# PROOF SEARCH ISSUES IN SOME NON-CLASSICAL LOGICS

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by
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# Abstract

This thesis develops techniques and ideas on proof search. Proof search is used with one of two meanings. Proof search can be thought of either as the search for a yes/no answer to a query (theorem proving), or as the search for all proofs of a formula (proof enumeration). This thesis is an investigation into issues in proof search in both these senses for some non-classical logics.

Gentzen systems are well suited for use in proof search in both senses. The rules of Gentzen sequent calculi are such that implementations can be directed by the top level syntax of sequents, unlike other logical calculi such as natural deduction. All the calculi for proof search in this thesis are Gentzen sequent calculi.

In Chapter 2, permutation of inference rules for Intuitionistic Linear Logic is studied. A focusing calculus, ILLF, in the style of Andreoli ([And92]) is developed. This calculus allows only one proof in each equivalence class of proofs equivalent up to permutations of inferences. The issue here is both theorem proving and proof enumeration.

For certain logics, normal natural deductions provide a proof-theoretic semantics. Proof enumeration is then the enumeration of all these deductions. Herbelin's cut-free LJT ([Her95], here called MJ) is a Gentzen system for intuitionistic logic allowing derivations that correspond in a 1–1 way to the normal natural deductions of intuitionistic logic. This calculus is therefore well suited to proof enumeration. Such calculi are called 'permutation-free' calculi. In Chapter 3, MJ is extended to a calculus for an intuitionistic modal logic (due to Curry) called Lax Logic. We call this calculus PFLAX. The proof theory of MJ is extended to PFLAX.

Chapter 4 presents work on theorem proving for propositional logics using a history mechanism for loop-checking. This mechanism is a refinement of one developed by Heuerding *et al* ([HSZ96]). It is applied to two calculi for intuitionistic logic and also to two modal logics: Lax Logic and intuitionistic S4. The calculi for intuitionistic logic are compared both theoretically and experimentally with other decision procedures for the logic.

Chapter 5 is a short investigation of embedding intuitionistic logic in Intuitionistic Linear Logic. A new embedding of intuitionistic logic in Intuitionistic Linear Logic is given. For the hereditary Harrop fragment of intuitionistic logic, this embedding induces the calculus MJ for intuitionistic logic.

In Chapter 6 a ‘permutation-free’ calculus is given for Intuitionistic Linear Logic. Again, its proof-theoretic properties are investigated. The calculus is proved to be sound and complete with respect to a proof-theoretic semantics and (weak) cut-elimination is proved.

Logic programming can be thought of as proof enumeration in constructive logics. All the proof enumeration calculi in this thesis have been developed with logic programming in mind. We discuss at the appropriate points the relationship between the calculi developed here and logic programming.

Appendix A contains presentations of the logical calculi used and Appendix B contains the sets of benchmark formulae used in Chapter 4.

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# Contents

# Notation

| | |
|---|---|
| P, Q, R, S, T | logical formulae |
| A, B, C | atomic formulae |
| $\Gamma, \Delta, \Psi, \Phi, \Xi, \Theta$ | multisets of logical formulae/labelled sets of logical formulae |
| $\Sigma, \Pi$ | sets of formulae |
| $\mathcal{H}$ | history, a set of formulae |
| $\Pi$ | a sequent proof |
| $x, y$ | variables |
| $P[y/x]$ | $y$ substituted for $x$ in $P$ |
| $\{x_i\}$ | the set of elements $x_1, .., x_n$ |
| $\overrightarrow{x}$ | the list of elements $x_1, ..., x_n$ |
| $\leadsto_r$ | a reduction |
| $\phi$ | empty set/empty multiset |

# Systems

| | |
|---|---|
| G3 | A sequent calculus for intuitionistic logic |
| G6 | A multiplicative calculus for intuitionistic logic |
| NJ | A natural deduction calculus for intuitionistic logic |
| NNJ | A natural deduction calculus for intuitionistic logic |
| MJ | A ‘permutation-free’ sequent calculus for Lax Logic |
| G4 | A contraction-free calculus for propositional intuitionistic logic |
| $\mathcal{I}$ | A backchaining calculus for hereditary Harrop logic |
| ILL$^{\Sigma}$ | A sequent calculus for Intuitionistic Linear Logic |
| ILLF | A focusing calculus for Intuitionistic Linear Logic |
| NLL | A natural deduction calculus for Lax Logic |
| LAX | A sequent calculus for Lax Logic |
| NLAX | A natural deduction calculus for Lax Logic |
| PFLAX | A ‘permutation-free’ sequent calculus for Lax Logic |
| LLP | A natural deduction calculus for a fragment of Lax Logic |
| G3$^{Hist}$ | A history calculus for propositional intuitionistic logic (two varieties) |
| MJ$^{Hist}$ | A history calculus for propositional intuitionistic logic (two varieties) |
| G3$^{D}$ | A sequent calculus for intuitionistic logic |
| MJ$^{D}$ | A sequent calculus for intuitionistic logic |
| S4 | A sequent calculus for S4 |
| S4$^{Hist}$ | A history calculus for S4 |
| PFLAX$^{Hist}$ | A history calculus for Lax Logic |
| IS4 | A sequent calculus for intuitionistic S4 |
| IS4$^{Hist}$ | A history calculus for IS4 |
| IU | A sequent calculus for hereditary Harrop logic |
| CLL | A sequent calculus for classical linear logic |
| CLL$^{2}$ | A sequent calculus for classical linear logic |
| ILL | A sequent calculus for Intuitionistic Linear Logic |
| NILL | A natural deduction calculus for Intuitionistic Linear Logic |
| NNILL | A natural deduction calculus for Intuitionistic Linear Logic |
| SILL | A sequent calculus for Intuitionistic Linear Logic |
| Lolli | A backchaining calculus for a fragment of Intuitionistic Linear Logic |

# Chapter 1

# Introduction and Background

This thesis develops a series of sequent calculus systems for some non-classical logics with computationally motivated properties. The calculi we develop here will be of two kinds: calculi for proving theorems, and calculi for enumerating proofs. The first kind of calculus solves problems – a yes/no answer to a query is given. The second kind of calculus tells in what ways something can be done – all useful solutions to a problem are given.

In this introduction we give background on intuitionistic logic and in particular the 'permutation-free' sequent calculus MJ. We also give background on linear logic and on logic programming. This serves as motivation for the calculi subsequently developed in this thesis, as well as giving some technical reference material.

## 1.1 The Permutation-free calculus MJ

Natural deduction ([Gen69], [Pra65]) is thought of as the 'real' proof system for intuitionistic logic. A normal form can be given for every proof in the natural deduction system – this normal form is standardly defined as a natural deduction to which no reduction rules, either eliminating introduction/elimination pairs or commuting inferences, are applicable. The normalisation process is confluent and strongly terminating. The normal form consists of a chain of elimination steps followed by a chain of introductions. Each minor premiss is again the conclusion of a normal natural deduction. Normal natural deductions are often thought of as the 'real' proofs of the logic.

