict ourselves in the sequel to the set $\mathcal{O}_3^+ = \left\{\mathbf{v} \in \mathbb{R}^3 : 0 \leqslant \mathbf{v}_1 \leqslant \mathbf{v}_2 \leqslant \mathbf{v}_3\right\}$.

A first gross approximation of $\Omega(\mathbf{v})$ is provided by $\|\mathbf{v}\|_\infty \leqslant \Omega(\mathbf{v}) \leqslant \|\mathbf{v}\|_1$ (see [**AAS97**]). Jamet and Toutant gave in [**JT09**] a procedure to compute $\Omega(\mathbf{v})$. It can be nicely expressed in terms of the ***fully subtractive algorithm*** $\mathbf{F} : \mathcal{O}_3^+ \to \mathcal{O}_3^+$ defined by

$$\mathbf{F}(\mathbf{v}) = \begin{cases} (\mathbf{v}_1, \mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } \mathbf{v}_1 \leqslant \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \\ (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1) & \text{if } \mathbf{v}_2 - \mathbf{v}_1 < \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 \\ (\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \mathbf{v}_1) & \text{if } \mathbf{v}_2 - \mathbf{v}_1 \leqslant \mathbf{v}_3 - \mathbf{v}_1 < \mathbf{v}_1. \end{cases}$$

This is the map $\mathbf{F}^{\mathsf{FS}}$ introduced on page 71. Similarly as with the Brun and Jacobi-Perron algorithms, we denote by $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots$ the sequence of vectors obtained by iterating $\mathbf{F}$ on a vector $\mathbf{v}$. The link between connecting thickness and the fully subtractive algorithm $\mathbf{F}$ is given in the next theorem.

Theorem 3.4.3 ([JT09],[DJT09]). *Let $\mathbf{v} \in \mathcal{O}_3^+$. The arithmetical discrete plane $\mathfrak{P}(\mathbf{v}, \omega)$ is 2-connected if and only if so is $\mathfrak{P}(\mathbf{F}(\mathbf{v}), \omega - \mathbf{v}_1)$. Consequently, $\Omega(\mathbf{v}) = \Omega(\mathbf{F}(\mathbf{v})) + \mathbf{v}_1$, and the following algorithm computes $\Omega(\mathbf{v})$.*

```
def FS(v):
    return sorted([v[0], v[1]-v[0], v[2]-v[0]])

def connecting_thickness(v):
    if v[0]+v[1] <= v[2]:
        return max(v)
    else:
        return v[0] + connecting_thickness(FS(v))
```

Moreover, if the algorithm never stops, then $\Omega(\mathbf{v}) = \sum_{n=0}^{\infty} \mathbf{v}_1^{(n)} = \|\mathbf{v}\|_1/2$.

Example 3.4.4. Let $\mathbf{v} = (1, \sqrt{13}, \sqrt{17})$. Iterating $\mathbf{F}$ yields

$$\begin{aligned}
\mathbf{v}^{(1)} &= (1, \sqrt{13} - 1, \sqrt{17} - 1) \\
\mathbf{v}^{(2)} &= (1, \sqrt{13} - 2, \sqrt{17} - 2) \\
\mathbf{v}^{(3)} &= (\sqrt{13} - 3, 1, \sqrt{17} - 3) \\
\mathbf{v}^{(4)} &= (4 - \sqrt{13}, \sqrt{17} - \sqrt{13}, \sqrt{13} - 3) \\
\mathbf{v}^{(5)} &= (\sqrt{17} - 4, 2\sqrt{13} - 7, 4 - \sqrt{13})
\end{aligned}$$

and stops because $\mathbf{v}_1^{(5)} + \mathbf{v}_2^{(5)} \leqslant \mathbf{v}_3^{(5)}$, so $\Omega(\mathbf{v}) = 1 + 1 + 1 + \sqrt{13} - 3 + 4 - \sqrt{13} + 4 - \sqrt{13} = 8 - \sqrt{13}$. Similarly, if $\mathbf{v} = (1, \sqrt[3]{10}, \pi)$, then the algorithm stops after 19 steps and $\Omega(\mathbf{v}) = 2\pi - 98\sqrt[3]{10} + 208$. It is also possible to exhibit some examples where the algorithm never stops, for example by choosing a right eigenvetor of one of the matrices of $\mathbf{F}$. This is the case for example with the vector $\mathbf{v} = (1, \alpha + 1, \alpha^2 + \alpha + 1) = (1, 1.54\ldots, 1.84\ldots)$, where $\alpha = 0.54\ldots$ is the real root of $x^3 + x^2 + x + 1$.

The set on which the algorithm of Theorem 3.4.3 never stops is defined by

$$\mathcal{F}_3 = \left\{ \mathbf{v} \in \mathcal{O}_3^+ : \mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} > \mathbf{v}_3^{(n)} \text{ for all } n \in \mathbb{N} \right\}.$$

It will play a crucial role in the characterization stated in Theorem 3.4.8: $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ is 2-connected if and only if $\mathbf{v} \in \mathcal{F}_3$. This set has been studied in [**Mee89**], and its properties are similar to that of the Rauzy gasket discussed in Remark 3.3.3. The next property relates the expansion of a vector $\mathbf{v}$ and its belonging to $\mathcal{F}_3$.

Lemma 3.4.5. *We have $\mathbf{v} \in \mathcal{F}_3$ if and only the expansion $(i_n)_{n\in\mathbb{N}}$ of $\mathbf{v}$ contains infinitely many occurrences of* 3.

Proof. Let $\mathbf{v} \in \mathcal{F}_3$, and assume by contradiction that $(i_n)_{n\in\mathbb{N}}$ does not take the value 3. One thus checks that $\lim_{n\to\infty} \mathbf{v}_1^{(n)} = \lim_{n\to\infty} \mathbf{v}_2^{(n)} = 0$, and hence, $\lim_{n\to\infty} \mathbf{v}_3^{(n)} = 0$. Furthermore, $\mathbf{v}_1^{(n+1)} + \mathbf{v}_2^{(n+1)} + \mathbf{v}_3^{(n+1)} + 2\mathbf{v}_1^{(n)} = \mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} + \mathbf{v}_3^{(n)}$, for all n. Hence

$$\frac{\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3}{2} = \sum_{n \geqslant 1} \mathbf{v}_1^{(n)}.$$

Note that the expansion of $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2)$ obtained by applying the fully subtractive algorithm $\mathbf{F}$ to $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2)$ coincides on the first two coordinates with the expansion of $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, that is, $\mathbf{F}^n(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{v}_1^{(n)}, \mathbf{v}_2^{(n)}, \mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)})$ for all $n \geqslant 1$. Hence, here again $\mathbf{v}_1 + \mathbf{v}_2 = \sum_{n \geqslant 1} \mathbf{v}_1^{(n)}$, which implies $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, a contradiction. Hence, the sequence $(i_n)_n$ takes the value 3 at least once, and by repeating the argument indefinitely.

Conversely, assume that $\mathbf{v} \notin \mathcal{F}_3$. If $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} \leqslant \mathbf{v}_3^{(n)}$ for some n, then $\mathbf{v}_1^{(m)} + \mathbf{v}_2^{(m)} \leqslant \mathbf{v}_3^{(m)}$ for all $m \geqslant n$, and in particular, $\mathbf{v}_3^{(m)} - \mathbf{v}_1^{(m)} \geqslant \mathbf{v}_2^{(m)} \geqslant \mathbf{v}_1^{(m)}$. This implies that the sequence $(i_m)_{m \geqslant n}$ will never take the value 3. ⬠

Remark 3.4.6. If $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} < \mathbf{v}_3^{(n)}$ for some n, then $\lim_{n\to\infty} \mathbf{v}^{(n)} \neq \mathbf{0}$. If $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} = \mathbf{v}_3^{(n)}$ for some n, we can say nothing concerning the fact that $\lim_{n\to\infty} \mathbf{v}^{(n)} = \mathbf{0}$. Indeed, take $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_1, 2\mathbf{v}_1)$ for some $\mathbf{v}_1 > 0$. Then $\lim_{n\to\infty} \mathbf{v}^{(n)} = (0, \mathbf{v}_1, \mathbf{v}_1) \neq \mathbf{0}$. Now take $\mathbf{v} = (1/\varphi^2, 1/\varphi, 1)$ with $1/\varphi + 1/\varphi^2 = 1$ and $\varphi > 0$. One checks that $\lim_{n\to\infty} \mathbf{v}^{(n)} = \mathbf{0}$.

By using methods similar as in [**AD13**], we can prove the following proposition.

Proposition 3.4.7. *If* $\mathbf{v} \in \mathcal{F}_3$, *then* $\dim_{\mathbb{Q}}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 3$.

Main result

We can now state the main result of this section, Theorem 3.4.8, which is a characterization of the vectors $\mathbf{v}$ for which the arithmetical discrete plane of thickness $\Omega(\mathbf{v})$ is 2-connected.

Proving the "⇒" direction of Theorem 3.4.8 can be done without too much effort (see the proof below). To prove the converse we need to introduce several ingredients: given a vector $\mathbf{v} \in \mathcal{F}_3$, we will construct two sequences $(\mathbf{P}_n)_{n\in\mathbb{N}}$ and $(\mathbf{T}_n)_{n\in\mathbb{N}}$ of subsetes os $\mathbb{Z}^3$, which play a crucial role in the proof of the theorem. We state and prove the theorem now, but the definitons and properties are given in the next section.

Theorem 3.4.8. *Let* $\mathbf{v} \in \mathcal{O}_3^+$. *The arithmetical discrete plane* $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ *is* 2*-connected if and only if* $\mathbf{v} \in \mathcal{F}_3$.

Proof. Let $\mathbf{v} \in \mathcal{F}_3$ and $\mathbf{x} \in \mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$. Theorem 3.4.3 we have $\Omega(\mathbf{v}) = \|\mathbf{v}\|_1/2$. If $\|\mathbf{v}\|_\infty \leqslant \langle \mathbf{x}, \mathbf{v}\rangle < \|\mathbf{v}\|_1/2$, then $\|\mathbf{v}\|_\infty - \mathbf{v}_1 \leqslant \langle \mathbf{x} - \mathbf{e}_1, \mathbf{v}\rangle < \|\mathbf{v}\|_1/2 - \mathbf{v}_1 < \|\mathbf{v}\|_\infty$, so $\mathbf{x} - \mathbf{e}_1 \in \mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty)$. In other words, an element $\mathbf{x}$ of $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ either belongs to $\mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty)$ or is 2-adjacent to an element of $\mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty)$.

Now, given $\mathbf{y} \in \mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$, both $\mathbf{x}$ and $\mathbf{y}$ belong or are adjacent to $\mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty)$, so they are 2-connected in $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ because:

- $\mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty) \subseteq \cup_{n=0}^\infty \mathbf{T}_n$, thanks to Propositions 3.4.13 and 3.4.14,
- $\cup_{n=0}^\infty \mathbf{T}_n$ is 2-connected: it is a increasing union of sets $\mathbf{T}_n$ which are 2-connected thanks to Proposition 3.4.11,
- $\cup_{n=0}^\infty \mathbf{T}_n \subseteq \mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$, thanks to Proposition 3.4.10.

We now prove the converse implication, and we assume that $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ is 2-connected. Assume $\dim_{\mathbb{Q}}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 1$, $\mathbf{v} \in \mathbb{Z}^3$ with $\gcd\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = 1$. Let $n \in \mathbb{N}$ such that $\mathbf{v}_1^{(n)} = 0$. The plane $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ is 2-connected if and only if so is $\mathfrak{P}(\mathbf{v}^{(n)}, \Omega(\mathbf{v}^{(n)}))$. But $\Omega(\mathbf{v}^{(n)}) = \mathbf{v}_2^{(n)} + \mathbf{v}_3^{(n)} - 1$ so $\mathfrak{P}(\mathbf{v}^{(n)}, \Omega(\mathbf{v}^{(n)}))$ is the translation along $\mathbf{e}_1$ of an arithmetical discrete line strictly thinner than a standard one and cannot be 2-connected. Hence $\dim_{\mathbb{Q}}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} > 1$. If $\mathbf{v} \notin \mathcal{F}_3$, there exists $n \in \mathbb{N}$ such that $\mathbf{v}_1^{(n)} + \mathbf{v}_2^{(n)} \leqslant \mathbf{v}_3^{(n)}$. The plane $\mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$ is 2-connected if and only if so is $\mathfrak{P}(\mathbf{v}^{(n)}, \Omega(\mathbf{v}^{(n)}))$. But $\Omega(\mathbf{v}^{(n)}) = \|\mathbf{v}^{(n)}\|_\infty$, so $\mathfrak{P}(\mathbf{v}^{(n)}, \|\mathbf{v}^{(n)}\|_\infty)$ cannot be 2-connected since $\mathbf{x}$ and $\mathbf{x}+\mathbf{e}_3$ cannot be both in $\mathfrak{P}(\mathbf{v}^{(n)}, \|\mathbf{v}^{(n)}\|_\infty)$. ⬠

Properties of the patterns $\mathbf{P}_n$ and $\mathbf{T}_n$

Let us express the action of $\mathbf{F}$ in terms of the dual substitutions $\Sigma_1^{\mathsf{FS}}, \Sigma_2^{\mathsf{FS}}, \Sigma_3^{\mathsf{FS}}$ defined in Section 2.6.8, similarly as done with the other continued fraction algorithms we have dealt with. We also define, in the same way, the ***F*-expansion** of a vector $\mathbf{v}$.

Definition 3.4.9 (Patterns $\mathbf{T}_n$). Let $\mathbf{v} \in \mathcal{F}_3$ be a vector with $\mathbf{F}$-expansion $(i_n)_{n\in\mathbb{N}}$. Denote by $\mathbf{M}_n$ the matrix ${}^{\mathsf{t}}\mathbf{M}_{\sigma^{\mathsf{FS}}_{i_n}}$, for all $n \geqslant 1$. The sequence $(\mathbf{T}_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{Z}^3$ defined as follows for all $n \geqslant 0$:

$$\mathbf{T}_0 = \{\mathbf{0}\}, \quad \mathbf{T}_1 = \{\mathbf{0}, \mathbf{e}_1\}, \quad \mathbf{T}_{n+1} = \mathbf{T}_n \cup \left(\mathbf{T}_n + {}^{\mathsf{t}}(\mathbf{M}_1 \ldots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1\right).$$

Note that the second initial condition $\mathbf{T}_1 = \{\mathbf{0}, \mathbf{e}_1\}$ is consistent with the usual convention that an empty product of matrices is equal to the identity matrix.

Proposition 3.4.10. *Let $\mathbf{v} \in \mathcal{F}_3$. We have $\cup_{n=0}^{\infty} \mathbf{T}_n \subseteq \mathfrak{P}(\mathbf{v}, \Omega(\mathbf{v}))$.*

Proof. Let us prove that for all $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbf{T}_n$, we have

$$\langle \mathbf{x}, \mathbf{v} \rangle < \sum_{i=0}^{n} \mathbf{v}_1^{(i)}.$$

The case $n \in \{0, 1\}$ can be checked easily. Assume that the inequality holds for some $n \geqslant 1$, and let $\mathbf{x} \in \mathbf{T}_{n+1} = \mathbf{T}_n \cup (\mathbf{T}_n + {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1)$. Then, two cases can occur:

(1) If $\mathbf{x} \in \mathbf{T}_n$ then $\langle \mathbf{x}, \mathbf{v} \rangle < \sum_{i=0}^{n} \mathbf{v}_1^{(i)} < \sum_{i=0}^{n+1} \mathbf{v}_1^{(i)}$.

(2) If $\mathbf{x} \in \mathbf{T}_n + {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1$, then let $\mathbf{y} \in \mathbf{T}_n$ such that $\mathbf{x} = \mathbf{y} + {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1$. We have

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{v} \rangle &= \langle \mathbf{y} + {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1, \mathbf{v} \rangle \\
&= \langle \mathbf{y}, \mathbf{v} \rangle + \langle {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1, \mathbf{v} \rangle \\
&= \langle \mathbf{y}, \mathbf{v} \rangle + \langle {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1, \mathbf{M}_1 \ldots \mathbf{M}_n \cdot \mathbf{v}^{(n)} \rangle \\
&= \langle \mathbf{y}, \mathbf{v} \rangle + \langle \mathbf{e}_1, \mathbf{v}^{(n)} \rangle \\
&= \langle \mathbf{y}, \mathbf{v} \rangle + \mathbf{v}_1^{(n)} < \sum_{0 \leqslant i \leqslant n} \mathbf{v}_1^{(i)} + \mathbf{v}_1^{(n)} = \sum_{0 \leqslant i \leqslant n+1} \mathbf{v}_1^{(i)}.
\end{aligned}$$

⬠

Proposition 3.4.11. *Let $\mathbf{v} \in \mathcal{F}_3$. For all $n \in \mathbb{N}$, the set $\mathbf{T}_n$ is 2-connected.*

Proof. With the same arguments as in proof of Proposition 3.4.10, and by using Proposition 3.4.7, we first get by induction that, for all $n \geqslant 1$:

$$\mathbf{T}_n = \left\{ \mathbf{x} \in \mathbb{Z}^3 : \langle \mathbf{x}, \mathbf{v} \rangle = \sum_{i=0}^{n-1} \varepsilon_i \mathbf{v}_1^{(i)} \text{with } \varepsilon_i \in \{0,1\} \text{ for all } i \right\}.$$

Now, for all $n \in \mathbb{N}$, let $\mathbf{x}_n \in \mathbf{T}_n$ such that $\langle \mathbf{x}_n, \mathbf{v} \rangle = \sum_{i=0}^{n-1} \mathbf{v}_1^{(i)}$ (we set $\mathbf{x}_0 = \mathbf{0}$). Let us prove by induction the following property: for all $n \geqslant 1$ there exists $i_n \in \{1,2,3\}$ such that $\mathbf{x}_n - \mathbf{e}_{i_n} \in \mathbf{T}_{n-1}$. This property implies that $\mathbf{x}_n$ is 2-adjacent to $\mathbf{T}_{n-1}$, which implies the 2-connectedness of $\mathbf{T}_n$.

The induction property is true for $n = 1$ with $\mathbf{x}_1 = \mathbf{e}_1$. Let us now assume that the induction hypothesis holds for $n \geqslant 1$. Let $u_1 \cdots u_n \in \{1,2,3\}^{\mathbb{N}}$ be such that ${}^t\mathbf{M}_{\sigma^{\mathsf{FS}}_{u_1}} \cdots {}^t\mathbf{M}_{\sigma^{\mathsf{FS}}_{u_n}} \mathbf{v}^{(n)} = \mathbf{v}$. We have $\langle \mathbf{x}_{n+1}, \mathbf{v} \rangle = \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_1^{(n)}$, and by definition of the fully subtractive algorithm:

$$\mathbf{v}_1^{(n)} = \begin{cases} \mathbf{v}_1^{(n-1)}, & \text{if } u_n = 1 \\ \mathbf{v}_2^{(n-1)} - \mathbf{v}_1^{(n-1)}, & \text{if } u_n \in \{2,3\}. \end{cases}$$

We conclude the proof by a case distinction of the values taken by $u_1 \cdots u_n$.

Case 1. If $u_n = 1$, then, $\langle \mathbf{x}_{n+1}, \mathbf{v} \rangle = \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_1^{(n-1)}$, and

$$\langle \mathbf{x}_{n+1} - \mathbf{e}_{i_n}, \mathbf{v} \rangle = \langle \underbrace{\mathbf{x}_n - \mathbf{e}_{i_n}}_{\in \mathbf{T}_{n-1}}, \mathbf{v} \rangle + \mathbf{v}_1^{(n-1)} = \sum_{i=1}^{n-2} \varepsilon_i \mathbf{v}_1^{(i)} + \mathbf{v}_1^{(n-1)},$$

where $\varepsilon_i \in \{0,1\}$ for $1 \leqslant i \leqslant n-2$, which implies that $\mathbf{x}_{n+1} - \mathbf{e}_{i_n} \in \mathbf{T}_n$, so taking $i_{n+1} = i_n$ yields the desired result.

Case 2. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = 1^k$, then we have $\mathbf{x}_{n+1} - \mathbf{e}_2 \in \mathbf{T}_n$, because

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} - \mathbf{v}_1^{(n-1)} = \langle \mathbf{x}_{n-1}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} \\ &= \langle \mathbf{x}_{n-2}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-2)} = \cdots = \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1-k)} = \mathbf{v}_2^{(0)}. \end{aligned}$$

Case 3. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = \cdots 21^k$ with $0 \leqslant k \leqslant n-2$, then $\mathbf{x}_{n+1} - \mathbf{e}_{i_{n-1-k}} \in \mathbf{T}_{n-1-k} \subseteq \mathbf{T}_n$, because

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_n, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} - \mathbf{v}_1^{(n-1)} = \langle \mathbf{x}_{n-1}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1)} \\ &= \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1-k)} = \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_1^{(n-2-k)}. \end{aligned}$$

Case 4. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = w31^k$ with $w \in \{1,2\}^\ell$ and $k \geqslant 0$, then so $\mathbf{x}_{n+1} - \mathbf{e}_3 \in \mathbf{T}_n$, because

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_{n-1-k}, \mathbf{v} \rangle + \mathbf{v}_2^{(n-1-k)} = \langle \mathbf{x}_{n-2-k}, \mathbf{v} \rangle + \mathbf{v}_3^{(n-2-k)} \\ &= \langle \mathbf{x}_{n-2-k-\ell}, \mathbf{v} \rangle + \mathbf{v}_3^{(n-2-k-\ell)} = \mathbf{v}_3^{(0)}. \end{aligned}$$

Case 5. If $u_n \in \{2,3\}$ and $u_1 \cdots u_{n-1} = \cdots 3w31^k$ with $w \in \{1,2\}^\ell$, $k \geqslant 0$, then $\mathbf{x}_{n+1} - \mathbf{e}_{i_{n-2-k-\ell}} \in \mathbf{T}_{n-2-k-\ell} \subseteq \mathbf{T}_n$, because

$$\begin{aligned} \langle \mathbf{x}_{n+1}, \mathbf{v} \rangle &= \langle \mathbf{x}_{n-2-k-\ell}, \mathbf{v} \rangle + \mathbf{v}_3^{(n-2-k-\ell)} \\ &= \langle \mathbf{x}_{n-2-k-\ell}, \mathbf{v} \rangle + \mathbf{v}_1^{(n-3-k-\ell)}. \end{aligned}$$

⬠

Definition 3.4.12 (Patterns $\mathbf{P}_n$). Let $\mathbf{v} \in \mathcal{F}_3$ be a vector with $\mathbf{F}$-expansion $(i_n)_{n\in\mathbb{N}}$. We define:

- $P_n = \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}(\mathcal{U})$ for $n \geqslant 1$ and $P_0 = \mathcal{U}$;
- $\mathbf{P}_n = \{\mathbf{x} : [\mathbf{x}, i]^\star \in P_n\}$ for $n \geqslant 0$.

Proposition 3.4.13. *Let* $\mathbf{v} \in \mathcal{F}_3$*. For every* $n \in \mathbb{N}$*, we have* $\mathbf{P}_n \subseteq \mathbf{T}_n$*.*

Proof. We first remark that $\Sigma^{\mathsf{FS}}_i(\mathcal{U}) = \mathcal{U} \cup [\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star =$  for all $i \in \{1, 2, 3\}$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} P_{n+1} &= \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n} \Sigma^{\mathsf{FS}}_{i_{n+1}}(\mathcal{U}) \\ &= \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}(\mathcal{U} \cup [\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star) \\ &= P_n \cup \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}([\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star), \end{aligned}$$

which implies $P_n \subseteq P_{n+1}$. We have $[\mathbf{e}_1, 2]^\star \cup [\mathbf{e}_1, 3]^\star \subseteq \mathbf{e}_1 + \mathcal{U}$, so

$$P_{n+1} \subseteq P_n \cup \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}(\mathbf{e}_1 + \mathcal{U}).$$

By Proposition 1.2.4 (1) and (2), it follows that

$$\begin{aligned} \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}(\mathbf{e}_1 + \mathcal{U}) &= \mathbf{M}^{-1}_{\sigma^{\mathsf{FS}}_{i_n}} \cdots \mathbf{M}^{-1}_{\sigma^{\mathsf{FS}}_{i_1}} \cdot \mathbf{e}_1 + \Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}(\mathcal{U}) \\ &= ({}^{\mathsf{t}}\mathbf{M}_n \cdots {}^{\mathsf{t}}\mathbf{M}_1)^{-1} \cdot \mathbf{e}_1 + P_n \\ &= {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1 + P_n, \end{aligned}$$

where $\mathbf{M}_n = {}^{\mathsf{t}}\mathbf{M}_{\sigma^{\mathsf{FS}}_{i_n}}$, which proves that $P_n \subseteq P_{n+1} \subseteq P_n \cup (P_n + {}^{\mathsf{t}}(\mathbf{M}_1 \cdots \mathbf{M}_n)^{-1} \cdot \mathbf{e}_1)$. The result now follows by induction. ⬠

Ultimately, we prove the following proposition thanks to the discrete plane generation properties of Chapter 2.

