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Feedback stabilization of one-dimensional parabolic systems related to formations

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Abstract. This paper is concerned with the problem of stabilizing one-dimensional parabolic systems related to formations by using finite-dimensional controllers of a modal type. The parabolic system is described by a Sturm-Liouville operator, and the boundary condition is different from any of Dirichlet type, Neumann type, and Robin type, since it contains the time derivative of boundary values. In this paper, it is shown that the system is formulated as an evolution equation with unbounded output operator in a Hilbert space, and further that it is stabilized by using an RMF (residual mode filter)-based controller which is of finite-dimension. A numerical simulation result is also given to demonstrate the validity of the finite-dimensional controller.

AMS subject classifications: Primary 93B52; Secondarys 93C20, 35B36.

Key words: Sturm-Liouville operator; parabolic system; unbounded output operator; stabilization; residual mode filter.

1. Introduction

In recent years, formations control has been actively investigated in the field of multi-agent systems such as mobile robots and vehicles by an approach based on the ODEs of distributed consensus (e.g. [3, 5–7, 10])

$$\dot{x}_i(t) = \varepsilon \sum_{j\in\mathcal{N}_i} (x_j(t) - x_i(t)), \tag{1}$$

$$\dot{y}_i(t) = \varepsilon \sum_{j\in\mathcal{N}_i} (y_j(t) - y_i(t)), \tag{2}$$

where ε is a positive constant, $(x_i(t), y_i(t))$ denotes the x-y coordinate of the agent with identifier i at time t, and $\mathcal{N}_i$ denotes the set of agents which communicate with the agent i (see Fig. 1). By the reference [6], Eqs. (1), (2) are formally approximated by the diffusion equations as follows:

$$x_t(t,\xi) = \varepsilon x_{\xi\xi}(t,\xi), \tag{3}$$

$$y_t(t,\xi) = \varepsilon y_{\xi\xi}(t,\xi), \tag{4}$$

where the parameter $\xi \in [0,1]$ is called the agent identity and it plays a role of an agent's identification number. Especially, when the agent with $\xi = 1$ plays a role of the leader, and the agent with $\xi = 0$ plays a role of the anchor, the boundary conditions for (3), (4) are given as follows:

$$x_t(t,0) = f_0(t), \qquad x_t(t,1) = f_1(t), \tag{5}$$

$$y_t(t,0) = g_0(t), \qquad y_t(t,1) = g_1(t). \tag{6}$$

Therefore, by producing appropriate inputs $f_0(t)$, $f_1(t)$ as well as $g_0(t)$, $g_1(t)$, we can control the agent with identifier $\xi \in (0,1)$ which is called follower. Since the diffusion system (3)–(6) has a stable equiribrium whose configuration is linear, we can achieve only a linear formation by adding controls for the anchor and leader to arrive at their goals.

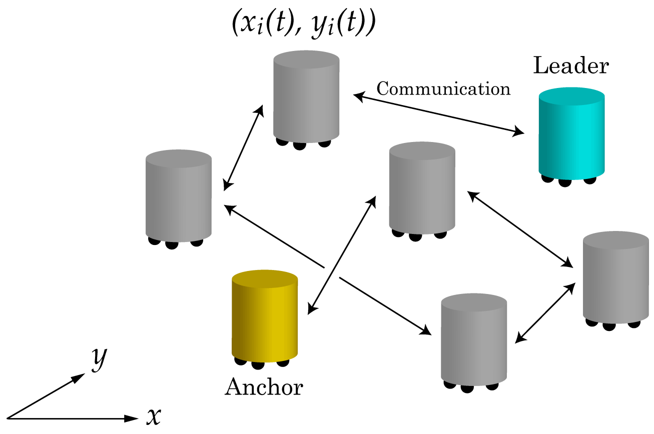

Fig. 1. Formation of mobile robots

In order to achieve more general formations except linear formations, Frihauf and Krstic [7] have considered the reaction-advection diffusion equations

$$x_t(t,\xi) = \varepsilon x_{\xi\xi}(t,\xi) + b x_\xi(t,\xi) + \lambda x(t,\xi), \tag{7}$$

$$y_t(t,\xi) = \varepsilon y_{\xi\xi}(t,\xi) + b y_\xi(t,\xi) + \lambda y(t,\xi), \tag{8}$$

instead of (3), (4), where $\varepsilon > 0$ and $b, \lambda \in \mathbf{R}$. In that paper, they have considered the corresponding Ginzburg-Landau PDE whose state takes a complex value, and constructed observer-based controllers of infinite-dimension under boundary observation by a backstepping approach.

In this paper, we generalize systems (7), (8), and consider the system

$$x_t(t,\xi) = -\mathcal{L}x(t,\xi), \tag{9}$$

$$y_t(t,\xi) = -\mathcal{L}y(t,\xi), \tag{10}$$

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with a Sturm-Liouville operator

$$(\mathcal{L}\varphi)(\xi) = \frac{1}{w(\xi)}\left(-\frac{d}{d\xi}\left(p(\xi)\frac{d\varphi(\xi)}{d\xi}\right) + q(\xi)\varphi(\xi)\right). \quad (11)$$

And, we give a design method of controllers containing finite-dimensional observers under the boundary observation

$$\Phi(t) = [\, x(t,0),\, x(t,1),\, x_\xi(t,1)\,]^T, \quad (12)$$

$$\Psi(t) = [\, y(t,0),\, y(t,1),\, y_\xi(t,1)\,]^T. \quad (13)$$

In the above, we assume that $w(\xi)$, $p(\xi)$, and $q(\xi)$ are real-valued, sufficiently smooth functions defined on $[0,1]$, and that $w(\xi) > 0$, $p(\xi) > 0$. Since the controls of x-direction and y-direction are independent each other, we consider, in this paper, only the control of x-direction, that is, the one of system (9), (5), and (12).