Natural deduction has a pragmatic drawback. In searching backwards for the proof of a formula, it is not always obvious which rule to apply. For instance in

$$\frac{\Gamma \vdash P \supset Q \quad \Gamma \vdash P}{\Gamma \vdash Q}\ (\supset_\varepsilon)$$

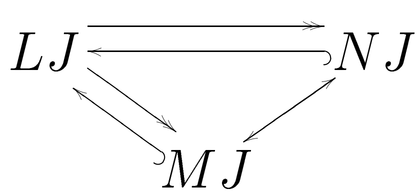
it is not obvious from the conclusion that we should apply $(\supset_\varepsilon)$. Even when this rule has been decided upon, what $P$ should be is hard to decide. The elimination rules do not introduce a connective. Cut-free Gentzen sequent calculus systems ([Gen69]) are much better from this point of view. When a principal formula has been chosen, the rules which can be applied to it are restricted. When the rule has also been chosen, the rule application is deterministic. The application of logical rules is directed by the syntax of the principal formula. Structural rules can be built into the sequent system. In such a system, when a principal formula has been chosen, the next rule application is determined exactly by the syntax of that formula. All logical rules of the sequent calculus are introduction rules (on the left or on the right).

There are well known translations ([Pra65]) between normal natural deductions and cut-free sequent proofs. Therefore we can search for proofs in sequent calculus systems and then translate the resulting proofs to normal natural deductions. The drawback here is that many sequent proofs translate to the same normal natural deduction. Hence when one is trying to enumerate all proofs of a formula, the same proof is found again and again.

This gives one motivation for the 'permutation-free' sequent calculus MJ for intuitionistic logic. This is a sequent calculus system for intuitionistic logic (enabling syntax directed proof search) whose proofs can be translated in a 1–1 way with the normal natural deductions for intuitionistic logic. MJ has the advantages of a sequent calculus system, whilst reflecting the structure of normal natural deductions.

The calculus originates with Herbelin ([Her95], [Her96]) and has also been investigated and developed by Dyckhoff and Pinto ([DP96], [DP98a]). Herbelin calls his calculus LJT, but here we follow Dyckhoff & Pinto in calling it MJ, as a calculus intermediate between natural deduction (NJ) and sequent calculus (LJ). (This nomenclature also avoids a clash with the calculus here called G4, but elsewhere also called LJT, [Dyc92]). MJ has two kinds of sequent. One looks like the usual kind of sequent; however, only right rules and contraction are applicable to this kind of sequent in backwards proof search. By backwards proof search we mean proof search starting from the root. The other kind of sequent has a formula (on the left) in a privileged position called the *stoup* (following [Gir91]). The formula in the stoup is always principal in the conclusion of an inference rule. Left rules are only applicable to stoup sequents. We display MJ in Figure 1.1.

We summarise the relationships between the systems in the following diagram:




Here LJ is the usual Gentzen system for intuitionistic logic (called G3 throughout the rest of this thesis) and NJ is the normal natural deduction calculus for intuitionistic calculus. There is an injection from the proofs of MJ into the proofs of LJ.

$$\frac{}{\Gamma \xrightarrow{P} P}\,(ax) \qquad \frac{}{\Gamma \xrightarrow{\perp} P}\,(-_{\mathcal{L}}) \qquad \frac{\Gamma, P \xrightarrow{P} R}{\Gamma, P \Rightarrow R}\,(C)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\,(\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \supset Q} R}\,(\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\,(\wedge_{\mathcal{R}}) \qquad \frac{\Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{P \wedge Q} R}\,(\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \wedge Q} R}\,(\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}_2}) \qquad \frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma \xrightarrow{P \vee Q} R}\,(\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P[y/x]}{\Gamma \Rightarrow \forall x.P}\,(\forall_{\mathcal{R}})\dagger \qquad \frac{\Gamma \xrightarrow{P[t/x]} R}{\Gamma \xrightarrow{\forall x.P} R}\,(\forall_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P[t/x]}{\Gamma \Rightarrow \exists x.P}\,(\exists_{\mathcal{R}}) \qquad \frac{\Gamma, P[y/x] \Rightarrow R}{\Gamma \xrightarrow{\exists x.P} R}\,(\exists_{\mathcal{L}})\dagger$$

† $y$ not free in $\Gamma, P, R$

**Figure 1.1:** The sequent calculus system MJ

The proofs of MJ can be seen as normal forms for proofs in LJ ([DP98b], [Min96]). Inferences in LJ can be permuted (see Chapter 2) to give different LJ proofs. Proofs in LJ that can be identified up to (semantically sound) permutations of inferences are those that translate to the same MJ proof (see [DP97], [DP98b]). Hence MJ is described as a 'permutation-free' sequent calculus – no semantically sound permutations of one MJ proof into another are possible. Another way to find the normal form of an LJ proof is to translate the LJ proof to a natural deduction, then translate it back again. The resulting proof will be a normal proof of LJ. These have the form of MJ proofs.

A major theme of this thesis is the extension of these calculi with permutation-free properties to other non-classical logics with normal natural deductions as their proof-theoretic semantics. To this end we study Lax Logic (where the extension is simple) and Intuitionistic Linear Logic (where the extension involves some new notions and a lot of complicated technical detail).

### 1.1.1 Technical Background

The technical details of the basic results on MJ are needed throughout this thesis, hence are included here in the introduction. We discuss cut (and its elimination) for MJ, we give term assignment systems for the intuitionistic calculi and we state some of the main theorems and important lemmas.

MJ has two judgement forms and as a result has four cut rules; these can be seen

$$\frac{\Gamma \xrightarrow{Q} P \quad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{Q} R}\ (cut_1) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{Q} R}{\Gamma \xrightarrow{Q} R}\ (cut_2)$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{P} R}{\Gamma \Rightarrow R}\ (cut_3) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow R}{\Gamma \Rightarrow R}\ (cut_4)$$

**Figure 1.2:** Cut rules for MJ

in Figure 1.2. We could give six cut rules, but these four suffice, the others being admissible in MJ plus the four rules given. In fact, if one adds the admissible rule of weakening as a primitive inference rule to MJ, the other cut rules are derivable.

The following theorem is proved in a variety of ways in [Her95] and [DP98a].

**Theorem 1.1** *The rules* $(cut_1)$, $(cut_2)$, $(cut_3)$, $(cut_4)$ *are admissible in* MJ.

We have the usual term assignment system for natural deduction via the Curry-Howard isomorphism ([How80]). We can give a restricted version of this for normal natural deductions, a term calculus in which only normal terms are grammatically correct. There are two kinds of proof terms, **N**, for normal proof terms (natural deductions) and **A**, for normal non-abstraction terms. We give this grammar and a presentation of a calculus for normal natural deductions with proof terms (Figure 1.3). Here V are the variables (proofs), U are the variables (individuals), T are the terms (in proof).