Proposition 3.4.14. *If* $\mathbf{v} \in \mathcal{F}_3$*, then* $\bigcup_{n=0}^{\infty} \mathbf{P}_n = \mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty)$*.*

Proof. Let $\mathbf{v} \in \mathcal{F}_3$ and let $(i_n)_{n \in \mathbb{N}}$ be its $\mathbf{F}$-expansion. By definition of the patterns $\mathbf{P}_n$, and thanks to the fact that $\{\mathbf{x} : [\mathbf{x}, i]^\star \in \Gamma_{\mathbf{v}}\} = \mathfrak{P}(\mathbf{v}, \|\mathbf{v}\|_\infty)$, it is enough to prove that the patterns $\Sigma^{\mathsf{FS}}_{i_1} \cdots \Sigma^{\mathsf{FS}}_{i_n}(\mathcal{U})$ have arbitrarily large combinatorial radius as $n \to \infty$.

This can be proved in exactly the same way as for the other families of substitutions we considered in Chapter 2, so the proof follows the lines of the reasonning of the proof of Theorem 2.5.1, by using Lemma 2.6.22 (to generate the first annulus), together with Proposition 2.3.5 (the annulus property), Proposition 2.6.20 (strong covering conditions), and Proposition 2.6.23 with Lemma 2.6.21 (Property A).

Note that since $\mathbf{v} \in \mathcal{F}_3$, Lemma 3.4.5 implies that $i_n = 3$ infinitely many often, a fact which was implicitely above in the statement of Lemma 2.6.22. ⬠

Chapter 4

Rauzy fractals with countable fundamental group

All the currently known examples of Rauzy fractals seem to suggest that the fundamental group of a Rauzy fractal is always either trivial or uncountable. In this chapter we show that such a "dichotomy" situation doesn't hold. We prove that *every* free group of finite rank can be realized as the fundamental group of the (planar) Rauzy fractal of a 4-letter unimodular cubic Pisot substitution. Our construction relies on two symbolic operations on substitutions: symbolic splitting and conjugacy by free group automorphisms. This is joint work with Benoît Loridant and Jun Luo [**JLL13**].

Contents

4.1 Introduction

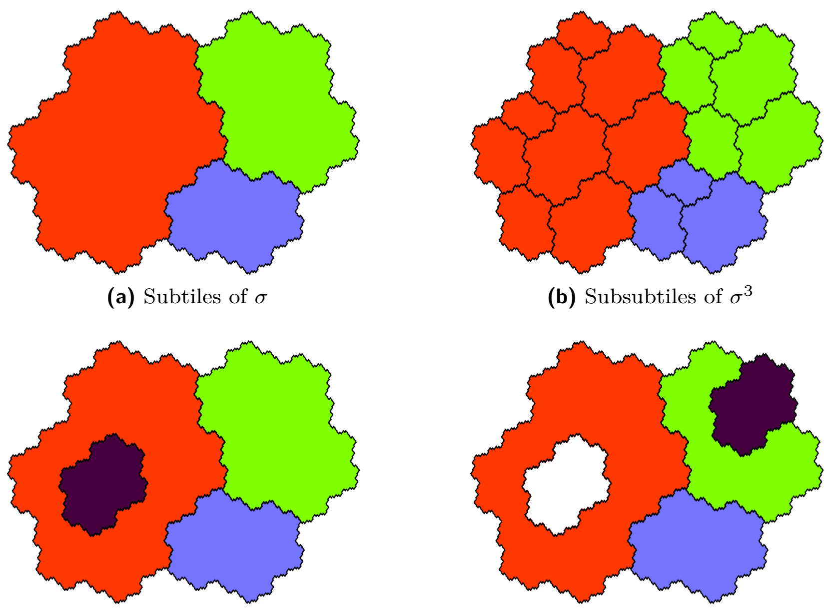
We have seen in Section 1.3 that Rauzy fractal topology is very rich and diverse. There are many explicit examples of fractals which are homeomorphic to a disc [**Mes98, Mes06**], or whose fundamental group is uncountable [**ST09**]. However, there is no known example of an "intermediate" constellation, where the fundamental group would be nontrivial, but countable. There are also examples of infinite families where the fundamental group jumps from trivial to uncountable when modifying parameters [**LMST13**].

In this chapter we will prove that such an intermediate situation is possible. We will first prove in Proposition 4.2.5 that if the fundamental group of a Rauzy fractal is countable, then it must be isomorphic to the free group F_k of rank k for some finite integer k. We will then give a method to construct, for any $k \in \mathbb{N}$, a 4-letter unimodular cubic Pisot substitution whose Rauzy fractal fundamental group is homeomorphic to the free group F_k of rank k (Theorem 4.5.3).

Our methods are based on symbolic operations on substitutions that induce manipulation of the subtiles of their Rauzy fractals, namely *symbol splittings* and *conjugation by free group* *automorphisms*. Questions about the effect of conjugation by free group automorphisms on Rauzy fractals have been raised in [**Gäh10**] and [**ABHS06**]. A consequence of this work is that the fundamental group of the Rauzy fractal of a substitution σ is not preserved by conjugating σ by free group automorphisms.

Strategy Rauzy fractals can naturally be decomposed into *subtiles* and *subsubtiles* (see Definitions 1.3.2 and 4.2.2). We will manipulate these tiles within a fractal in order to obtain the desired topological properties. This will be done using tools consisting of two symbolic operations on substitutions, *state splittings* and *conjugacy by free group automorphisms*. The strategy is more precisely described in Figure 4.1 below:

(a) Start with a 3-letter substitution σ whose Rauzy fractal and its subtiles are disklike.
(b) Take n large enough such that the subtiles of σ^n consist of sufficiently many subsubtiles for the next two steps to be applicable (Proposition 4.2.7).
(c) Split a symbol to isolate one subsubtile and turn it into a subtile of the Rauzy fractal of a new substitution τ on four symbols (Proposition 4.3.2).
(d) Conjugate τ by a suitable free group automorphism ρ. The Rauzy fractal associated with $\rho\tau\rho^{-1}$ now has a hole (Proposition 4.4.1).



(a) Subtiles of σ

(b) Subsubtiles of σ^3

(c) Subtiles after splitting a symbol in σ^3

(d) Subtiles after conjugating by a free group automorphism

Figure 4.1: The main strategy.

4.2 Basic properties and definitions

We will need several definitions introduced in Section 1.3, including Rauzy fractals and their subtiles (Definition 1.3.2), and the projection and renormalization maps π_σ and $\mathbf{h}_\sigma$ associated with a substitution σ.

We insist again on the point already made in Remark 1.3.3, that the norm of the chosen eigenvector $\mathbf{v}_\beta$ used to define π_σ (and hence to define Rauzy fractals) affects the fractal tiles up to an affine mapping. This fact will be used in Propositions 4.3.2 and 4.4.1, by carefully choosing the eigenvectors to relate some Rauzy fractals coming from substitutions which do not have the same number of letters, but have the same Pisot eigenvalue. The *a priori* difficulty that such fractals "do not live in the same space" and cannot be compared is solved, because we have chosen the representation space $\mathbb{R}^r \times \mathbb{C}^s$ for Rauzy fractals, which only depends on the degree of β and not on the number of letters of the substitution.

If $w \in \mathcal{A}^\star$ is a word, we denote by w_i the i-th symbol of w. A pair $(j,k) \in \mathcal{A} \times \mathbb{N}$ is an ***occurrence*** of the symbol i in σ if $\sigma(j)_k = i$, that is if the k-th letter of $\sigma(j)$ is i. We will denote occurrences by $(j;k)$ to emphasize the fact that j is an element of $\mathcal{A}$ and k is an index. The ***set of occurrences*** of i in σ is denoted by $\operatorname{occ}(\sigma, i)$.

Example 4.2.1. Let $\sigma : 1 \mapsto 11213, 2 \mapsto 331, 3 \mapsto 1$. We have $\operatorname{occ}(\sigma,1) = \{(1;1),\ (1;2),\ (1;4),\ (2;3),\ (3;1)\}$, $\operatorname{occ}(\sigma,2) = \{(1;3)\}$ and $\operatorname{occ}(\sigma,3) = \{(1;5),\ (2;1),\ (2;2)\}$.

It is possible to decompose Rauzy fractals one step further than the subtiles $\mathcal{T}_\sigma(i)$: each subtile $\mathcal{T}_\sigma(i)$ can be decomposed into its ***subsubtiles*** $\mathcal{T}_\sigma(i,j;k)$, defined below.

Definition 4.2.2. Let $(j;k) \in \operatorname{occ}(\sigma, i)$. The ***subsubtile*** $\mathcal{T}_\sigma(i,j;k)$ is defined by

$$\mathcal{T}_\sigma(i,j;k) = \mathbf{h}_\sigma(\mathcal{T}_\sigma(j)) + \pi_\sigma \mathbf{P}(\sigma(j)_1 \cdots \sigma(j)_{k-1}).$$

Note that $\mathcal{T}_\sigma(i,j;k)$ is defined only if $(j;k) \in \operatorname{occ}(\sigma, i)$. Sirvent and Wang [**SW02**] have proved that the tiles $\mathcal{T}_\sigma(i)$ are the solution of a graph-directed iterated function system, which can be conveniently expressed in terms of subsubtiles and symbol occurrences in the following theorem. Proofs of this result can also be found in [**ST09**] or [**BR10**].

Theorem 4.2.3. *Let σ be primitive unimodular Pisot substitution on alphabet $\mathcal{A}$. For every $i \in \mathcal{A}$ we have*

$$\mathcal{T}_\sigma(i) = \bigcup_{(j;k)\in\operatorname{occ}(\sigma,i)} \mathcal{T}_\sigma(i,j;k).$$

Example 4.2.4. Let $\sigma : 1 \mapsto 21, 2 \mapsto 31, 3 \mapsto 1$. The subsubtiles of $\sigma^3 : 1 \mapsto 1213121, 2 \mapsto 213121, 3 \mapsto 3121$ are plotted in Figure 4.1 (b). The 9 subsubtiles of $\mathcal{T}_\sigma(1)$ correspond to the 9 occurrences of 1 in σ^3; the 5 subsubtiles of $\mathcal{T}_\sigma(2)$ correspond to the 5 occurrences of 2 in σ^3; the 3 subsubtiles of $\mathcal{T}_\sigma(3)$ correspond to the 3 occurrences of 3 in σ^3.

We now prove that free groups of finite rank are the only possible countable fundamental groups of Rauzy fractals. Let us recall the following basics in topology [**WD79**]. A topological space X is a ***continuum*** if it is compact and connected. It is ***locally connected*** if it has a base of connected sets. A ***path from x to y*** in X is a continuous function $f : [0,1] \to X$ with $f(x) = 0, f(y) = 1$. X is ***path-connected*** if every two points of X are joined by a path, and ***locally path-connected*** if it has a base of path-connected sets. It follows from the theorem of Hahn-Mazurkiewicz that any locally connected continuum is path-connected. Moreover, in metric spaces, every locally connected continuum is locally path-connected by results of Mazurkiewicz, Moore and Menger (see [**Kur68**, Section 50, Chapter II, p.254]).

Proposition 4.2.5. *Let σ be a primitive unimodular Pisot substitution and let $\mathcal{T}_\sigma$ be its Rauzy fractal. Suppose that $\mathcal{T}_\sigma$ and its subtiles are planar locally connected continua. If the fundamental group of $\mathcal{T}_\sigma$ is countable, then it is isomorphic to the free group F_k on k generators for some finite rank k.*

Proof. This result can be proved in two different ways. A theorem of Shelah [**She88**] states that for every compact, path-connected and locally path-connected metric space K, if the fundamental group of K is not finitely generated then it is uncountable, so the result follows directly by contraposition.

Another result by Conner and Lamoreaux [**CL05**] states that if K is a connected, locally path-connected set of the plane, then its fundamental group is not free if and only if it is uncountable, which also directly implies the result. ⬠

Remark 4.2.6. For a given primitive unimodular Pisot substitution, a sufficient condition for the associated Rauzy fractal and its subtiles to be locally connected continua can be found in [**ST09**, Theorem 4.13]. This condition can be checked algorithmically.

Proposition 4.2.7. *Let $p \geqslant 1$. Let σ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A}$ with dominant cubic Pisot eigenvalue, such that $\mathcal{T}_\sigma$ and its subtiles are homeomorphic to a closed disc. Suppose that there exist $a, c \in \mathcal{A}$ such that for every $j \in \{1,2,3\}$, there exists $n \in \mathbb{N}$ such that $\sigma(j)$ contains at least one occurrence of the word ca. Then there exists $N \geqslant 1$ and $I \subseteq \operatorname{occ}(\sigma^N, a)$ such that*

$$\bigcup_{(j;k)\in\operatorname{occ}(\sigma^N,a)\setminus I} \mathcal{T}_{\sigma^N}(a;j;k)$$

is homeomorphic to a closed disc minus the union of p disjoint open discs that do not intersect the boundary of $\mathcal{T}_\sigma(a)$.

Proof sketch. This result can be proved thanks to the fact that iterating σ produces tiles with smaller and smaller diameter (because they are scaled down by $\mathbf{h}_\sigma$ at each step, which is a contraction), and thanks to primitivity of σ together with the hypothesis on the occurrences of the words ca in the images of each letter. ⬠

4.3 Symbol splittings

We define a symbolic operation, *symbol splitting*, that we use in Proposition 4.3.2 in order to select a subsubtile and assign it to a new symbol.

Definition 4.3.1. Let σ be a substitution on alphabet $\mathcal{A}$, let $a \in \mathcal{A}$, let $b \notin \mathcal{A}$ be a new symbol and let $I \subseteq \operatorname{occ}(\sigma, a)$. The ***splitting*** of symbol a to the new symbol b with occurrences I is the substitution τ defined by

$$\tau(i) = \begin{cases} \text{the word } \sigma(i) \text{ in which } \sigma(i)_k \text{ is replaced by } b \text{ for every } (i;k) \in I, \text{ if } i \neq b, \\ \tau(a), \text{ if } i = b. \end{cases}$$

Note that if σ is a primitive unimodular Pisot substitution, then so also is any splitting τ arising from σ, provided that $\mathrm{occ}(a,\tau) \neq \varnothing$, which we will always assume to be the case in the following. Moreover, we have $\chi_\tau(x) = x \cdot \chi_\sigma(x)$, where χ_σ and χ_τ are the characteristic polynomials of $\mathbf{M}_\sigma$ and $\mathbf{M}_\tau$, respectively. The action of state splittings on the Rauzy fractal of a substitution σ is described in the next proposition.

Proposition 4.3.2. *Let*

- *σ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A} = \{1, \dots, n\}$,*
- *τ be obtained by splitting of σ from symbol a to a new symbol $b = n+1$ with occurrences $I \subseteq \mathrm{occ}(\sigma, a)$,*
- *$\mathbf{v}_\beta = (v_1, \dots, v_n) \in \mathbb{R}^n$ be a left β-eigenvector of $\mathbf{M}_\sigma$,*
- *$\mathbf{w}_\beta = (v_1, \dots, v_n, v_a) \in \mathbb{R}^{n+1}$ (which is a β-eigenvector of $\mathbf{M}_\tau$, see Lemma 4.3.4),*
- *$\mathcal{T}_\sigma$ be the Rauzy fractal of σ (with respect to the β-eigenvector $\mathbf{v}_\beta$),*
- *$\mathcal{T}_\tau$ be the Rauzy fractal of τ (with respect to the β-eigenvector $\mathbf{w}_\beta$).*

We have

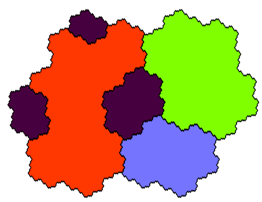
(1) $\mathcal{T}_\tau(i) = \mathcal{T}_\sigma(i)$ *if* $i \notin \{a, b\}$,

(2) $$\mathcal{T}_\tau(a) = \bigcup_{(j;k)\in \mathrm{occ}(\sigma,a)\setminus I} \mathcal{T}_\sigma(a, j; k),$$

(3) $$\mathcal{T}_\tau(b) = \bigcup_{(j;k)\in I} \mathcal{T}_\sigma(a, j; k).$$

Example 4.3.3. Let $\sigma : 1 \mapsto 1213121, 2 \mapsto 213121, 3 \mapsto 3121$. We split the symbol $a = 1$ to the new symbol $b = 4$ with occurrences $I = \{(1;1), (2;6), (3;2)\}$ of 1 in σ. The resulting substitution τ and its Rauzy fractal are shown below. (The tiles associated with 4 are shown in dark.)

$$\tau : \begin{cases} 1 \mapsto 4213121 \\ 2 \mapsto 213124 \\ 3 \mapsto 3421 \\ 4 \mapsto 4213121 \end{cases}$$



In order to prove Proposition 4.3.2 we need Lemma 4.3.4 and Lemma 4.3.5 below.

Lemma 4.3.4. *Under the hypotheses of Proposition 4.3.2, β is an eigenvalue of $\mathbf{M}_\tau$ and $\mathbf{w}_\beta$ a left β-eigenvector of $\mathbf{M}_\tau$. Hence, the Rauzy fractal $\mathcal{T}_\tau$ mentioned in the statement of Proposition 4.3.2 is well defined.*

Proof. Denote by $C_1, \dots, C_n$ the columns of $\mathbf{M}_\sigma$ and by $C'_1, \dots, C'_{n+1}$ the columns of $\mathbf{M}_\tau$. By definition of symbol splittings we have

- $C'_{i,j} = C_{i,j}$ for all $i \neq n+1$ and $j \notin \{a, n+1\}$,
- $C'_{n+1,j} = C_{a,j}$ for all $j \notin \{a, n+1\}$,
- $C'_{i,a} + C'_{i,n+1} = C_{i,a}$ for all $i \neq n+1$.

Hence, by definition of $\mathbf{w}_\beta$ we have $(\mathbf{w}_\beta \mathbf{M}_\tau)_i = (\mathbf{v}_\beta \mathbf{M}_\sigma)_i$ if $i \neq n+1$ and $(\mathbf{w}_\beta \mathbf{M}_\tau)_{n+1} = (\mathbf{v}_\beta \mathbf{M}_\sigma)_a$, so $\mathbf{w}_\beta \mathbf{M}_\tau = \beta \mathbf{w}_\beta$, which proves the lemma. ⬠

Lemma 4.3.5. *Under the hypotheses of Proposition 4.3.2, let* $i \in \{1, \dots, n, b = n+1\}$ *and* $(j; k) \in \operatorname{occ}(\tau, i)$*, and let* $i' = a$ *if* $i = b$ *and* $i' = i$ *otherwise. We have* $\mathcal{T}_\tau(i, j; k) = \mathcal{T}_\sigma(i', j; k)$ *if* $j \notin \{a, b\}$*, and* $\mathcal{T}_\tau(i, a; k) \cup \mathcal{T}_\tau(i, b; k) = \mathcal{T}_\sigma(i', a; k)$*.*

Proof. First, note that $\mathbf{h}_\sigma = \mathbf{h}_\tau$ by definition, because these two maps both depend on the same eigenvalue β. Also, by Lemma 4.3.4, for $j = 1, \dots, r+s$, β_j is an eigenvalue and $\mathbf{w}_{\beta_j}$ is a left β_j-eigenvector of $\mathbf{M}_\tau$, where $\mathbf{w}_{\beta_j}$ is obtained by replacing β by β_j in the coordinates of $\mathbf{w}_\beta$. It follows that $\pi_\tau \mathbf{P}(\tau(i)) = \pi_\sigma \mathbf{P}(\sigma(i))$ for all $i \neq b$ and $\pi_\tau \mathbf{P}(\tau(b)) = \pi_\sigma \mathbf{P}(\sigma(a))$. We will use these facts later in the proof.

Next, we claim that $\mathcal{T}_\tau(i) = \mathcal{T}_\sigma(i)$ if $i \notin \{a, b\}$ and $\mathcal{T}_\tau(a) \cup \mathcal{T}_\tau(b) = \mathcal{T}_\sigma(a)$. Indeed, let u be a periodic point of τ, and let u' be defined by $u'_m = a$ if $u_m = b$ and $u'_m = u_m$ otherwise. Then it is easy to check that u' is a periodic point of σ, and that $\pi_\tau \mathbf{P}(u_1 \cdots u_m) = \pi_\sigma \mathbf{P}(u'_1 \cdots u'_m)$ for all $m \geqslant 1$, so our claim follows from Definition 1.3.2 of Rauzy fractals. Finally, we have

$$\begin{aligned}
\mathcal{T}_\tau(i, j; k) &= \mathbf{h}_\tau \mathcal{T}_\tau(j) + \pi_\tau \mathbf{P}(\tau(j)_1 \cdots \tau(j)_{k-1}) \\
&= \mathbf{h}_\sigma \mathcal{T}_\sigma(j) + \pi_\sigma \mathbf{P}(\sigma(j)_1 \cdots \sigma(j)_{k-1}) \\
&= \mathcal{T}_\sigma(i', j; k) \text{ for } j \notin \{a, b\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_\tau(i, a; k) \cup \mathcal{T}_\tau(i, b; k) &= \mathbf{h}_\tau(\mathcal{T}_\tau(a) \cup \mathcal{T}_\tau(b)) + \pi_\tau \mathbf{P}(\tau(a)_1 \cdots \tau(a)_{k-1}) \\
&= \mathbf{h}_\tau \mathcal{T}_\sigma(a) + \pi_\tau \mathbf{P}(\tau(a)_1 \cdots \tau(a)_{k-1}) \\
&= \mathbf{h}_\sigma \mathcal{T}_\sigma(a) + \pi_\sigma \mathbf{P}(\sigma(a)_1 \cdots \sigma(a)_{k-1}) \\
&= \mathcal{T}_\sigma(i', a; k),
\end{aligned}$$

which proves the lemma. ⬠

Proof of Proposition 4.3.2. Let $(j; k) \in \operatorname{occ}(\tau, i)$, and let $i' = a$ if $i = b$ and $i' = i$ otherwise. We have

$$\begin{aligned}
\mathcal{T}_\tau(i) &= \bigcup_{(j;k) \in \operatorname{occ}(\tau, i)} \mathcal{T}_\tau(i, j; k) \qquad \text{by Theorem 4.2.3} \\
&= \bigcup_{\substack{(j;k) \in \operatorname{occ}(\tau, i) \\ j \notin \{a,b\}}} \mathcal{T}_\tau(i, j; k) \ \cup \bigcup_{(a;k) \in \operatorname{occ}(\tau, i)} \mathcal{T}_\tau(i, a; k) \ \cup \bigcup_{(b;k) \in \operatorname{occ}(\tau, i)} \mathcal{T}_\tau(i, b; k) \\
&= \bigcup_{\substack{(j;k) \in \operatorname{occ}(\tau, i) \\ j \notin \{a,b\}}} \mathcal{T}_\tau(i, j; k) \ \cup \bigcup_{(a;k) \in \operatorname{occ}(\tau, i)} \mathcal{T}_\tau(i, a; k) \cup \mathcal{T}_\tau(i, b; k) \\
&= \bigcup_{\substack{(j;k) \in \operatorname{occ}(\tau, i) \\ j \notin \{a,b\}}} \mathcal{T}_\sigma(i', j; k) \ \cup \bigcup_{(a;k) \in \operatorname{occ}(\tau, i)} \mathcal{T}_\sigma(i', a; k) \qquad \text{by Lemma 4.3.5} \\
&= \bigcup_{\substack{(j;k) \in \operatorname{occ}(\tau, i) \\ j \neq b}} \mathcal{T}_\sigma(i', j; k).
\end{aligned}$$

The third line of the above equation follows from the second line because $(a; k) \in \operatorname{occ}(\tau, i)$ if and only if $(b; k) \in \operatorname{occ}(\tau, i)$, by definition of symbol splittings. Statements (1), (2), and (3) can now be proved by combining the above equality and the fact that the condition "$(j;k) \in \mathrm{occ}(\tau, i)$ and $j \neq b$" is equivalent to

- $(j;k) \in \mathrm{occ}(\sigma, i)$ if $i \notin \{a, b\}$, which proves (1);
- $(j;k) \in \mathrm{occ}(\sigma, a) \setminus I$ if $i = a$, which proves (2);
- $(j;k) \in I$ if $i = b$, which proves (3).

Note that Proposition 4.3.2 (1) was already proved in the proof of Lemma 4.3.5. ⬠

4.4 Conjugacy by free group automorphisms

In this section we describe the action of a particular family of free group automorphisms on the Rauzy fractal of a substitution in Proposition 4.4.1, which will be used to prove our main result, Theorem 4.5.3.

A ***free group morphism*** on the alphabet $\mathcal{A}$ is a non-erasing morphism of the free group generated by $\mathcal{A}$, consisting of the finite words made of symbols a and a^{-1} for $a \in \mathcal{A}$. Substitutions can be seen as a particular case of free group automorphisms, where no "$^{-1}$" appears in the image of each letter. The ***inverse*** of a free group automorphism ρ is the unique morphism (denoted by ρ^{-1}) such that $\rho\rho^{-1} = \rho^{-1}\rho$ is the identity. For example, the inverse of $\rho : 1 \mapsto 1, 2 \mapsto 211$, is $\rho^{-1} : 1 \mapsto 1, 2 \mapsto 21^{-1}1^{-1}$. The fundamental operation we will perform on a substitution σ is ***conjugacy by a free group automorphism*** ρ, *i.e.*, the product $\rho^{-1}\sigma\rho$ where ρ is an automorphism. In the specific cases that we will consider, σ and $\rho^{-1}\sigma\rho$ will always both be substitutions (*i.e.*, contain no "$^{-1}$"). The particular family of free group automorphisms we will need consists of the mappings ρ_{ij} defined below.

$$\rho_{ij}(k) = \begin{cases} ij & \text{if } k = j \\ k & \text{if } k \neq j \end{cases} \qquad \rho_{ij}^{-1}(k) = \begin{cases} i^{-1}j & \text{if } k = j \\ k & \text{if } k \neq j \end{cases}$$

The next proposition describes how the Rauzy fractal of a substitution is affected when it is conjugated by a free group automorphism ρ_{ij}.

