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Almost balanced and uncorrelated quaternary sequence pairs of even length

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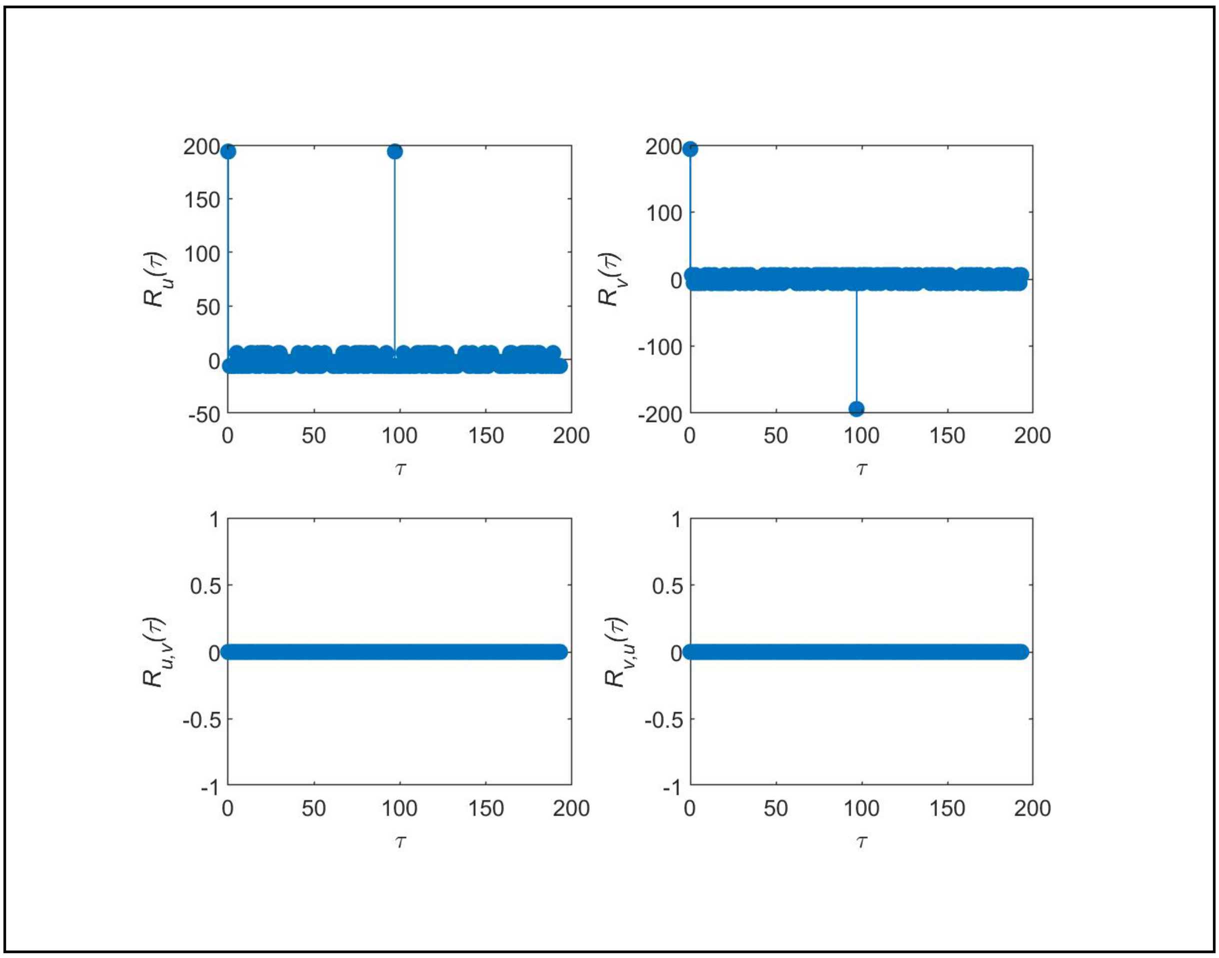
Graphical abstract



The autocorrelation $R_u(\tau), R_v(\tau)$ and cross-correlation $R_{u,v}(\tau), R_{v,u}(\tau)$ of sequence pair (u, v).

Public summary

■ Given a partition of $\mathbb{Z}_N^*$ into four subsets, a generic construction of uncorrelated quaternary sequence pairs of length $2N$ was proposed.

■ Choose the partition of $\mathbb{Z}_N^*$ from cyclotomic classes of order 4 and 8. The sequence pairs obtained are uncorrelated, almost balanced and with low autocorrelation, except at a few positions.

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Abstract: Given a partition of $\mathbb{Z}_N^*$ into four subsets, we present a generic construction of uncorrelated quaternary sequence pairs of length $2N$ using the interleaved technique based on this partition. By choosing partitions arising from cyclotomic classes of order 4 and 8 over $\mathbb{Z}_p$, we construct uncorrelated quaternary sequence pairs of length $2p$, which are almost balanced and have low autocorrelation, except at a few positions.

Keywords: quaternary sequence pair; interleaved technique; cyclotomic class

CLC number: O236.2 **Document code:** A

2020 Mathematics Subject Classification: 94A55; 94A60

1 Introduction

Sequences and sequence pairs that are optimally autocorrelated, largely complex, and balanced are important in many applications, such as measurement, digital communication, and continuous-wave radar[1−3]. In recent years, owing to their simple implementation, they have an amount of attention. Sequences and sequence pairs with low correlation can be generalized to quasi-complementary sequence sets and a few weight codes. Luo et al. proposed three new constructions of asymptotically optimal periodic quasi-complementary sequence sets using new parameters and small alphabet sizes[4]. Shi et al. obtained a family of abelian binary five-weight codes using a Gray map[5]. Several studies have been conducted on binary sequence pairs[6−9]. In this study, we focus on quaternary sequence pairs.