**N**::=

$$an(A) \mid \lambda V.N \mid efq(A) \mid pr(N,N) \mid i(N) \mid j(N) \mid wn(A, V.N, V.N)$$

$$\lambda U.N \mid prq(T,N) \mid ee(A, U.V.N)$$

**A**::=

$$var(V) \mid ap(A,N) \mid fst(A) \mid snd(A) \mid apn(A,T)$$

We can also give a term system for derivations in MJ. We give the grammar for this, including terms for proofs which are not cut-free. There are two kinds of proof terms corresponding to the two kinds of sequent. V are variables (proofs). U are variables (individuals) and T are the terms. Note that the cut terms are parameterised by the cut formula.

**M**::=

$$(V; Ms) \mid \lambda V.M \mid pair(M,M) \mid inl(M) \mid inr(M) \mid \lambda U.M \mid pairq(T,M)$$

$$cut_3^P(M, Ms) \mid cut_4^P(M, V.M)$$

$$\frac{}{\Gamma, x : P \rhd var(x) : P}\ (ax) \qquad \frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P}\ (M)$$

$$\frac{\Gamma \rhd\!\!\rhd A : -}{\Gamma \rhd\!\!\rhd efq(A) : P}\ (-_{\varepsilon})$$

$$\frac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd \lambda x.N : P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \rhd A : P \supset Q \quad \Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd ap(A, N) : Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd N_1 : P \quad \Gamma \rhd\!\!\rhd N_2 : Q}{\Gamma \rhd\!\!\rhd pr(N_1, N_2) : P \wedge Q}\ (\wedge_{\mathcal{I}})$$

$$\frac{\Gamma \rhd A : P \wedge Q}{\Gamma \rhd fst(A) : P}\ (\wedge_{\varepsilon_1}) \qquad \frac{\Gamma \rhd A : P \wedge Q}{\Gamma \rhd snd(A) : Q}\ (\wedge_{\varepsilon_2})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd i(N) : P \vee Q}\ (\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd j(N) : P \vee Q}\ (\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \rhd A : P \vee Q \quad \Gamma, x_1 : P \rhd\!\!\rhd N_1 : R \quad \Gamma, x_2 : Q \rhd\!\!\rhd N_2 : R}{\Gamma \rhd\!\!\rhd wn(A, x_1.N_1, x_2.N_2) : R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P[u/x]}{\Gamma \rhd\!\!\rhd \lambda u.N : \forall x.P}\ (\forall_{\mathcal{I}})\dagger \qquad \frac{\Gamma \rhd A : \forall x.P}{\Gamma \rhd apn(A, t) : P[t/x]}\ (\forall_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P[t/x]}{\Gamma \rhd\!\!\rhd prq(t, N) : \exists x.P}\ (\exists_{\mathcal{I}}) \qquad \frac{\Gamma \rhd A : \exists x.P \quad \Gamma, x : P[u/x] \rhd\!\!\rhd N : R}{\Gamma \rhd\!\!\rhd ee(A, u.x.N) : R}\ (\exists_{\varepsilon})\dagger$$

† $u$ not free in $\Gamma, R$

**Figure 1.3:** NNJ: Normal Natural Deduction Calculus for intuitionistic logic, with proof term annotation

**Ms**::=

$$[\,] \mid ae \mid (M :: Ms) \mid p(Ms) \mid q(Ms) \mid when(V.M, V.M)$$
$$apq(T, Ms) \mid spl(U.V.M) \mid cut_1^P(Ms, Ms) \mid cut_2^P(M, V.Ms)$$

Figure 1.4 shows these terms typed by sequents.

We now note an important point. Calculi with multisets and calculi with term assignments are not the same. With respect to enumerating proofs, the systems are not equivalent. To take a very simple example, consider the sequent $P, P \Rightarrow P$. With the context a multiset of formulae, this has one MJ proof:

$$\dfrac{\dfrac{}{P, P \xrightarrow{P} P}\ (ax)}{P, P \Rightarrow P}\ (C)$$

whereas when the context has labelled formulae, there are two proofs:

$$\dfrac{\dfrac{}{x_1 : P, x_2 : P \xrightarrow{P} [\,] : P}\ (ax)}{x_1 : P, x_2 : P \Rightarrow (x_1; [\,]) : P}\ (C) \qquad \dfrac{\dfrac{}{x_1 : P, x_2 : P \xrightarrow{P} [\,] : P}\ (ax)}{x_1 : P, x_2 : P \Rightarrow (x_2; [\,]) : P}\ (C)$$

In this thesis, unless stated otherwise, we use calculi with proof terms for proof enumeration (whether the terms have been included or not).

We give translations between the proof terms for normal natural deductions and those for MJ proofs. Along with proofs of the soundness (Lemma 1.3) and adequacy (Lemma 1.4) of the term annotations, this gives us a proof not only of the soundness and completeness of MJ (Corollary 1.1), but also (via Lemmas 1.1 and 1.2) that proofs of MJ correspond in a 1–1 to the normal natural deductions for intuitionistic logic. We give the translations here, and state the lemmas and theorems, all of which can be found in [DP96], [DP98a], [Her95], [Her96].

**Sequent Calculus → Natural Deduction:**

$\theta$ : M → N

$$\theta(x; Ms) = \theta'(var(x), Ms)$$
$$\theta(\lambda x.M) = \lambda x.\theta(M)$$
$$\theta(pair(M_1, M_2)) = pr(\theta(M_1), \theta(M_2))$$
$$\theta(inl(M)) = i(\theta(M))$$
$$\theta(inr(M)) = j(\theta(M))$$
$$\theta(\lambda u.M) = \lambda u.(\theta(M))$$
$$\theta(pairq(T, M)) = prq(T, \theta(M))$$

$$\frac{}{\Gamma \xrightarrow{P} [\,] : P}\,(ax) \qquad \frac{}{\Gamma \xrightarrow{\perp} ae : P}\,(-_{\mathcal{L}}) \qquad \frac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\,(C)$$

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \supset Q}\,(\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} (M :: Ms) : R}\,(\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma \Rightarrow M_2 : Q}{\Gamma \Rightarrow pair(M_1, M_2) : P \wedge Q}\,(\wedge_{\mathcal{R}})$$