Proposition 4.4.1. *Let*

- *τ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A}$,*
- *$b \in \mathcal{A}$ be such that there exists a unique $c \in \mathcal{A}$ such that for every $(j;k) \in \mathrm{occ}(\tau, b)$, we have $k \geqslant 2$ and $\tau(j)_{k-1} = c$,*
- *$\theta = \rho_{cb}^{-1}\tau\rho_{cb}$,*
- *$\mathbf{w}_\beta \in \mathbb{R}^{n+1}$ be a left β-eigenvector of $\mathbf{M}_\tau$,*
- *$\mathbf{z}_\beta = \mathbf{w}_\beta \mathbf{M}_{\rho_{cb}} \in \mathbb{R}^{n+1}$ (which is a β-eigenvector of $\mathbf{M}_\theta$),*
- *$\mathcal{T}_\tau$ be the Rauzy fractal of τ (with respect to β-eigenvector $\mathbf{w}_\beta$),*
- *$\mathcal{T}_\theta$ be the Rauzy fractal of θ (with respect to β-eigenvector $\mathbf{z}_\beta$).*

We have

(1) $\mathcal{T}_\theta(i) = \mathcal{T}_\tau(i)$ if $i \notin \{b, c\}$,
(2) $\mathcal{T}_\theta(b) \cup \mathcal{T}_\theta(c) = \mathcal{T}_\tau(c)$,

and in particular $\mathcal{T}_\theta = \bigcup_{i \neq b} \mathcal{T}_\tau(i)$.

Proof. We first check that $\mathbf{z}_\beta$ is a left β-eigenvector of $\mathbf{M}_\theta$.

$$\mathbf{z}_\beta \mathbf{M}_\theta = \mathbf{w}_\beta \mathbf{M}_{\rho_{cb}} \mathbf{M}_\theta = \mathbf{w}_\beta \mathbf{M}_{\rho_{cb}} \mathbf{M}_{\rho_{cb}}^{-1} \mathbf{M}_\tau \mathbf{M}_{\rho_{cb}} = \mathbf{w}_\beta \mathbf{M}_\tau \mathbf{M}_{\rho_{cb}} = \beta \mathbf{w}_\beta \mathbf{M}_{\rho_{cb}} = \beta \mathbf{z}_\beta.$$

Hence, the Rauzy fractal $\mathcal{T}_\theta$ mentioned in the statement of the Proposition is well defined. Let u be a periodic point of θ (*i.e.*, there exists $k \geqslant 1$ such that $\theta^k(u) = u$), and let $v = \rho_{cb}(u)$. It is easy to check that v is a periodic point of τ:

$$\tau^k(\rho_{cb}(u)) = (\rho_{cb}\theta\rho_{cb}^{-1})^k \rho_{cb}(u) = \rho_{cb}\theta^k \rho_{cb}^{-1} \rho_{cb}(u) = \rho_{cb}\theta^k(u) = \rho_{cb}(u).$$

Let $\ell : \mathbb{N} \to \mathbb{N}$ be the unique function defined by induction as follows for $m \geqslant 2$:

$$\ell(1) = \begin{cases} 1 \text{ if } u_1 \neq b \\ 2 \text{ if } u_1 = b, \end{cases} \qquad \ell(m) = \begin{cases} \ell(m-1) + 1 \text{ if } u_m \neq b \\ \ell(m-1) + 2 \text{ if } u_m = b. \end{cases}$$

In particular, we have $u_m = v_{\ell(m)}$ for all $m \in \mathbb{N}$. The definition of ℓ is illustrated below, on an example where $u_3 = v_4 = b$, $v_3 = c$ and $u_1, u_2, u_4, v_1, v_2, v_5 \notin \{b, c\}$.

$$\begin{array}{rclllll} u & = & u_1 & u_2 & & u_3 = b & u_4 & \cdots \\ \rho_{cb}(u) = v & = & v_1 & v_2 & v_3 = c & v_4 = b & v_5 & \cdots \\ & = & v_{\ell(1)} & v_{\ell(2)} & v_{\ell(2)+1} & v_{\ell(3)} & v_{\ell(4)} & \cdots \end{array}$$

The equality $\mathbf{z}_\beta = \mathbf{w}_\beta \mathbf{M}_{\rho_{cb}}$ implies that $\langle \mathbf{z}_\beta, e_i \rangle = \langle \mathbf{w}_\beta, e_i \rangle$ if $i \neq b$ and $\langle \mathbf{z}_\beta, e_b \rangle = \langle \mathbf{w}_\beta, e_c \rangle + \langle \mathbf{w}_\beta, e_b \rangle$. Hence, by definition of π_τ and π_θ we have $\pi_\theta \mathbf{P}(i) = \pi_\tau \mathbf{P}(i)$ if $i \neq b$ and $\pi_\theta \mathbf{P}(b) = \pi_\tau \mathbf{P}(c) + \pi_\tau \mathbf{P}(b)$. It follows that for all $m \in \mathbb{N}$,

$$\begin{aligned} \pi_\theta \mathbf{P}(u_1 \cdots u_m) &= \sum_{\substack{1 \leqslant k \leqslant m \\ u_k \neq b}} \pi_\theta \mathbf{P}(u_k) + \sum_{\substack{1 \leqslant k \leqslant m \\ u_k = b}} \pi_\theta \mathbf{P}(b) \\ &= \sum_{\substack{1 \leqslant k \leqslant m \\ u_k \neq b}} \pi_\tau \mathbf{P}(v_{\ell(k)}) + \sum_{\substack{1 \leqslant k \leqslant m \\ u_k = b}} (\pi_\tau \mathbf{P}(c) + \pi_\tau \mathbf{P}(b)) \\ &= \sum_{\substack{1 \leqslant k \leqslant m \\ u_k \neq b}} \pi_\tau \mathbf{P}(v_{\ell(k)}) + \sum_{\substack{1 \leqslant k \leqslant m \\ u_k = b}} \pi_\tau \mathbf{P}(v_{\ell(k)-1} v_{\ell(k)}) \\ &= \pi_\tau \mathbf{P}(v_{\ell(1)} \cdots v_{\ell(m)}). \end{aligned}$$

Finally, it is easy to verify that for all $i \notin \{b, c\}$, ℓ is a bijection between $\{m \in \mathbb{N} : u_{m+1} = i\}$ and $\{m \in \mathbb{N} : v_{m+1} = i\}$, so

$$\begin{aligned} \mathcal{T}_\theta(i) &= \{\pi_\theta \mathbf{P}(u_1 \cdots u_m) : u_{m+1} = i\} \\ &= \{\pi_\tau \mathbf{P}(v_1 \cdots v_{\ell(m)}) : v_{\ell(m)+1} = i\} \\ &= \mathcal{T}_\tau(i), \end{aligned}$$

and ℓ is also a bijection between $\{m \in \mathbb{N} : u_{m+1} = b \text{ or } c\}$ and $\{m \in \mathbb{N} : v_{m+1} = c\}$, so

$$\begin{aligned} \mathcal{T}_\theta(b) \cup \mathcal{T}_\theta(c) &= \{\pi_\theta \mathbf{P}(u_1 \cdots u_m) : u_{m+1} = b \text{ or } c\} \\ &= \{\pi_\tau \mathbf{P}(v_1 \cdots v_{\ell(m)}) : v_{\ell(m)+1} = c\} \\ &= \mathcal{T}_\tau(c), \end{aligned}$$

so the proposition is proved. ⬠

Remark 4.4.2. Statement (2) of Proposition 4.4.1, stating that $\mathcal{T}_\theta(b) \cup \mathcal{T}_\theta(c) = \mathcal{T}_\tau(c)$ will be sufficient for our purposes in Proposition 4.4.1. By using similar arguments as in the proof of Proposition 4.3.2, we could have been more precise and give a subsubtile decomposition of the subtiles $\mathcal{T}_\theta(b)$ and $\mathcal{T}_\theta(c)$ as follows:

$$\mathcal{T}_\theta(b) = \bigcup_{(j;k)\in I} \mathcal{T}_\tau(c, j; k-1), \qquad \mathcal{T}_\theta(c) = \bigcup_{\substack{(j;k)\in \mathrm{occ}(\tau,c)\\ (j;k+1)\notin I}} \mathcal{T}_\tau(c, j; k).$$

Example 4.4.3. Let $\sigma : 1 \mapsto 21, 2 \mapsto 31, 3 \mapsto 1$. First we split $\sigma^3 : 1 \mapsto 1213121, 2 \mapsto 213121, 3 \mapsto 3121$ from $a = 1$ to $b = 4$ with occurrences $I = \{(1; 7)\}$ to obtain $\tau : 1 \mapsto 1213124, 2 \mapsto 213121, 3 \mapsto 3121, 4 \mapsto 1213124$. Then we conjugate τ with $\rho_{24} : 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 24$:

$$\rho_{24}^{-1}\tau\rho_{24} : \begin{cases} 1 \mapsto 1 \mapsto 1213124 \mapsto 121314 \\ 2 \mapsto 2 \mapsto 213121 \mapsto 213121 \\ 3 \mapsto 3 \mapsto 3121 \mapsto 3121 \\ 4 \mapsto 24 \mapsto 2131211213124 \mapsto 213121121314 \end{cases}$$

The effect of these operations on the Rauzy fractals are shown in Figure 4.1 and in Figure 4.2.

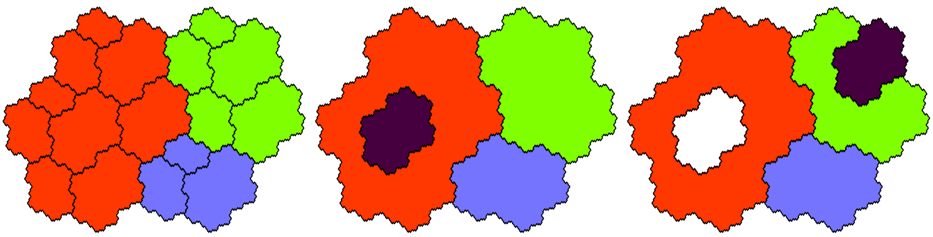


Figure 4.2: Rauzy fractals of the substitutions defined in Example 4.4.3. The subsubtiles of $\mathcal{T}_{\sigma^3}$ (left), the subtiles of $\mathcal{T}_\tau$ (center), and the subtiles of $\mathcal{T}_\theta$ (right).

Example 4.4.4. Let $\sigma : 1 \mapsto 21, 2 \mapsto 31, 3 \mapsto 1$. Let τ be the splitting of σ^6 from 1 to 4 with occurrences $I = \{(1; 24); (1; 31); (1; 33); (1; 40)\}$. (Note that $\sigma(1)_{k-1} = 2$ for all $(1, k) \in I$.) Let $\theta = \rho_{24}^{-1}\tau\rho_{24}$ with $\rho_{24} : 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 24$. The effect of these operations on the Rauzy fractal are shown in Figure 4.3.

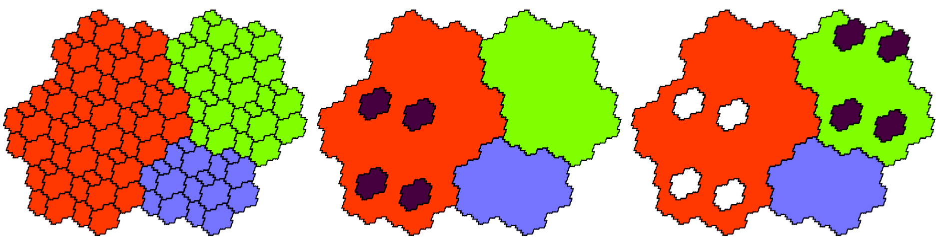


Figure 4.3: Rauzy fractals of the substitutions defined in Example 4.4.4. The subsubtiles of $\mathcal{T}_{\sigma^6}$ (left), the subtiles of $\mathcal{T}_\tau$ (center), and the subtiles of $\mathcal{T}_\theta$ (right).

4.5 Main results

We now combine the results of Section 4.4 (state splittings) and Section 4.4 (conjugacies by free group automorphisms) in order to prove our main result, Theorem 4.5.3.

First, we shall prove that the subtiles of the Rauzy fractals obtained from state splittings and conjugacies by free group automorphisms have disjoint interiors. A sufficient condition ensuring this is the strong coincidence condition (see [**AI01**] for the irreducible case and [**BS05**] for the reducible case). We recall that a substitution σ on alphabet $\mathcal{A}$ satisfies the ***strong coincidence condition*** if for every $(j_1, j_2) \in \mathcal{A}^2$, there exists $k \in \mathbb{N}$ and $i \in \mathcal{A}$ such that $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$ or $\mathbf{P}(s_1) = \mathbf{P}(s_2)$.

Lemma 4.5.1. *Let σ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A} = \{1, \dots, n\}$, let $a \in \mathcal{A}$ and let τ be obtained by splitting of σ from symbol a to a new symbol $b = n+1$ with occurrences $I \subseteq \operatorname{occ}(\sigma, a)$. If σ satisfies the strong coincidence condition, then τ also satisfies it.*

Proof. Let $(j_1, j_2) \in \mathcal{A}^2$. Because σ satisfies the strong coincidence condition, there exists $k \in \mathbb{N}$ and $i \in \mathcal{A}$ such that $\sigma^k(j_1) = p_1 i s_1$ and $\sigma^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$ or $\mathbf{P}(s_1) = \mathbf{P}(s_2)$ (without loss of generality we assume $\mathbf{P}(p_1) = \mathbf{P}(p_2)$). By definition of state splittings, we have

$$\begin{aligned} \tau^{k+1}(j_1) &= \tau(p_1)\tau(i)\tau(s_1) \\ \tau^{k+1}(j_2) &= \tau(p_2)\tau(i)\tau(s_2), \end{aligned}$$

and $\mathbf{P}(\tau(p_1)) = \mathbf{P}(\tau(p_2))$ so τ satisfies the strong coincidence condition. We have omitted the case where $j_1 = b$ or $j_2 = b$, because it follows directly from the fact that $\tau(a) = \tau(b)$. ⬠

Lemma 4.5.2. *Let τ be a primitive unimodular Pisot substitution on alphabet $\mathcal{A}$, let $b \in \mathcal{A}$ be such that there exists a unique $c \in \mathcal{A}$ such that for every $(j; k) \in \operatorname{occ}(\tau, b)$, we have $k \geqslant 2$ and $\tau(j)_{k-1} = c$. Let $\theta = \rho_{cb}^{-1} \tau \rho_{cb}$. If τ satisfies the strong coincidence condition, then θ also satisfies it.*

Proof. Let $(j_1, j_2) \in \mathcal{A}^2$. Because τ satisfies the strong coincidence condition, there exists $k \in \mathbb{N}$ and $i \in \mathcal{A}$ such that $\tau^k(j_1) = p_1 i s_1$ and $\tau^k(j_2) = p_2 i s_2$ with $\mathbf{P}(p_1) = \mathbf{P}(p_2)$ or $\mathbf{P}(s_1) = \mathbf{P}(s_2)$ (without loss of generality we assume $\mathbf{P}(p_1) = \mathbf{P}(p_2)$). Suppose first that $j_1 \neq b$ and $j_2 \neq b$. We have

$$\begin{aligned} \theta^k(j_1) &= \rho_{cb}^{-1} \tau^k \rho_{cb}(j_1) = \rho_{cb}^{-1} \tau^k(j_1) = \rho_{cb}^{-1}(p_1 i s_1) \\ \theta^k(j_2) &= \rho_{cb}^{-1} \tau^k \rho_{cb}(j_2) = \rho_{cb}^{-1} \tau^k(j_2) = \rho_{cb}^{-1}(p_2 i s_2), \end{aligned}$$

so τ satisfies the strong coincidence condition because $\mathbf{P}(\rho_{cb}^{-1}(p_1)) = \mathbf{P}(\rho_{cb}^{-1}(p_2))$. In the remaining case where j_1 or j_2 is equal to b (say, j_1), we have

$$\theta^k(j_1) = \theta^k(b) = \rho_{cb}^{-1} \tau^k \rho_{cb}(b) = \rho_{cb}^{-1} \tau^k(cb) = \rho_{cb}^{-1} \tau^k(c) \rho_{cb}^{-1} \tau^k(b),$$

so result follows because the case $j_1 = b$ is now reduced to the case $j_1 = c$, which has been treated above. ⬠

We can now prove our main theorem, using the tools developed in this chapter.

Theorem 4.5.3. *Let $k \geqslant 1$ be an integer. We have:*

(1) There exists a 4-letter primitive unimodular Pisot substitution τ such that the fundamental group of the subtile $\mathcal{T}_\tau(a)$ is homeomorphic to F_k (for some $a \in \{1,2,3\}$), and such that the subtiles of $\mathcal{T}_\tau$ have disjoint interiors.

(2) There exists a 4-letter primitive unimodular Pisot substitution θ such that the fundamental group of the Rauzy fractal $\mathcal{T}_\theta$ is homeomorphic to F_k and such that the subtiles of $\mathcal{T}_\tau$ have disjoint interiors.

Proof. Let σ be a primitive unimodular on alphabet $\{1,2,3\}$ whose dominant eigenvalue is a cubic Pisot number, and such that $\mathcal{T}_\sigma$ and its subtiles are homeomorphic to a disc and have disjoint interiors. Assume that there exists $a, c \in \{1,2,3\}$ such that for every $j \in \{1,2,3\}$, there exists $n \in \mathbb{N}$ such that $\sigma(j)$ contains at least one occurrence of ca. Such a substitution exists, as can easily be verified for the substitution σ given in Example 4.4.3 or Example 4.4.4. By Proposition 4.2.7, there exists $N \geqslant 1$ and $I \subseteq \mathrm{occ}(\sigma^N, a)$ such that

$$\bigcup_{(j;k) \in \mathrm{occ}(\sigma^N, a) \setminus I} \mathcal{T}_\sigma(a; j; k)$$

is homeomorphic to a closed disc minus the union of k disjoint open discs, and such that for every $(j;k) \in I$, we have $k \geqslant 2$ and $\sigma(j)_{k-1} = c$.

Performing a splitting of the occurrences I of a to a new symbol $b \notin \{1,2,3\}$ yields a substitution τ. Statement (1) now follows by Proposition 4.3.2. The subtiles of $\mathcal{T}_\tau$ have disjoint interior because τ satisfies the strong coincidence condition thanks to Lemma 4.5.1 and [**ST09**, Theorem 2.10]. (This fact also follows from the fact that every subtile is a interior-disjoint union of subsubtiles, and subsubtiles always have disjoint interior [**ST09**, Theorem 2.6].)

Let $\theta = \rho_{cb}^{-1} \tau \rho_{cb}$. Statement (2) follows from Proposition 4.4.1, and the subtiles of $\mathcal{T}_\theta$ have disjoint interior because θ satisfies the strong coincidence condition thanks to Lemma 4.5.2 and [**ST09**, Theorem 2.10]. ⬠

4.6 Conclusion

Note that these results are obtained with a fixed number of symbols, 4, so there is no bound of the number of holes by the number of symbols (which answers a question asked by Minervino). It is not known if there exists a 3-letter irreducible Pisot substitution with nontrivial but countable fundamental group.

Further possible developments include establishing higher homology/homotopy realization results for Rauzy fractals associated with Pisot numbers of degree $\geqslant 4$. This is out of reach with the current tools at the current state because we have no tools to prove, for example, that a tile is homeomorphic to a ball. An illustration is given in Figure 4.4. Like in the proof of Theorem 4.5.3, Proposition 4.3.2 and Proposition 4.4.1 do imply that the subsubtiles corresponding to the occurrences of θ in Figure 4.4 have been “removed” from the Rauzy fractal of σ. However, we cannot deduce that we have managed to “drill a hole” in the fractal, because the fact that it is 3-dimensional makes it difficult to prove, for example, that the subtiles are homeomorphic to a ball, which is an essential preliminary fact. Note that this fractal is three-dimensional because its associated Pisot eigenvalue $\beta \approx 1.926$ is of degree 4.

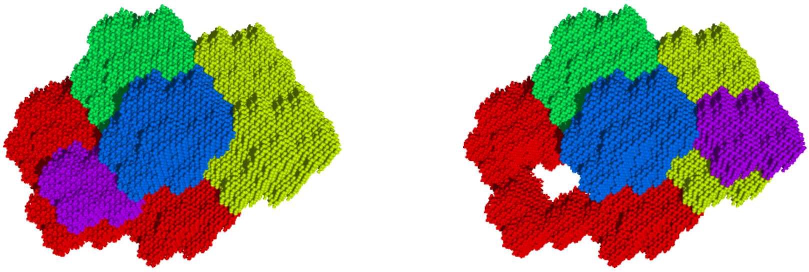


Figure 4.4: Drilling holes in the "Quadribonacci" substitution $\sigma : 1 \mapsto 21, 2 \mapsto 31, 3 \mapsto 41, 4 \mapsto 1$. We have splitted the occurrences $\{(1;8),(2;7)\}$ of 1 to a new symbol 5 in σ^3 to obtain a new substitution τ. Conjugating τ by ρ_{25} yields $\theta : 1 \mapsto 4121315, 2 \mapsto 121315, 3 \mapsto 213121, 4 \mapsto 3121, 5 \mapsto 1213154121315$. The subtiles of $\mathcal{T}_\tau$ and $\mathcal{T}_\theta$ are plotted left and right, respectively. A "hole" (or "tunnel") can be observed in $\mathcal{T}_\theta$.

Another perspective is trying to describe the *uncountable* fundamental groups of some simple examples, such as the fractal shown in Figure 4.5. This has successfully been done for some fractals such as the Hawaiian earring or the Sierpiński triangle [**CC00, ADTW09**].

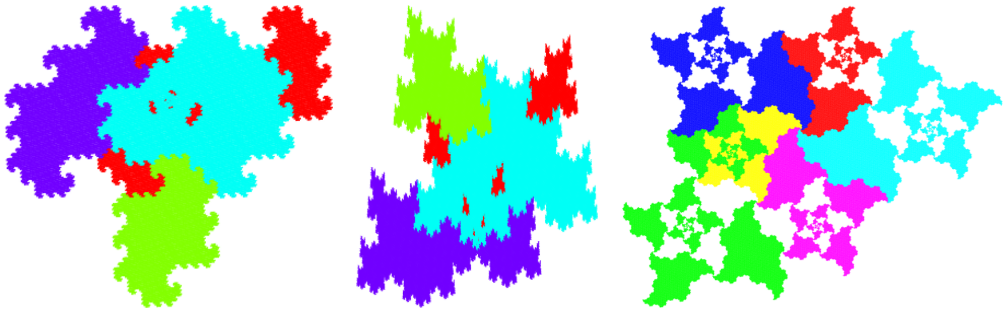


Figure 4.5: The Rauzy fractals of $1 \mapsto 2413, 2 \mapsto 43, 3 \mapsto 2433, 4 \mapsto 1$ (left), $1 \mapsto 2413, 2 \mapsto 1, 3 \mapsto 2433, 4 \mapsto 43$ (middle), $1 \mapsto 2, 2 \mapsto 4, 3, 3 \mapsto 4, 4 \mapsto 53, 5 \mapsto 6, 6 \mapsto 1$ (right). In the first examples, we are tempted to guess that one of the tiles is homeomorphic to a disc from which infinitely discs have been removed, which would make it homeomorphic to the Hawaiian earring.

Chapter 5

Examples and counterexamples of Rauzy fractals

We conclude the first four chapters, where the focus was directly on one-dimensional Pisot substitutions, by giving a collection of examples of Rauzy fractals with various properties.

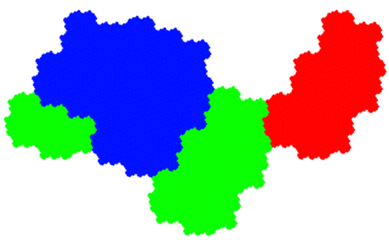
Contents

5.1 A connected fractal with a disconnected subtile

The following observation is made in Section 4.5 of the book *Topological properties of Rauzy fractals* [**ST09**].

> Notice that it is not easy to provide a necessary and sufficient condition for the connectivity of the central tile. The reason is that the central tile $\mathcal{T}$ might be connected even if some of the subtiles are disconnected. However, we were not able to find an example with this constellation.

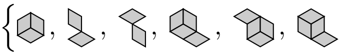
Such counterexamples can actually be found by computer exploration (by plotting many Rauzy fractals and watching them all). The following picture strongly suggests that we have found a good candidate, and Proposition 5.1.1 below confirms that it is the case.



Proposition 5.1.1. *Let $\sigma : 1 \mapsto 212, 2 \mapsto 3, 3 \mapsto 3123$ be the substitution associated with the Rauzy fractal plotted above.*

(1) The subtiles $\mathcal{T}_\sigma(1)$ and $\mathcal{T}_\sigma(3)$ are connected, but $\mathcal{T}_\sigma(2)$ is disconnected.
(2) The Rauzy fractal $\mathcal{T}_\sigma(1) \cup \mathcal{T}_\sigma(2) \cup \mathcal{T}_\sigma(3)$ is connected.