Li et al. constructed quaternary sequences and sequence pairs using binary complete sequences and sequence pairs[10]. Peng et al. proposed quaternary sequence pairs with an even length and three-level correlation using inverse Gray mapping and two difference set pairs of the same modulus[11]. Yang et al. presented two lower bounds on the maximum cross-correlation magnitude of balanced quaternary sequence pairs with (almost) optimal autocorrelation, and constructed a balanced quaternary sequence pair whose autocorrelation and cross-correlation achieve the lower bound[12]. Zhou et al. presented two generic constructions of quaternary periodic complementary pairs[13]: first, using the known binary odd periodic complementary pairs and Gray mapping; and second, based on the product sequences of a known quaternary sequence and a perfect quaternary sequence.

The remainder of this paper is organized as follows. In Section 2, we introduce quaternary sequences and their correlations, partitions of $\mathbb{Z}_p$ by cyclotomic classes, and permutations of $\mathbb{Z}_4$. In Section 3, we describe the construction of an uncorrelated quaternary sequence pair. In Section 4, we present almost balanced and uncorrelated sequence pairs with low autocorrelation, except at a few positions, and provide some examples to illustrate this. Finally, we summarize this study in Section 5.

2 Preliminaries

2.1 Quaternary sequences and their correlations

Let $\mathbb{Z}_4=\{0,1,2,3\}$. A quaternary sequence is a sequence with the alphabet $\mathbb{Z}_4$. The periodic cross-correlation function between two quaternary sequences $a=(a(0), a(1), \cdots, a(N-1))$ and $b=(b(0), b(1), \cdots, b(N-1))$ of period N is the complex function:

$$R_{a,b}(\tau):=\sum_{t=0}^{N-1}\omega^{a(t)-b(t+\tau)}\quad(0\leqslant\tau<N) \tag{1}$$

where $\omega=\sqrt{-1}$, and after the addition, $t+\tau$ is taken as modulo N. We say that the sequence pair (a,b) is uncorrelated if $R_{a,b}(\tau)=0$ for $0\leqslant\tau<N$.

Suppose $G_i=\{i:a(t)=i\}, 0\leqslant i\leqslant 3$. For $i\neq j$, a is called balanced if $|G_i|-|G_j|=0$ and $4\mid N$, or $|G_i|-|G_j|=1$ and $4\nmid N$. a is called almost balanced if $|G_i|-|G_j|\leqslant 2$ for $i\neq j$.

Let $\{\alpha_0,\alpha_1,\cdots,\alpha_{M-1}\}$ be a sequence set that consists of M sequences of length N. An $N\times M$ matrix R is formed by placing the sequence α_i in the ith column, $0\leqslant i<M$, that is, $R=[\alpha_0,\alpha_1,\cdots,\alpha_{M-1}]$. The interleaved sequence u is the sequence concatenating the successive rows of matrix R, denoted as

$$u=I(\alpha_0, \alpha_1, \cdots, \alpha_{M-1}).$$

I is called the interleaving operator, and $\alpha_0, \alpha_1, \cdots, \alpha_{M-1}$ are called the column sequences of u.

Lemma 2.1. Suppose $u = I(\alpha_0, \alpha_1, \cdots, \alpha_{M-1})$ and $v = I(\beta_0, \beta_1, \cdots, \beta_{M-1})$. For $0 \leqslant \tau < MN$, we write $\tau = \tau' M + \tau''$, where $0 \leqslant \tau' < N$ and $0 \leqslant \tau'' < M$. The cross-correlation function between u and v is given by:

$$R_{u,v}(\tau) = \sum_{k=0}^{M-\tau''-1} R_{\alpha_k, \beta_{k+\tau''}}(\tau') + \sum_{k=M-\tau''}^{M-1} R_{\alpha_k, \beta_{k+\tau''-M}}(\tau'+1) \qquad (2)$$

2.2 Partitions of $\mathbb{Z}_p$ by cyclotomic classes

Recall that for a set X, a partition of X is a family of subsets $\{X_i\}_{i\in I}$ such that $\bigcup_i X_i = X$ and $X_i \cap X_j = \emptyset$ if $i \neq j$. Cyclotomic classes are typically used to obtain the partitions of $\mathbb{Z}_N$.

We first recall the definition[3] of cyclotomic classes over $\mathbb{Z}_p$.

Definition 2.1. Let p be an odd prime, $p-1 = ef$, and γ a generator of $\mathbb{Z}_p^*$. For $0 \leqslant l < e$, $D_l^{(e,p)} = \{\gamma^{l+ek} : 0 \leqslant k < f\}$ is the set of cyclotomic classes of order e over $\mathbb{Z}_p$. For $0 \leqslant l \neq l' < e$, the cyclotomic number

$$(l, l')_e^p = |(D_l^{(e,p)} + 1) \cap D_{l'}^{(e,p)}|.$$

We drop e and p from the notation when they are implied by the context. We let $D_e = D_e^{(e,p)} = \{0\}$.

Cyclotomic classes give partitions of $\mathbb{Z}_p^*$ and $\mathbb{Z}_p$:

$$\mathbb{Z}_p^* = \bigcup_{l=0}^{e-1} D_l^{(e,p)}, \quad \mathbb{Z}_p = \bigcup_{l=0}^{e} D_l^{(e,p)}.$$

Lemma 2.2. [3] Suppose $p-1 = ef$, $0 \leqslant l \neq l' < e$, then

$$(l, l') = \begin{cases} (l', l), & \text{if } f \text{ is even;} \\ \left(l' + \dfrac{e}{2}, l + \dfrac{e}{2}\right), & \text{if } f \text{ is odd.} \end{cases}$$

We need the following information about cyclotomic numbers[3]:

(i) If $e = 4$, then $p = 4f + 1 = x^2 + 4y^2$ where $x \equiv 1 \pmod 4$ is unique and y is unique up to a certain point depending on the choice of γ. Table 1 lists the cyclotomic numbers of order four.