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{P \wedge Q} p(Ms) : R}\,(\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \wedge Q} q(Ms) : R}\,(\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow inl(M) : P \vee Q}\,(\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow M : Q}{\Gamma \Rightarrow inr(M) : P \vee Q}\,(\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, x_1 : P \Rightarrow M_1 : R \quad \Gamma, x_2 : Q \Rightarrow M_2 : R}{\Gamma \xrightarrow{P \vee Q} when(x_1.M_1, x_2.M_2) : R}\,(\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P[y/x]}{\Gamma \Rightarrow \lambda y.M : \forall x.P}\,(\forall_{\mathcal{R}})\dagger \qquad \frac{\Gamma \xrightarrow{P[t/x]} Ms : R}{\Gamma \xrightarrow{\forall x.P} apq(t, Ms) : R}\,(\forall_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P[t/x]}{\Gamma \Rightarrow pairq(t, M) : \exists x.P}\,(\exists_{\mathcal{R}}) \qquad \frac{\Gamma, P[y/x] \Rightarrow M : R}{\Gamma \xrightarrow{\exists x.P} spl(y, x.M) : R}\,(\exists_{\mathcal{L}})\dagger$$

$$\frac{\Gamma \xrightarrow{Q} Ms_1 : P \quad \Gamma \xrightarrow{P} Ms_2 : R}{\Gamma \xrightarrow{Q} cut_1^P(Ms_1, Ms_2) : R}\,(cut_1)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma, x : P \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{Q} cut_2^P(M, x.Ms) : R}\,(cut_2)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \Rightarrow cut_3^P(M, Ms) : R}\,(cut_3)$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma, x : P \Rightarrow M_2 : R}{\Gamma \Rightarrow cut_4^P(M_1, x.M_2) : R}\,(cut_4)$$

† $y$ not free in $\Gamma, R$

**Figure 1.4:** The sequent calculus system MJ with term assignments

$\theta'$ : A $\times$ Ms $\rightarrow$ N

$$\theta'(A, [\,]) = an(A)$$
$$\theta'(A, (M :: Ms)) = \theta'(ap(A, \theta(M)), Ms)$$
$$\theta'(A, ae) = efq(A)$$
$$\theta'(A, p(Ms)) = \theta'(fst(A), Ms)$$
$$\theta'(A, q(Ms)) = \theta'(snd(A), Ms)$$
$$\theta'(A, when(x_1.M_1, x_2.M_2)) = wn(A, x_1.\theta(M_1), x_2.\theta(M_2))$$
$$\theta'(A, apq(T, Ms)) = \theta'(apn(A, T), Ms)$$
$$\theta'(A, spl(u.x.M)) = ee(A, u.x.\theta(M))$$

**Natural Deduction to Sequent Calculus:**

$\psi$ : N $\rightarrow$ M

$$\psi(an(A)) = \psi'(A, [\,])$$
$$\psi(\lambda x.N) = \lambda x.\psi(N)$$
$$\psi(efq(A)) = \psi'(A, ae)$$
$$\psi(pr(N_1, N_2)) = pair(\psi(N_1), \psi(N_2))$$
$$\psi(i(N)) = inl(\psi(N))$$
$$\psi(j(N)) = inr(\psi(N))$$
$$\psi(wn(A, x_1.N_1, x_2.N_2)) = \psi'(A, when(x_1.\psi(N_1), x_2.\psi(N_2)))$$
$$\psi(\lambda u.N) = \lambda u.\psi(N)$$
$$\psi(prq(T, N)) = pairq(T, \psi(N))$$
$$\psi(ee(A, u.x.N) = \psi'(A, spl(u.x.\psi(N)))$$

$\psi'$ : A $\times$ Ms $\rightarrow$ M

$$\psi'(var(x), Ms) = (x; Ms)$$
$$\psi'(ap(A, N), Ms) = \psi'(A, (\psi(N) :: Ms))$$
$$\psi'(fst(A), Ms) = \psi'(A, p(Ms))$$
$$\psi'(snd(A), Ms) = \psi'(A, q(Ms))$$
$$\psi'(apn(A, T), Ms) = \psi'(A, apq(T, Ms))$$

**Lemma 1.1**

**i)** $\psi(\theta(M)) \; = \; M$

**ii)** $\psi(\theta'(A, Ms)) \; = \; \psi'(A, Ms)$

**Lemma 1.2**

**i)** $\theta(\psi(N)) \; = \; N$

**ii)** $\theta(\psi'(A, Ms)) \; = \; \theta'(A, Ms)$

**Lemma 1.3 (**SOUNDNESS) *The following rules are admissible:*

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \rhd\!\!\rhd \theta(M) : P} \qquad \frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(A, Ms) : R}$$

**Lemma 1.4 (**ADEQUACY) *The following rules are admissible:*

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \Rightarrow \psi(N) : P} \qquad \frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \Rightarrow \psi'(A, Ms) : R}$$

**Corollary 1.1** *The calculus* MJ *is sound and complete.*

Finally, by study of the cut-elimination reductions and the associated term reductions (neither of which have been included here), the following theorem can be proved (again from [DP96], [DP98a]):

**Theorem 1.2 (**STRONG NORMALISATION) *Every cut-elimination strategy terminates (in a cut-free proof).*

### 1.1.2 Advantages of MJ

As discussed above, the proofs of MJ represent a normal form for proofs in a more usual sequent system: all proofs can be permuted to one with the form of an MJ proof. The proofs of this systems are also in 1–1 correspondence with the normal natural deductions of intuitionistic logic.

MJ's focusing (see [And92]) on the stoup formula (that is, its avoidance of permutations) makes the calculus more direct for finding proofs of a formula. As discussed below, MJ can be seen as a logic programming language. Again, this is related to its proof search properties.

There are both practical and theoretical reasons to be interested in MJ and other ‘permutation-free’ calculi. MJ provides a refinement of the notion of sequent, bringing the sequent calculus closer to its proof-theoretic semantics of normal natural deductions. Indeed the structure of a normal natural deduction can be seen in the structure of an MJ proof.

One of the main themes of this thesis is the use of the ideas and techniques developed for MJ with other constructive logics, namely Lax Logic (Chapter 3) and Intuitionistic Linear Logic (see Chapter 6). We also use MJ as the basis for proving intuitionistic formulae (as opposed to enumerating all proofs), and argue that for this purpose too, MJ is a better calculus than some more usual formulations (Chapter 4). In Chapter 5 we will discuss embedding intuitionistic logic in linear logic, with especial attention to MJ.

## 1.2 Theorem Proving

Whilst for many purposes one may be interested in enumerating all proofs of a formula, for others a simple provable/unprovable answer will do. In this case we are interested in the quickest way of getting this answer (and in its correctness). Propositional logics are usually decidable (although propositional linear logic is a notable exception to this, see [LMSS92]) and therefore we are interested in finding these decision procedures, in particular we would like quick decision procedures.

The contraction rule is a major obstacle to finding decision procedures for non-classical logics. Duplication of a formula means that on backwards proof search the sequents become more complicated, not less. We have no obvious way of seeing that we should terminate the search. Leaving contraction out usually leaves an incomplete calculus. One can either try and find a calculus that duplicates resources in a more subtle way (leading to G4 for intuitionistic logic) or study the nature of non-terminating backwards search to see where one can stop the search.

In Chapter 4 we develop a technique for detecting loops using a history mechanism, building on work of Heuerding *et al* ([HSZ96], [Heu98]). We apply it to some non-classical logics, giving useful decision procedures.