Proof. Statement (1) can be checked thanks to the algorithm given in [**ST09**, Theorem 4.9], but Statement (2) cannot be proved in the same way because the tools of [**ST09**] do not apply to this case. However, we can prove (2) thanks to the $\mathbf{E}_1^\star$ definition of the Rauzy fractal of σ and $\mathcal{L}$-coverings (described in Section 2.2). Let

$\mathcal{L} = \{$  $\}.$

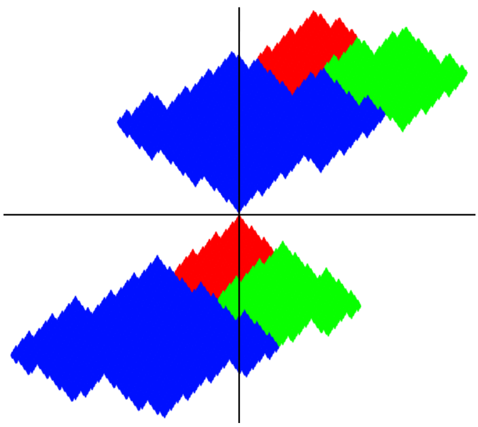
We can easily check that $\mathcal{U}$ is $\mathcal{L}$-covered and that $\mathbf{E}_1^\star(\sigma)(P)$ is $\mathcal{L}$-covered for every $P \in \mathcal{L}$, where $\mathcal{U} = [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star$ is the "starting pattern" used in Definition 1.3.5 to define Rauzy fractals as renormalized patches. Thanks to Proposition 2.2.2, it follows that $\mathbf{E}_1^\star(\sigma)^n(\mathcal{U})$ is $\mathcal{L}$-covered for all $n \geqslant 0$. Hence, in the same way as in Theorem 3.2.1, the connectedness of the patterns in $\mathcal{L}$ imply that the Rauzy fractal of σ is a Hausdorff limit of connected sets, so it is connected. ⬠

Note that the above proof using $\mathcal{L}$-coverings is currently the only known way to prove the connectedness of this Rauzy fractal.

5.2 Rauzy fractals with common matrix but single point intersection

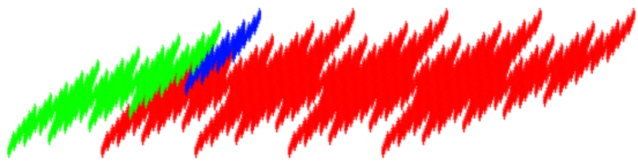
Let σ and τ be two substitution that have the same incidence matrix. Tarek Sellami proved [**Sel12**] that if $\mathbf{0}$ is an inner point of the Rauzy fractal of σ or τ, then the intersection of the two Rauzy fractals has positive Lebesgue measure, and it is equal to the Rauzy fractal of a substitution which can be algorithmically constructed.

The following example suggests that the assumption of having $\mathbf{0}$ as an interior point cannot be dropped. Let $\sigma : 1 \mapsto 13, 2 \mapsto 133, 3 \mapsto 233$ and let $\tau : 1 \mapsto 31, 2 \mapsto 331, 3 \mapsto 332$ (the mirror image of σ). The Rauzy fractals of σ and τ are plotted below left and right, respectively. These two substitutions have the same incidence matrix, but the Rauzy fractals seem to intersect at a single point only (we do not prove this here).



5.3 A non-simply connected Arnoux-Rauzy fractal

We know that the Rauzy fractal of a finite product of Arnoux-Rauzy substitutions is always connected by Theorem 3.2.1. However, such fractals are not always simply connected. Indeed, let $\sigma = (\sigma_1^{\mathsf{AR}})^4(\sigma_2^{\mathsf{AR}})^4(\sigma_3^{\mathsf{AR}})^4$, and let $\tau = 1 \mapsto 21111, 2 \mapsto 31111, 3 \mapsto 1$. It can be proved that $\mathcal{T}_\tau$ is not simply connected using the algorithms given in [**ST09**]. Moreover we have $\sigma = \tau^3$, so $\mathcal{T}_\sigma = \mathcal{T}_\tau$. It follows that σ is an example of a product of Arnoux-Rauzy substitutions whose Rauzy fractal is not simply connected.

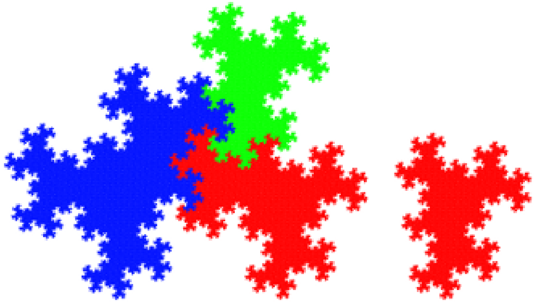


This example also shows that there cannot be a "reasonable" link between the simple connectedness of $\mathcal{T}_\sigma$ and the factor complexity of the language of σ, because of the well known fact that Arnoux-Rauzy sequences have minimal factor complexity $2n+1$.

5.4 Invertibility and connectedness

In the of 2-letter substitutions, it has been shown that the Rauzy fractal of σ is connected if and only if σ is invertible [**EI98, Lam98, BEIR07**]; see also [**BFS12**] for a survey of many equivalent formulations of invertibility in the 2-letter case. (Recall that a substitution σ is ***invertible*** if there exists an automorphism σ' of the free group such that $\sigma\sigma'$ is the identity.) Both implications of the above statement are false in the 3-letter case, as shown by the following two examples.

Let $\sigma = \varepsilon_{1,2}\varepsilon_{3,1}\varepsilon_{2,3}\varepsilon_{1,3} : 1 \mapsto 31, 2 \mapsto 12, 3 \mapsto 31123$, where for every distinct $i, j \in \{1,2,3\}$, $\varepsilon_{i,j}$ is the ***elementary substitution*** $j \mapsto ji$, $k \mapsto k$ if $k \neq j$. This substitution is invertible (as a product of elementary substitutions) but its Rauzy fractal (shown below left) is not connected, as can be checked using the algorithms given in [**ST09**]. Conversely, there also exist examples of non-invertible substitutions with a connected Rauzy fractal (see the examples found by Edmund Harriss [**HFa, HFc**]).

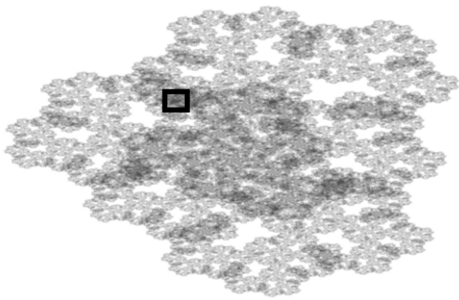


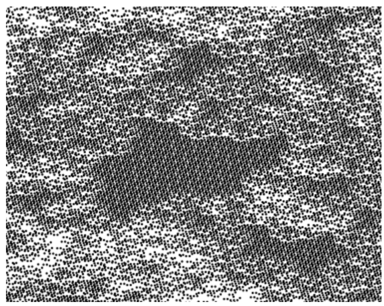
5.5 Hokkaido mysteries

The *Hokkaido tile* (plotted below left) is one of the first examples of fractal tiles appearing in numeration, as early as 1989 in the work of Thurston [**Thu89**], and it is known to be homeomorphic to a closed disk [**Luo02**]. It owes its name to its resemblance with the northern island of Japan, Hokkaido (plotted below right).

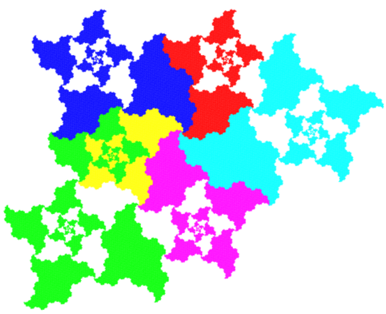
This tile is naturally linked with the smallest cubic Pisot number $\beta_{\mathsf{min}} \approx 1.325$ (the real root of $x^3 - x - 1$), since it can be defined as the Rauzy fractal of the beta-substitution $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$, whose dominant Pisot eigenvalue is equal to β_{min}.

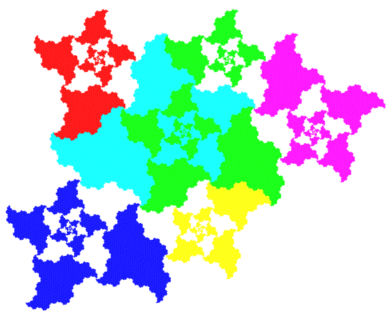
There are many other tiles related in the same way with β_{min}, as for example the Rauzy fractal of $\sigma : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1, 2$, whose dominant eigenvalue is also β_{min}. The "mystery" is that by zooming in the Rauzy fractal of σ, we can see a copy of the Hokkaido tile, even though the two fractals do not seem to share any similarities, as shown below. (The framed area in the fractal below left is shown zoomed at the right. Plots with several million points are needed to allow such visualization.)





Such an observation was first made by Julien Bernat, who glimpsed a Hokkaido tile between three translated copies of the Rauzy fractal of σ. The Hokkaido tile also appears in many other Rauzy fractals of substitutions with dominant eigenvalue β_{min}, such as $1 \mapsto 2, 2 \mapsto 43, 3 \mapsto 4, 4 \mapsto 53, 5 \mapsto 6, 6 \mapsto 1$, which yields the following surprising fractal shown below (together with its associated domain exchange).



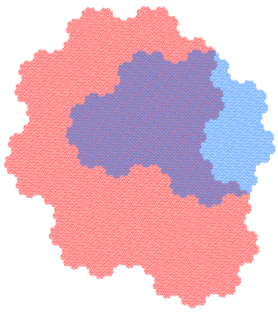


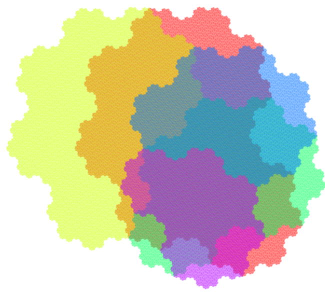
Question. Can any of the above observations be rigorously proved? That is, do copies of the Hokkaido tile actually appear in the Rauzy fractals shown above? Similar questions for other examples than the Hokkaido tile also seem to be unsolved.

5.6 Reducible substitutions with overlapping tiles

A weak formulation of the Pisot conjecture is that the subtiles of the Rauzy fractal of a Pisot *irreducible* substitution do not overlap. This formulation is wrong if we remove the irreducibility assumption. The first counterexample was given in [**BBK06**]: a 6-letter reducible substitutions (with quadratic dominant eigenvalue) whose subtiles overlap on a set of positive measure. The tiles in this example are one-dimensional because the eigenvalue is quadratic.

It is also possible to find examples where the overlapping tiles are two-dimensional, which allows a proper visualization of the overlapping tiles. For example let $\sigma : 1 \mapsto 213, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 1, 5 \mapsto 21$. Below left are plotted the tiles $\mathcal{T}_\sigma(2)$ and $\mathcal{T}_\sigma(4)$, which clearly seem to overlap, and things are worse when all the tiles are plotted together (below right). Note that the incidence matrix of σ is the square of the incidence matrix of the Hokkaido substitution.





Many other examples can be found thanks to computer exploration. All the above claims about overlapping tiles can be rigorously proved thanks to the algorithms described in [**ST09**] or in Chapter 5 of [**BR10**].

Since the Pisot conjecture is wrong in the reducible case and believed to be true in the irreducible case, it would be interesting to replace the irreducibility assumption by a weaker one, which includes all the known examples of reducible substitutions that satisfy the Pisot conjecture. Efforts in this direction can be found for example in [**BBJS12**], where the irreducibility assumption is replaced by conditions on the action of the substitution on the cohomology of the tiling space.

Chapter 6

Combinatorial substitutions

In this chapter we consider a general notion of multidimensional substitution acting on words defined over the integer lattice $\mathbb{Z}^d$, where the words are not necessarily rectangles. This requires a specific description of the way the images of a symbol are glued together. Two problems can arise when defining a substitution in such a way: it can fail to be consistent, and the patterns in an image by the substitution might overlap.

We prove that it is undecidable whether a two-dimensional substitution is consistent or overlapping, and we provide practical algorithms to decide these properties in some particular cases.

The original motivating examples that led to this work come from the dual $\mathbf{E}_1^\star$ substitutions defined in Section 1.2; the link between the two is described in Section 6.5. This is joint work with Jarkko Kari [**JK12**].

Contents

6.1 Introduction

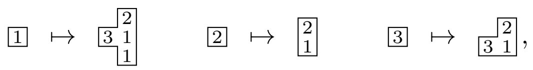
The aim of this chapter is to discuss a general notion of multidimensional substitutions acting not on one-dimensional words, but on words defined over the integer lattice $\mathbb{Z}^d$.

In the one-dimensional setting we have the natural relation $\sigma(uv) = \sigma(u)\sigma(v)$, which allows the computation of the image of a word by *concatenating* the images of each letter. For example, $\sigma(1321) = \sigma(1)\sigma(3)\sigma(2)\sigma(1) = 1211312$ if σ is the substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$. The main challenge of defining multidimensional substitutions is to generalize the relation $\sigma(uv) = \sigma(u)\sigma(v)$ to higher dimensional words. This can be done easily if, for example, the images of the letters are squares of the same size, like in the two-dimensional Thue-Morse substitution $1 \mapsto \begin{smallmatrix}1&2\\2&1\end{smallmatrix}$, $2 \mapsto \begin{smallmatrix}2&1\\1&2\end{smallmatrix}$, which gives:

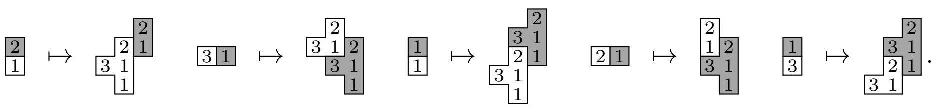
$$1 \mapsto \begin{smallmatrix}1&2\\2&1\end{smallmatrix} \mapsto \begin{smallmatrix}1&2&2&1\\2&1&1&2\\2&1&1&2\\1&2&2&1\end{smallmatrix} \mapsto \begin{smallmatrix}1&2&2&1&2&1&1&2\\2&1&1&2&1&2&2&1\\2&1&1&2&1&2&2&1\\1&2&2&1&2&1&1&2\\1&2&2&1&2&1&1&2\\2&1&1&2&1&2&2&1\\2&1&1&2&1&2&2&1\\1&2&2&1&2&1&1&2\end{smallmatrix} \mapsto \cdots$$

Similar generalizations are possible with rectangular shapes that have compatible edge lengths; see the survey [**Fra08**], or [**Moz89**].

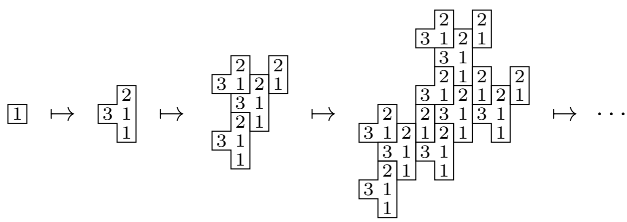
We are interested in the more general case where the images of the letters have arbitrary shapes (not necessarily rectangular), which are *a priori* not compatible with concatenation. For example, if σ is a substitution defined by:



how can we define the image by σ of patterns that consist of more than one cell? A natural approach is to give explicit *concatenation rules*, such as:



Using these rules, we can compute the image of every pattern that can be "covered" by the two-cell patterns on the left-hand sides of the rules, as shown below.



Our results In this chapter we study the two problems that can arise when multidimensional concatenation is defined as above:

(1) The resulting substitution can fail to be *consistent*: depending on the sequence of concatenation rules that are used, a pattern might have two different images.
(2) The resulting substitution can fail to be *non-overlapping*: the images of the cells of a pattern might overlap.

The substitutions which are consistent and non-overlapping correspond to "well defined" substitutions, and we would like to be able to detect them algorithmically. We will prove that consistency and non-overlapping are undecidable properties for two-dimensional combinatorial substitutions (Theorems 6.3.1 and 6.3.3), but that these properties are decidable in the case of *domino-complete* substitutions, that is, when the concatenation rules are given for every possible domino (Theorems 6.4.1 and 6.4.4). This answers the decidability question raised in [**Fer07a**].

As an application of the methods developed for the decidability results, we provide combinatorial proofs of the consistency and non-overlapping of some particular two-dimensional substitutions, using a slightly more general definition of domino-completeness (Theorem 6.4.2, Section 6.5). Such proofs have been requested in [**ABS04, Fra08**].

It is important to remark that the properties studied in this chapter (consistency and overlapping) only concern a *single* iteration of the substitution. This is a necessary first step for the further study of properties related with iterating a substitution indefinitely on a given pattern (see the related open problems in the conclusion).

Links with dual substitutions Combinatorial substitutions as defined in this chapter were introduced in [**ABS04**] in order to study the particular example of the dual $\mathbf{E}_1^\star(\sigma)$ substitution of $\sigma : 1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 3$ (see Section 1.2). They were later defined more generally in [**Fer07a**], and they are related to the substitutions found in [**Fra03**] defined using the dual graph of a pattern. (See [**Fra08**] for a survey about multidimensional substitutions in general.)

The idea that originally led to the introduction of combinatorial substitutions was that of *local rules*. In the context of $\mathbf{E}_1^\star$ substitutions, this idea consists in computing images by "multidimensional concatenation", instead of computing the new position of each cell individually by multiplication with a matrix (like in Definition 1.2.3 of $\mathbf{E}_1^\star$ substitutions). This idea was initially exploited in [**IO93, IO94**] and then in [**ABI02, ABS04**], and it is one of the key ingredients of the results in Chapters 2 and 3 of this thesis, where it is implicitly used in the form of $\mathcal{L}$-coverings; see Section 2.2. See Section 6.5.

6.2 Definitions

Cells and patterns

Let $\mathcal{A}$ denote a finite set of symbols. A ***d-dimensional cell*** is a couple $c = [v, t]$, where $v \in \mathbb{Z}^d$ is the ***vector*** of c and $t \in \mathcal{A}$ is the ***type*** of c. A ***d-dimensional pattern*** is a finite union of d-dimensional cells with distinct vectors. Translation $v+P$ of a pattern P by $v \in \mathbb{Z}^d$ is defined in the natural way. The ***support*** of a pattern is $\mathrm{supp}(P) = \{v : [v, t] \in P\}$.

Many of the substitutions we will encounter later use ***dominoes***, which are two-dimensional patterns that consists of two cells of vectors v and v' such that $v' - v \in \{(\pm 1, 0), (0, \pm 1))\}$.

Substitutions

A ***d-dimensional substitution*** σ on alphabet $\mathcal{A}$ is defined by:

- a ***base rule***: an application σ_{base} from $\mathcal{A}$ to the set of d-dimensional patterns,
- a finite set of ***concatenation rules*** $(t, t', u) \mapsto v$, where $t, t' \in \mathcal{A}$ and $u, v \in \mathbb{Z}^d$.

The way to interpret this definition is the following: σ_{base} replaces each cell of a pattern by a pattern, and the concatenation rules describe how to place the images of the cells relatively to each other. The intuitive meaning of "$(t, t', u) \mapsto v$" is: two cells of types t and t' separated by u must be mapped by σ to the two patterns $\sigma_{\text{base}}(t)$ and $\sigma_{\text{base}}(t')$ separated by v. (A precise definition is given below.) From now on we only consider ***deterministic*** substitutions, which means that if a rule has a left-hand side (t, t', u), then there is no other rule with left-hand side either (t, t', u) or $(t', t, -u)$. We will need the following notation.

- We extend σ_{base} from $\mathcal{A}$ to the set of cells naturally: $\sigma_{\text{base}}(c) = \sigma_{\text{base}}(t)$ for a cell $c = [v, t]$. (Only the type of c is taken into account by σ_{base}.)
- The set of the ***starting patterns*** of σ is

$$\mathcal{C}_\sigma = \{\{[0, t], [u, t']\} : (t, t', u) \mapsto v \text{ is a rule of } \sigma \text{ for some } v\}.$$

  It corresponds to the patterns at the left-hand sides of the concatenation rules of σ. (This set contains only patterns of size two.)
- If $c = [u, t]$ and $c' = [u', t']$ are cells, we denote

$$\sigma_{\text{rule}}(c, c') \;=\; \begin{cases} v & \text{if } (t, t', u' - u) \mapsto v \text{ is a rule of } \sigma \\ -v & \text{if } (t', t, u - u') \mapsto v \text{ is a rule of } \sigma \end{cases},$$

  and σ_{rule} is not defined otherwise.

A ***domino substitution*** is a two-dimensional substitution such that for every rule $(t, t', u) \mapsto v$, we have $u \in \{\pm(1, 0), \pm(0, 1)\}$. A ***domino-to-domino*** substitution is a domino substitution such that for every rule $(t, t', u) \mapsto v$, we have $v \in \{\pm(1, 0), \pm(0, 1)\}$, and the patterns $\sigma_{\text{base}}(t)$ and $\sigma_{\text{base}}(t')$ both consist of a single cell of vector $(0, 0)$.

Example 6.2.1. The combinatorial substitution given in Section 6.1 is formally defined as follows: it is the two-dimensional substitution on alphabet $\{1, 2, 3\}$ with the following base rule (on the left) and concatenation rules (on the right).

$$\begin{aligned} 1 &\mapsto \{[(0,0),1],[(0,1),1],[(0,2),2],[(-1,1),3]\} \\ 2 &\mapsto \{[(0,0),1],[(0,1),2]\} \\ 3 &\mapsto \{[(0,0),3],[(1,0),1],[(1,1),2]\} \end{aligned} \qquad \begin{aligned} &(1,2,(0,1)) \mapsto (1,2) \\ &(3,1,(1,0)) \mapsto (2,-2) \\ &(1,1,(0,1)) \mapsto (1,2) \\ &(2,1,(1,0)) \mapsto (1,-2) \\ &(3,1,(0,1)) \mapsto (2,1). \end{aligned}$$

Paths, covers, image vectors

Let $\mathcal{C}$ be a finite set of patterns each consisting of two cells. A ***$\mathcal{C}$-path from*** c_1 ***to*** c_n is a finite sequence of cells $\gamma = (c_1, \ldots, c_n)$ such that $\{c_i, c_{i+1}\}$ is a translated copy of an element of $\mathcal{C}$ for all $1 \leqslant i \leqslant n-1$, and such that $c_i = c_j$ if c_i and c_j have the same vector. (Paths are hence allowed to self-overlap, but the overlapping cells must agree.) If $c_1 = c_n$, then γ is called a ***$\mathcal{C}$-loop***. A path (or a loop) is ***simple*** if it does not self-intersect. If all the cells of γ are contained in P then it is called a ***$\mathcal{C}$-path of*** P.

A pattern P is ***$\mathcal{C}$-covered*** if for every $c, c' \in P$, there exists a $\mathcal{C}$-path γ of P from c to c'. Let σ be a substitution and $\gamma = (c_1, \ldots, c_n)$ be a $\mathcal{C}_\sigma$-path. We denote by $\omega_\sigma(\gamma)$ the

image vector of γ defined by

$$\omega_\sigma(\gamma) \;=\; \sum_{i=1}^{n-1} \sigma_{\text{rule}}(c_i, c_{i+1}).$$

Consistency and non-overlapping

Let σ be a substitution and P be a $\mathcal{C}_\sigma$-covered pattern. We say that σ is:

- ***consistent on*** P if for every cells $c, c' \in P$ and for every $\mathcal{C}_\sigma$-paths γ, γ' of P from c to c', we have $\omega_\sigma(\gamma) = \omega_\sigma(\gamma')$, *i.e.*, if the placement of the images does not depend on the path used.
- ***non-overlapping on*** P if for every cells $c, c' \in P$ such that $c \neq c'$ and for every $\mathcal{C}_\sigma$-path γ of P from c to c', we have $\text{supp}(\sigma_{\text{base}}(c)) \cap (\text{supp}(\omega_\sigma(\gamma) + \sigma_{\text{base}}(c'))) = \varnothing$, *i.e.*, if two distinct cells have non-overlapping images.

If σ is consistent on every $\mathcal{C}_\sigma$-covered pattern, then σ is said to be ***consistent***. (The same goes for non-overlapping.) Examples of inconsistent and overlapping substitutions will be given in Examples 6.2.4 and 6.2.5.

Proposition 6.2.2. *Let σ be a combinatorial substitution and P be a pattern. The following statements are equivalent.*

(1) σ is consistent on P.
(2) For every $\mathcal{C}_\sigma$-loop γ of P, we have $\omega_\sigma(\gamma) = 0$.
(3) For every simple $\mathcal{C}_\sigma$-loop γ of P, we have $\omega_\sigma(\gamma) = 0$.