(ii) If $e = 8$, then $p = 8f + 1 = x^2 + 4y^2 = a^2 + 2b^2$ with a unique $x \equiv a \equiv 1 \pmod 4$. Table 2 lists the cyclotomic numbers of order eight when $p \equiv 1 \bmod 16$ (equivalently f is even).

2.3 Permutations of $\mathbb{Z}_4$

A permutation of $\mathbb{Z}_4$ is written as $\sigma = (\sigma(0), \sigma(1), \sigma(2), \sigma(3))$. It is well-known that 24 permutations of $\mathbb{Z}_4$ exist, which are ordered as follows:

Table 1. Cyclotomic numbers of order 4.

	f is even	f is odd
$16\times(0,0)$	$p-11-6x$	$p-7+2x$
$16\times(0,1)$	$p-3+2x+8y$	$p+1+2x-8y$
$16\times(0,2)$	$p-3+2x$	$p+1-6x$
$16\times(0,3)$	$p-3+2x-8y$	$p+1+2x+8y$
$16\times(1,1)$	$p-3+2x-8y$	$p-3-2x$
$16\times(1,2)$	$p+1-2x$	$p+1+2x+8y$

Table 2. Cyclotomic numbers of order 8 for $p \equiv 1 \bmod 16$.

	If 2 is a quartic residue	If 2 is not a quartic residue
$64\times(0,0)$	$p-23-18x-24a$	$p-23+6x$
$64\times(0,1)$	$p-7+2x+4a+16y+16b$	$p-7+2x+4a$
$64\times(0,2)$	$p-7+6x+16y$	$p-7-2x-8a-16y$
$64\times(0,3)$	$p-7+2x+4a-16y+16b$	$p-7+2x+4a$
$64\times(0,4)$	$p-7-2x+8a$	$p-7-10x$
$64\times(0,5)$	$p-7+2x+4a+16y-16b$	$p-7+2x+4a$
$64\times(0,6)$	$p-7+6x-16y$	$p-7-2x-8a+16y$
$64\times(0,7)$	$p-7+2x+4a-16y-16b$	$p-7+2x+4a$
$64\times(1,2)$	$p+1+2x-4a$	$p+1-6x+4a$
$64\times(1,3)$	$p+1-6x+4a$	$p+1+2x-4a-16b$
$64\times(1,4)$	$p+1+2x-4a$	$p+1+2x-4a+16y$
$64\times(1,5)$	$p+1+2x-4a$	$p+1+2x-4a-16y$
$64\times(1,6)$	$p+1-6x+4a$	$p+1+2x-4a+16b$
$64\times(2,4)$	$p+1-2x$	$p+1+6x+8a$
$64\times(2,5)$	$p+1+2x-4a$	$p+1-6x+4a$

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$$\begin{aligned}&\sigma_1=(0,1,2,3),\ \sigma_2=(0,3,2,1),\\&\sigma_3=(1,0,3,2),\ \sigma_4=(1,2,3,0),\\&\sigma_5=(2,3,0,1),\ \sigma_6=(2,1,0,3),\\&\sigma_7=(3,2,1,0),\ \sigma_8=(3,0,1,2),\\&\sigma_9=(0,1,3,2),\ \sigma_{10}=(0,3,1,2),\\&\sigma_{11}=(1,0,2,3),\ \sigma_{12}=(1,2,0,3),\\&\sigma_{13}=(2,3,1,0),\ \sigma_{14}=(2,1,3,0),\\&\sigma_{15}=(3,2,0,1),\ \sigma_{16}=(3,0,2,1),\\&\sigma_{17}=(0,2,3,1),\ \sigma_{18}=(0,2,1,3),\\&\sigma_{19}=(1,3,0,2),\ \sigma_{20}=(1,3,2,0),\\&\sigma_{21}=(2,0,1,3),\ \sigma_{22}=(2,0,3,1),\\&\sigma_{23}=(3,1,2,0),\ \sigma_{24}=(3,1,0,2).\end{aligned}$$

3 New construction of perfect quaternary sequences pairs

In this section, we provide a generic construction of uncorrelated quaternary sequences pairs.

Suppose N is an odd integer. We need a special case for Lemma 2.1:

Lemma 3.1. Let a,b,c be quaternary sequences of length N. Let u *and* v be the interleaved sequences

$$u=I\left(a,L^{\frac{N+1}{2}}(a)\right)\text{ and }v=I\left(b,L^{\frac{N+1}{2}}(c)\right)$$

where L is the left shift operator. Subsequently, the auto- and cross-correlation functions of u and v are given by:

$$R_u(\tau)=\begin{cases}2R_a(\tau'), & \tau''=0;\\ 2R_a\left(\tau'+\dfrac{N+1}{2}\right), & \tau''=1\end{cases}\tag{3}$$

$$R_v(\tau)=\begin{cases}R_b(\tau')+R_c(\tau'), & \tau''=0;\\ R_{b,c}\left(\tau'+\dfrac{N+1}{2}\right)+R_{c,b}\left(\tau'+\dfrac{N+1}{2}\right), & \tau''=1\end{cases}\tag{4}$$

$$R_{u,v}(\tau)=\begin{cases}R_{a,b}(\tau')+R_{a,c}(\tau'), & \tau''=0;\\ R_{a,c}\left(\tau'+\dfrac{N+1}{2}\right)+R_{a,b}\left(\tau'+\dfrac{N+1}{2}\right), & \tau''=1\end{cases}\tag{5}$$

where $0\leqslant\tau<2N$, $\tau=2\tau'+\tau''$, $0\leqslant\tau'<N$ and $0\leqslant\tau''<2$.