Let d_0 and d_1 be the destinations of x-direction of anchor and leader, respectively. Then, the equillibrium state $\overline{x}(\xi)$ is determined by solving the boundary value problem

$$\mathcal{L}\overline{x}(\xi) = 0, \qquad \overline{x}(0) = d_0, \qquad \overline{x}(1) = d_1. \quad (14)$$

Introducing the error variable

$$z(t,\xi) := x(t,\xi) - \overline{x}(\xi)$$

expressing the difference between the state $x(t,\xi)$ of system (9), (5), (12) and the equillibrium state $\overline{x}(\xi)$, we have the error system

$$\begin{cases} z_t(t,\xi) = -\mathcal{L}z(t,\xi), \quad (t,\xi) \in (0,\infty)\times(0,1), \\ z_t(t,0) = f_0(t), \quad z_t(t,1) = f_1(t), \quad t > 0, \\ z(0,\xi) = x(0,\xi) - \overline{x}(\xi) =: z_0(\xi), \quad \xi \in [0,1], \end{cases} \quad (15)$$

$$\begin{aligned} \overline{\Phi}(t) &:= [\, z(t,0),\, z(t,1),\, z_\xi(t,1)\,]^T \\ &= \Phi(t) - [\, d_0,\, d_1,\, \overline{x}_\xi(1)\,]^T, \qquad t > 0. \end{aligned} \quad (16)$$

Therefore, we have only to construct the control law $f_0(t)$ and $f_1(t)$ for the stabilization of the error system (15), (16) in order to achieve formations. In this paper, we formulate system (15), (16) in a Hilbert space, and construct stabilizing controllers containing finite-dimensional observers by using modal expansion method. Hereafter, we assume that $x(0,\cdot) \in C[0,1]$, i.e. $z_0 \in C[0,1]$ for the initial condition.

This paper is organized as follows: In Sec. 2, we formulate the one-dimensional parabolic system with boundary inputs and boundary outputs, whose boundary conditions contain the time derivative of boundary values, as an evolution equation with unbounded output operator in a Hilbert space. In Sec. 3, in order to stabilize the system, we first decompose the modes into three parts and derive a finite-dimensional model which is controllable and observable. Then, we give a design method of finite-dimensional controllers based on RMFs, where the usual RMF (e.g. [1, 14, 15]) is modified to fit our system. In Sec. 4, we prove the closed-loop stability with the RMF-based controller. Finally, in Sec. 5, a numerical simulation result is also given to demonstrate the validity of the finite-dimensional controller.

2. System description and its formulation

2.1. System description. Let $\mathcal{L}$ be a Sturm-Liouville operator

$$(\mathcal{L}\varphi)(\xi) = \frac{1}{w(\xi)}\left(-\frac{d}{d\xi}\left(p(\xi)\frac{d\varphi(\xi)}{d\xi}\right) + q(\xi)\varphi(\xi)\right),$$

where $w(\xi)$, $p(\xi)$, $q(\xi)$ are real-valued, sufficiently smooth functions on $[0,1]$, and $w(\xi) > 0$, $p(\xi) > 0$. As stated in Sec. 1, we consider the following one-dimensional parabolic system:

$$\begin{cases} z_t(t,\xi) = -\mathcal{L}z(t,\xi), \quad (t,\xi) \in (0,\infty)\times(0,1), \\ z_t(t,0) = u(t), \quad z_t(t,1) = v(t), \quad t > 0, \\ z(0,\xi) = z_0(\xi), \quad \xi \in [0,1], \end{cases} \quad (17)$$

with the output equation

$$\overline{\Phi}(t) = [z(t,0), z(t,1), z_\xi(t,1)]^T, \qquad t > 0, \quad (18)$$

where $[u(t), v(t)]^T \in \mathbf{R}^2$ is the control input and $\overline{\Phi}(t) \in \mathbf{R}^3$ is the measured output[1]. For the initial condition, we assume that $z_0 \in C[0,1]$. In this paper, we first formulate system (17), (18) as a system in a Hilbert space and then construct finite-dimensional stabilizing controllers based on RMFs by using modal expansion techniques.

2.2. Formulation of the system. In order to transform system (17), (18) to a system with homogeneous boundary condition, we first rewrite the system (17) as follows:

$$\begin{cases} z_t(t,\xi) = -\mathcal{L}z(t,\xi), \quad (t,\xi) \in (0,\infty)\times(0,1), \\ z(t,0) = r(t), \quad z(t,1) = s(t), \quad t > 0, \\ \dot{r}(t) = u(t), \quad \dot{s}(t) = v(t), \quad t > 0, \\ z(0,\xi) = z_0(\xi), \quad \xi \in [0,1], \\ r(0) = z_0(0), \quad s(0) = z_0(1). \end{cases} \quad (19)$$

Now, we introduce a variable (e.g. [11, 12])

$$\theta(t,\xi) = z(t,\xi) - \varphi(\xi)r(t) - \psi(\xi)s(t), \quad (20)$$

where $\varphi(\xi)$, $\psi(\xi)$ are smooth functions determined later. By using (20), the first equation of (19) becomes

$$\begin{aligned} &\theta_t(t,\xi) + \varphi(\xi)\dot{r}(t) + \psi(\xi)\dot{s}(t) \\ &= -\mathcal{L}(\theta(t,\xi) + \varphi(\xi)r(t) + \psi(\xi)s(t)) \\ &= -\mathcal{L}\theta(t,\xi) - \mathcal{L}\varphi(\xi)r(t) - \mathcal{L}\psi(\xi)s(t). \end{aligned} \quad (21)$$

Let $L^2_w(0,1)$ be the weighted L^2-space with inner product defined by

$$\langle f, g\rangle_w = \int_0^1 f(\xi)g(\xi)w(\xi)d\xi \qquad \text{for} \quad f, g \in L^2_w(0,1),$$

[1]The boundary condition of system (17), (18) is different from any of Dirichlet type, Neumann type, and Robin type, since it contains the time derivative of boundary values.

and let us define the operator $A : D(A) \subset L^2_w(0,1) \to L^2_w(0,1)$ by

$$A\varphi = \mathcal{L}\varphi, \qquad \varphi \in D(A),$$

$$\begin{aligned} D(A) &= \{ \varphi \in H^2(0,1); \varphi(0) = \varphi(1) = 0 \} \\ &= H^1_0(0,1) \cap H^2(0,1). \end{aligned} \tag{22}$$

Then, the operator A is closed and self-adjoint in $L^2_w(0,1)$, and it has compact resolvent and is bounded from below. Therefore, A has the eigenvalues $\{\lambda_i\}_{i=1}^\infty$ such that

$$-\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to \infty,$$

and the corresponding eigenfunctions $\{\varphi_i\}_{i=1}^\infty$ forms a complete orthonormal system in $L^2_w(0,1)$. Especially, in this paper, let $\lambda_1 < 0$ be assumed. Now, we choose a positive number c such that $\lambda_1 + c > 0$ and hereafter fix it.