Proof. (1) $\Leftrightarrow$ (2). Suppose that σ is consistent, and let $\gamma = (c_1, \ldots, c_n)$ be a $\mathcal{C}_\sigma$-loop of P. Let $\gamma' = (c_1, \ldots, c_{n-1})$ and $\gamma'' = (c_n, c_{n-1})$. Because $c_n = c_1$ and σ is consistent we have $\omega_\sigma(\gamma') = \omega_\sigma(\gamma'')$, so

$$\begin{aligned}\omega_\sigma(\gamma) \;&=\; \sum_{i=1}^{n-1} \sigma_{\text{rule}}(c_i, c_{i+1}) \;=\; \sum_{i=1}^{n-2} \sigma_{\text{rule}}(c_i, c_{i+1}) - \sigma_{\text{rule}}(c_n, c_{n-1}) \\ &=\; \omega_\sigma(\gamma') - \omega_\sigma(\gamma'') \;=\; 0.\end{aligned}$$

Conversely, let $\gamma = (c_1, \ldots, c_n)$ and $\gamma' = (c'_1, \ldots c'_m)$ be two $\mathcal{C}_\sigma$-paths of P with $c_1 = c'_1$ and $c_n = c'_m$. Now, $\gamma'' := (c_1, \ldots, c_{n-1}, c_n, c'_{m-1}, \ldots, c'_1)$ is a $\mathcal{C}_\sigma$-loop, so $\omega_\sigma(\gamma'') = 0$, which implies that $\omega_\sigma(\gamma) = \omega_\sigma(\gamma) - \omega_\sigma(\gamma'') = \omega_\sigma(\gamma) - (\omega_\sigma(\gamma) - \omega_\sigma(\gamma')) = \omega_\sigma(\gamma')$.

(2) $\Leftrightarrow$ (3). Implication "$\Rightarrow$" is trivial. For the converse, suppose that there exists a non-simple $\mathcal{C}_\sigma$-loop $\gamma = (c_1, \ldots, c_n)$ of P such that $\omega_\sigma(\gamma) \neq 0$, and let $i < j < n$ such that $c_i = c_j$. Let $\gamma' = (c_i, \ldots, c_j)$ and $\gamma'' = (c_1, \ldots, c_i, c_{j+1}, \ldots, c_n)$. We have $\omega_\sigma(\gamma') + \omega_\sigma(\gamma'') = \omega_\sigma(\gamma) \neq 0$, so $\omega_\sigma(\gamma') \neq 0$ or $\omega_\sigma(\gamma'') \neq 0$. Repeating this operation inductively yields the existence of a simple loop (strictly smaller than γ) which does not overlap itself and which has a nonzero image vector. (The loop cannot reduce to a single cell because we assumed that $\omega_\sigma(\gamma) \neq 0$.) ⬠

Image by a substitution

Let σ be a non-overlapping substitution. Let P be a $\mathcal{C}_\sigma$-covered pattern and c_0 be a cell of P. An ***image of P by σ computed from*** c_0 is a pattern

$$\bigcup_{c \in P} (\sigma_{\text{base}}(c) \ + \ \omega_\sigma(\gamma_c)),$$

where for each $c \in P$, γ_c is a $\mathcal{C}_\sigma$-path from c_0 to c. This union of patterns is indeed a pattern because the cells have distinct positions (σ is non-overlapping).

If σ is consistent, this pattern is uniquely defined because it does not depend on the choice of the paths γ_c, by consistency of σ. In this case, the image of P by σ computed from c_0 is denoted by $\sigma(P, c_0)$, but because c_0 only affects by a translation we will use the simpler notation $\sigma(P)$ when the translation is irrelevant.

In practice, to compute the image of an $\mathcal{C}_\sigma$-covered pattern P by σ, we start by constructing a covering tree whose vertices are the cells of P, such that two cells are connected must belong to a same pattern of $\mathcal{C}_\sigma$. (Such a tree exists because P is $\mathcal{C}_\sigma$-covered.) We then choose a "root cell" (c_0 in the definition above), and we construct the image of P incrementally, starting from c_0 and following the edges in the tree. This is shown in Example 6.2.3 below.

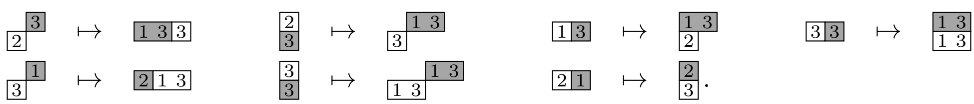
Example 6.2.3. Let σ be the two-dimensional substitution defined on the alphabet $\{1, 2, 3\}$ with the base rule

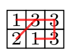
$$\begin{array}{llll} 1 & \mapsto & \{[(0,0),2]\} & = \ \boxed{2} \\ 2 & \mapsto & \{[(0,0),3]\} & = \ \boxed{3} \\ 3 & \mapsto & \{[(0,0),1],[(1,0),3]\} & = \ \boxed{1\ 3} \end{array}$$

and the concatenation rules

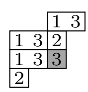
$$\begin{array}{llll} (2,3,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) & (3,2,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right) & (1,3,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) & (3,3,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \\ (3,1,\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}-1\\0\end{smallmatrix}\right) & (3,3,\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}-2\\-1\end{smallmatrix}\right) & (2,1,\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \mapsto \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) & \end{array}$$

which can also be represented as follows

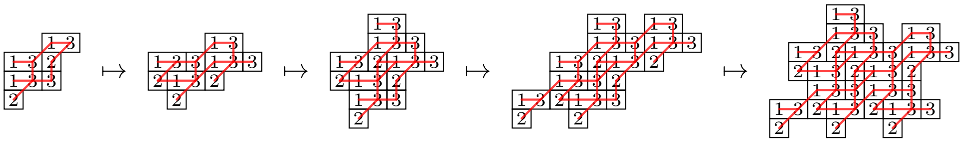


To compute the image of $\begin{array}{|c|c|c|}\hline 1&3&3\\ \hline 2&1&3\\ \hline\end{array}$ by σ, we can use the $\mathcal{C}_\sigma$-covering :

$$\begin{aligned} \sigma\Big(\begin{array}{|c|c|c|}\hline 1&3&3\\ \hline 2&1&3\\ \hline\end{array}\Big) = \{[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), 3]\} & \\ \cup \{[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),1],[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),3]\} + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) & \\ \cup \{[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),2]\} + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right) & \\ \cup \{[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),1],[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),3]\} + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) & \\ \cup \{[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),1],[\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),3]\} + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) + \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right) & \\ \cup \{[\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right),2]\} + \left(\begin{smallmatrix}-2\\0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) + \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right) + \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right) = & \end{aligned}$$



We chose to place the image of the cell $\boxed{2}$ first, at position $(0, 0)$. This cell and its image are shown in gray. In this case it is possible to iterate σ on its images; a few iterations are shown below. (The $\mathcal{C}_\sigma$-covering is drawn on each pattern.)



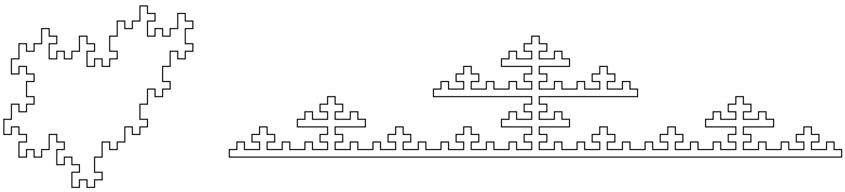
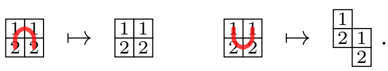
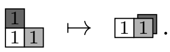


Figure 6.1: The boundaries of the patterns $\sigma^{12}(\{[(0,0),3],[(1,0),1],[(1,1),2]\}$ of Example 6.2.3 (left), and $\sigma^4(\{[(0,0),1]\})$ of Example 6.4.5 (right).

Example 6.2.4. The substitution defined by the base rule $1 \mapsto \{[(0,0),1\}$, $2 \mapsto \{[(0,0),2\}$ and the concatenation rules $\boxed{1}\boxed{1} \mapsto \boxed{1}\boxed{1}$, $\genfrac{}{}{0pt}{}{\boxed{1}}{\boxed{2}} \mapsto \genfrac{}{}{0pt}{}{\boxed{1}}{\boxed{2}}$, $\boxed{2}\boxed{2} \mapsto \genfrac{}{}{0pt}{}{\boxed{2}}{\quad\boxed{2}}$ is not consistent: the $\mathcal{C}_\sigma$-loop $\gamma = \{[(0,0),2],[(1,0),2],[(1,1),1],[(0,1),1],[(0,0),2]\}$ has a nonzero image vector $\omega_\sigma(\gamma) = (1,-1)+(0,1)-(1,0)-(0,1) = (0,-1)$, so σ in inconsistent by Proposition 6.2.2. This is also illustrated by the following:



Example 6.2.5. The substitution defined by the base rule $1 \mapsto \{[(0,0),1\}$ and the concatenation rules $\boxed{1}\boxed{1} \mapsto \boxed{1}\boxed{1}$, $\genfrac{}{}{0pt}{}{\boxed{1}}{\boxed{1}} \mapsto \boxed{1}\boxed{1}$ is overlapping: the images of the cells $[(0,1),1]$ and $[(1,0),1]$ overlap in the image of the pattern $\{[(0,0),1],[(1,0),1],[(0,1),1]\}$:



Example 6.2.6. The following example concerns the definition of consistency. Let σ be the substitution on the alphabet $\mathcal{A} = \{1,2,3,4,5,6\}$, with the base rule $\sigma_{\mathrm{base}}(a) = [(0,0),1]$ for every $a \in \mathcal{A}$, and the concatenation rules

$$\begin{array}{lcl} (a,b,(1,0)) & \mapsto & (1,0) \quad \text{for} \quad (a,b) \in \{(1,1),(2,2),(3,1),(4,2),(1,5),(2,6)\} \\ (3,4,(0,1)) & \mapsto & (0,1) \\ (5,6,(0,1)) & \mapsto & (0,-1). \end{array}$$

No $\mathcal{C}_\sigma$-covered-pattern can have more than two rows. Also, the only patterns that admit more than one possible $\mathcal{C}_\sigma$-covering are the patterns of the form $\begin{smallmatrix} 4 & 2 & 2 & \cdots & 2 & 6 \\ 3 & 1 & 1 & \cdots & 1 & 5 \end{smallmatrix}$, but all their possible images are equal up to translation. Hence, every $\mathcal{C}_\sigma$-covered pattern admits only one image by σ up to translation.

This naturally suggests that σ is consistent, but it is in fact not the case: the two paths going from the cell of type 4 to the cell of type 5 in the above pattern do not have the same image vector.

Therefore, the property "every $\mathcal{C}_\sigma$-covered pattern admits only one image up to translation" is not equivalent to the consistency of σ, which is an important fact to mention because it is another natural candidate to define consistency. Note that our definition of consistency is actually stronger. Also, the undecidability proofs of Section 6.3 can easily be adapted to yield the undecidability of the alternative notion of consistency.

6.3 Undecidability results

Wang tiles are unit square tiles with colored edges, which are oriented and may not be rotated. We say that a set of Wang tiles T admits a ***valid tiling of a cycle*** if there exists a nontrivial sequence $(a_1, \dots, a_n)$ of translates of tiles of T such that a_i and a_{i+1} share exactly one edge and their colors agree on it for all $1 \leqslant i < n$, and such that $a_n = a_1$ (the other tiles a_i cannot overlap and are distinct). Note that cycle are required to be nontrivial so we must have $n \geqslant 5$.

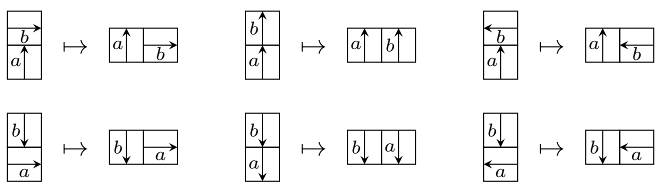
In [**Kar02**], it is proved that the following problem is undecidable: "Does a given finite set of Wang tiles admit a valid tiling of a cycle?" This problem is called the ***weak cycle tiling problem*** (the strong version of the same problem requires that *any* two adjacent tiles in the cycle match in color, and not only a_i and a_{i+1}). We will use the fact that this problem is undecidable in order to prove Theorems 6.3.1 and 6.3.3.

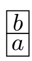
The undecidability results below are proved for two-dimensional substitutions. The proofs can easily be modified to get undecidability in higher dimensions, but dimension 1 has to be ruled out.

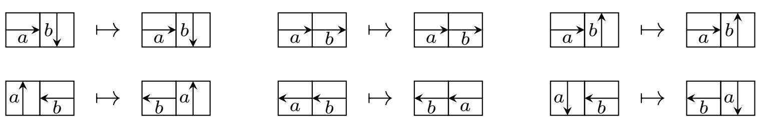
Consistency

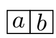
Theorem 6.3.1. *It is undecidable whether a two-dimensional combinatorial substitution is consistent.*

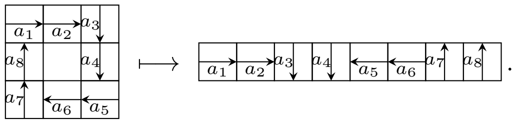
Proof. We are going to reduce the weak cycle tiling problem for Wang tiles to the consistency problem for substitutions. The former is undecidable so the result follows. Let T be a set of Wang tiles. Let $\mathcal{A} = T \times \{\rightarrow, \uparrow, \leftarrow, \downarrow\}$, and σ be the substitution over alphabet $\mathcal{A}$ defined by $\sigma_{\text{base}}(t) = [(0,0), t]$ for all $t \in \mathcal{A}$, with the rules



for all the tiles $a, b \in T$ such that the edges match in , and the rules



for all the tiles $a, b \in T$ such that the edges match in . By definition of the above rules, the set $\mathcal{C}_\sigma$ consists of all the valid dominoes of tiles of T, where in each domino, exactly one of the two tiles points at the other. The image of a domino by σ_{base} is then the concatenation of the pointing tile and the pointed tile, from left to right respectively, as illustrated below in an image by σ of a σ-loop of length 8:

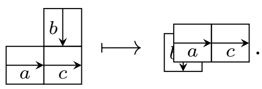


We can now finish the proof by showing that T admits a valid tiling of a cycle if and only if σ is not consistent. Indeed, suppose that T admits a valid tiling of a cycle $(a_1, \dots, a_n)$. To this cycle corresponds a $\mathcal{C}_\sigma$-loop $\gamma = (c_1, \dots, c_n)$ where the type of each c_i is (a_i, d_i) and the arrow d_i points at the cell c_{i+1}, for $1 \leqslant i < n$ (and $d_n = d_1$ so d_n points at c_2). However, we have $\omega_\sigma(\gamma) = (n-1, 0) \neq (0,0)$, so σ is not consistent, by Proposition 6.2.2.

Conversely, if T does not admit a valid tiling of a cycle then there cannot exist any simple $\mathcal{C}_\sigma$-loop, so σ is consistent thanks to Proposition 6.2.2. ⬠

Remark 6.3.2. The above proof yields a stronger version of Theorem 6.3.1: consistency is undecidable for two-dimensional domino-to-domino substitutions.

We can also prove that undecidability of consistency holds for non-overlapping substitutions, by modifying the above reduction slightly. Let us first note that the substitution produced in the above reduction is not non-overlapping, as can be seen for example if two cells point at a same arrow, in which case the two pointing cells will overlap in an image by σ:

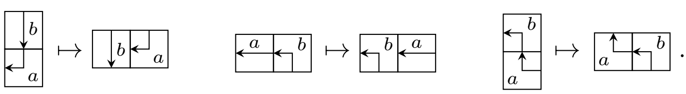


Hence, we want to make sure that the image of a pattern P can be computed only if a cell of P is pointed by *at most one* other cell of P.

The new reduction is then the following given a tile set T let σ be the two-dimensional substitution defined on the alphabet

$$\mathcal{A} = T \times \{\longrightarrow, \uparrow, \longleftarrow, \downarrow, ↳, ⬑, ↲, ⬏, ↱, ⬐, ↰, ⬎.\}$$

and such that $\mathcal{C}_\sigma$ consist of all the valid dominoes of tiles of T in which exactly one tile points at the other, but a tile is allowed to point at another if and only if the tip of the arrow of the pointing tile matches with the tail of the arrow of the pointed tile. The rules of σ behave similarly as in the above reduction: the pointed tile is put at the right of the pointing tile, as shown in the following three examples:



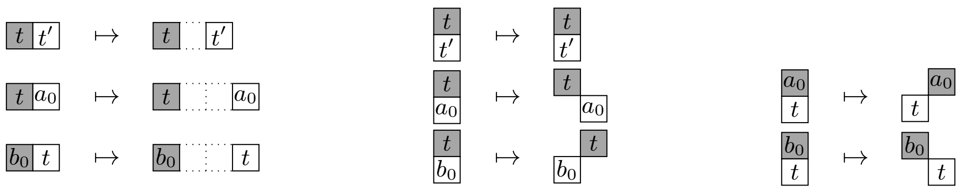
The substitution is non-overlapping, because the only patterns that admit an image by σ are paths or cycles decorated by matching arrows, whose images are necessarily made of non-overlapping cells.

Overlapping

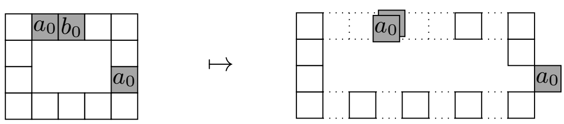
The weak cycle tiling problem can also be used to prove the undecidability of overlapping for consistent two-dimensional substitutions.

Theorem 6.3.3. *It is undecidable whether a two-dimensional consistent combinatorial substitution is overlapping.*

Proof. We will reduce the weak cycle tiling problem. Let T be a set of Wang tiles. Given two tiles $a, b \in T$ whose colors match in $\boxed{a}\boxed{b}$, let $\sigma_{a,b}$ be the two-dimensional substitution defined on the alphabet $\mathcal{A} = T \cup \{a_0, b_0\}$ where a_0 and b_0 are two new states, with the base rule $t \mapsto \{[(0,0), t]\}$ for all $t \in \mathcal{A}$ and the concatenation rules



for the $t, t' \in \mathcal{A} \setminus \{a_0, b_0\}$ such that the tiles match in left-hand sides of the above rules (we require a_0 and b_0 to match in the same way as a and b). On patterns without a_0 or b_0, the image of a pattern by $\sigma_{a,b}$ is a copy expanded horizontally by a factor of two (leaving one horizontal gap between horizontal neighbors). When a_0 or b_0 is in the pattern, the action of $\sigma_{a,b}$ is the same, and in addition it shifts every occurrence of a_0 to the right and every occurrence of b_0 to the left, as illustrated below. Note that the images of a_0 and b_0 are shifted but that the images of a and b are not (they are treated like the other cells).



The rule is hence consistent, and an overlap can happen only between the images of a cell of type a_0 and a cell of type b_0: an overlap occurs if and only if the image of $\boxed{a_0}\boxed{b_0}$ is computed (as shown in the above picture). It follows that $\sigma_{a,b}$ is overlapping if and only if there exists a Wang tile cycle of T that contains $\boxed{a}\boxed{b}$. Indeed, if there exists such a cycle, then the image of the corresponding pattern in which exactly one occurrence of $\boxed{a}\boxed{b}$ is replaced by $\boxed{a_0}\boxed{b_0}$ can be computed (it is $\mathcal{C}_{\sigma_{a,b}}$-covered) and will cause an overlap. Conversely, an overlap can only be caused by the computation of the image of $\boxed{a_0}\boxed{b_0}$. This is possible only if a cycle of T containing $\boxed{a}\boxed{b}$ exists, because $\boxed{a_0}\boxed{b_0}$ is not a starting pattern of $\sigma_{a,b}$.

Now we can finish the reduction. Given a set of Wang tiles T, compute $\sigma_{a,b}$ for all the tiles a, b whose colors match in $\boxed{a}\boxed{b}$. One of the substitutions $\sigma_{a,b}$ is overlapping if and only if T admits a tiling of a cycle. ⬠

Remark 6.3.4. The above proof yields a stronger version of Theorem 6.3.3: non-overlapping is undecidable for consistent two-dimensional domino substitutions. One can also prove that this holds even for domino-to-domino substitutions.

6.4 Decidability results

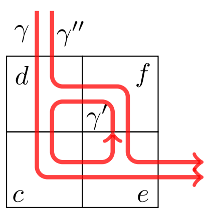
In this section we give algorithms to decide the consistency or the non-overlappingness of a natural class of substitutions: the substitutions σ that are ***domino-complete***, that is, such that the set of starting patterns $\mathcal{C}_\sigma$ is the set of *all* the possible dominoes.

Consistency for domino-complete substitutions

Theorem 6.4.1. *It is decidable whether a given two-dimensional domino-complete substitution is consistent. More precisely, such a substitution is consistent if and only if it is consistent on every* 2×2 *pattern.*

Proof. The "only if" implication is trivial. For the "if" implication, suppose that σ is not consistent. By Proposition 6.2.2, there exists a simple $\mathcal{C}_\sigma$-loop $\gamma = (c_1, \ldots, c_n)$ such that $\omega_\sigma(\gamma) \neq 0$. We will prove that there exists a 2×2 pattern on which σ is not consistent by "reducing" γ inductively.

Let c be the lowest cell on the leftmost column of γ. Since γ does not overlap itself, there exist two cells $d, e \in \gamma$ such that d is above c and e is at the right of c. We suppose, without loss of generality, that d, c, e appear in this order in γ, *i.e.*, $\gamma = (c_1, \ldots, c_i, d, c, e, c_{i+4}, \ldots, c_n)$. Let $f = [v + (1,1), t]$, where v is the vector of c and $t \in \mathcal{A}$ is arbitrary (or t agrees with γ if γ already contains a cell of vector $v + (1,1)$). Let $\gamma' = (c, e, f, d, c)$ and $\gamma'' = (c_1, \ldots, c_i, d, f, e, c_{i+4}, \ldots, c_n)$, as shown below.



We have $\omega_\sigma(\gamma') + \omega_\sigma(\gamma'') = \omega_\sigma(\gamma) \neq 0$, so $\omega_\sigma(\gamma') \neq 0$ or $\omega_\sigma(\gamma'') \neq 0$, which implies the existence of a $\mathcal{C}_\sigma$-loop (γ' or γ'') with nonzero image vector which surrounds strictly less cells than γ (unless γ consists 4 cells already). Now, in the same way as in the second part of the proof of Proposition 6.2.2, γ' or γ'' must contain a simple loop with nonzero image vector. Applying this reasoning inductively eventually leads to a 2×2 loop γ such that $\omega_\sigma(\gamma) \neq 0$, which concludes the proof. ⬠

Generalization to domino-completeness within a set of patterns We now want to generalize Theorem 6.4.1 to substitutions that are domino-complete only within a particular set of patterns. Let us first state a few definitions. If $\mathcal{P}$ is a set of patterns, we say that σ is ***$\mathcal{P}$-domino-complete*** if $\mathcal{C}_\sigma$ is equal to the set of dominoes that appear in the patterns of $\mathcal{P}$. If $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^2}$, we denote by $\text{patt}(\mathcal{S})$ the set of the patterns that appear in the elements of $\mathcal{S}$. (That is, a pattern is in $\text{patt}(\mathcal{S})$ if it can be extended to an element of $\mathcal{S}$.)

Theorem 6.4.2 below gives a simple criterion to determine if a $\mathcal{P}$-domino-complete substitution is consistent when $\mathcal{P}$ is the set of all the 2×2 patterns of some $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^2}$. Note that to decide this property, we must be able to compute the 2×2 patterns of $\mathcal{S}$, which is not necessarily possible. Note that Theorem 6.4.1 can be seen as the particular case of Theorem 6.4.2 when $\mathcal{S} = \mathcal{A}^{\mathbb{Z}^2}$.

Theorem 6.4.2. *Let $\mathcal{S} \subseteq \mathcal{A}^{\mathbb{Z}^2}$ and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be the list of the 2×2 patterns that appear in the elements of $\mathcal{S}$. A $\mathcal{P}$-domino-complete substitution is consistent on* $\mathrm{patt}(\mathcal{S})$ *if and only if it is consistent on the patterns $P_1, \dots, P_n$.*

Proof. Let P be a pattern of $\mathrm{patt}(\mathcal{S})$ that contains a $\mathcal{C}_\sigma$-loop γ such that $\omega_\sigma(\gamma) \neq 0$. We cannot directly reduce the loop as in the proof of Theorem 6.4.1 because σ is not domino-complete. However, there exists $c \in \mathcal{S}$ that contains P and σ is $\mathcal{P}$-domino complete, so we can then reduce γ within c to a 2×2 loop, as explained in the proof of Theorem 6.4.1. It follows that σ is consistent on $\mathrm{patt}(\mathcal{S})$ if and only if it is consistent on $P_1, \dots, P_n$. ⬠

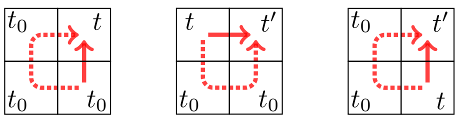
Overlapping for consistent domino-complete substitutions

We now focus on the domino-complete substitutions that are consistent. Proposition 6.4.3 below tells us that such substitutions are simple: there exists $\alpha, \beta \in \mathbb{Z}^2$ such that the image of the cell placed at (x, y) is placed at $(x\alpha, y\beta)$ in the lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$. We will use this proposition to give an algorithm that decides if a consistent domino-complete is overlapping (Theorem 6.4.4).

Unfortunately, we are not able to give an analogue of Theorem 6.4.2 for the non-overlapping property, because the associated decision problems seem to become too difficult to track.