## 1.3 Linear Logic

Girard's linear logic ([Gir87]) is a powerful 'constructive' logic. It is a substructural (resource sensitive) logic – weakening and contraction are not generally valid. The logic takes the usual logical connectives and breaks them into multiplicative (context splitting) and additive (context sharing) versions. Hence we have two conjunctions (tensor '$\otimes$' and with '$\&$'); two disjunctions (par '$\invamp$' and plus '$\oplus$'); and it is possible to give two implications (lollipop '$\multimap$'. Additive implication, '$\leadsto$' can be defined, but is rarely included). We also have four logical constants, multiplicative: '$I$', '$-$', additive: '$\top$', '$0$'. A logic without any structural rules at all is very weak. The main novelty of linear logic is that the structural rules are reintroduced,

$$\frac{}{P \Rightarrow P}\,(ax) \qquad \frac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow R}{\Gamma, \Delta \Rightarrow R}\,(cut)$$

$$\frac{}{\Rightarrow I}\,(I_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow R}{\Gamma, I \Rightarrow R}\,(I_{\mathcal{L}})$$

$$\frac{}{\Gamma \Rightarrow \top}\,(\top_{\mathcal{R}}) \qquad \frac{}{\Gamma, 0 \Rightarrow R}\,(0_{\mathcal{L}})$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \multimap Q}\,(\multimap_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P \quad \Delta, Q \Rightarrow R}{\Gamma, \Delta, P \multimap Q \Rightarrow R}\,(\multimap_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Delta \Rightarrow Q}{\Gamma, \Delta \Rightarrow P \otimes Q}\,(\otimes_{\mathcal{R}}) \qquad \frac{\Gamma, P, Q \Rightarrow R}{\Gamma, P \otimes Q \Rightarrow R}\,(\otimes_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \& Q}\,(\&_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow R}{\Gamma, P \& Q \Rightarrow R}\,(\&_{\mathcal{L}_1}) \qquad \frac{\Gamma, Q \Rightarrow R}{\Gamma, P \& Q \Rightarrow R}\,(\&_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \oplus Q}\,(\oplus_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \oplus Q}\,(\oplus_{\mathcal{R}_2}) \qquad \frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma, P \oplus Q \Rightarrow R}\,(\oplus_{\mathcal{L}})$$

$$\frac{!\Gamma \Rightarrow P}{!\Gamma \Rightarrow !P}\,(P) \qquad \frac{\Gamma, P \Rightarrow R}{\Gamma, !P \Rightarrow R}\,(D)$$

$$\frac{\Gamma \Rightarrow R}{\Gamma, !P \Rightarrow R}\,(W) \qquad \frac{\Gamma, !P, !P \Rightarrow R}{\Gamma, !P \Rightarrow R}\,(C)$$

**Figure 1.5:** Sequent Calculus system for ILL

but for marked formulae (the exponential formulae) only. For this purpose two extra logical connectives are needed: ofcourse (or bang, allowing weakening and contraction on the left) '!', and whynot (query, allowing weakening and contraction on the right) 'Γ'. Full classical linear logic (CLL) is completely symmetric, and is often presented as a single-sided sequent calculus. Both single-sided and two-sided presentations of linear logic can be found in the Appendix A. There are several good introductions to linear logic: amongst them are Girard's original paper ([Gir87]), [Gir95], [Ale93] and [Tro92].

In this thesis we are mainly interested in *Intuitionistic Linear Logic* (ILL). This system is the single succedent restriction of the two-sided presentation of linear logic. This leads to a logic without the ⅋ and Γ connectives, as well as the logical constant $-$. Another way of looking at ILL is as a deconstruction of intuitionistic logic, a refinement of the understanding of intuitionistic connectives (hence the nomenclature). Intuitionistic logic has no structural rules on the right, and both weakening and contraction on the left. ILL restricts structural rules to certain marked formulae on the left. The logical connectives are then split into additive and multiplicative connectives as before. The sequent calculus system (which we refer to simply as ILL) can be seen in Figure 1.5. (Note that, as observed by Schellinx in [Sch94], CLL is not a conservative extension of ILL. The system of Full Intuitionistic Linear Logic (FILL) is therefore of interest – CLL is a conservative extension of this system. See [dPH93]).

$$\frac{}{P \vdash P}\ (ax) \qquad \frac{\Gamma \vdash P \quad \Delta, P \vdash R}{\Gamma, \Delta \vdash R}\ (subs)$$

$$\frac{}{\vdash I}\ (I_{\mathcal{I}}) \qquad \frac{\Gamma \vdash I \quad \Delta \vdash R}{\Gamma, \Delta \vdash R}\ (I_{\varepsilon})$$

$$\frac{\Gamma_1 \vdash P_1 \quad \ldots \quad \Gamma_n \vdash P_n}{\Gamma_1, \ldots, \Gamma_n \vdash \top}\ (\top_{\mathcal{I}}) \qquad \frac{\Gamma_1 \vdash P_1 \quad \ldots \quad \Gamma_n \vdash P_n \quad \Delta \vdash 0}{\Gamma_1, \ldots, \Gamma_n, \Delta \vdash R}\ (0_{\varepsilon})$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \multimap Q}\ (\multimap_{\mathcal{I}}) \qquad \frac{\Gamma \vdash P \multimap Q \quad \Delta \vdash P}{\Gamma, \Delta \vdash Q}\ (\multimap_{\varepsilon})$$

$$\frac{\Gamma \vdash P \quad \Delta \vdash Q}{\Gamma, \Delta \vdash P \otimes Q}\ (\otimes_{\mathcal{I}}) \qquad \frac{\Gamma \vdash P \otimes Q \quad \Delta, P, Q \vdash R}{\Gamma, \Delta \vdash R}\ (\otimes_{\varepsilon})$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \& Q}\ (\&_{\mathcal{I}}) \qquad \frac{\Gamma \vdash P \& Q}{\Gamma \vdash P}\ (\&_{\varepsilon_1}) \qquad \frac{\Gamma \vdash P \& Q}{\Gamma \vdash Q}\ (\&_{\varepsilon_2})$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \oplus Q}\ (\oplus_{\mathcal{I}_1}) \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \oplus Q}\ (\oplus_{\mathcal{I}_2})$$

$$\frac{\Gamma \vdash P \oplus Q \quad \Delta, P \vdash R \quad \Delta, Q \vdash R}{\Gamma, \Delta \vdash R}\ (\oplus_{\varepsilon})$$

$$\frac{\Gamma_1 \vdash !Q_1 \quad \ldots \quad \Gamma_n \vdash !Q_n \quad !Q_1, \ldots, !Q_n \vdash P}{\Gamma_1, \ldots, \Gamma_n \vdash !P}\ (P)$$

$$\frac{\Gamma \vdash !P}{\Gamma \vdash P}\ (D)$$

$$\frac{\Gamma \vdash !P \quad \Delta \vdash R}{\Gamma, \Delta \vdash R}\ (W) \qquad \frac{\Gamma \vdash !P \quad \Delta, !P, !P \vdash R}{\Gamma, \Delta \vdash R}\ (C)$$

**Figure 1.6:** NILL: Sequent Style Natural Deduction Calculus for ILL

We are interested in ILL because of its relationship to intuitionistic logic (as well as with logic programming). We can use similar machinery for studying ILL to that used for intuitionistic logic, whereas, despite being constructive, CLL has to be understood in new ways. For example, CLL has as semantics: proof nets or coherence spaces or certain categories or perhaps games. For ILL we can (amongst other semantics) study natural deduction, which for intuitionistic logic has a long history and is well understood. The natural deduction calculus we primarily consider is that of Benton, Bierman, de Paiva and Hyland from ([BBdPH92], [BBdPH93b], [BBdPH93a], [Bie94]). This calculus can be seen in a sequent style in Figure 1.6. We call this calculus NILL. There are several other natural deduction systems for ILL in the literature. Some are perfectly satisfactory alternatives to the one we consider; others less so. We leave discussion of these alternative systems, as well as commentary on NILL, until Chapter 6.