Let $\overline{\varphi} : \{0,1\} \to \mathbf{R}$ and $\overline{\psi} : \{0,1\} \to \mathbf{R}$ be the functions defined by

$$\overline{\varphi}(\xi) = \begin{cases} 1, & \text{if } \xi = 0, \\ 0, & \text{if } \xi = 1, \end{cases} \qquad \overline{\psi}(\xi) = \begin{cases} 0, & \text{if } \xi = 0, \\ 1, & \text{if } \xi = 1. \end{cases}$$

In the variable transformation (20), we choose $\varphi, \psi \in H^2(0,1)$ such that they become the unique solutions of the boundary value problems

$$(\mathcal{L} + c)\varphi = 0 \quad \text{in } (0,1), \qquad \varphi = \overline{\varphi} \quad \text{on } \{0,1\}, \tag{23}$$

$$(\mathcal{L} + c)\psi = 0 \quad \text{in } (0,1), \qquad \psi = \overline{\psi} \quad \text{on } \{0,1\}. \tag{24}$$

Then, it follows that

$$\begin{aligned} \theta(t,0) &= z(t,0) - \varphi(0)r(t) - \psi(0)s(t) \\ &= z(t,0) - r(t) = 0, \end{aligned}$$

$$\begin{aligned} \theta(t,1) &= z(t,1) - \varphi(1)r(t) - \psi(1)s(t) \\ &= z(t,1) - s(t) = 0, \qquad t > 0, \end{aligned}$$

which implies that $\theta(t,\cdot) \in D(A)$ for all $t > 0$. Consequently, from (21) we have

$$\theta_t(t,\cdot) = -A\theta(t,\cdot) + c\varphi r(t) + c\psi s(t) - \varphi \dot{r}(t) - \psi \dot{s}(t), \quad t > 0. \tag{25}$$

Noting that

$$\dot{r}(t) = u(t), \qquad \dot{s}(t) = v(t), \tag{26}$$

we have

$$\theta_t(t,\cdot) = -A\theta(t,\cdot) + c\varphi r(t) + c\psi s(t) - \varphi u(t) - \psi v(t), \quad t > 0. \tag{27}$$

For the initial condition of (17), we have assumed that $z_0 \in C[0,1]$. Therefore, we have

$$\begin{aligned} \theta(0,\xi) &= z_0(\xi) - \varphi(\xi)z_0(0) - \psi(\xi)z_0(1) \\ &=: \theta_0(\xi) \in C[0,1]. \end{aligned} \tag{28}$$

Next, we shall express the output Eq. (18) by using the variables $r(t)$, $s(t)$ and $\theta(t,\cdot)$. First, we represent $z_\xi(t,1)$ as follows:

$$z_\xi(t,1) = \int_0^1 (h(\xi)z_\xi(t,\xi)p(\xi))_\xi \, d\xi, \tag{29}$$

where h is a smooth function that satisfies

$$h(0) = 0, \qquad h(1) = \frac{1}{p(1)}. \tag{30}$$

And, we define the function $\overline{h} : \{0,1\} \to \mathbf{R}$ by

$$\overline{h}(\xi) = \begin{cases} 0, & \text{if } \xi = 0, \\ \dfrac{1}{p(1)}, & \text{if } \xi = 1. \end{cases}$$

On the other hand, Eq. (29) is calculated as

$$\begin{aligned} z_\xi(t,1) &= \int_0^1 h'(\xi)p(\xi)z_\xi(t,\xi)d\xi \\ &+ \int_0^1 h(\xi)p'(\xi)z_\xi(t,\xi)d\xi + \int_0^1 h(\xi)z_{\xi\xi}(t,\xi)p(\xi)d\xi \\ &= \Big[h'(\xi)p(\xi)z(t,\xi)\Big]_{\xi=0}^{\xi=1} - \int_0^1 (h'(\xi)p(\xi))'z(t,\xi)d\xi \\ &+ \int_0^1 h(\xi)p'(\xi)z_\xi(t,\xi)d\xi + \int_0^1 h(\xi)z_{\xi\xi}(t,\xi)p(\xi)d\xi \\ &= h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ &- \int_0^1 (p(\xi)h''(\xi) + p'(\xi)h'(\xi) - q(\xi)h(\xi))z(t,\xi)d\xi \\ &+ \int_0^1 h(\xi)(p(\xi)z_{\xi\xi}(t,\xi) + p'(\xi)z_\xi(t,\xi) - q(\xi)z(t,\xi))d\xi \\ &= h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ &+ \int_0^1 \frac{1}{w(\xi)}\Big[-(p(\xi)h'(\xi))' + q(\xi)h(\xi)\Big]z(t,\xi)w(\xi)d\xi \\ &- \int_0^1 h(\xi)\frac{1}{w(\xi)}\Big[-(p(\xi)z_\xi(t,\xi))' + q(\xi)z(t,\xi)\Big]w(\xi)d\xi \\ &= h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ &+ \int_0^1 (\mathcal{L}h)(\xi)z(t,\xi)w(\xi)d\xi \\ &- \int_0^1 h(\xi)\mathcal{L}z(t,\xi)w(\xi)d\xi, \end{aligned} \tag{31}$$

where we have used integration by parts and the boundary condition of (19). Here, by using the notation $\langle \cdot,\cdot \rangle_w$ of L^2_w-inner product, Eq. (31) becomes

$$\begin{aligned} z_\xi(t,1) &= h'(1)p(1)s(t) - h'(0)p(0)r(t) + \langle \mathcal{L}h, z(t,\cdot)\rangle_w \\ &- \langle h, \mathcal{L}z(t,\cdot)\rangle_w = h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ &+ \langle (\mathcal{L}+c)h, z(t,\cdot)\rangle_w - \langle h, (\mathcal{L}+c)z(t,\cdot)\rangle_w. \end{aligned} \tag{32}$$

Hereafter, let $h \in H^2(0,1)$ be the unique solution of the boundary value problem

$$(\mathcal{L}+c)h = 0 \quad \text{in} \quad (0,1), \qquad h = \overline{h} \quad \text{on} \quad \{0,1\}. \tag{33}$$