Definition 3.1. Suppose $\pi=\{A_0,A_1,A_2,A_3\}$ is a partition of $\mathbb{Z}_N^*$, $\sigma=(\sigma(0),\sigma(1),\sigma(2),\sigma(3))$ is a permutation of $\mathbb{Z}_4$, and $s\in\{0,1,2,3\}$. The quaternary sequence $\beta=\beta_{\pi,\sigma,s}$ associated with π, σ, and s of length N is defined as

$$\beta(t)=\begin{cases}\sigma(s), & \text{if } t=0;\\ k, & \text{if } t\in A_{\sigma(k)},\ k=0,1,2,3\end{cases}\tag{6}$$

If π implied by context and $\sigma=\sigma_i$, then we write β_i^s for $\beta_{\pi,\sigma_i,s}$.

The first result of this study is the following:

Theorem 3.1. Suppose the permutations of $\mathbb{Z}_4$ are ordered as in Chapter 2.3. Subsequently, for any partition π of $\mathbb{Z}_N^*$ and any $s\in\{0,1,2,3\}$, if i, j, $j+4$ are in the same interval $[1,8]$, $[9,16]$, or $[17,24]$, then

$$R_{\beta_i^s,\beta_j^s}(\tau)+R_{\beta_i^s,\beta_{j+4}^s}(\tau)=0,\ 0\leqslant\tau<N.$$

Consequently, the interleaved sequence pair

$$u=I\left(\beta_i^s,L^{\frac{N+1}{2}}(\beta_i^s)\right),\quad v=I\left(\beta_j^s,L^{\frac{N+1}{2}}(\beta_{j+4}^s)\right)\tag{7}$$

is an uncorrelated pair, that is, $R_{u,v}(\tau)=0$.

Proof. We prove the case in which $i=j=1$ and $s=0$. The other cases can be proved in the same manner.

We write $\beta_i^0=\beta_i$, and we must show that $R_{\beta_1}(\tau)+R_{\beta_1,\beta_5}(\tau)=0$. For subsets X,Y of $\mathbb{Z}_N$, the difference function $d_\tau(X,Y)$ is the function

$$d_\tau(X,Y)=|X\cap(Y+\tau)|.$$

By definition, the real and imaginary parts of the autocorrelation function $R_{\beta_1}(\tau)$ and the cross-correlation function $R_{\beta_1,\beta_5}(\tau)$ are given as follows:

$$\begin{aligned}\mathrm{Re}R_{\beta_1}(\tau)=&|\{t:\beta_1(t)-\beta_1(t+\tau)=0\}|-|\{t:\beta_1(t)-\beta_1(t+\tau)=2\}|-|\{t:\beta_1(t)-\beta_1(t+\tau)=-2\}|=\\&d_\tau(\{0\}\cup A_0,\{0\}\cup A_0)+d_\tau(A_1,A_1)+d_\tau(A_2,A_2)+d_\tau(A_3,A_3)-d_\tau(\{0\}\cup A_0,A_2)-d_\tau(A_1,A_3)-d_\tau(A_2,\{0\}\cup A_0)-d_\tau(A_3,A_1),\\ \mathrm{Im}R_{\beta_1}(\tau)=&|\{t:\beta_1(t)-\beta_1(t+\tau)=1\}|+|\{t:\beta_1(t)-\beta_1(t+\tau)=-3\}|-|\{t:\beta_1(t)-\beta_1(t+\tau)=-1\}|-|\{t:\beta_1(t)-\beta_1(t+\tau)=3\}|=\\&d_\tau(\{0\}\cup A_0,A_1)+d_\tau(A_1,A_2)+d_\tau(A_2,A_3)+d_\tau(A_3,\{0\}\cup A_0)-d_\tau(A_1,\{0\}\cup A_0)-d_\tau(A_2,A_1)-d_\tau(A_3,A_2)-d_\tau(\{0\}\cup A_0,A_3),\\ \mathrm{Re}R_{\beta_1,\beta_5}(\tau)=&|\{t:\beta_1(t)-\beta_5(t+\tau)=0\}|-|\{t:\beta_1(t)-\beta_5(t+\tau)=2\}|-|\{t:\beta_1(t)-\beta_5(t+\tau)=-2\}|=\\&d_\tau(A_2,\{0\}\cup A_0)+d_\tau(A_3,A_1)+d_\tau(\{0\}\cup A_0,A_2)+d_\tau(A_1,A_3)-d_\tau(A_2,A_2)-d_\tau(A_3,A_3)-d_\tau(\{0\}\cup A_0,\{0\}\cup A_0)-d_\tau(A_1,A_1),\\ \mathrm{Im}R_{\beta_1,\beta_5}(\tau)=&|\{t:\beta_1(t)-\beta_5(t+\tau)=1\}|+|\{t:\beta_1(t)-\beta_5(t+\tau)=-3\}|-|\{t:\beta_1(t)-\beta_5(t+\tau)=-1\}|-|\{t:\beta_1(t)-\beta_5(t+\tau)=3\}|=\\&d_\tau(A_2,A_1)+d_\tau(A_3,A_2)+d_\tau(\{0\}\cup A_0,A_3)+d_\tau(A_1,\{0\}\cup A_0)-d_\tau(A_3,\{0\}\cup A_0)-d_\tau(\{0\}\cup A_0,A_1)-d_\tau(A_1,A_2)-d_\tau(A_2,A_3),\end{aligned}$$

where $0\leq t<N$. If the subsets $X_1\cap X_2=\emptyset$ and $Y_1\cap Y_2=\emptyset$, we verify that

$$d_\tau(X_1\cup X_2,Y_1\cup Y_2)=d_\tau(X_1,Y_1)+d_\tau(X_1,Y_2)+d_\tau(X_2,Y_1)+d_\tau(X_2,Y_2).$$

Therefore, $R_{\beta_1}(\tau)+R_{\beta_1,\beta_5}(\tau)=0$ for $0\leqslant\tau<N$.