We will be interested in the normal forms of natural deductions in ILL. A notion of ($\beta$, $c$)-normal form can be defined for the natural deductions of ILL. It is these ($\beta$, $c$)-normal deductions that are investigated further in Chapter 6.

## 1.4 Logic Programming

The final piece of background included in this introduction is the link between permutation-free sequent calculi and logic programming.

One view of logic programming is that it is about backwards proof search (as in proof enumeration) in constructive logics. This view is laid out by Miller *et al.* in [MNPS91] (see also [Har94]). We describe goal-directed proof search in the Horn formula and hereditary Harrop formula fragments of first-order intuitionistic logic, as given in [MNPS91]. We also present calculi for goal-directed proof search in these fragments.

### 1.4.1 Uniform Proofs and Abstract Logic Programming Languages

We give the definitions of uniform proof and of abstract logic programming language from [MNPS91].

**Definition 1.1** *A* **uniform proof** *of a single succedent sequent in a fragment of intuitionistic logic is a sequent calculus proof in which every occurrence of a sequent with a non-atomic goal is the conclusion of a right rule.*

**Definition 1.2** *An* **abstract logic programming language** *is a triple* $(D, G, \vdash)$ *(where* $D$ *is the set of valid context formulae and* $G$ *is the set of valid goal formulae and* $\vdash$ *is the consequence relation), such that for any subset* $D'$ *of* $D$ *and any element* $G'$ *of* $G$*,* $D' \vdash G'$ *iff there is a uniform proof of* $G'$ *from* $D'$*.*

### 1.4.2 Horn formulae

Horn formulae (**D**) are given by the following grammar (where **G** stands for Horn goal formula and **D** for Horn definite formula);

**G**::=

$$\top \mid A \mid G \wedge G \mid G \vee G \mid \exists V.G$$

**D**::=

$$A \mid G \supset A \mid D \wedge D \mid \forall V.D$$

It is known that $\vdash_{IL} Ds \Rightarrow G$ iff $\vdash_{CL} Ds \Rightarrow G$. Moreover, the Horn definite formulae are classically equivalent to the ‘Horn clauses’ of theorem proving (modulo issues to do with quantifiers).

$$\frac{}{\Gamma, A \Rightarrow A}\,(ax) \qquad \frac{}{\Gamma \Rightarrow \top}\,(\top)$$

$$\frac{\Gamma, D \Rightarrow G}{\Gamma \Rightarrow D \supset G}\,(\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow G_1 \quad \Gamma \Rightarrow G_2}{\Gamma \Rightarrow G_1 \wedge G_2}\,(\wedge_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow G_1}{\Gamma \Rightarrow G_1 \vee G_2}\,(\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow G_2}{\Gamma \Rightarrow G_1 \vee G_2}\,(\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma \Rightarrow G[y/x]}{\Gamma \Rightarrow \forall x.G}\,(\forall_{\mathcal{R}})* \qquad \frac{\Gamma \Rightarrow G[t/x]}{\Gamma \Rightarrow \exists x.G}\,(\exists_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow G_1 \quad .... \quad \Gamma \Rightarrow G_n}{\Gamma \Rightarrow A}\,(BC)\dagger$$

* $y$ not free in $\Gamma$

† where $n \geq 0$, $D \in \Gamma$ and $< \{G_1, ..., G_n\}, A > \in |D|$

**Figure 1.7:** The system $\mathcal{I}$ for a fragment of intuitionistic logic

### 1.4.3 Hereditary Harrop formulae

Hereditary Harrop formulae are given by the following grammar (where **G** stands for hereditary Harrop goal formula and **D** for hereditary Harrop definite formula):

**G**::=

$$\top \mid A \mid D \supset G \mid G \wedge G \mid G \vee G \mid \forall V.G \mid \exists V.G$$

**D**::=

$$A \mid G \supset D \mid D \wedge D \mid \forall V.D$$

Note that the Horn formulae defined in the previous section are hereditary Harrop definite formulae. We give a calculus (the backchaining calculus, $\mathcal{I}$) for hereditary Harrop logic (and hence one that can be restricted to one for Horn formulae) which gives exactly the uniform proofs. The soundness and completeness of this calculus tells us that hereditary Harrop goal formulae, hereditary Harrop definite formulae and the intuitionistic consequence relation form an abstract logic programming language. The backchaining calculus for intuitionistic logic can be seen in Figure 1.7. This is taken from [HM94]. We need the following definition:

**Definition 1.3** *Where $P$ is a* **D** *formula, we define $|P|$ to be the smallest set of pairs such that:*

1. $< \phi, P > \in |P|$
2. *if* $< \Delta, P_1 \wedge P_2 > \in |P|$ *then* $< \Delta, P_1 > \in |P|$ *and* $< \Delta, P_2 > \in |P|$
3. *if* $< \Delta, \forall x.P' > \in |P|$ *then for all closed terms* $t$, $< \Delta, P'[t/x] > \in |P|$
4. *if* $< \Delta, G \supset P' > \in |P|$ *then* $< \Delta \cup \{G\}, P' > \in |P|$

It has been noted by Dyckhoff & Pinto ([DP94], [Dyc98]) that the proofs produced by goal-directed proof search in the backchaining calculus correspond to normal natural deductions in the fragment of intuitionistic logic being studied. Note that this correspondence is only for certain restricted fragments of the logic.

### 1.4.4 MJ and Logic Programming

The backchaining calculus results from the development of the view of logic programming as the backwards search for a proof of a formula in a constructive logic. The hereditary Harrop formula fragment of intuitionistic logic can be seen as the maximal fragment of intuitionistic logic for which goal directed proof search is complete ([Har94]).

Logic programming is not just about what is provable, but about how something is proved – proof enumeration, not just theorem proving. If one holds the view that the proofs that should be enumerated are normal natural deductions, then one would like a suitable system for enumerating these proofs. As discussed above, MJ is such a system.