Then, Eq. (32) becomes

$$\begin{aligned} z_\xi(t,1) &= h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ -\langle h, (\mathcal{L}+c)z(t,\cdot)\rangle_w &= h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ &\quad -\langle h, (\mathcal{L}+c)(z(t,\cdot) - \varphi r(t) - \psi s(t))\rangle_w \\ &= h'(1)p(1)s(t) - h'(0)p(0)r(t) - \langle h, (\mathcal{L}+c)\theta(t,\cdot)\rangle_w \\ &= h'(1)p(1)s(t) - h'(0)p(0)r(t) - \langle h, (A+c)\theta(t,\cdot)\rangle_w, \end{aligned} \tag{34}$$

where we have used (23), (24) and the fact that $z(t,\cdot) - \varphi r(t) - \psi s(t) = \theta(t,\cdot) \in D(A)$ for all $t > 0$. Moreover, by [8], noting that

$$H^2(0,1) \subset H^1(0,1) \subset H^{\frac{1}{2}-2\epsilon}(0,1) = D((A+c)^{\frac{1}{4}-\epsilon}),$$
$$0 < \epsilon < \frac{1}{4},$$

Eq. (34) becomes

$$\begin{aligned} z_\xi(t,1) &= h'(1)p(1)s(t) - h'(0)p(0)r(t) \\ &\quad -\langle (A+c)^{\frac{1}{4}-\epsilon}h, (A+c)^{\frac{3}{4}+\epsilon}\theta(t,\cdot)\rangle_w. \end{aligned} \tag{35}$$

In this way, we finally obtain

$$\begin{aligned} \overline{\Phi}(t) &= \begin{bmatrix} z(t,0) \\ z(t,1) \\ z_\xi(t,1) \end{bmatrix} \\ &= \begin{bmatrix} r(t) \\ s(t) \\ -h'(0)p(0)r(t) + h'(1)p(1)s(t) \\ +\langle -(A+c)^{\frac{1}{4}-\epsilon}h, (A+c)^{\frac{3}{4}+\epsilon}\theta(t,\cdot)\rangle_w \end{bmatrix}. \end{aligned} \tag{36}$$

Therefore, from (26), (27) and (36), we have the following system:

$$\begin{cases} \dot{r}(t) = u(t), \quad r(0) = z_0(0), \\ \dot{s}(t) = v(t), \quad s(0) = z_0(1), \\ \theta_t(t,\cdot) = -A\theta(t,\cdot) + c\varphi r(t) \\ \qquad + c\psi s(t) - \varphi u(t) - \psi v(t), \quad \theta(0,\cdot) = \theta_0, \\ \overline{\Phi}(t) = \begin{bmatrix} r(t) \\ s(t) \\ -h'(0)p(0)r(t) + h'(1)p(1)s(t) \\ +\langle -(A+c)^{\frac{1}{4}-\epsilon}h, (A+c)^{\frac{3}{4}+\epsilon}\theta(t,\cdot)\rangle_w \end{bmatrix}. \end{cases} \tag{37}$$

Moreover, by defining the bounded operators $D_1 : \mathbf{R} \to L^2_w(0,1)$, $D_2 : \mathbf{R} \to L^2_w(0,1)$, $B_1 : \mathbf{R} \to L^2_w(0,1)$, $B_2 : \mathbf{R} \to L^2_w(0,1)$ and $C : L^2_w(0,1) \to \mathbf{R}$ as

$$\begin{aligned} &D_1 v = c\varphi v, \quad v \in \mathbf{R}, \qquad D_2 v = c\psi v, \quad v \in \mathbf{R}, \\ &B_1 v = -\varphi v, \quad v \in \mathbf{R}, \qquad B_2 v = -\psi v, \quad v \in \mathbf{R}, \\ &C\xi = \langle -(A+c)^{\frac{1}{4}-\epsilon}h, \xi\rangle_w, \quad \xi \in L^2_w(0,1), \end{aligned} \tag{38}$$

system (37) is expressed as

$$\begin{cases} \dot{r}(t) = u(t), \quad r(0) = z_0(0), \\ \dot{s}(t) = v(t), \quad s(0) = z_0(1), \\ \theta_t(t,\cdot) = -A\theta(t,\cdot) + D_1 r(t) + D_2 s(t) \\ \qquad + B_1 u(t) + B_2 v(t), \quad \theta(0,\cdot) = \theta_0, \\ \overline{\Phi}(t) = \begin{bmatrix} r(t) \\ s(t) \\ \alpha r(t) + \beta s(t) + C(A+c)^\gamma \theta(t,\cdot) \end{bmatrix}. \end{cases} \tag{39}$$

In this, we have set $\alpha = -h'(0)p(0)$, $\beta = h'(1)p(1)$, and $\gamma = \dfrac{3}{4} + \epsilon \in \left(\dfrac{3}{4}, 1\right)$.

Remark 2.1. In the above, we obtained the abstract system (39), which is also expressed as

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} r(t) \\ s(t) \\ \theta(t,\cdot) \end{bmatrix} = \overline{A}\begin{bmatrix} r(t) \\ s(t) \\ \theta(t,\cdot) \end{bmatrix} + \overline{B}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \\ \begin{bmatrix} r(0) \\ s(0) \\ \theta(0,\cdot) \end{bmatrix} = \begin{bmatrix} z_0(0) \\ z_0(1) \\ \theta_0 \end{bmatrix}, \\ \overline{\Phi}(t) = \overline{C}\begin{bmatrix} r(t) \\ s(t) \\ \theta(t,\cdot) \end{bmatrix}, \end{cases} \tag{40}$$

where

$$\overline{A} := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_1 & D_2 & -A \end{bmatrix}, \qquad \overline{B} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ B_1 & B_2 \end{bmatrix},$$

$$\overline{C} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & C(A+c)^\gamma \end{bmatrix}.$$

The unbounded operator $\overline{A}$ generates an analytic semigroup $e^{t\overline{A}}$ on the real Hilbert space $\overline{Z} := \mathbf{R} \times \mathbf{R} \times L^2_w(0,1)$. $\overline{B}$ is a bounded operator, and $\overline{C}$ is an unbounded operator with domain $D(\overline{C}) := \mathbf{R} \times \mathbf{R} \times D((A+c)^\gamma)$ with $\gamma \in \left(\dfrac{3}{4}, 1\right)$. Therefore, the system (40) is well-posed (see e.g. [9]).