In practice, the pair (u,v), given by Eq. (7), is preferrable as be an uncorrelated sequence pair, which is almost balanced and with low autocorrelation, except at a few positions. Note that

(i) If the partition components A_i are of the same size (that is, $N\equiv 1\bmod 4$), then β_i^s is balanced, that is, u and v are almost balanced.

(ii) By Lemma 3.1,

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$$R_u(\tau)=\begin{cases}2R_{\beta_i^s}(\tau'), & \tau''=0;\\ 2R_{\beta_i^s}\left(\tau'+\dfrac{N+1}{2}\right), & \tau''=1;\end{cases}$$

$$R_v(\tau)=\begin{cases}2R_{\beta_j^s}(\tau'), & \tau''=0;\\ -2R_{\beta_j^s}\left(\tau'+\dfrac{N+1}{2}\right), & \tau''=1.\end{cases}$$

Hence, the low autocorrelation of u and v at $\tau\neq 0, N$ is guaranteed by the low autocorrelation of β_i^s and β_j^s at $\tau\neq 0$, respectively. We also note that $R_u(0)=R_v(0)=R_u(N)=2N$, $R_v(N)=-2N$.

For a prime p, we choose appropriate partitions of $\mathbb{Z}_p^*$ by cyclotomic classes of orders 4 and 8 to obtain (u,v) with good properties. Here, we provide a toy example.

Suppose $N\equiv 1 \pmod 4$ with the partition given by:

$$A_0=\left\{1,\cdots,\frac{N-1}{4}\right\},\qquad A_1=\left\{\frac{N+3}{4},\cdots,\frac{N-1}{2}\right\},$$
$$A_2=\left\{\frac{N+1}{2},\cdots,\frac{3(N-1)}{4}\right\},\qquad A_3=\left\{\frac{3N+1}{4},\cdots,N-1\right\}.$$

This provides a perfect and almost balanced pair with high autocorrelation.

Example 3.1. Let $N=13$, $A_0=\{1,2,3\}$, $A_1=\{4,5,6\}$, $A_2=\{7,8,9\}$, and $A_3=\{10,11,12\}$. Let $(i,j)=(1,2)$ and $s=1$. Subsequently,

$$\beta_1^1=1000111222333,$$
$$\beta_2^1=3000333222111,$$
$$\beta_6^1=1222111000333.$$

The quaternary sequences pair (u,v) is

$$u=12020203131311202020313131,$$
$$v=30000003333331222222111111.$$

We can see that $R_{u,v}(\tau)=0$ for $0\leqslant\tau<26$; however, the sequences u,v have a large autocorrelation.

4 Construction of sequence pairs using cyclotomic classes of $\mathbb{Z}_p$

Let p be an odd prime. We choose the partition of $\mathbb{Z}_p^*$ from cyclotomic classes of orders 4 and 8. This makes the sequences β_i^s balanced such that the sequence pairs obtained are uncorrelated, almost balanced, and have low autocorrelation, except at a few positions.

Theorem 4.1. Suppose $p=4f+1=x^2+4y^2$ with $x\equiv 1 \bmod 4$. Let D_k be the cyclotomic class of order 4 and π the partition $A_k=D_k$. Subsequently, β_i^s is balanced and

① if f is even,

$$R_{\beta_i^s}(\tau)\in\{-1+\Delta_1,-1+y+\Delta_1,-1-y+\Delta_1\},$$

where $\Delta_1\in\{0,\pm 2\}$;

② if f is odd and $1\leqslant i\leqslant 8$,

$$R_{\beta_i^s}(\tau)\in\{-1+\Delta_1+\Delta_2\omega,-1+y+\Delta_1+\Delta_2\omega,-1-y+\Delta_1+\Delta_2\omega\},$$

where $\Delta_1\in\{0,\pm 1\}$, $\Delta_2\in\{0,\pm 1,\pm 2\}$.

Therefore, if y is small, then β_i^s has low autocorrelation, except at the in-phase position.

Proof. The real and imaginary parts of the autocorrelation function $R_{\beta_i^s}(\tau)$ are given as follows:

$$\begin{aligned}\mathrm{Re}R_{\beta_i^s}(\tau)=&d_\tau(D_{\sigma_i(0)},D_{\sigma_i(0)})+d_\tau(D_{\sigma_i(1)},D_{\sigma_i(1)})+d_\tau(D_{\sigma_i(2)},D_{\sigma_i(2)})+\\&d_\tau(D_{\sigma_i(3)},D_{\sigma_i(3)})-d_\tau(D_{\sigma_i(0)},D_{\sigma_i(2)})-d_\tau(D_{\sigma_i(1)},D_{\sigma_i(3)})-\\&d_\tau(D_{\sigma_i(2)},D_{\sigma_i(0)})-d_\tau(D_{\sigma_i(3)},D_{\sigma_i(1)})+\Delta_1,\\ \mathrm{Im}R_{\beta_i^s}(\tau)=&d_\tau(D_{\sigma_i(0)},D_{\sigma_i(1)})+d_\tau(D_{\sigma_i(1)},D_{\sigma_i(2)})+d_\tau(D_{\sigma_i(2)},D_{\sigma_i(3)})+\\&d_\tau(D_{\sigma_i(3)},D_{\sigma_i(0)})-d_\tau(D_{\sigma_i(1)},D_{\sigma_i(0)})-d_\tau(D_{\sigma_i(2)},D_{\sigma_i(1)})-\\&d_\tau(D_{\sigma_i(3)},D_{\sigma_i(2)})-d_\tau(D_{\sigma_i(0)},D_{\sigma_i(3)})+\Delta_2,\end{aligned}$$