Proposition 6.4.3. *Let σ be a consistent two-dimensional domino-complete substitution. There exist two vectors $\alpha, \beta \in \mathbb{Z}^2$ and a vector $v_t \in \mathbb{Z}^2$ for every $t \in \mathcal{A}$ such that for every $\mathcal{C}_\sigma$-path γ from a cell $[(0,0), t]$ to a cell $[(x,y), t']$, we have $\omega_\sigma(\gamma) = x\alpha + y\beta - v_t + v_{t'}$. Moreover, α, β and v_t can be obtained effectively.*

Proof. Let $t_0 \in \mathcal{A}$ be arbitrary, where $\mathcal{A}$ is the alphabet of σ. Let $\alpha = \omega_\sigma(\gamma)$ and $\beta = \omega_\sigma(\gamma')$, where $\gamma = ([(0,0), t_0], [(1,0), t_0])$ and $\gamma' = ([(0,0), t_0], [(0,1), t_0])$. For $t \in \mathcal{A}$, define $v_t = \omega_\sigma(\gamma) - \alpha$, where $\gamma = ([(0,0), t_0], [(1,0), t])$. We first prove the theorem for paths of length two. Because σ is consistent and domino-complete, we can for example use the following patterns



to compute the values

$$
\begin{aligned}
\omega_\sigma([(0,0), t_0], [(0,1), t]) &= -\alpha + \beta + \alpha + v_t &&= \beta + v_t \\
\omega_\sigma([(0,0), t], [(1,0), t']) &= -\beta - v_t + \alpha + \beta + v_{t'} &&= \alpha - v_t + v_{t'} \\
\omega_\sigma([(0,0), t], [(0,1), t']) &= -\alpha - v_t + \beta + \alpha + v_{t'} &&= \beta - v_t + v_{t'}
\end{aligned}
$$

which proves the statement for paths of length two. The statement for arbitrary paths follows directly, by adding the consecutive dominoes along the path. ⬠

Theorem 6.4.4. *It is decidable whether a given two-dimensional consistent domino-complete substitution σ is overlapping.*

Proof. By Proposition 6.4.3, we can compute $\alpha, \beta, v_t, v_{t'} \in \mathbb{Z}^2$ such that $\omega_\sigma(\gamma) = x\alpha + y\beta - v_t + v_{t'}$ for every $\mathcal{C}_\sigma$-path γ from a cell $[(0,0), t]$ to a cell $[(x,y), t']$.

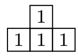
Denote $A_t = \mathrm{supp}(\sigma_{\mathrm{base}}(t))$ for $t \in \mathcal{A}$. By definition, σ is overlapping if and only if there exists $t, t' \in \mathcal{A}$ and $x, y \in \mathbb{Z}$ such that $A_t \cap \{b + x\alpha + y\beta - v_t + v_{t'} : b \in A_{t'}\} \neq \varnothing$. This leaves a finite number of linear equations to check: for each $(t, t') \in \mathcal{A}^2$, we check if there exists $a \in A_t$ and $b \in A_{t'}$ such that the following equation has a nonzero solution $(x, y) \in \mathbb{Z}^2$:

$$a \;=\; b + x\alpha + y\beta - v_t + v_{t'}$$

This can be done algorithmically and σ is overlapping if and only if such a solution exists. ⬠

Note that some conditions equivalent to consistency and non-overlapping have been given in some particular examples in [**Del11**].

Example 6.4.5. Let σ be the two-dimensional substitution on the alphabet $\{1\}$ defined by the base rule

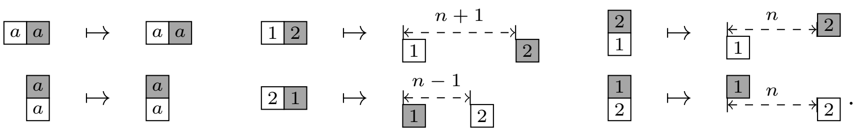
$$1 \;\mapsto\; \{[(0,0),1],[(1,0),1],[(2,0),1],[(1,1),1]\} \;=$$ 

and the concatenation rules $(1, 1, (1, 0)) \mapsto (3, 0)$ and $(1, 1, (0, 1)) \mapsto (0, 2)$. This substitution is domino-complete and consistent. Proposition 6.4.3 applied to σ gives $\omega_\sigma(\gamma) = (3x, 0) + (0, 2y)$ for every $\mathcal{C}_\sigma$-path γ from a cell $[(0, 0), 1]$ to a cell $[(x, y), 1]$. Note that in this case, $v_1 = (0, 0)$.

Example 6.4.6. For $n \geqslant 0$, let σ_n be the two-dimensional substitution on the alphabet $\{1, 2\}$ defined by the base rule $1 \mapsto \{[(0, 0), 1]\}$, $2 \mapsto \{[(0, 0), 2]\}$ and

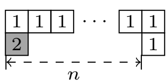
$$\begin{array}{lll@{\qquad\qquad}lll}
(1,1,(1,0)) & \mapsto & (1,0) & (1,2,(1,0)) & \mapsto & (n+1,0)\\
(1,1,(0,1)) & \mapsto & (0,1) & (1,2,(-1,0)) & \mapsto & (n-1,0)\\
(2,2,(1,0)) & \mapsto & (1,0) & (1,2,(0,1)) & \mapsto & (n,1)\\
(2,2,(0,1)) & \mapsto & (0,1) & (1,2,(0,-1)) & \mapsto & (n,-1),
\end{array}$$

or more graphically:



The substitution σ_n is domino-complete and consistent for all n. Proposition 6.4.3 applied to σ_n gives $\omega_\sigma(\gamma) = (x, y) - v_t + v_{t'}$ for every $\mathcal{C}_\sigma$-path γ from a cell $[(0, 0), t]$ to a cell $[(x, y), t']$, where $v_1 = (0, 0)$ and $v_2 = (n, 0)$. This gives an example of a substitution with at least one nonzero v_t.

This example is also interesting because it is overlapping, but only on sufficiently large patterns. Indeed, it is non-overlapping on patterns of horizontal diameter smaller than n, but overlapping on larger patterns such as

$$P_n \;=$$  $$=\; [(0,0),2] \cup [(n+1,0),1] \cup \{[(i,1),1] : 0 \leqslant i \leqslant n+1\}.$$

This shows that the overlapping property cannot be decided as simply as consistency, where looking at the 2×2 patterns was sufficient. Now the size of the rules has to be taken into account, which explains why the algorithm of Theorem 6.4.4 is not as simple as the algorithm of Theorem 6.4.1.

6.5 Links with dual substitutions

The subshift of discrete surfaces

As mentioned earlier, combinatorial substitution as introduced in this chapter were originally motivated by the study of dual $\mathbf{E}_1^\star$ substitutions. A formal correspondence between patterns of $\{1,2,3\}^{\mathbb{Z}^2}$ and $\mathbf{E}_1^\star$ patterns made of unit faces can be found in [**ABI02, ABS04**], but it is intuitively summarized by Figure 6.2: point the $\mathbf{E}_1^\star$ faces so that all the pointings lie on a two-dimensional lattice, and label the lattice vertices with the types of the faces.

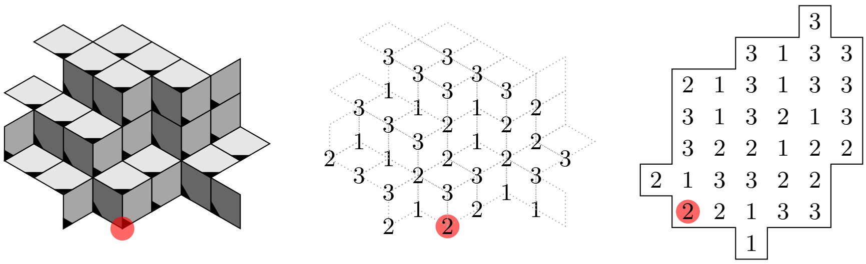
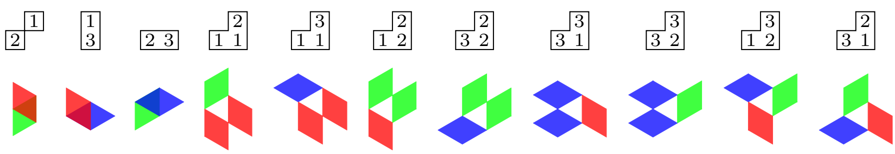


Figure 6.2: It is possible to define a bijective mapping φ from the patterns that are contained in the set $\mathcal{S}_{\text{surf}} \subseteq \{1,2,3\}^{\mathbb{Z}^2}$ defined below and the patterns of lozenges that are codings of discrete surfaces. A detailed definition can be found in [**ABI02, ABS04**].

Let $\mathcal{S}_{\text{surf}} \subseteq \{1,2,3\}^{\mathbb{Z}^2}$ be the set configurations whose set of allowed 2×2 patterns is the following set $\mathcal{P}_{\text{surf}}$ of 28 patterns

$\begin{smallmatrix}1&1\\1&1\end{smallmatrix}$	$\begin{smallmatrix}2&1\\1&1\end{smallmatrix}$	$\begin{smallmatrix}3&1\\1&1\end{smallmatrix}$	$\begin{smallmatrix}1&1\\1&2\end{smallmatrix}$	$\begin{smallmatrix}2&1\\1&2\end{smallmatrix}$	$\begin{smallmatrix}3&1\\1&2\end{smallmatrix}$	$\begin{smallmatrix}1&2\\1&3\end{smallmatrix}$	$\begin{smallmatrix}1&3\\1&3\end{smallmatrix}$	$\begin{smallmatrix}2&2\\1&3\end{smallmatrix}$	$\begin{smallmatrix}3&2\\1&3\end{smallmatrix}$	$\begin{smallmatrix}3&3\\1&3\end{smallmatrix}$	$\begin{smallmatrix}1&2\\2&1\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&1\end{smallmatrix}$	$\begin{smallmatrix}2&2\\2&1\end{smallmatrix}$
$\begin{smallmatrix}3&2\\2&1\end{smallmatrix}$	$\begin{smallmatrix}3&3\\2&1\end{smallmatrix}$	$\begin{smallmatrix}1&2\\2&2\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&2\end{smallmatrix}$	$\begin{smallmatrix}2&2\\2&2\end{smallmatrix}$	$\begin{smallmatrix}3&2\\2&2\end{smallmatrix}$	$\begin{smallmatrix}3&3\\2&2\end{smallmatrix}$	$\begin{smallmatrix}2&1\\3&1\end{smallmatrix}$	$\begin{smallmatrix}3&1\\3&1\end{smallmatrix}$	$\begin{smallmatrix}2&1\\3&2\end{smallmatrix}$	$\begin{smallmatrix}3&1\\3&2\end{smallmatrix}$	$\begin{smallmatrix}2&2\\3&3\end{smallmatrix}$	$\begin{smallmatrix}3&2\\3&3\end{smallmatrix}$	$\begin{smallmatrix}3&3\\3&3\end{smallmatrix}$

(each pattern can be extended to a configuration in $\mathcal{S}_{\text{surf}}$). Equivalently, $\mathcal{S}_{\text{surf}}$ is the set of elements of $\{1,2,3\}^{\mathbb{Z}^2}$ that do not contain any of the following 11 forbidden patterns. For each pattern we have plotted the corresponding $\mathbf{E}_1^\star$ pattern (using the correspondence described above). We see that for each forbidden pattern, there is either an overlap, or a pattern that cannot be extended to an infinite configuration.



It has been proved [**Jam04**] that $c \in \mathcal{S}_{\text{surf}}$ if and only if c is the coding of a discrete surface made of $\mathbf{E}_1^\star$ faces. (See [**ABFJ07**] for more about these *stepped surfaces*.) Intuitively, sufficiently many forbidden patterns are enough, because in a “problematic” configuration, the problematic patterns (causing overlaps or gaps) can be reduced inductively to patterns of minimal size. The 11 patterns above are all possible “normal forms” after reduction.

From dual substitutions to combinatorial substitutions

We now describe one way of constructing examples of combinatorial substitutions, by starting from a dual substitution: take a dual substitution $\Sigma = \mathbf{E}_1^\star(\sigma)$ and compute (using Definition 1.2.3) the images of all the two-face patterns corresponding to the allowed dominoes in $\mathcal{S}_{\text{surf}}$. This yields a set of concatenation rules, and the three base rules are given by the image by Σ of each face. This is illustrated in Example 6.5.2 for the substitution $1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2$, an example also given in [**Fer07a**].

Remark 6.5.1. Constructing a combinatorial substitution from $\mathbf{E}_1^\star(\sigma)$ as described above can be done for every unimodular 3-letter substitution σ. However, to be able to iterate the resulting combinatorial substitution, we need its image patterns to be covered by its starting patterns (as specified in the definition of the image of a combinatorial substitution in Section 6.2). This property is far from being true in general. For example, if the image patterns are disconnected (which can happen with $\mathbf{E}_1^\star(\sigma)$ substitutions), they will not be covered by dominoes (or any set of connected patterns). In Example 6.5.2 below, we are "lucky" because the forward images of 1 are all covered by the starting patterns of the substitution, which allows us to iterate it indefinitely.

Note that finding such a set of rules which allows iteration is equivalent to finding a finite set $\mathcal{L}$ of $\mathbf{E}_1^\star$ patterns such that the forward images of $\mathbf{E}_1^\star(\sigma)$ are all $\mathcal{L}$-covered (see Section 2.2).

Now, using Proposition 1.2.4, it can be proved that the resulting combinatorial substitution is consistent and non-overlapping. (We actually need a stronger variant of the proposition that holds in the more general case of discrete surfaces, which exists in [**ABFJ07**]). However, such a proof is specific to substitutions constructed in this way, and relies on some "scalar product inequalities" arguments about discrete surfaces and dual substitutions. The authors of [**ABS04**] have asked for a more generic, "purely combinatorial" proof in the particular case of the substitution of Example 6.5.2:

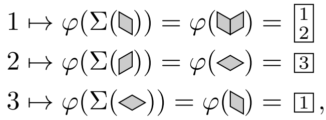
> Unfortunately, although this theorem appears to be a purely combinatorial result, we do not know any combinatorial proof of it, and we would be very interested in such a proof.

Such generic proofs have also been requested in [**PF02**]. The results presented in Section 6.4 partially answer these questions. The situation is different for consistency and non-overlapping, as described below.

Consistency in $\mathcal{S}_{\text{surf}}$ Checking if a given $\mathcal{P}_{\text{surf}}$-domino-complete substitution on patt $\mathcal{S}_{\text{surf}}$, is consistent can be reduced to simply checking its consistency on the 28 patterns in $\mathcal{P}_{\text{surf}}$, thanks to Theorem 6.4.2.

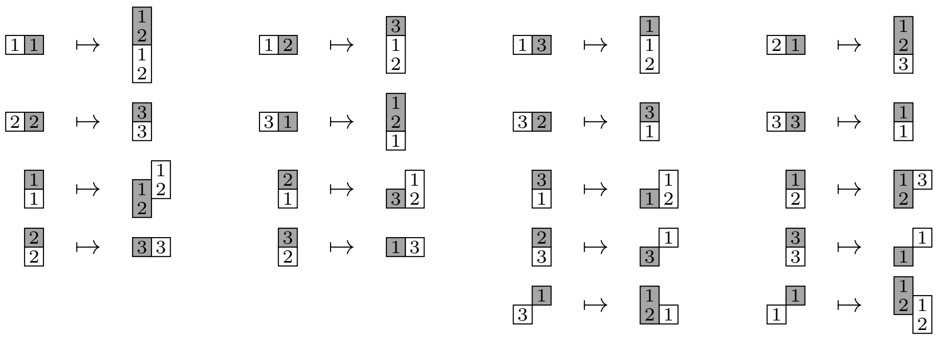
Overlapping in $\mathcal{S}_{\text{surf}}$ Checking the overlapping property on patt$(\mathcal{S}_{\text{surf}})$ for $\mathcal{P}_{\text{surf}}$-domino-complete substitutions is not as easy as checking consistency, because we have not proved an "enhanced" version of Theorem 6.4.4 as we did with Theorem 6.4.2 for Theorem 6.4.1. We do it "by hand" in Proposition 6.5.3 for the particular substitution of Example 6.5.2.

Example 6.5.2. Let $\Sigma = \mathbf{E}_1^\star(1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2)$. Using the correspondence φ described in Figure 6.2, we define a two-dimensional substitution σ on patterns of $\mathcal{S}_{\text{surf}} \subseteq \{1,2,3\}^{\mathbb{Z}^2}$ corresponding to the dual substitution Σ. The base is rule is given by



and the concatenation rules can be obtained by computing the images of the $\mathbf{E}_1^\star$ patterns corresponding to dominoes in $\mathcal{S}_{\text{surf}}$, using the mapping φ. For example the images of the dominoes $\boxed{2}\boxed{1}$ and $\genfrac{}{}{0pt}{}{\boxed{3}}{\boxed{1}}$ are computed in the following way:

In total this gives the 16 concatenation rules below.



This substitution is the main object of study of [**ABS04**] and it also appears in [**Fer07a**]. It is $\mathcal{P}_{\text{surf}}$-domino-complete, and it is consistent and non-overlapping on $\text{patt}(\mathcal{S}_{\text{surf}})$. Note that the last two rules $\genfrac{}{}{0pt}{}{\quad\boxed{1}}{\boxed{3}\quad}$ and $\genfrac{}{}{0pt}{}{\quad\boxed{1}}{\boxed{1}\quad}$ have been given in order for the substitution to be iterable indefinitely when starting from any valid domino.

We can now recover the following properties of the substitution of Example 6.5.2 in a combinatorial way, without having to use the inherent to $\mathbf{E}_1^\star$ substitutions.

Proposition 6.5.3. *The substitution of Example 6.5.2 is consistent and non-overlapping on* $\text{patt}(\mathcal{S}_{\text{surf}})$.

Proof. Consistency can be checked algorithmically, thanks to Theorem 6.4.2. To prove that it is non-overlapping, let $c = [(0,0), t]$ and $c' = [(x,y), t']$ be two distinct cells belonging to a pattern $P \in \text{patt}(\mathcal{S}_{\text{surf}})$. There exists a $\mathcal{C}_\sigma$-path from c to c' that consists of a horizontal segment followed by a vertical segment, because σ is $\mathcal{P}_{\text{surf}}$-domino-complete and P is extendible to an element of $\mathcal{S}_{\text{surf}}$. Along the horizontal segment, all image vectors are of the form $(0, i)$, because the right-hand sides of the rules of σ for horizontal dominoes have the image patterns aligned vertically. Along the vertical segment, all image vectors are of the form $(-1, i)$ when moving upward, because the images of the vertical dominoes are aligned that way. It follows that $\omega_\sigma(\gamma) = (-y, i)$ with $i \in \mathbb{Z}$. Since the image patterns have width 1, it is clear that there is no overlap between the images of c and c' if $y \neq 0$. If $y = 0$ then we are considering a horizontal path, and it is clear that such path induces no overlap either because the images are stacked on top of each other. ⬠

6.6 Conclusion

In this chapter we focused on consistency and overlapping properties, because they are the first properties that one should look at, in order to make sure that a substitution is "well defined". There are many other interesting properties whose decidability status has not been investigated yet. Some of them are:

- Given a substitution σ and a pattern P, can σ be iterated on the successive images of P by σ? (That is, are the successive images of P by σ all $\mathcal{C}_\sigma$-covered?)
- Given a substitution σ and a pattern P, are the iterates of σ on P (simply) connected?
- One property that would be very interesting to decide is whether the iterates of a given substitution σ on a given pattern P contains arbitrarily large balls? As we have seen in Section 1.4, in the case of a dual $\mathbf{E}_1^\star(\sigma)$ substitution this property is equivalent to the Pisot conjecture for σ. Even more interesting would be to be able to decide such properties for the iterations of a finite set of substitutions, hence tackling questions similar to the discrete plane generation questions addressed in Chapter 2.

More generally, as pointed out in [**Fra08**], there are many things to discover about the emerging class of combinatorial substitutions, whose study is still at its beginnings. It would be interesting to find some families of relevant examples which do not come from dual $\mathbf{E}_1^\star$ substitutions.

Towards topological substitutions Let us mention some work in progress with Nicolas Bédaride and Arnaud Hilion [**BHJ13**], where we study some particular examples of *topological substitutions* which are a less "rigid" and more geometrical version of combinatorial substitutions.

We aim to describe how the examples of combinatorial substitutions coming from $\mathbf{E}_1^\star$ substitutions can sometimes be expressed in the form of a topological substitution. This provides us with an alternative viewpoint of the combinatorial properties of the original combinatorial substitution, but also gives new interesting examples of topological substitutions, for which not much examples are known at the moment.

This has been carried out for the combinatorial substitution of Example 6.5.2; as illustrated in Figure 6.3.

Topological substitutions have been introduced in [**BH12**], where it is proved that no such substitution can be primitive and generate a tiling of the hyperbolic plane. (An explicit non-primitive example for which such a tiling exists is also given in [**BH12**].)

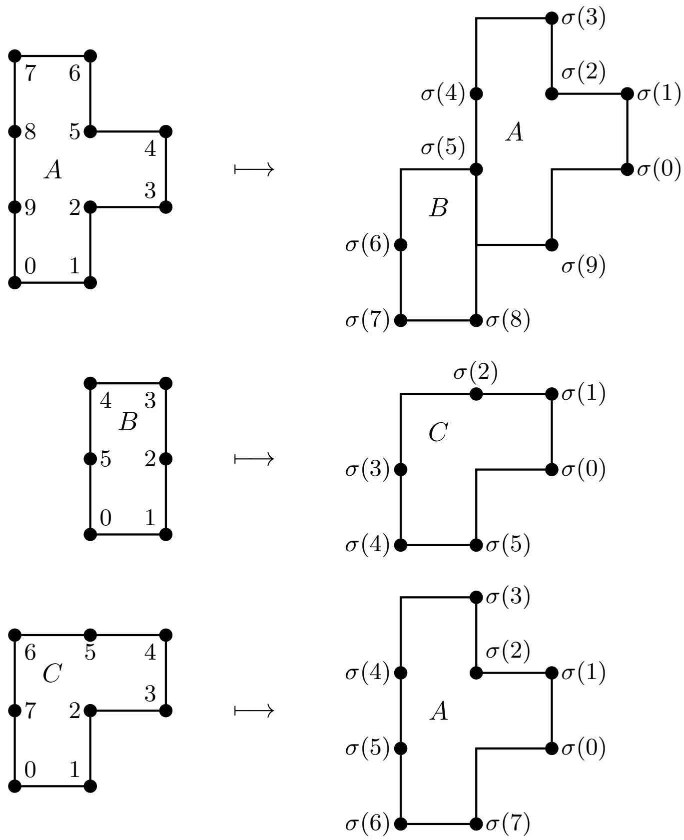


Figure 6.3: Topological substitution associated with $\mathbf{E}_1^\star(1 \mapsto 13, 2 \mapsto 1, 3 \mapsto 2)$, obtained from the combinatorial substitution given in Example 6.5.2.

Chapter 7

Undecidable properties of self-affine sets

This final chapter is devoted to establishing the undecidability of some properties of self-affine sets specified by a graph-directed iterated function system (GIFS). We focus on topological properties such as interior emptiness. The situation contrasts with the case of Rauzy fractals, for which many such properties are decidable. This is joint work with Jarkko Kari [**JK13**].

Contents

7.1 Introduction

Iterated function systems

One of the most common ways to define fractals is to use an ***iterated function system (IFS)***, defined by a finite collection of maps $f_1, \ldots, f_n : \mathbb{R}^d \to \mathbb{R}^d$ which are all ***contracting***: there exists $0 \leqslant c < 1$ such that $\|f_i(x) - f_i(y)\| \leqslant c\|x - y\|$ for all $x, y \in \mathbb{R}^d$. The associated fractal, called the ***attractor*** of the IFS, is the unique nonempty compact set R such that

$$R = \bigcup_{i=1}^{n} f_i(R).$$

Such a set R always exists and is unique thanks to a famous result of Hutchinson [**Hut81**], based on an application of Banach fixed-point theorem; see also [**Fal03**] or [**Bar93**]. For example, the classical Cantor set can be defined as the unique compact set $X \subseteq \mathbb{R}$

satisfying the set equation $X = \frac{1}{3}X \cup (\frac{1}{3}X + \frac{2}{3})$, and the Sierpiński triangle can defined as the unique compact set $X \subseteq \mathbb{R}^2$ satisfying $X = \frac{1}{2}X \cup (\frac{1}{2}X + (1/2, 0)) \cup (\frac{1}{2}X + (0, 1/2))$.

A natural generalization of IFS can be obtained by restricting which infinite sequences of maps $(f_{i_n})_{n\in\mathbb{N}}$ we are allowed to iterate. One of the simplest such restrictions is to require the set of allowed sequence $(i_n)_{n\in\mathbb{N}}$ to be the language of the infinite paths of a finite graph. Doing so we can give a new definition: a d-dimensional ***graph-directed iterated function system (GIFS)*** is a directed graph in which each edge e is labelled by a contracting mapping $f_e : \mathbb{R}^d \to \mathbb{R}^d$. The ***attractors*** of the GIFS are the unique nonempty compact sets $\{R_q\}_{q\in\mathcal{Q}}$ such that

$$R_q = \bigcup_{q\in\mathcal{Q}} \bigcup_{e\in E_{q,r}} f_e(R_r),$$

where $\mathcal{Q}$ is the set of vertices of the directed graph defining the GIFS, and $E_{q,r}$ denote the set of edges from vertex q to vertex r. Again, such a collection of compact set $\{R_q\}_{q\in\mathcal{Q}}$ exist and are unique [**Fal97**].