3. Construction of finite-dimensional controllers

3.1. Partitioned system. In order to derive a finite-dimensional model for the system (39), we use the orthogonal projection P_k defined by

$$P_k f = \sum_{i=1}^{k} \langle f, \varphi_i\rangle_w \varphi_i.$$

Let κ be a given positive number. First of all, we choose an integer $l\,(l \geq 1)$ such that $-\lambda_{l+1} < -\kappa$. Moreover, we choose another integer n larger than l. Using the operators P_l and P_n ($n > l$), we decompose the state variable $\theta(t,\cdot)$ as

$$\theta(t,\cdot) = \theta_1(t) + \theta_2(t) + \theta_3(t),$$

where $\theta_1(t) := P_l\theta(t,\cdot)$, $\theta_2(t) := (P_n - P_l)\theta(t,\cdot)$, and $\theta_3(t) := (I - P_n)\theta(t,\cdot)$. Also, the space $L^2_w(0,1)$ is expressed as

$$L^2_w(0,1) = P_l L^2_w(0,1) \oplus (P_n - P_l)L^2_w(0,1) \oplus (I - P_n)L^2_w(0,1),$$

and their dimensions are given by

$$\dim P_l L^2_w(0,1) = l, \qquad \dim(P_n - P_l)L^2_w(0,1) = n - l,$$
$$\dim(I - P_n)L^2_w(0,1) = \infty.$$

Therefore, the system (39) is equivalently expressed as follows:

$$\begin{cases} \dot r(t) = u(t), \quad r(0) = z_0(0), \\ \dot s(t) = v(t), \quad s(0) = z_0(1), \\ \dot\theta_1(t) = -A_1\theta_1(t) + D_{1,1}r(t) + D_{2,1}s(t) \\ \qquad + B_{1,1}u(t) + B_{2,1}v(t), \quad \theta_1(0) = \theta_{01}, \\ \dot\theta_2(t) = -A_2\theta_2(t) + D_{1,2}r(t) + D_{2,2}s(t) \\ \qquad + B_{1,2}u(t) + B_{2,2}v(t), \quad \theta_2(0) = \theta_{02}, \\ \dot\theta_3(t) = -A_3\theta_3(t) + D_{1,3}r(t) + D_{2,3}s(t) \\ \qquad + B_{1,3}u(t) + B_{2,3}v(t), \quad \theta_3(0) = \theta_{03}, \\ \overline{\Phi}(t) = \begin{bmatrix} r(t) \\ s(t) \\ \alpha r(t) + \beta s(t) + C_1(A_1+c)^\gamma\theta_1(t) \\ + C_2(A_2+c)^\gamma\theta_2(t) + C_3(A_3+c)^\gamma\theta_3(t) \end{bmatrix}, \end{cases} \tag{41}$$

where

$$A_1 := P_l A P_l, \qquad A_2 := (P_n - P_l)A(P_n - P_l),$$
$$A_3 := (I - P_n)A(I - P_n),$$
$$D_{1,1} := P_l D_1, \qquad D_{1,2} := (P_n - P_l)D_1,$$
$$D_{1,3} := (I - P_n)D_1,$$
$$D_{2,1} := P_l D_2, \qquad D_{2,2} := (P_n - P_l)D_2,$$
$$D_{2,3} := (I - P_n)D_2,$$
$$B_{1,1} := P_l B_1, \qquad B_{1,2} := (P_n - P_l)B_1,$$
$$B_{1,3} := (I - P_n)B_1,$$
$$B_{2,1} := P_l B_2, \qquad B_{2,2} := (P_n - P_l)B_2,$$
$$B_{2,3} := (I - P_n)B_2,$$
$$C_1 := CP_l, \qquad C_2 := C(P_n - P_l),$$
$$C_3 := C(I - P_n),$$
$$\theta_{01} := P_l\theta_0, \qquad \theta_{02} := (P_n - P_l)\theta_0,$$
$$\theta_{03} := (I - P_n)\theta_0.$$

In the above, the operators A_3 and $(A_3 + c)^\gamma$ are unbounded, whereas all the other operators are bounded.

Hereafter, we identify the finite-dimensional Hilbert space $P_l L^2_w(0,1)$ with the Euclidean space $\mathbf{R}^l$ with respect to the basis $\{\varphi_1, \varphi_2, \ldots, \varphi_l\}$. In this way, each element in $P_l L^2_w(0,1)$ is identified with an l-dimensional vector, and the operators A_1, $D_{1,1}$, $D_{2,1}$, $B_{1,1}$, $B_{2,1}$, and C_1 are identified with matrices with appropriate size. Similarly, each element in $(P_n - P_l)L^2_w(0,1)$ is identified with an $(n-l)$-dimensional vector, and the operators A_2, $D_{1,2}$, $D_{2,2}$, $B_{1,2}$, $B_{2,2}$, and C_2 are identified with matrices with appropriate size.

3.2. Finite-dimensional controllers using RMFs.

By the partitioned system (41), we consider the finite-dimensional system

$$\begin{cases} \dot r(t) = u(t), \\ \dot s(t) = v(t), \\ \dot\theta_1(t) = -A_1\theta_1(t) + D_{1,1}r(t) + D_{2,1}s(t) \\ \qquad + B_{1,1}u(t) + B_{2,1}v(t), \\ y(t) = \begin{bmatrix} r(t) \\ s(t) \\ \alpha r(t) + \beta s(t) + C_1(A_1+c)^\gamma\theta_1(t) \end{bmatrix} \end{cases} \tag{42}$$

as a finite-dimensional model for system (39). The model (42) is equivalently rewritten as

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} r(t) \\ s(t) \\ \theta_1(t) \end{bmatrix} = \mathcal{A}_1 \begin{bmatrix} r(t) \\ s(t) \\ \theta_1(t) \end{bmatrix} + \mathcal{B}_1 \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \\ y(t) = \mathcal{C}_1 \begin{bmatrix} r(t) \\ s(t) \\ \theta_1(t) \end{bmatrix}, \end{cases} \tag{43}$$

where

$$\mathcal{A}_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_{1,1} & D_{2,1} & -A_1 \end{bmatrix}, \qquad \mathcal{B}_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ B_{1,1} & B_{2,1} \end{bmatrix},$$

$$\mathcal{C}_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & C_1(A_1+c)^\gamma \end{bmatrix}.$$

Then, we have the following fact for the model (43):

Lemma 3.1. The pair $(\mathcal{A}_1, \mathcal{B}_1)$ is controllable, and the pair $(\mathcal{C}_1, \mathcal{A}_1)$ is observable.