If one restricts MJ to the hereditary Harrop formula fragment, one can see that this semantically motivated calculus matches the pragmatically motivated backchaining calculus. As MJ extends the backchaining calculus to a calculus for the whole of first-order intuitionistic logic, it might be thought of as a logic programming language. MJ can then be thought of as suggesting a natural extension to the notion of abstract logic programming language, one bringing it away from the syntactic notion of goal-directed proof search and instead basing it on semantics. MJ is then an extension of the backchaining calculus to a calculus for a logic programming language with disjunction and the existential quantifier on the left, that is, the whole of intuitionistic logic.

In [FMW97] a backchaining calculus for a fragment of an intuitionistic modal logic, Lax Logic, is given as an abstract language for constraint logic programming. As Lax Logic is a simple extension of intuitionistic logic, this seems an appropriate case to apply permutation-free techniques to.

In [HM94] the ideas of abstract logic programming language, uniform proof and backchaining calculi are applied to a fragment of ILL. This results in the calculus/programming language Lolli. Linear logic programming languages provide a more refined language than the usual ones, increasing the expressivity of logic programming languages. In Chapter 6 we develop an MJ like calculus for ILL with the aim of giving a natural extension to Lolli in the same way that MJ extends the backchaining calculus. A more detailed discussion and overview of Lolli, as well as all details, are left to Chapter 6.

# Chapter 2

# Permutations

This chapter is an investigation of the permutability properties of the rules of the two-sided Gentzen system for Intuitionistic Linear Logic (see Figure 1.5). We give background on the permutability of the rules for intuitionistic logic and single-sided linear logic, as well as definitions of permutation of inferences and of inference rules. We tabulate the permutations in ILL and give a calculus, ILLF, for the logic adapted from Andreoli's work on focusing proofs ([And92]). ILLF finds only one proof in each equivalence class of proofs equivalent up to permutations.

## 2.1 Background

### 2.1.1 Intuitionistic Logic

Kleene studies the permutability properties of sequent calculi in [Kle52b]. Kleene considers the permutability properties of classical and intuitionistic first-order logic. He defines a notion of permutability of inferences $I_1$ and $I_2$, where $I_2$ is immediately below (as in closer to the root) $I_1$ in the proof tree. The results of this investigation (for the propositional connectives) can be seen in Table 2.1. (Note that this table would be slightly different under the definition of permutation we give in section 2.1.3).

A similar table can be found in [DP97], [DP98b]. Mints also studies permutability of inferences in intuitionistic logic ([Min96]). These papers give a more detailed account of permutability of proofs in intuitionistic logic. Curry studies permutations for classical logic in [Cur52b].

The calculus MJ can be studied as a calculus avoiding permutations. The derivations in this system can be seen as canonical forms for intuitionistic proofs with respect to permutation of inferences. Every proof in the usual sequent formulation (G3) of the logic can, by permutation of inferences, be (weakly) normalised to the structure of an MJ proof (see [DP98b]). Strong normalisation of permutation of proofs is

| | | $I_1$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | $\supset_\mathcal{R}$ | $\supset_\mathcal{L}$ | $\wedge_\mathcal{R}$ | $\wedge_\mathcal{L}$ | $\vee_\mathcal{R}$ | $\vee_\mathcal{L}$ |
| $I_2$ | $\supset_\mathcal{R}$ | n | x | n | p | n | p |
| | $\supset_\mathcal{L}$ | p | p | p | p | p | p |
| | $\wedge_\mathcal{R}$ | n | p | n | p | n | p |
| | $\wedge_\mathcal{L}$ | p | p | p | p | p | p |
| | $\vee_\mathcal{R}$ | n | p | n | p | n | p |
| | $\vee_\mathcal{L}$ | x | x | x | p | x | p |

Key:
p stands for permutable
x stands for non-permutable
n stands for not possible.

Table 2.1: Permutability of inference rules in propositional intuitionistic logic, G3

investigated in [DP98b], [Sch98]. It should be noted that MJ doesn't avoid all permutations – some of those involving $(\vee_\mathcal{L})$ can still be performed on the image of MJ derivations inside G3. However, Dyckhoff & Pinto claim that these permutations are not semantically sound. There are no corresponding equivalences of proof in natural deduction for intuitionistic logic. Hence these permutations are not interesting (an alternative point of view would be that this suggests that natural deduction is a poor semantics outside of hereditary Harrop logic, [GLT89]). Indeed, the table of the permutability of inference rules in a sequent system is dependent on exactly which sequent system for the logic we look at. Kleene and Dyckhoff & Pinto study the system G3, an additive system, and allow liberal use of structural rules to ensure the permuted proofs are valid. However, one could easily consider a multiplicative intuitionistic calculus (such as G6), where the structural rules would be more important. In this calculus the table of the permutability of inferences would be significantly different. For example, the $(\vee_\mathcal{L})$ rule no longer permutes down past $(\supset_\mathcal{L})$. The permutabilities in G6 can be seen in Table 2.2. Permutation of inference rules in a sequent system seems to be a syntactic notion – its relationship to semantics is not a straightforward issue.

### 2.1.2 Linear Logic

Permutation of inferences in linear logic has also been studied, notably by Bellin ([Bel93]) and by Galmiche & Perrier ([GP94]). These studies consider full classical linear logic with a one-sided sequent presentation. In Tables 2.3 and 2.4 we present the results of Bellin and Galmiche & Perrier respectively (restricting to the propositional fragment).

We are interested in the permutation properties of Intuitionistic Linear Logic, presented as a two-sided sequent calculus with implication as a connective.

| | | $I_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | W | C | $\supset_{\mathcal{R}}$ | $\supset_{\mathcal{L}}$ | $\wedge_{\mathcal{R}}$ | $\wedge_{\mathcal{L}}$ | $\vee_{\mathcal{R}}$ | $\vee_{\mathcal{L}}$ |
| $I_2$ | W | p | p | p | p | p | p | p | p |
| | C | p | p | p | x | x | p | p | x |
| | $\supset_{\mathcal{R}}$ | p | p | n | x | n | p | n | p |
| | $\supset_{\mathcal{L}}$ | p | p | p | p | p | p | p | x |
| | $\wedge_{\mathcal{R}}$ | p | p | n | p | n | p | n | x |
| | $\wedge_{\mathcal{L}}$ | p | p | p | x | x | p | p | x |
| | $\vee_{\mathcal{R}}$ | p | p | n | p | n | p | n | p |
| | $\vee_{\mathcal{L}}$ | p | p | x | x | x | p | x | p |

Key:
p stands for permutable
x stands for non-permutable
n stands for not possible

Table 2.2: Permutability of inference rules in propositional intuitionistic logic, G6

| | | $I_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | W | C | $\otimes$ | $\invamp$ | & | $\oplus$ | $P$ | $D$ |
| $I_2$ | W | p | p | p | p | p | p | p | p |
| | C | p | p | x | p | p | p | p | p |
| | $\otimes$ | p | p | p | p | p | p | x | p |
| | $\invamp$ | p | p | x | p | p | p | x | p |
| | & | x | x | x | p | p | x | x | x |
| | $\oplus$ | p | p | p | p | p | p | x | p |
| | $P$ | p | p | x | x | x | x | x | x |
| | $D$ | p | p | p | p | p | p | p | p |