Self-affine and self-similar sets

Many works are focused on the more specific family of ***self-affine attractors***, in which the contractions f_i must be affine (of the form $M_i x + v_i$ where M_i is a $d \times d$ matrix and $v_i \in \mathbb{R}^d$), or the even more constrained family of ***self-similar attractors***, in which the f_i must be similarities (of the form $ax + v_i$ where $a \in [0, 1[$ and $v_i \in \mathbb{R}^d$).

Self-affine attractors are intensively studied, and many results are known about some particular families. For example the Hausdorff dimension of Bedford-McMullen carpets (which are described in Example 7.3.5) admits an exact simple formula [**Bed84, McM84**], and similar results about the fractal dimension or the Lebesgue measure of some other classes exist [**LG92, FW05, Bar07, BK11, Fra12**]. Moreover, there is an "almost sure" formula for the packing and Hausdorff dimension in the self-similar case [**Fal88**].

Despite all the positive results stated above, the notorious difficulty of self-affine sets suggests that there cannot exist any simple criteria to decide such properties in full generality. From a computer-theoretical point of view, this would correspond to undecidability results of the type: "there cannot be an algorithm that, given input an IFS specified by rational coefficients, determines if Property X holds for the IFS attractor", where "Property X" can be any IFS attractor property we are interested in.

A first undecidability result has been established by Dube [**Dub93**]: it is undecidable if the attractor of a rational 2-dimensional affine IFS intersects the diagonal $\{(x, x) : x \in [0, 1]\}$, or if each point in the attractor has a unique address. (This latter property is referred to as "totally disconnected" in [**Dub93**], but it is not the same the attractor being totally disconnected in the topological sense.)

Our results

The aim of this chapter is to prove the undecidability of some topological properties of self-affine graph-directed iterated function systems. Most notably, we prove that it is undecidable if the attractor of a 2-dimensional, 3-state affine GIFS has empty interior (Theorem 7.4.2). Other related GIFS attractor undecidability results are obtained in Theorems 7.4.1 and 7.4.3.

To do so, we take the approach of Dube [**Dub93**] and we associate self-affine sets with computational devices called *multitape automata*, which are finite automata acting on several tapes, with an independent head reading each tape. Then we relate some properties of the automaton with topological properties of its associated attractor, and we obtain the undecidability of the latter by proving the undecidability of the former; see 7.3.

All the GIFS that will appear in our constructions are ***box-like***, in the sense that all the considered affine maps take $[0,1]^d$ to a shrinked copy of itself, with faces parallel to the axes. Our undecidability results hence yield undecidability results for the particular class of box-like self-affine sets specified by rational coefficients. The class of box-like affine sets include all the families mentioned above, for which "positive" results exit.

Links with Rauzy fractals

The undecidability results of this chapter can be interestingly compared with the case of Rauzy fractal, where much more things are decidable.

In Section 1.3 we have seen that many topological properties of Rauzy fractals can be algorithmically checked. The Hausdorff dimension of the boundary of a planar Rauzy fractal can be computed if its corresponding Pisot eigenvalue has complex conjugates, because the associated GIFS is then self-similar. In contrast, no formula is known if the two conjugates of the Pisot eigenvalue are real, because then their norms are not equal, so the GIFS is self-affine but not self-similar. This reflects the difficulty of the study of self-affine sets.

There are several algorithms to decide if the tiles of the Rauzy fractal of a unimodular Pisot substitution σ do not overlap, that is, if they intersect on a set of Lebesgue measure zero. In the case of Rauzy fractals, this is equivalent to having intersection with empty interior. It follows that the undecidable property that we stated in Theorem 7.4.3 is actually decidable for the case of Rauzy fractals.

Finally, note that the family of Rauzy fractals GIFS is disjoint from the family of the GIFS associated with multitape automata, for the simple reason that negative powers of integers cannot be the expansion factors of a Rauzy fractal GIFS, which, in opposition, is always the case for multitape automata GIFS; see Remark 7.4.4.

7.2 Multitape automata

Definitions

A ***d-tape automaton*** $\mathcal{M}$ on alphabet $\mathcal{A} = A_1 \times \cdots \times A_d$ is defined by:

- a finite set of ***states*** $\mathcal{Q}$ of $\mathcal{M}$,
- a finite set of ***transitions*** $\mathcal{R} \subseteq \mathcal{Q} \times \mathcal{Q} \times (A_1^+ \times \cdots \times A_d^+)$.

A d-tape automaton on state $\mathcal{Q}$ is conveniently represented by a directed graph with vertex set $\mathcal{Q}$ and an edge (q,r) labelled by $w_1|\cdots|w_d$ for every transition $(q,r,(w_1,\ldots,w_d))$. This is illustrated in Example 7.2.1.

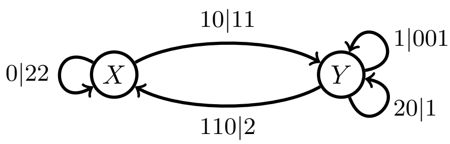
A ***configuration*** is an infinite sequence $c \in \mathcal{A}^{\mathbb{N}} = (A_1 \times \cdots \times A_d)^{\mathbb{N}}$. For $k \in \{1,\ldots,d\}$, the ***kth tape*** of c refers to the infinite sequence $((c_n)_k)_{n\in\mathbb{N}}$, which is an infinite concatenation of words in $A_k^\star$. For convenience, configurations will be denoted by writing the tape components separated the symbol "|". For example, $00\cdots\,|\,11\cdots\,|\,00\cdots$ denotes the 3-tape configuration $(0,1,0),(0,1,0),\ldots \in (\{0,1\}\times\{0,1\}\times\{0,1\})^{\mathbb{N}}$.

Let q be a state of $\mathcal{M}$. A configuration $c \in A^{\mathbb{N}}$ is ***q-accepted*** by $\mathcal{M}$ if there exists an infinite sequence of transitions

$$\begin{aligned}&(q_1, r_1, (w_{1,1}, \ldots, w_{1,d})),\\ &(q_2, r_2, (w_{2,1}, \ldots, w_{2,d})),\\ &\ldots\end{aligned}$$

such that $q_1 = q$, $r_n = q_{n+1}$ for all $n \geqslant 1$, and for every $k \in \{1, \ldots, d\}$, the infinite word $w_{1,k}w_{2,k}\ldots$ is equal to the kth tape of c (that is, $w_{1,k}w_{2,k}\ldots = (c_1)_k(c_2)_k\ldots$). Such an infinite sequence of transitions will sometimes be referred to as a ***run of $\mathcal{M}$ starting at** q*. Note that the above definition of acceptance forbids ε-transitions because the words $w_1, \ldots, w_d$ used in transitions must be nonempty.

Example 7.2.1. Consider the following 2-tape, 2-state automaton on alphabet $\mathcal{A} = \{0,1\}\times\{0,1,2\}$, with state set $\mathcal{Q} = \{X, Y\}$ and transitions given by the following.



It is easy to check that the configuration $00\cdots|22\cdots$ is not Y-accepted but is X-accepted by $\mathcal{M}$ (by repeatedly using the transition $(X, X, (0, 22)))$. However, giving a precise description of the set of configurations which are accepted by $\mathcal{M}$ seems difficult.

Remark 7.2.2. Multitape automata are very powerful computational devices because of the fact that the words $w_1, \ldots, w_d$ in a transition are allowed to have different lengths. This is the fundamental feature that will allow us to establish several undecidability results about multitape automata later in this section. On the other hand, if the words $w_1, \ldots, w_d$ all have the same length in every transition, then it is easy to see that the automaton is not more powerful than a classical finite automaton on a product alphabet.

Post correspondence problems

The undecidability results of this chapter are all derived from the undecidability of the following decision problems. The ***Post correspondence problem (PCP)*** is defined by:

- **Instance:** A list of pairs of nonempty words $(u_1, v_1), \ldots, (u_n, v_n)$.
- **Question:** Do there exist $m \geqslant 1$ and a word $i_1\cdots i_m$ such that $u_{i_1}\cdots u_{i_m} = v_{i_1}\cdots v_{i_m}$?

PCP is a well-known undecidable problem [**Pos46**]. We will actually need a slight variant of PCP, the ***prefix Post correspondence problem (prefix-PCP)***, defined by:

- **Instance:** A list of pairs of nonempty words $(u_1, v_1), \ldots, (u_n, v_n)$.
- **Question:** Do there exist $m, m' \geqslant 1$ and two words $i_1\cdots i_m$ and $i_1\cdots i_{m'}$ such that $u_{i_1}\cdots u_{i_m} = v_{i_1}\cdots v_{i_{m'}}$ and one of the two words $i_1\cdots i_m$ and $i_1\cdots i_{m'}$ is a prefix of the other?

A valid PCP always yields a valid prefix-PCP instance (by taking $m = m'$), but the converse is not always true. For example, the instance $(u_1, v_1) = (a, abb)$, $(u_2, v_2) = (bb, aa)$ admits the prefix-PCP solution given by $u_1u_2u_1u_1 = v_1v_2 = aabbaa$, that is, $m = 4$, $m' = 2$ and the two words $i_1i_2i_3i_4 = 1211$ and $i_1i_2 = 12$. However, this instance cannot admit any PCP solution because no pair of words ends by the same symbol.

Lemma 7.2.3. *Prefix-PCP is undecidable.*

Proof. We reduce PCP to prefix-PCP. Let $(u_1, v_1), \ldots, (u_n, v_n)$ be an instance of PCP on alphabet $\mathcal{A}$. Let $\mathcal{B} = \mathcal{A} \cup \{\texttt{\#}, \texttt{*}\}$ be a new alphabet, where # and * are two new symbols not contained in $\mathcal{A}$. We construct a prefix-PCP instance (A_1, B_1), ..., (A_n, B_n), (U_1, V_1), ..., (U_n, V_n), (Y, Z) on the new alphabet $\mathcal{B}$, defined by

$$
\begin{aligned}
A_i &= \texttt{\#}x_1\texttt{*}x_2\texttt{*}\cdots\texttt{*}x_k & \qquad U_i &= \texttt{*}x_1\texttt{*}x_2\texttt{*}\cdots\texttt{*}x_k & \qquad Y &= \texttt{*\#} \\
B_i &= \texttt{\#}y_1\texttt{*}y_2\texttt{*}\cdots\texttt{*}y_\ell\texttt{*} & V_i &= \phantom{\texttt{*}}y_1\texttt{*}y_2\texttt{*}\cdots\texttt{*}y_\ell\texttt{*} & Z &= \phantom{\texttt{*}}\texttt{\#}
\end{aligned}
$$

for all $i \in \{1, \ldots, n\}$, where $u_i = x_1 \cdots x_n$ and $v_i = y_1 \cdots y_\ell$ and the x_j, y_j are in $\mathcal{A}$. We now prove that the PCP instance has a solution if and only if the prefix-PCP instance has a solution. Suppose that there exists a solution $i_1 \cdots i_m$ to the PCP instance, that is $u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_m}$. Then clearly the prefix-PCP also has a solution, given by $A_{i_1}U_{i_2} \cdots U_{i_m}Y = B_{i_1}V_{i_2} \cdots V_{i_m}Z$.

Conversely, suppose that the prefix-PCP instance has a solution. By construction, because of # and *, there must exist a prefix-PCP solution of the form $A_{i_1}U_{i_2} \cdots V_{i_m}Y = B_{i_1}V_{i_2} \cdots V_{i_{m'}}Z$, where $i_1 \cdots i_m$ is a prefix of $i_1 \cdots i_{m'}$ or vice-versa. But the pairs (U_i, V_i) do not contain any #, so the pair (Y, Z) is used exactly once, both after mth pair and the m'th pair, so $m = m'$ and the PCP instance has a solution. ⬠

Undecidable properties of multitape automata

Let $\mathcal{M}$ be a d-tape automaton on alphabet $\mathcal{A}$, and let q be a state of $\mathcal{M}$. State q is ***universal*** if every sequence in $\mathcal{A}^{\mathbb{N}}$ is q-accepted by $\mathcal{M}$. A finite sequence $x \in \mathcal{A}^\star$ is a ***universal prefix*** for state q if for every infinite sequence $y \in \mathcal{A}^{\mathbb{N}}$, the infinite sequence xy is q-accepted by $\mathcal{M}$.

Example 7.2.4. Let $\mathcal{M}$ be a 1-tape, 1-state automaton on alphabet $\{0, 1\}$ with three transitions labelled by 1, 10 and 00. This automaton is not universal because every sequence starting with 01 is rejected, but the word 1 is a universal prefix: any sequence starting with 1 is accepted, because any finite segment 10^n1 is accepted by transitions 1, $00 \times k$, 1 if $n = 2k$ or 10, $00 \times k$, 1 if $n = 2k + 1$, and any infinite tail of 0's of 1's is obviously accepted. Hence there exist some multitape automata without universal states but that admit universal prefixes. The self-affine set associated with this automaton is discussed in Example 7.3.6.

Theorem 7.2.5. *It is undecidable whether a given state of a given d-tape automaton is universal. This problem remains undecidable if we restrict to 2-tape automata with 3 states.*

Proof. We reduce prefix-PCP, which is undecidable thanks to Lemma 7.2.3. Let (u_1, v_1), ..., (u_n, v_n) be an instance of prefix-PCP where the u_i, v_i are words over $\mathcal{B}$. We define a 2-tape automaton $\mathcal{M}$ on 3 states (denoted by X, U, V). The alphabet of $\mathcal{M}$ is $A_1 \times A_2$, with $A_1 = \{1, \ldots, n\}$ and $A_2 = \mathcal{B} \cup \{\#\}$, where n is the size of the prefix-PCP instance, $\mathcal{B}$ is the alphabet of words u_i, v_i and # is a new symbol not in $\mathcal{B}$. The transitions of $\mathcal{M}$ are

(1) $X \xrightarrow{i|u_i} U$ and $U \xrightarrow{i|u_i} U$ for every $i \in A_1$;

(2) $X \xrightarrow{i|v_i} V$ and $V \xrightarrow{i|v_i} V$ for every $i \in A_1$;

(3) $U \xrightarrow{i|u} X$ for every $i \in A_1$ and $u \in A_2^+$ such that

  (i) $|u| \leqslant |u_i|$,
  (ii) u is not a prefix of u_i,
  (iii) u does not begin with #;

(4) $X \xrightarrow{i|u} X$ for every $i \in A_1$ and $u \in A_2^+$ such that (i) and (ii) above hold;

(5) $V \xrightarrow{i|v} X$ for every $i \in A_1$ and $v \in A_2^+$ such that

  (i) $|v| \leqslant |v_i|$,
  (ii) v is not a prefix of v_i,
  (iii) v does not begin with #;

(6) $X \xrightarrow{i|v} X$ for every $i \in A_1$ and $v \in A_2^+$ such that (i) and (ii) above hold.

We now prove that there exists a solution to the prefix-PCP instance $(u_1, v_1), \ldots, (u_n, v_n)$ if and only if state X is **not** universal in $\mathcal{M}$.

($\Rightarrow$) Suppose that the prefix-PCP instance admits a solution: there exist $m, m' \geqslant 1$ and two words $i_1 \cdots i_m$ and $i_1 \cdots i_{m'}$ such that $u_{i_1} \cdots u_{i_m} = v_{i_1} \cdots v_{i_{m'}}$ and one of the two words $i_1 \cdots i_m$ and $i_1 \cdots i_{m'}$ is a prefix of the other. Without loss of generality we can assume that $m \geqslant m'$ and $i_1 \cdots i_{m'}$ is a prefix of $i_1 \cdots i_m$. We prove that $\mathcal{M}$ cannot accept any infinite sequence in $(A_1 \times A_2)^{\mathbb{N}}$ beginning with

$$i_1 \cdots i_m \mid u_{i_1} \cdots u_{i_m} \#$$

when starting from state X, so $\mathcal{M}$ is not universal. Indeed, let us describe the evolution of $\mathcal{M}$ when reading such a sequence.

- We start from X, so $\mathcal{M}$ necessarily uses a transition defined in (1) and (2) and moves to state U or V after having read $i_1|u_{i_1}$ or $i_1|v_{i_1}$, respectively. (The other transitions (4) and (6) cannot be used because of the conditions (i) and (ii).) Note that both u_{i_1} and v_{i_1} are prefixes of the content of the second tape.
- Now if $\mathcal{M}$ is in state U, the remaining input starts with some i on the first tape and starts with u_i on the second tape. So $\mathcal{M}$ must use transition (1): stay in state U and read $i|u_i$. (Transition (3) cannot be used because of the conditions (i) and (ii).) The same holds if $\mathcal{M}$ is in state V.

It follows that when $\mathcal{M}$ reads $i_1, \ldots, i_{m'}$ on the first tape, then it is either in state U and has read $u_{i_1} \cdots u_{i_{m'}}$ on the second tape, or it is in state V and has read $v_{i_1} \cdots v_{i_{m'}} = u_{i_1} \cdots u_{i_m}$ on the second tape. In the second case, the next symbol on the second tape is #, so $\mathcal{M}$ is "blocked" on this input (there is no suitable transition for this sequence because of (iii)). In the first case, the computation must continue in the same way as before, so eventually $\mathcal{M}$ is still in state U and has read $i_1 \cdots i_m | u_{i_1} \cdots u_{i_m}$, and again, $\mathcal{M}$ is blocked because the next symbol on the second tape is #.

($\Leftarrow$) Suppose that no solution exists for the prefix-PCP instance. The following strategy shows that a move by the automaton can always be made, whatever its tape contents. If

$\mathcal{M}$ is in state U or V, any move is possible. In state X, if no move is possible, then in the current configuration $(i_1i_2\cdots|w)$, both u_{i_1} and v_{i_1} must be prefixes of w, otherwise (4) or (6) could have been used. Write $w = u_{i_1}w' = v_{i_1}w''$. Then:

(a) if $u_{i_1}\cdots u_{i_k}$# is a prefix of w for some k, then go to state V by reading $i_1|v_{i_1}$;
(b) if $v_{i_1}\cdots v_{i_k}$# is a prefix of w for some k, then go to state U by reading $i_1|u_{i_1}$;
(c) if none of the two above cases hold, go either to U or V.

The only possible ways to be stuck at this point are:

- $\mathcal{M}$ is in state U or V and the next symbol on the second tape is #;
- $\mathcal{M}$ is in state X and (i), (ii) prevent from moving to U or V.

The second case cannot happen because it implies the existence of a prefix-PCP solution. If we are in the first case, we can assume by symmetry that we are in state U. In the last step where $\mathcal{M}$ went from X to U, the configuration must start with $i_1i_2\cdots|u_{i_1}u_{i_2}\cdots u_{i_k}\#\cdots$ for some k, because this is the only way to get stuck in U some k steps later. However, this contradicts the choice made in (a) above, because $\mathcal{M}$ should have moved to V instead of state U. ⬠

Theorem 7.2.6. *It is undecidable whether a given state of a given d-tape automaton admits a universal prefix. This problem remains undecidable if we restrict to 2-tape automata with 3 states.*

Proof. We modify the prefix-PCP reduction made in the proof of Theorem 7.2.5. Let $(u_1, v_1), \dots, (u_n, v_n)$ be an instance of prefix-PCP where the u_i, v_i are words over $\mathcal{B}^\star$. First we modify the u_i, v_i by adding a new symbol $*$ not in $\mathcal{B}$ after each letter of each u_i and each v_i (a word $x_1x_2\cdots x_k$ becomes $x_1{*}x_2{*}\cdots x_k{*}$). This modified instance is clearly equivalent to the original one, so we denote it again by $(u_1, v_1), \dots, (u_n, v_n)$.

We now define a 2-tape automaton $\mathcal{M}$ on 3 states X, U, V. We take the same alphabet $A_1 \times A_2$ as in the other reduction, with a new symbol & for both A_1 and A_2, and the symbol $*$ for A_2. This gives $A_1 = \{1, \dots, n\} \cup \{\&\}$ and $A_2 = \mathcal{B} \cup \{\#, \&, *\}$, where n is the size of the prefix-PCP instance, $\mathcal{B}$ is the alphabet of words u_i, v_i and $\#, \&, *$ are new symbol not in $\mathcal{B}$. The transitions of $\mathcal{M}$ consist of

- (1) and (2) like in the proof of Theorem 7.2.5, without allowing any symbol & or $*$;
- (3), (4), (5), (6) like in the proof of Theorem 7.2.5, where symbols & or $*$ are allowed, except in the first letter of u or v;

plus the following transitions:

(7) $X \xrightarrow{a|\&} X$, $U \xrightarrow{a|\&} X$ and $V \xrightarrow{a|\&} X$ for every $a \in A_1$;
(8) $X \xrightarrow{\&|a} X$, $U \xrightarrow{\&|a} X$ and $V \xrightarrow{\&|a} X$ for every $a \in A_2 \setminus \{*\}$;
(9) $X \xrightarrow{a|*b} X$, $U \xrightarrow{a|*b} X$ and $V \xrightarrow{a|*b} X$ for every $a \in A_1$ and $b \in A_2$.

We now prove that there exists a solution to the prefix-PCP instance $(u_1, v_1), \dots, (u_n, v_n)$ if and only if state X does **not** have any unversal prefix.

($\Rightarrow$) Suppose that the prefix-PCP instance has a solution: there exist $m, m' \geqslant 1$ and two words $i_1\cdots i_m$ and $i_1\cdots i_{m'}$ such that $u_{i_1}\cdots u_{i_m} = v_{i_1}\cdots v_{i_{m'}}$ and one of the two words $i_1\cdots i_m$ and $i_1\cdots i_{m'}$ is a prefix of the other. Consider the following claim.

Claim. Let $x \in A_1^\star$ and $y \in A_2^\star$ be such that $x\&\&\cdots|y\&\&\cdots$ is X-accepted by at most $k \geqslant 1$ different runs of $\mathcal{M}$. Then there exist $x' \in A_1^\star$ and $y' \in A_2^\star$ such that $xx'\&\&\cdots|yy'\&\&\cdots$ is X-accepted by at most $k-1$ different runs.

This claim implies that X does not have any universal prefix, *i.e.*, that for every finite words $x \in A_1^\star$ and $y \in A_2^\star$, there exists a configuration starting with $x|y$ that is not X-accepted. Indeed, for every such x, y, there can be only finitely many different accepting runs (say k), because $\mathcal{M}$ eventually loops on state X with transition $\&|\&$. So it suffices to apply the claim k times to obtain a configuration starting with $x|y$ which is not X-accepted.

We now prove the claim, using the prefix-PCP solution. Let $x \in A_1^\star$ and $y \in A_2^\star$ be such that $x\&\&\cdots|y\&\&\cdots$ is X-accepted by k different runs. Denote by $R_1, \ldots, R_k$ the finite prefixes of the k runs, each cut when $\mathcal{M}$ reaches the $\&\&\cdots|\&\&\cdots$ part. Let $s = i_1 \cdots i_m \in A_1^\star$ and let $t = u_1 \cdots u_{i_m}$, which can be written in the form $t = a_1 * a_2 * \cdots * a_{|t|-1} * \in A_2^\star$, where each a_i is in $A_2 \setminus \{\#, \&, *\}$, thanks to the modification made to the instance.

Let ℓ be the distance between the two tapes heads when $\mathcal{M}$ has completed the finite run R_1. (Note that the first head is always behind the second one because it can only move by one cell at at time.) Without loss of generality we can assume that R_1 is the run for which such an ℓ is minimal. Let $L, L' \geqslant 0$ such that s (on the first tape) begins ℓ positions behind t (on the second tape) in the configuration

$$c \;=\; x\&^L s\&\&\cdots \;|\; y\&^{L'} t\#\&\&\cdots,$$

so that during any run starting with R_1, $\mathcal{M}$ starts reading s and $t\#$ exactly at the same time. It follows that R_1 cannot be extended to an accepting run for c, because s, t corresponds to a prefix-PCP solution, similarly as in the proof of Theorem 7.2.5. The same is true for any other run R_i for which such an ℓ is the same as R_1.

Let us now consider another accepting run R_i. By minimality of ℓ, the distance between the two tapes heads when $\mathcal{M}$ first reaches $\&\&\cdots|\&\&\cdots$ during run R_i is strictly larger than ℓ. We now prove that R_i can be extended in a unique way to an accepting run for c. Indeed, any run of $\mathcal{M}$ starting with R_i must evolve in the following way:

- when t starts being read the second tape, s is not yet being read on the first tape, so at this time $\mathcal{M}$ is reading $\&$ on the first tape and a_1 on the second tape;
- the only possible transition is (8), so $\mathcal{M}$ moves one step on both tapes, and is now reading $*$ on the second tape;
- the only possible transition is (9), so $\mathcal{M}$ moves one step on the first tape and two steps on the second, and is again reading $*$ on the second tape;
- this continues until the whole $t = a_1 * a_2 * \cdots * a_{|t|-1} *$ has been read on the second tape, and $\mathcal{M}$ is deterministically looping on $\&|\&$.