Proof. Let φ, ψ, h be the solutions of the boundary value problems (23), (24) and (33). Then, we can express the matrices A_1, $B_{1,1}$, $B_{2,1}$, and $C_1(A_1 + c)^\gamma$ as follows:

$$A_1 = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_l), \qquad B_{1,1} = [b^1_1, b^1_2, \cdots, b^1_l]^T,$$
$$B_{2,1} = [b^2_1, b^2_2, \cdots, b^2_l]^T,$$
$$C_1(A_1+c)^\gamma = [c_1(\lambda_1+c)^\gamma, c_2(\lambda_2+c)^\gamma, \cdots, c_l(\lambda_l+c)^\gamma],$$

where

$$b_i^1 = -\frac{p(0)\varphi_i'(0)}{\lambda_i + c} \neq 0, \qquad b_i^2 = \frac{p(1)\varphi_i'(1)}{\lambda_i + c} \neq 0,$$

$$c_i(\lambda_i + c)^\gamma = \varphi_i'(1) \neq 0, \quad 1 \leq i \leq l.$$

See Appendix A for the derivation of b_i^1, b_i^2, and $c_i(\lambda_i + c)^\gamma$. Therefore, the assertion of the lemma directly follows by using the results of finite-dimensional systems theory.

Since the pair $(\mathcal{A}_1, \mathcal{B}_1)$ is controllable by Lemma 3.1, we can choose a matrix F_1 such that $\mathcal{A}_1 - \mathcal{B}_1 F_1$ is Hurwitz. Similarly, since the pair $(\mathcal{C}_1, \mathcal{A}_1)$ is observable by Lemma 3.1, we can choose a matrix G_1 such that $\mathcal{A}_1 - G_1\mathcal{C}_1$ is Hurwitz. Here, let us consider the finite-dimensional controller

$$\begin{cases} \dot{w}_2(t) = -A_2 w_2(t) + D_{1,2} r(t) + D_{2,2} s(t) \\ \qquad + B_{1,2} u(t) + B_{2,2} v(t), \quad w_2(0) = w_{20}, \\ \widehat{y}_2(t) = C_2(A_2 + c)^\gamma w_2(t), \end{cases} \tag{44}$$

$$\begin{cases} \dot{w}_1(t) = (\mathcal{A}_1 - G_1\mathcal{C}_1) w_1(t) \\ \qquad + G_1 \left\{ \overline{\Phi}(t) - \begin{bmatrix} 0 \\ 0 \\ \widehat{y}_2(t) \end{bmatrix} \right\} + \mathcal{B}_1 \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \\ \qquad w_1(0) = w_{10}, \\ \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = -F_1 w_1(t). \end{cases} \tag{45}$$

The first part (44) is a modified version of the usual residual mode filter (RMF) (e.g. [1, 14, 15]), which plays an important role in the construction of finite-dimensional stabilizing controllers for unstable distributed parameter systems. The second part (45) is an observer-based controller of Luenberger type. Note that the output $\widehat{y}_2(t)$ from the RMF (44) is used in (45) such as $\overline{\Phi}(t) - [0 \;\, 0 \;\, \widehat{y}_2(t)]^T$.

The following theorem is our main result in this paper.

Theorem 3.1. For a given positive number κ, let an integer $l\,(l \geq 1)$ be chosen such that $-\lambda_{l+1} < -\kappa$. Moreover, let another integer n be chosen such that $n > l$. Then, the control law consisting of (44) and (45) becomes a finite-dimensional stabilizing controller for system (39), if the integer n is chosen sufficiently large. In addition, the decay rate of C_0-semigroup describing the closed-loop system approaches $-\kappa$ as n goes to infinity.

4. Proof of Theorem 3.1

Let us set $\overline{\theta}_1(t) := \begin{bmatrix} r(t) \\ s(t) \\ \theta_1(t) \end{bmatrix}$ and introduce the new variables $e_1(t) := \overline{\theta}_1(t) - w_1(t)$ and $e_2(t) := \theta_2(t) - w_2(t)$. Then, the closed-loop system is described as follows:

$$\frac{d\xi(t)}{dt} = (\mathcal{A} + \Delta\mathcal{A})\xi(t), \qquad \xi(0) = \xi_0, \tag{46}$$

where the state vector

$$\xi(t) := \begin{bmatrix} \overline{\theta}_1(t) \\ e_1(t) \\ \theta_2(t) \\ e_2(t) \\ \theta_3(t) \end{bmatrix}$$

is in the real Hilbert space $Z := \mathbf{R}^{l+2} \times \mathbf{R}^{l+2} \times \mathbf{R}^{n-l} \times \mathbf{R}^{n-l} \times (I - P_n) L_w^2(0,1)$ with inner product defined by

$$\langle \xi \, , \widetilde{\xi} \rangle_Z = \overline{\theta}_1^T \widetilde{\overline{\theta}}_1 + e_1^T \widetilde{e}_1 + \theta_2^T \widetilde{\theta}_2 + e_2^T \widetilde{e}_2 + \langle \theta_3, \widetilde{\theta}_3 \rangle_w,$$

$$\xi = \begin{bmatrix} \overline{\theta}_1 \\ e_1 \\ \theta_2 \\ e_2 \\ \theta_3 \end{bmatrix}, \qquad \widetilde{\xi} = \begin{bmatrix} \widetilde{\overline{\theta}}_1 \\ \widetilde{e}_1 \\ \widetilde{\theta}_2 \\ \widetilde{e}_2 \\ \widetilde{\theta}_3 \end{bmatrix} \in Z,$$

and the operators $\mathcal{A}$ and $\Delta\mathcal{A}$ are defined as

$$\mathcal{A} = \left[\begin{array}{ccc|cc} \mathcal{A}_1 - \mathcal{B}_1 F_1 & \mathcal{B}_1 F_1 & 0 & 0 & 0 \\ 0 & \mathcal{A}_1 - G_1\mathcal{C}_1 & 0 & -G_{1c} C_2 (A_2 + c)^\gamma & -G_{1c} C_3 (A_3 + c)^\gamma \\ [D_{1,2} \;\, D_{2,2} \;\, 0] - [B_{1,2} \;\, B_{2,2}] F_1 & [B_{1,2} \;\, B_{2,2}] F_1 & -A_2 & 0 & 0 \\ \hline 0 & 0 & 0 & -A_2 & 0 \\ 0 & 0 & 0 & 0 & -A_3 \end{array}\right],$$

$$\Delta\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ [D_{1,3} \;\, D_{2,3} \;\, 0] - [B_{1,3} \;\, B_{2,3}] F_1 & [B_{1,3} \;\, B_{2,3}] F_1 & 0 & 0 & 0 \end{bmatrix},$$

where G_{1c} is the $(1,3)$-block of the matrix $G_1 = \begin{bmatrix} G_{1a} & G_{1b} & G_{1c} \end{bmatrix}$. In the above, the operator $\mathcal{A}$ is unbounded since it contains the unbounded operator A_3 and $(A_3+c)^\gamma$, whereas the operator $\Delta\mathcal{A}$ is bounded. By using the similar techniques as in the previous work (e.g. [15,16]), the norm of C_0-semigroup $e^{t\mathcal{A}}$ generated by $\mathcal{A}$ can be estimated as

$$\|e^{t\mathcal{A}}\|_{\mathcal{L}(Z)} \le \overline{M}e^{-\kappa t}, \qquad t \ge 0, \tag{47}$$

where $\overline{M}$ (≥ 1) is a constant independent of n.