where if $\sigma_i(s)=k$,

$$\begin{aligned}\Delta_1=&|D_{\sigma_i(k)}\cap\tau|+|\{0\}\cap(D_{\sigma_i(k)}+\tau)|-\\&|\{0\}\cap(D_{\sigma_i(k+2)}+\tau)|-|D_{\sigma_i(k+2)}\cap\tau|,\\ \Delta_2=&|D_{\sigma_i(k+3)}\cap\tau|+|\{0\}\cap(D_{\sigma_i(k+1)}+\tau)|-\\&|\{0\}\cap(D_{\sigma_i(k+3)}+\tau)|-|D_{\sigma_i(k+1)}\cap\tau|.\end{aligned}$$

If f is even, then $-1\in D_0$, and $\Delta_1\in\{0,\pm 2\},\Delta_2=0$, by calculation, for $\tau\in\mathbb{Z}_p^*$, $\mathrm{Re}R_{\beta_i^s}(\tau)\in\{-1+\Delta_1,-1+y+\Delta_1,-1-y+\Delta_1\}$, $\mathrm{Im}R_{\beta_i^s}(\tau)=0$, that is, $R_{\beta_i^s}(\tau)\in\{-1+\Delta_1,-1+y+\Delta_1,-1-y+\Delta_1\}$.

If f is odd, then $-1\in D_2$, and $\Delta_1\in\{0,\pm 1\},\Delta_2\in\{0,\pm 1,\pm 2\}$, by calculation, for $\tau\in\mathbb{Z}_p^*$, $\mathrm{Re}R_{\beta_i^s}(\tau)\in\{-1+\Delta_1,-1+y+\Delta_1,-1-y+\Delta_1\}$, we can verify that $\mathrm{Im}R_{\beta_i^s}(\tau)=\Delta_2$ if $1\leqslant i\leqslant 8$, that is, $R_{\beta_i^s}(\tau)\in\{-1+\Delta_1+\Delta_2\omega,-1+y+\Delta_1+\Delta_2\omega,-1-y+\Delta_1+\Delta_2\omega\}$.

Example 4.1. Suppose $p=37=4\times 9+1$, $\mathbb{F}_{37}^{\times}=\langle 2\rangle$. The cyclotomic classes of order 4 over $\mathbb{Z}_{37}$ are

$$D_0=\{1,7,9,10,12,16,26,33,34\},$$
$$D_1=\{2,14,15,18,20,24,29,31,32\},$$
$$D_2=\{3,4,11,21,25,27,28,30,36\},$$
$$D_3=\{5,6,8,13,17,19,22,23,35\}.$$

Let $A_k=D_k, 0\leqslant k\leqslant 3$, $i=2$, $j=3$ and $s=1$, then,

$$\beta_2^1=3032211010020133013132113202232330012,$$
$$\beta_3^1=0103322121131200120203220313303001123,$$
$$\beta_7^1=2321100303313022302021002131121223301.$$

The quaternary sequence pair (u,v) of length 74 is

$$u=31033221211312001202032203133030011233103322121131200120203220313303001123,$$
$$v=00120130302221132111123112220303102102230231212000331033330133000212132032.$$

By calculation, $R_{u,v}(\tau)=0$, and $R_u(\tau)$ and $R_v(\tau)$ are given by:

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$$\begin{aligned}R_u(\tau)_{\tau=0}^{73}=&\{74,-2,-2-4\omega,-2,-2,-2+4\omega,-2+4\omega,-2,-2+4\omega,-2,-2,-2,\\&-2,-2+4\omega,-2-4\omega,-2-4\omega,-2,-2+4\omega,-2-4\omega,-2+4\omega,\\&-2-4\omega,-2,-2+4\omega,-2+4\omega,-2-4\omega,-2,-2,-2,-2,-2-4\omega,\\&-2,-2-4\omega,-2-4\omega,-2,-2,-2+4\omega,-2,74,-2,-2-4\omega,-2,-2,\\&-2+4\omega,-2+4\omega,-2,-2+4\omega,-2,-2,-2,-2,-2+4\omega,-2-4\omega,\\&-2-4\omega,-2,-2+4\omega,-2-4\omega,-2+4\omega,-2-4\omega,-2,-2+4\omega,\\&-2+4\omega,-2-4\omega,-2,-2,-2,-2,-2-4\omega,-2,-2-4\omega,-2-4\omega,\\&-2,-2,-2+4\omega,-2\},\end{aligned}$$

$$\begin{aligned}R_v(\tau)_{\tau=0}^{73}=&\{74,2,-2-4\omega,2,-2,2-4\omega,-2+4\omega,2,-2+4\omega,2,-2,2,-2,2-4\omega,\\&-2-4\omega,2+4\omega,-2,2-4\omega,-2-4\omega,2-4\omega,-2-4\omega,2,-2+4\omega,\\&2-4\omega,-2-4\omega,2,-2,2,-2,2+4\omega,-2,2+4\omega,-2-4\omega,2,-2,2-4\omega,\\&-2,-74,-2,2+4\omega,-2,2,-2+4\omega,2-4\omega,-2,2-4\omega,-2,2,-2,2,\\&-2+4\omega,2+4\omega,-2-4\omega,2,-2+4\omega,2+4\omega,-2+4\omega,2+4\omega,-2,\\&2-4\omega,-2+4\omega,2+4\omega,-2,2,-2,2,-2-4\omega,2,-2-4\omega,2+4\omega,-2,\\&2,-2+4\omega,2\}.\end{aligned}$$