From this analysis, it follows that R_i can be extended in a *unique* way to an accepting run for c. Hence c is a configuration starting with $x|y$ with at most $k-1$ accepting runs, because every accepting run for c must start with an R_i, each of which can be extended in at most one way if $i \in \{2, \ldots, k\}$, or in no way at all if $i = 1$. Thus the claim is proved by taking $x' = \&^L s$ and $y' = \&^{L'} t\#$.

($\Leftarrow$) Suppose that no solution exists for the prefix-PCP instance. The strategy described in the "$\Leftarrow$" direction of the proof of Theorem 7.2.5 can be applied to prove that every sequence must be accepted, with the additional case that if the tape begins by $\&$ or $*$, then the transition (7), (8) or (9) can always be used. ⬠

Remark 7.2.7. In the reduction made in the above proof of Theorem 7.2.6, if state X has a universal prefix, then in fact X is universal. Also, in this case, it is easy to see that any finite word satisfying (i), (ii) and (iii) of transition (3) is a universal prefix for U (and V), so X, U (and V) have a common universal prefix Hence we have the following: given a 2-tape automaton $\mathcal{M}$ on 3 states and two states q, r of $\mathcal{M}$, it is undecidable if q and r have a common universal prefix.

7.3 Affine GIFS associated with multitape automata

Let $\mathcal{M}$ be a d-tape automaton on alphabet $\mathcal{A} = A_1 \times \ldots \times A_d$. We want to give a "numerical interpretation" to a finite word $u \in \mathcal{A}^\star$ or to an infinite configuration $c \in \mathcal{A}^{\mathbb{N}}$.

We must first specify, for each $k \in \{1, \ldots, n\}$, a numerical interpretation of the letters of A_k by choosing a bijection $\delta_k : A_k \to \{0, \ldots, |A_k| - 1\}$. We then define $\Delta_k : A_k^\star \to \mathbb{R}$ by

$$\Delta_k(u) = \sum_{1 \leqslant i \leqslant |u|} \delta_k(u_i) |A_k|^{-i}.$$

Equivalently, for $u = u_1 \cdots u_n \in A_k^n$, the number $\Delta_k(u)$ is represented by $0.\delta_k(u_1) \cdots \delta_k(u_n)$ in base $|A_k|$. The domain of definition of Δ_k can naturally be extended to configurations $c \in A_k^{\mathbb{N}}$. Finally, for every $w_1, \ldots, w_d \in A_1^+ \times \ldots \times A_d^+$, we write $\Delta(w_1, \ldots, w_d) = (\Delta_1(w_1), \ldots, \delta_d(w_d))$. The mappings Δ_k and Δ can naturally be extended to infinite sequences in $A_k^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{N}}$, respectively.

In the examples that will follow, if the alphabets A_k are all of the form $\{0, \ldots, |A_k| - 1\}$ and the maps $\delta_k : A_k \to \{0, \ldots, |A_k| - 1\}$ are not specified, we will assume for convenience that they are identity mappings.

Definition 7.3.1. Let $\mathcal{M}$ be a d-tape automaton on state $\mathcal{Q}$ and alphabet $\mathcal{A} = A_1 \times \cdots \times A_n$. The ***GIFS associated with*** $\mathcal{M}$ is the GIFS defined by the graph G with vertex set $\mathcal{Q}$ and, for every transition $R = (q, r, (w_1, \ldots, w_d))$ of $\mathcal{M}$, an edge (q, r) labelled by the map $f : [0, 1]^d \to [0, 1]^d$ defined by

$$f_R(x) \;=\; \begin{pmatrix} |A_1|^{-|w_1|} & & 0 \\ & \ddots & \\ 0 & & |A_d|^{-|w_d|} \end{pmatrix} x \;+\; \Delta(w_1, \ldots, w_d).$$

Example 7.3.2. Let $\mathcal{M}$ be a 2-tape automaton on alphabet $\mathcal{A} = \{0, 1\} \times \{0, 1\}$, and let $c \in \mathcal{A}^{\mathbb{N}}$ be configuration. If $\mathcal{M}$ contains a transition $R = (q, r, (1011, 11))$, then applying the contracting map f_R on $\Delta(c) = (0.x_1x_2\ldots, 0.y_1y_2\ldots) \in [0, 1]^2$ has the following effect:

$$\begin{aligned} f_R(\Delta(c)) &= \begin{pmatrix} 1/16 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0.x_1x_2\ldots \\ 0.y_1u_2\ldots \end{pmatrix} + \Delta(1011, 11) \\ &= \begin{pmatrix} 0.0000x_1x_2\ldots \\ 0.00y_1u_2\ldots \end{pmatrix} + \begin{pmatrix} 0.1011 \\ 0.11 \end{pmatrix} \\ &= \begin{pmatrix} 0.1011x_1x_2\ldots \\ 0.11y_1u_2\ldots \end{pmatrix}. \end{aligned}$$

Similarly, applying a sequence of mappings $f_{R_1} \cdots f_{R_n}(\Delta(c))$ corresponds to concatenating the words associated with the transitions R_n in the numerical interpretation $\Delta(c)$ of a configuration c. This is the key thanks to which a correspondence between the GIFS of an automaton and its accepted sequences can be established. This is formalized in the next proposition.

Proposition 7.3.3. *Let $\mathcal{M}$ be a 2-tape automaton and let q be a state of $\mathcal{M}$. The GIFS attractor of $\mathcal{M}$ associated with q is equal to the set $\{\Delta(c) : c$ is q-accepted by $\mathcal{M}\}$.*

Proof. Let $x \in [0,1]^d$. A standard fact in the theory of iterated function systems [**Fal03**, Chapter 9] is that $x \in R_q$ if and only if there is an infinite sequence of mappings $(f_{R_n})_{n\geqslant 1}$ of the GIFS such that

$$x = \bigcap_{n\geqslant 1} f_{R_1}\cdots f_{R_n}([0,1]^d)$$

and such that f_{R_1} is the labelling of an outgoing edge from q. Moreover, by definition of the GIFS of $\mathcal{M}$, for every such sequence of transitions $R_1, R_2, \ldots$, the configuration

$$c \;=\; w_{1,1}w_{2,1}\cdots \mid \cdots \mid w_{1,d}w_{2,d}\cdots$$

is such that $x = \Delta(c)$, where the $w_{n,k}$ are given by the transitions $(q_n, r_n, (w_{n,1}, \ldots, w_{n,d}))$ for all $n \geqslant 1$, so the proposition is proved because c is a q-accepted configuration. ⬠

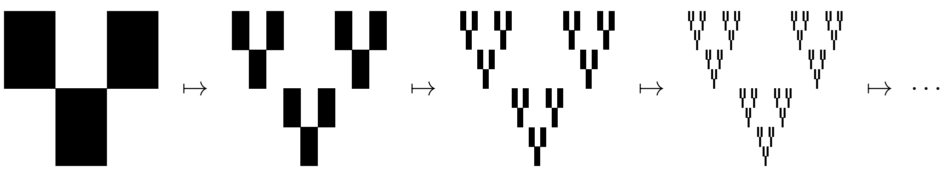
Example 7.3.4. Let $\mathcal{M}$ be the 2-tape GIFS on alphabet $\{0,1\}$ with one state and transitions $0|0$, $0|1$, $1|0$. The iterated function system associated with $\mathcal{M}$ consists of the maps $x \mapsto x/2$, $x \mapsto x/2 + (0,1/2)$, $x \mapsto x/2 + (1/2,0)$ and it can easily be seen that the associated attractor the Sierpiński triangle.

Example 7.3.5. Let $n, m \geqslant 1$ and let $I \subseteq \{1,\ldots,n\} \times \{1,\ldots,m\}$. The ***Bedford-McMullen carpet*** of size (n,m) and parameters I is the attractor the IFS specified the maps

$$f_{i,j} : x \mapsto \begin{pmatrix} 1/n & 0 \\ 0 & 1/m \end{pmatrix} x + \begin{pmatrix} i/n \\ j/m \end{pmatrix}$$

for every $(i,j) \in I$. Such an attractor can easily be realized by the attractor of the one-state, 2-tape automaton on alphabet $\{1,\ldots,n\} \times \{1,\ldots,m\}$, with the transition $i|j$ for every $(i,j) \in I$.

Bedford-McMullen carpets have been introduced in [**Bed84, McM84**] and an explicit formula is known to compute their Hausdorff dimension. The original motivating example for the study of this class was the carpet with parameters $(n,m) = (3,2)$ and $I = \{(0,1),(1,0),(2,1)\}$: cut the square in 3 columns and 2 lines, choose the sub-squares with coordinates I and replace each square of with a renormalized union of the chosen squares:



Example 7.3.6. The 1-tape, 1-state automaton $\mathcal{M}$ on alphabet $\{0,1\}$ with three transitions 1, 10 and 00 (described in Example 7.2.4) is an example of a non-universal automaton which admits universal prefixes. This reflects in the attractor associated with $\mathcal{M}$ in the following way: it is not equal to $[0,1]$ but it has nonempty interior. This can be proved

either by Proposition 7.3.8, or by proving directly that a configuration $x \in \{0,1\}^{\mathbb{N}}$ is accepted by $\mathcal{M}$ if and only if it does not start with $0^{2k+1}1$ for some $k \geqslant 0$, which implies that the attractor is equal to $\bigcup_{k\geqslant 0}[2^{-2k-1}, 2^{-2k}]$.

Remark 7.3.7. Given a d-tape automaton and a point $x \in [0,1]^d$, if there exists two distinct configuration c, c' such that $x = \Delta(c) = \Delta(c')$, then every tape component of c and c' must end by a stationary sequence 0^ω or $(|A_k|-1)^\omega$. In particular, Δ is finite-to-one.

The next proposition establishes the desired correspondence between word-theoretical properties of multitape automata and topological properties of the associated self-affine attractors.

Proposition 7.3.8. *Let $\mathcal{M}$ be a d-tape automaton on alphabet $\mathcal{A}$, let q, r be two states of $\mathcal{M}$, and let R_q, R_r be their associated GIFS attractor. We have:*

(1) q is universal if and only if $R_q = [0,1]^d$,
(2) q has a universal prefix if and only if R_q has nonemtpy interior.
(3) q and r have a common universal prefix if and only if $R_q \cap R_r$ has nonemtpy interior.

Proof. (1) If state q is universal the expansion of every element of $[0,1]^d$ must be q-accepted so $R_q = [0,1]^d$ thanks to Proposition 7.3.3. Conversely, suppose that there exists an infinite sequence c that is not q-accepted. By a compactness argument, there must exist a prefix w of c such that wc' is not q-accepted for every infinite sequence c'. Thanks to Remark 7.3.7, by choosing c' with no tape components ending by 0^ω or $(|A_k|-1)^\omega$, the sequence wc' is the *only* sequence such that $x = \Delta(wc')$, so $\Delta(wc') \notin R_q$ because otherwise wc' would be q-accepted otherwise. It follows that $R_q \neq [0,1]^d$.

(2) For a finite word $w \in \mathcal{A}^\star$, define the ***cylinder*** $[w]$ to be equal to the set of configurations that start with w. If q admits a universal prefix w, then $\Delta([w]) \subseteq R_q$ by Proposition 7.3.3, so R_q has nonempty interior. Conversely, suppose that there exists a nonempty open set $U \subseteq R_q$, and let $w \in \mathcal{A}^\star$ be a finite word such that $\Delta([w]) \subseteq U$. By a reasoning similar as in the proof of (1), we can prove that w is a universal prefix for q. The proof of (3) is analogous. ⬠

7.4 Undecidability results

Thanks to the undecidability results obtained for multitape automata in Theorem 7.2.5 and to the correspondence between word-theoretical and topological properties in Proposition 7.3.8, we obtain the following undecidability results about topological properties of self-affine attractors.

The first result below states that it is undecidable if an attractor "takes up the whole space", that is, equals $[0,1]^d$.

Theorem 7.4.1. *The following problem is undecidable. Instance: a d-dimensional affine GIFS $\mathcal{G}$ specified by maps with rational coefficients, and a state q of $\mathcal{G}$. Question: is $R_q = [0,1]^d$? This problem remains undecidable if we restrict to 2-dimensional GIFS with 3 states.*

Proof. Follows directly from Theorem 7.2.5 and Proposition 7.3.8, (1). ⬠

The next result states the undecidability of a fundamental topological property for self-affine sets: having empty interior.

Theorem 7.4.2. *The following problem is undecidable. Instance: a d-dimensional affine GIFS $\mathcal{G}$ specified by maps with rational coefficients, and a state q of $\mathcal{G}$. Question: does R_q have empty interior? This problem remains undecidable if we restrict to 2-dimensional GIFS with 3 states.*

Proof. Follows directly from Theorem 7.2.6 and Proposition 7.3.8, (2). ⬠

The next theorem, stating the undecidability of interior emptiness of attractor intersections, is interesting to compare with the case of Rauzy fractals, where this property is decidable. (See Chapter 1 for various examples of Rauzy fractals with non-overlapping attractors, and see Section 5.6 for an example of a Rauzy fractal with overlapping tiles.)

Theorem 7.4.3. *The following problem is undecidable. Instance: a d-dimensional affine GIFS $\mathcal{G}$ specified by maps with rational coefficients, and two states q, q' of $\mathcal{G}$. Question: does $R_q \cap R_{q'}$ have empty interior? This problem remains undecidable if we restrict to 2-dimensional GIFS with 3 states.*

Proof. Follows directly from Remark 7.2.7 and Proposition 7.3.8, (3). ⬠

Remark 7.4.4. All the undecidability results above have been obtained via a reduction using affine GIFS associated with a multitape automaton. Hence it follows that undecidability holds even if we restrict to affine GIFS in which the linear part of the contractions f_i are diagonal matrices whose entries are negative powers of integers. By adding dummy duplicate symbols, undecidability holds even if the entries are negative powers of two.

The diagonal entries are not always equal, so the attractors are not always self-similar, but we can observe that the family of affine GIFS in question is still quite a restricted one, and it belongs to the class of box-like self-affine sets mentioned in the introduction.

7.5 Conclusion

The undecidability results of this chapter have been obtained by linking word-theoretical properties of multitape automata with topological properties of their associated attractors. Examples of such correspondence are given in the table below. The first three are the ones treated in this chapter (Proposition 7.3.8), the fourth is the original one studied in [**Dub93**]. The fifth and sixth rows correspond to natural topological properties for which the corresponding language-theoretical properties do not seem easy to state at first glance.

Property of the d-tape automaton	Topological property of the attractor
Is universal	Is equal to $[0,1]^d$
Admits universal prefixes	Has nonempty interior
States q, r have a common univ. prefix	$R_q \cap R_r$ has nonempty intersection
Accepts a configuration with identical tape contents	Intersects the diagonal [**Dub93**]
?	Is connected
?	Is totally disconnected
Compute language entropy	Compute Hausdorff dimension

Concerning the last row in the table above, we hope to relate the automaton language entropy with the Hausdorff dimension of its attractor in order to prove that Hausdorff dimension is uncomputable for such sets (or more precisely, to prove that it is undecidable if the Hausdorff dimension of a given IFS equals 2). Two possible approaches are either by establishing some new word-theoretical undecidability results, or by adapting the reductions of this chapter in such a way that the Hausdorff dimension can be controlled in the reductions. We also would like to reduce the number of states in our reductions from 3 to 1, in order for the results to hold for the more restricted class of IFS (and not GIFS).

Another perspective of work in the context of multitape automata attractors is the study of some other "fatness properties", such as having nonempty interior, having positive Lebesgue measure and having full fractal dimension. Note that there are some examples of self-similar sets with empty interior but nonzero Lebesgue measure [**CJP+06**].

Conclusion

We conclude the work done in this thesis by stating some perspectives for further research. Such perspectives have already been given in Section 4.6 about fundamental groups of Rauzy fractals, in Section 6.6 about several questions concerning the notion of combinatorial substitution, and in Section 7.5 about some decidability issues concerning self-affine sets, such as computability of the Hausdorff dimension.

Higher-dimensional Rauzy fractals

Figure 7.1 shows the two 3-dimensional Rauzy fractals associated with two 4-letter substitutions. The first one, which can be referred to as the "Quadribonacci" fractal, is suspected to be homeomorphic to a three-dimensional ball. However, no proof of this fact is known. The two main difficulties that we are facing are:

- We cannot use the techniques used in the two-dimensional case, because they rely on the fact that the boundaries of the tiles are one-dimensional curves.
- Even if we managed to prove that the boundaries of the three-dimensional tiles are homeomorphic to two-dimensional spheres, this would not directly imply that the tiles are homeomorphic to a ball because of counterexamples such as Alexander's horned sphere (a set $S \subseteq \mathbb{R}^3$ which is homeomorphic to the two-dimensional sphere, but such that the unbounded component of $\mathbb{R}^3 \setminus S$ is not simply connected [**Hat02**]).

Thus, new techniques have to be developed for the study of such fractals. Despite these difficulties, we expect that the *connectedness* of these fractals can be proved in the same way as we proved the connectedness results in Theorem 3.2.1 and Proposition 5.1.1, by exhibiting suitable $\mathcal{L}$-coverings (which are a flexible combinatorial tool which can be used in higher dimensions as well).

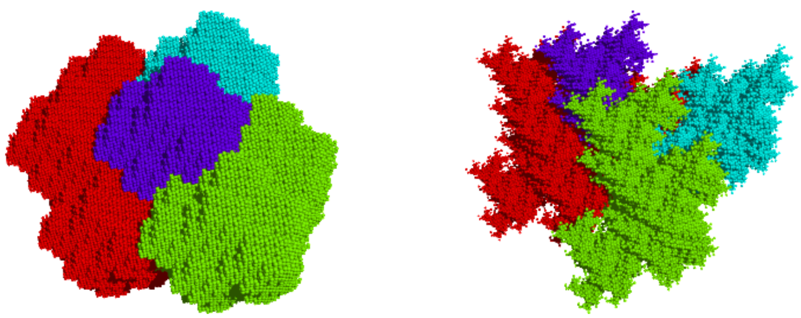


Figure 7.1: The Rauzy fractals of $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 14, 4 \mapsto 1$ (left) and $1 \mapsto 12, 2 \mapsto 31, 3 \mapsto 14, 4 \mapsto 1$ (right). They are three-dimensional because their associated Pisot eigenvalue $\beta \approx 1.926$ is of degree 4 (it is the dominant root of $x^4 - x^3 - x^2 - x - 1$).

Combinatorial tools for a more general algebraic framework

The flexibility of dual substitutions has allowed us to prove many results about arbitrary compositions of substitutions in the *irreducible* case. Even though a general $\mathbf{E}_k^\star$ theory has been developed for unimodular substitutions [**SAI01**], no definition of Rauzy fractals similar to Definition 1.3.5 (using dual substitutions) has been given for the *reducible* case.

> *Can such a definition be given (even for some particular classes only), in a way that allows proving results about arbitrary compositions of reducible substitutions?*

Several results in this direction already exist:

- the *quotient mapping condition* is introduced in [**ST09**] to get some periodic Rauzy fractal tilings from some reducible Pisot subsitutions;
- some "dual-like" substitutions coming from reducible substitutions are studied in [**EIR06**], [**EI05**], and also in [**EEFI07, FIR06, FIR09, AFHI11**] for some morphisms whose incidence matrix is hyperbolic but not Pisot in the strict sense;
- a full theory of dual substitutions in the *non-unit* irreducible case has been developed in [**MT13**].

Bratteli diagrams for 2-dimensional Sturmian subshifts

Giordano, Matui, Putnam and Skau proved in [**GMPS10**] that for every $d \geqslant 1$, every minimal $\mathbb{Z}^d$-action on the Cantor set is realized by a minimal $\mathbb{Z}$-action, up to orbit equivalence.

However, it is not known which minimal $\mathbb{Z}$-actions are orbit-equivalent to a minimal $\mathbb{Z}^d$ action for some $d \geqslant 2$. A realization result which partially answers this question is contained in [**CP13**], where it is proved that for every $d \geqslant 1$ and for every one-dimensional Toeplitz subshift X, there exists a d-dimensional Toeplitz subshift that is orbit equivalent to X.

We aim to use the combinatorial tools developed in this thesis in order to prove a realization result of the same flavor, namely that every 2-dimensional Sturmian subshift in $\{1,2,3\}^{\mathbb{Z}^2}$ is orbit-equivalent to a 1-dimensional subshift in $\{1,2,3\}^{\mathbb{Z}}$ by explicitly constructing a Bratteli diagram encoding the dynamics of the two subshifts up to orbit equivalence. Such a result would rely on the duality between a substitution σ and $\mathbf{E}_1^\star(\sigma)$, and on the fact that every two-dimensional Sturmian sequence can be generated by infinite compositions of Brun or Jacobi-Perron dual substitutions.

Note that 2-dimensional Sturmian subshifts can equivalently be seen either as codings of discrete planes, or as codings of two irrational rotations on the unit circle [**BV00**].

Totally real eigenvalues of products of Brun matrices

By running computer simulations, we can make the following empirical observation about the eigenvalues of products of Brun substitutions.

> For every product $\sigma = \sigma_{i_1}^{\mathsf{Brun}} \cdots \sigma_{i_n}^{\mathsf{Brun}}$ of Brun substitutions for which there exists a cycle $\bullet \xrightarrow{i_1} \bullet \xrightarrow{i_2} \cdots \xrightarrow{i_n} \bullet$ in the directed graph of Theorem 2.5.2, the eigenvalues of $\mathbf{M}_\sigma$ are all real.

This is a surprising fact because complex and real eigenvalues seem to appear in non-predictable ways in the case of arbitrary products. Can this observation be proved?

Note that there are some products for which $\mathbf{M}_\sigma$ has totally real eigenvalues but which do not belong to the language described above, for example $\sigma_2^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}\sigma_3^{\mathsf{Brun}}$.

Decision problems in symbolic dynamics

Several interesting decision problems related with substitutions subshifts can be raised. For instance:

> *Is the conjugacy problem decidable for one-dimensional substitutive subshifts? That is, given two substitutions σ and τ, is it decidable if their associated systems (X_σ, S) and (X_τ, S) are topologically conjugate?*

Partial answers to this question are given in [**CKL08, CDKL13**], where for example one can find a characterization of the systems which are topologically conjugate to the Thue-Morse substitution subshift. Note that the decidability of the conjugacy problem is still unknown for the class of subshifts of finite type in the one-dimensional two-sided case [**Boy08**].

The same question can be asked for the higher dimensional $\mathbb{Z}^d$-actions generated by higher dimensional substitutions.

Another interesting question is the periodicity problem for the fixed points of d-dimensional substitutions in the case $d \geqslant 2$. (This problem is decidable in the one-dimensional case; see Remark 1.1.4.)

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Abstract

Substitutions are mappings which replace each symbol of a given alphabet by a word over the same alphabet. They naturally act on infinite sequences of symbols, and produce highly ordered systems with many properties. This thesis concerns a particular class with algebraic restrictions, Pisot substitutions, and their related objects of dynamical, fractal or combinatorial nature.

We begin with the study of some qualitative properties of the two-dimensional patterns generated by iterating a two-dimensional "dual" version of Pisot substitutions. We then use these combinatorial tools to study the infinite families of substitutions obtained by taking arbitrary products over a finite set of Pisot substitutions. We obtain several applications: dynamical properties of the associated symbolic systems, some language theoretical characterizations of some topological properties of the associated Rauzy fractals, some number-theoretical properties of the associated Pisot numbers, and some results in discrete geometry. Particular focus is set on the substitutions associated with the Arnoux-Rauzy, Brun and Jacobi-Perron multidimensional continued fraction algorithms.

Next we give explicit constructions to describe all the possible fundamental groups of planar Rauzy fractals in the case where the group is free of finite rank (*i.e.*, countable).

In the last two chapters, we "step back" from the Pisot algebraic assumption to study some more general objects arising from the combinatorial tools used in the previous chapters, focusing on some computational (un)decidability questions.

Résumé

Les substitutions sont des applications qui remplacent chaque lettre d'un alphabet par un mot sur le même alphabet. Elles agissent naturellement sur des suites infinies de symboles, et produisent des systèmes très structurés ayant beaucoup de propriétés. Cette thèse porte sur l'étude d'une classe particulière vérifiant des restriction algébriques, les substitutions de type Pisot, et leurs objets associés, de nature dynamique, fractale ou combinatoire.

On commence par l'étude de certaines propriétés qualitatives des motifs bi-dimensionnels engendrés par une version « duale » des substitutions de type Pisot. On applique ensuite ces outils combinatoires à l'étude des familles infinies de substitutions définies comme l'ensemble des produits finis sur un ensemble fini de susbtitutions. On obtient des propriétés dynamiques des systèmes symboliques associés, des caractérisations de certaines propriétes topologiques des fractals de Rauzy associés, des propriétes des nombres Pisot associés, et des applications en géométrie discrète. Une attention particulière est portée sur les substitutions associées aux algorithmes de fraction continues multidimensionnels d'Arnoux-Rauzy, Brun et Jacobi-Perron.

Ensuite, des constructions explicites sont données pour fournir une description complète des groupes fondamentaux de fractals de Rauzy planaires, dans le cas où le groupe est libre de rang fini (*i.e.*, dénombrable).

Dans les deux derniers chapitres, on s'affranchit de la condition algébrique Pisot, pour étudier des objets plus généraux provenant des outils combinatoires utilisés dans les précédents chapitres, en insistant sur des questions d'(in)décidabilité calculatoire.