For the C_0-semigroup $e^{t(\mathcal{A}+\Delta\mathcal{A})}$ generated by the operator $\mathcal{A}+\Delta\mathcal{A}$, we can estimate the operator norm by using the well-known perturbation result of semigroups (e.g. [17, Theorem 3.4.1], [13, Theorem 3.1.1], [4, Theorem 3.2.1]) as follows:

$$\|e^{t(\mathcal{A}+\Delta\mathcal{A})}\|_{\mathcal{L}(Z)} \le \overline{M}e^{-\sigma t}, \qquad t \ge 0, \tag{48}$$

where

$$\sigma := \kappa - \overline{M}\|\Delta\mathcal{A}\|_{\mathcal{L}(Z)}.$$

Here, noting that $\|D_{1,3}\| \to 0$, $\|D_{2,3}\| \to 0$, $\|B_{1,3}\| \to 0$, $\|B_{2,3}\| \to 0$ as n goes to infinity, and that $\|F_1\|$ does not depend on n, we see that

$$\|\Delta\mathcal{A}\|_{\mathcal{L}(Z)} \le \|D_{1,3}\| + \|D_{2,3}\| + 2(\|B_{1,3}\| + \|B_{2,3}\|)\|F_1\| \to 0$$

$$\text{as} \quad n \to \infty.$$

Accordingly, there exists a positive integer n_1 such that

$$\sigma = \kappa - \overline{M}\|\Delta\mathcal{A}\|_{\mathcal{L}(Z)} > 0, \quad \forall\, n \ge n_1. \tag{49}$$

That is, the C_0-semigroup $e^{t(\mathcal{A}+\Delta\mathcal{A})}$ becomes exponentially stable, if the integer n is chosen such that $n \ge n_1$. And, the decay rate $-\sigma$ of the C_0-semigroup approaches $-\kappa$ as n goes to infinity. The proof of the theorem is thus complete.

Remark 4.1. When the integer n is chosen such that $n \ge n_1$, the state $z(t,\cdot) = \theta(t,\cdot) + \varphi r(t) + \psi s(t)$ is also stabilized exponentially with the same decay rate, since the following inequalities hold:

$$\|\theta(t,\cdot)\| = (\|\theta_1(t)\|^2 + \|\theta_2(t)\|^2 + \|\theta_3(t)\|^2)^{\frac{1}{2}} \le \|\xi(t)\|_Z,$$
$$|r(t)| \le \|\xi(t)\|_Z, \qquad |s(t)| \le \|\xi(t)\|_Z.$$

5. Numerical simulation

In the Sturm-Liouville operator (11), we set $w(\xi) = 1$, $p(\xi) = \varepsilon$ and $q(\xi) = -\nu$, where it is assumed that $\varepsilon > 0$, $\nu > 0$ and $\nu > \varepsilon\pi^2$. That is, system (9), (5), (12) is unstable under zero boundary inputs. The equillibrium state $\overline{x}(\xi)$ is determined by solving the boundary value problem (14) as follows:

$$\overline{x}(\xi) = \frac{d_1 - d_0\cos\omega}{\sin\omega}\sin\omega\xi + d_0\cos\omega\xi, \quad \omega := \sqrt{\frac{\nu}{\varepsilon}}. \tag{50}$$

The set of eigenpairs $\{\lambda_i, \varphi_i\}_{i=1}^\infty$ of A is given by

$$\lambda_i = \varepsilon i^2\pi^2 - \nu, \qquad \varphi_i(\xi) = \sqrt{2}\sin i\pi\xi, \qquad i \ge 1. \tag{51}$$

Especially, we choose the constant c as $c = \nu$, as a result, $\lambda_1 + c = \varepsilon\pi^2 > 0$ holds. Then, three kinds of boundary value problems (23), (24) and (33) are concretely solved as follows:

$$\varphi(\xi) = 1 - \xi, \qquad \psi(\xi) = \xi, \qquad h(\xi) = \frac{1}{\varepsilon}\xi. \tag{52}$$

Therefore, by (51) and (52), we have the matrix representation for the finite-dimensional model (42), i.e. (43), as a result, we have the finite-dimensional controller (44), (45) for the control of x-direction. For the control of y-direction, although we have an equilibrium state $\overline{y}(\xi)$ such that

$$\mathcal{L}\overline{y}(\xi) = 0, \qquad \overline{y}(0) = d_0', \qquad \overline{y}(1) = d_1',$$

that is,

$$\overline{y}(\xi) = \frac{d_1' - d_0'\cos\omega}{\sin\omega}\sin\omega\xi + d_0'\cos\omega\xi, \quad \omega := \sqrt{\frac{\nu}{\varepsilon}},$$

which is different from (50), we can also use the same controller as that of x-direction.

Now, let us set $\varepsilon = 0.1$, $\nu = 1.5$, and $\epsilon = 0.1$, i.e. $\gamma = 3/4 + \epsilon = 0.85$. Also, we choose the integer l as $l = 2$. Since the pair $(\mathcal{A}_1, \mathcal{B}_1)$ is controllable and the pair $(\mathcal{C}_1, \mathcal{A}_1)$ is observable (Lemma 3.1 assures this fact), we can determine the matrices G_1 and F_1 such that

$$\sigma(\mathcal{A}_1 - G_1\mathcal{C}_1) = \{-9, -10, -11, -12\},$$

$$\sigma(\mathcal{A}_1 - \mathcal{B}_1F_1) = \{-5, -6, -7, -8\}.$$

Note that Matlab Control System Toolbox can be used for the actual computation for G_1 and F_1. For the RMF, we choose the integer n as $n = 10$. Finally, we set $d_0 = 3$, $d_1 = 3$, $d_0' = 1$, $d_1' = -1$.