Theorem 4.2. Suppose $p=m^4+16$ is a prime. Let the partition π be given by $A_0=D_0\cup D_4, A_1=D_1\cup D_7, A_2=D_2\cup D_6$, $A_3=D_3\cup D_5$, where the D_ks are cyclotomic classes of order 8. Then, the sequence β_i^s is balanced and has low autocorrelation except at the in-phase position: $R_{\beta_i^s}(\tau)\in\{-7,\pm1,\pm3\}$ for $0<\tau<p$.

Proof. If f is even, $x=m^2, y=\pm2, a=m^2-4$. We prove this when $i=1, s=0$, and the other cases are provable in the same manner.

The real and imaginary parts of $R_{\beta_1^0}(\tau)$ are as follows:

$$\mathrm{Re}R_{\beta_1^0}(\tau)=d_\tau(\{0\}\cup A_0,\{0\}\cup A_0)+d_\tau(A_1,A_1)+d_\tau(A_2,A_2)+d_\tau(A_3,A_3)-d_\tau(\{0\}\cup A_0,A_2)-d_\tau(A_1,A_3)-d_\tau(A_2,\{0\}\cup A_0)-d_\tau(A_3,A_1),$$

$$\mathrm{Im}R_{\beta_1^0}(\tau)=d_\tau(\{0\}\cup A_0,A_1)+d_\tau(A_1,A_2)+d_\tau(A_2,A_3)+d_\tau(A_3,\{0\}\cup A_0)-d_\tau(A_1,\{0\}\cup A_0)-d_\tau(A_2,A_1)-d_\tau(A_3,A_2)-d_\tau(\{0\}\cup A_0,A_3).$$

For $\tau\in D_0$, we have

$$\begin{aligned}\mathrm{Re}R_{\beta_1^0}(\tau)=&d_\tau(A_0,A_0)+d_\tau(A_1,A_1)+d_\tau(A_2.A_2)+d_\tau(A_3,A_3)-\\&d_\tau(A_0,A_2)-d_\tau(A_1,A_3)-d_\tau(A_2,A_0)-d_\tau(A_3,A_1)+\\&|A_0\cap\tau|+|\{0\}\cap(A_0+\tau)|-|\{0\}\cap(A_2+\tau)|-|A_2\cap\tau|=\\&(0,0)+(0,1)-(0,2)+(0,3)+3(0,4)+(0,5)-(0,6)+(0,7)+\\&2(1,2)+2(2,5)-2(2,4)-2(1,3)-2(1,5)-2(1,4)-2(1,6)+2=\\&\begin{cases}1, & \text{if 2 is a quartic residue;}\\1-x+a, & \text{if 2 is not a quartic residue;}\end{cases}\end{aligned}$$

$$\begin{aligned}\mathrm{Im}R_{\beta_1^0}(\tau)=&d_\tau(A_0,A_1)+d_\tau(A_1,A_2)+d_\tau(A_2,A_3)+d_\tau(A_3,A_0)-d_\tau(A_1,A_0)-d_\tau(A_2,A_1)-d_\tau(A_3,A_2)-d_\tau(A_0,A_3)+\\&|\{0\}\cap(A_1+\tau)|+|A_3\cap\tau|-|A_1\cap\tau|-|\{0\}\cap(A_3+\tau)|=0.\end{aligned}$$

Therefore, $R_{\beta_1^0}(\tau)=\mathrm{Re}R_{\beta_1^0}(\tau)$ if $\tau\in D_0$. Similarly, we obtain the values $R_{\beta_1^0}(\tau)$ for $\tau\in D_l, 0\leqslant l<8$ as follows: when 2 is a quartic residue,

$$R_{\beta_1^0}(\tau)=\begin{cases}1, & \tau\in D_0\cup D_4,\\-1-y+\frac{1}{2}(x-a)=-1,3, & \tau\in D_1\cup D_5,\\-3-x+a=-7, & \tau\in D_2\cup D_6,\\-1+y+\frac{1}{2}(x-a)=-1,3, & \tau\in D_3\cup D_7;\end{cases}$$

and when 2 is not a quartic residue,

$$R_{\beta_1^0}(\tau)=\begin{cases}1-x+a=-3, & \tau\in D_0\cup D_4,\\-1+y+\frac{1}{2}(x-a)=-1,3, & \tau\in D_1\cup D_5,\\-3, & \tau\in D_2\cup D_6,\\-1-y+\frac{1}{2}(x-a)=-1,3, & \tau\in D_3\cup D_7.\end{cases}$$

Example 4.2. For $p=97=3^4+16$, let $\mathbb{F}_{97}^{\times}=\langle 5\rangle$; then, the cyclotomic classes of order 8 in $\mathbb{Z}_{97}$ are

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$$
\begin{aligned}
D_0 &= \{1, 6, 16, 22, 35, 36, 61, 62, 75, 81, 91, 96\},\\
D_1 &= \{5, 13, 14, 17, 19, 30, 67, 78, 80, 83, 84, 92\},\\
D_2 &= \{2, 12, 25, 27, 32, 44, 53, 65, 70, 72, 85, 95\},\\
D_3 &= \{10, 26, 28, 34, 37, 38, 59, 60, 63, 69, 71, 87\},\\
D_4 &= \{4, 9, 24, 33, 43, 47, 50, 54, 64, 73, 88, 93\},\\
D_5 &= \{20, 21, 23, 29, 41, 45, 52, 56, 68, 74, 76, 77\},\\
D_6 &= \{3, 8, 11, 18, 31, 48, 49, 66, 79, 86, 89, 94\},\\
D_7 &= \{7, 15, 39, 40, 42, 46, 51, 55, 57, 58, 82, 90\}.
\end{aligned}
$$

Let $i = 1$, $j = 3$ and $s = 0$, then,

$$
\begin{aligned}
\beta_1^0 &= 0022010120322111012133030232331220300331131023102201320131133003022133232030331210111223021010220,\\
\beta_3^0 &= 1133101031233000103022121323220331211220020132013310231020022111213302232312122030100033213010 1331,\\
\beta_7^0 &= 3311323213011222321200303101002113033002202310231132013202200330311200101303002123222110312323113.
\end{aligned}
$$

The quaternary sequence pair (u, v) of length 194 is

$$
\begin{aligned}
u = {}&02002123021001132101332320101310021221133332033200233023333112212\\
&00131010232331012311001203212002020021230210011321013323201013100\\
&2122113333203320023302333311221200131010232331012311001203212002,\\
v = {}&11133230110312003212203033030003110132002021102113302330202201323\\
&31222121121200301220312332101133333110123321302210300212112122213\\
&3231022020332033112011202002310113000303330302212300213011032 3311.
\end{aligned}
$$

Subsequently, $R_{u,v}(\tau) = 0$, $R_u(\tau)$ and $R_v(\tau)$ are as follows:

$$
\begin{aligned}
R_u(\tau)_{\tau=0}^{193} = \{&194, -6, -6, -6, -6, 6, -6, -2, -6, -6, -2, -6, -6, 6, 6, -2, -6, 6, -6,\\
&6, 6, 6, -6, 6, -6, -6, -2, -6, -2, 6, 6, -6, -6, -6, -2, -6, -6, -2, -2,\\
&-2, -2, 6, -2, -6, -6, 6, -2, -6, -6, -6, -6, -2, 6, -6, -6, -2, 6, -2,\\
&-2, -2, -2, -6, -6, -2, -6, -6, -6, 6, 6, -2, -6, -2, -6, -6, 6, -6, 6,\\
&6, 6, -6, 6, -6, -2, 6, 6, -6, -6, -2, -6, -6, -2, -6, 6, -6, -6, -6, -6,\\
&194, -6, -6, -6, -6, 6, -6, -2, -6, -6, -2, -6, -6, 6, 6, -2, -6, 6, -6,\\
&6, 6, 6, -6, 6, -6, -6, -2, -6, -2, 6, 6, -6, -6, -6, -2, -6, -6, -2, -2,\\
&-2, -2, 6, -2, -6, -6, 6, -2, -6, -6, -6, -6, -2, 6, -6, -6, -2, 6, -2,\\
&-2, -2, -2, -6, -6, -2, -6, -6, -6, 6, 6, -2, -6, -2, -6, -6, 6, -6, 6,\\
&6, 6, -6, 6, -6, -2, 6, 6, -6, -6, -2, -6, -6, -2, -6, 6, -6, -6, -6, -6\},
\end{aligned}
$$

$$
\begin{aligned}
R_v(\tau)_{\tau=0}^{193} = \{&194, 6, -6, 6, -6, -6, -6, 2, -6, 6, -2, 6, -6, -6, 6, 2, -6, -6, -6, -6,\\
&6, -6, -6, -6, -6, 6, -2, 6, -2, -6, 6, 6, -6, 6, -2, 6, -6, 2, -2, 2, -2, -6,\\
&-2, 6, -6, -6, -2, 6, -6, 6, -6, 2, 6, 6, -6, 2, 6, 2, -2, 2, -2, 6, -6, 2, -6,\\
&6, -6, -6, 6, -2, -6, 2, -6, 6, 6, 6, 6, -6, 6, 6, 6, 6, -2, -6, 6, 6, -6, 2, -6,\\
&6, -2, 6, 6, 6, -6, 6, -6, -194, -6, 6, -6, 6, 6, 6, -2, 6, -6, 2, -6, 6, 6, -6,\\
&-2, 6, 6, 6, 6, -6, 6, 6, 6, 6, -6, 2, -6, 2, 6, -6, -6, 6, -6, 2, -6, 6, -2, 2,\\
&-2, 2, 6, 2, -6, 6, 6, 2, -6, 6, -6, 6, -2, -6, -6, 6, -2, -6, -2, 2, -2, 2,\\
&-6, 6, -2, 6, -6, 6, 6, -6, 2, 6, -2, 6, -6, -6, -6, -6, 6, -6, -6, -6, -6,\\
&2, 6, -6, -6, 6, -2, 6, -6, 2, -6, -6, -6, 6, -6, 6\}.
\end{aligned}
$$

Remark 4.1. Note that the numbers of primes of the form x^2+4, x^2+9, x^4+16, etc., are all assumed to be infinite by the Bouniakowsky conjecture[14].

5 Conclusions

In this study, we constructed uncorrelated quaternary sequence pairs of length $2N$ by using the cyclotomic classes. The result shows that the sequence pairs are almost balanced and have low autocorrelation, except at a few positions. In the future work, studying other partitions of $\mathbb{Z}_N^*$ would be interesting. For example, if $N = pq$, the partitions of $\mathbb{Z}_{pq}^*$ may be choose as combinations of the generalized cyclotomic classes[15] introduced by Shen et al..

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Conflict of interest

The authors declare that they have no conflict of interest.

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