In the numerical simulation, we considered the case where the total number of the agents including anchor and leader was 21. In connection with the number of all agents, to solve the two parabolic equations numerically, we used the finite difference method with mesh width $\Delta\xi = 1/20$, and the Runge-Kutta method of the fourth order with time step $\Delta t = 0.0001$ for its time integration. For the finite-dimensional controller (44), (45) for the control of x-direction and the one for the control of y-direction, we used the Runge-Kutta method of the fourth order with the same time step Δt. In the initial time, we placed all agents at the point $x = -2.5$, $y = 0$. Figures 2 and 3 show the 2D-plot and 3D-plot of the same numerical simulation result, respectively. From Fig. 3, we see that the formation to "U" curve is successfully achieved by our method.

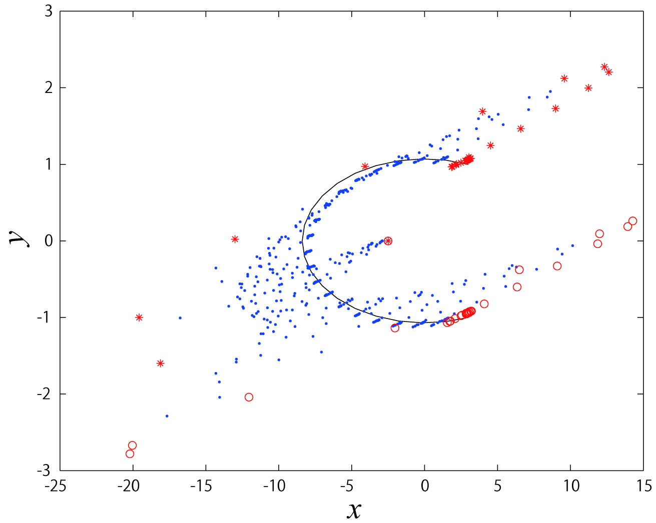

Fig. 2. Formation to "U" curve (anchor "$*$", leader "$\circ$", follower "$\cdot$") 2D-plot

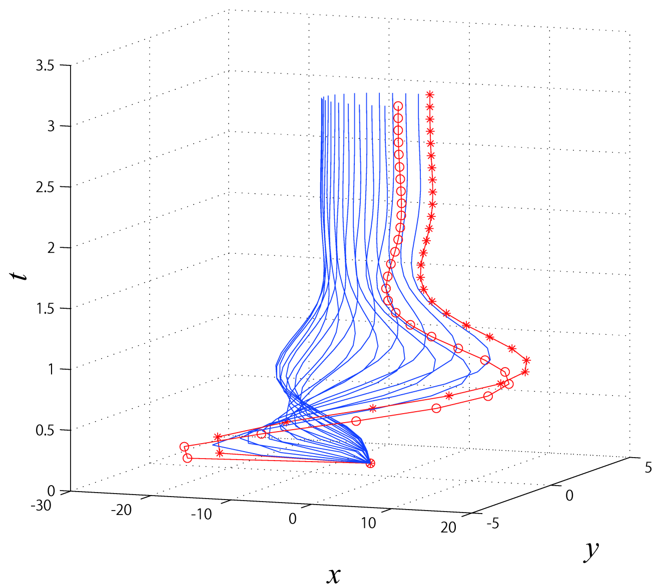

Fig. 3. Formation to "U" curve (anchor "$*$", leader "$\circ$", follower "$\cdot$") 3D-plot

Appendix A

From the definition of the operators B_1, B_2, C of (38) and the fact that the eigenfunctions $\{\varphi_i\}_{i=1}^{\infty}$ of A forms a complete orthonormal system in $L_w^2(0,1)$, we have

$$\begin{aligned} b_i^1 &= -\langle \varphi, \varphi_i \rangle_w, \quad b_i^2 = -\langle \psi, \varphi_i \rangle_w, \\ c_i &= -(\lambda_i + c)^{\frac{1}{4}-\epsilon} \langle h, \varphi_i \rangle_w, \quad 1 \leq i \leq l. \end{aligned} \tag{A.1}$$

First, let us calculate the value of inner product $\langle \varphi, \varphi_i \rangle_w$. Since φ is the solution of the boundary value problem (23), it satisfies

$$\frac{1}{w(\xi)}\left(-\frac{d}{d\xi}\left(p(\xi)\frac{d\varphi(\xi)}{d\xi}\right) + q(\xi)\varphi(\xi)\right) = -c\varphi(\xi). \tag{A.2}$$

Here, multiplying $\varphi_i(\xi)w(\xi)$ on both sides of (A.2) and integrating over $[0,1]$, we have

$$\begin{aligned} &\int_0^1 -\frac{d}{d\xi}\left(p(\xi)\frac{d\varphi(\xi)}{d\xi}\right)\varphi_i(\xi)d\xi \\ &+ \int_0^1 q(\xi)\varphi(\xi)\varphi_i(\xi)d\xi = -c\langle \varphi, \varphi_i \rangle_w. \end{aligned} \tag{A.3}$$

Moreover, using integration by parts for the first term of the left-hand side of (A.3) and noting that $\varphi(0) = 1$, $\varphi(1) = 0$, we have

$$(\lambda_i + c)\langle \varphi, \varphi_i \rangle_w = p(0)\varphi_i'(0),$$

which leads to

$$\langle \varphi, \varphi_i \rangle_w = \frac{p(0)\varphi_i'(0)}{\lambda_i + c}, \tag{A.4}$$

since $\lambda_i + c > 0$. Similarly, from the boundary value problems (24) and (33), we have

$$\langle \psi, \varphi_i \rangle_w = -\frac{p(1)\varphi_i'(1)}{\lambda_i + c}, \qquad \langle h, \varphi_i \rangle_w = -\frac{\varphi_i'(1)}{\lambda_i + c}. \tag{A.5}$$

Now, note that $\varphi_i'(0) \neq 0$, $\varphi_i'(1) \neq 0$ for all $i \geq 1$, since $w(\xi)$, $p(\xi)$, $q(\xi)$ are sufficiently smooth and $p(0) > 0$, $p(1) > 0$. Therefore, from (A.1), (A.4), (A.5) as well as $p(0) > 0$, $p(1) > 0$, we obtain

$$\begin{aligned} &b_i^1 = -\frac{p(0)\varphi_i'(0)}{\lambda_i + c} \neq 0, \qquad b_i^2 = \frac{p(1)\varphi_i'(1)}{\lambda_i + c} \neq 0, \\ &c_i(\lambda_i + c)^{\gamma} = \varphi_i'(1) \neq 0, \quad 1 \leq i \leq l, \end{aligned}$$

where $\gamma = \frac{3}{4} + \epsilon$.

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