Key:
p stands for permutable
x stands for non-permutable

Table 2.3: Permutability of inference rules in propositional linear logic, Bellin

| | | $I_1$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | W | C | $\otimes$ | $\invamp$ | & | $\oplus$ | $P$ | $D$ |
| $I_2$ | W | p | p | p | p | p | p | p | p |
| | C | p | p | x | p | p | p | p | p |
| | $\otimes$ | p | p | p | p | p | p | x | p |
| | $\invamp$ | p | p | x | p | p | p | x | p |
| | & | x | x* | x | x* | x* | x | x | x |
| | $\oplus$ | p | p | p | p | p | p | x | p |
| | $P$ | p | p | n | n | n | n | n | x |
| | $D$ | p | p | p | p | p | p | p | p |

Key:
p stands for permutable
x stands for non-permutable
n stands for not possible
x* stands for permutable, depending on the definition of permutability

Table 2.4: Permutability of inference rules in propositional linear logic, Galmiche & Perrier

### 2.1.3 Permutation

In this section we define what we mean by a permutation, taking our terminology from Kleene ([Kle52b]), Galmiche & Perrier ([GP94]) and Troelstra & Schwichtenberg ([TS96]). We define permutation of inferences (as specific rule instances), and permutation of inference rules. We give a table of the permutabilities of inference rules in ILL, and a discussion of its content.

**Definition 2.1** *The* **principal** *formula of an inference $I$ is the formula in the conclusion in which the logical symbol is introduced, or which is the result of a contraction or a weakening.*

**Definition 2.2** *The* **active** *formulae of an inference $I$ are those formulae in the premiss(es) from which the principal formula derives.*

**Definition 2.3** *The* **side** *formulae of an inference $I$ are those formulae that are unchanged from premiss(es) to conclusion (that is, those that are not principal or active).*

Having given terms of reference to the formulae in an inference, we give some definitions of positional relationships of inferences in a proof.

**Definition 2.4** *Inference $I_2$ is an* **immediate ancestor** *of inference $I_1$ (and $I_1$ is an* **immediate descendant** *of $I_2$) if the conclusion of $I_1$ is a premiss of $I_2$. (Notice that an inference has only one immediate ancestor, but may have many immediate descendants.)*

**Definition 2.5** *Inferences $I_1$ and $I_2$ are in* **permutation position** *if $I_1$ is an immediate descendant of $I_2$ and if the principal formula of $I_1$ is not active in $I_2$.*

**Definition 2.6** *Let inferences $I_1$ and $I_2$ be in permutation position. Let inference $I_1$ be an instance of rule $R_1$ with premisses $\mathcal{P}_1$ and conclusion $C_1$. Let inference $I_2$ be an instance of rule $R_2$ with premisses $\mathcal{P}_2 \cup \{C_1\}$ and conclusion $C_2$. Inference $I_1$* **permutes over** *inference $I_2$ if there is a deduction of $C_2$ from $\mathcal{P}_1 \cup \mathcal{P}_2$, with instances of rules $R_1$ and $R_2$ as the only primitive rule instances used, $C_2$ the conclusion of the only instance of rule $R_1$ and with one or more instances of rule $R_2$ (or admissible rules) also used. If instances of $R_1$ and $R_2$ are the only inferences, then we say that $I_1$* **strictly permutes over** *$I_2$.*

**Definition 2.7** *Rule $R_1$* **permutes over** *(***strictly permutes over***) rule $R_2$ if for every occurrence of these rules as inferences $I_1$, $I_2$ in permutation position, $I_1$ permutes over $I_2$ ($I_1$ strictly permutes over $I_2$).*

**Definition 2.8** *If in proof $\Pi_1$, we permute an inference $I_1$ over an inference $I_2$, to get proof $\Pi_2$, then we call $\Pi_1$ the* **permutation object** *and $\Pi_2$ the* **permutation result**.

We have made a distinction between permutation and strict permutation. Inferences strictly permute if the permutation is simply a case of swapping the order two inferences, whereas they simply permute if an admissible rule (an inversion, or a structural rule) is needed. An example of a permutation where $R_1$ permutes over $R_2$, but doesn't strictly permute is the following permutation in G3 (where weakening is needed):

$$\dfrac{\Gamma, A \supset B \Rightarrow A \quad \dfrac{\Gamma, A \supset B, B, C \Rightarrow D}{\Gamma, A \supset B, B \Rightarrow C \supset D}\,(\supset_{\mathcal{R}})}{\Gamma, A \supset B \Rightarrow C \supset D}\,(\supset_{\mathcal{L}})$$

permutes to

$$\dfrac{\dfrac{\dfrac{\Gamma, A \supset B \Rightarrow A}{\Gamma, A \supset B, C \Rightarrow A}\,(W) \quad \Gamma, A \supset B, B, C \Rightarrow D}{\Gamma, A \supset B, C \Rightarrow D}\,(\supset_{\mathcal{L}})}{\Gamma, A \supset B \Rightarrow C \supset D}\,(\supset_{\mathcal{R}})$$

When the admissible rule is an inversion, it is less obvious that we should allow such a permutation. We introduced the distinction since it explains the differences in the tables of permutation of inference for single-sided classical linear logic owing to Bellin and Galmiche & Perrier that we gave earlier.

## 2.2 Permutations and Intuitionistic Linear Logic

### 2.2.1 Invertibility

As described below, invertibility of inference rules is related to permutability. In this section we give some results about the invertibility of the inference rules of two-sided ILL. Some of these results can be found in [Tro92]. We give illustrative proofs and counterexamples as well as the invertibilities themselves.

**Proposition 2.1** *The following primitive inference rules of* ILL *are invertible:* $(\otimes_{\mathcal{L}})$, $(\oplus_{\mathcal{L}})$, $(\multimap_{\mathcal{R}})$, $(\&_{\mathcal{R}})$, $(C)$, $(P)$, $(I_{\mathcal{L}})$. *The following primitive inference rules of* ILL *are not invertible:* $(\multimap_{\mathcal{L}})$, $(\&_{\mathcal{L}})$, $(\otimes_{\mathcal{R}})$, $(\oplus_{\mathcal{R}})$, $(W)$, $(D)$.

PROOF: We prove the invertibilities by showing that the inverse rules are admissible in ILL. This may be done in either of two ways. Firstly we may proceed by induction on the height of the derivation of the premiss. For example, we show that:

$$\frac{\Gamma, P \otimes Q \Rightarrow R}{\Gamma, P, Q \Rightarrow R}\ (\otimes_{\mathcal{L}}^{inv})$$

is admissible in ILL by case analysis of the last rule of the derivation of the premiss. For each possible rule we either get the conclusion or we can perform the rule at a lesser height, and we get the result by induction. We omit the long and repetitive detail. This proof can be useful because of its independence from cut elimination.

Unlike the following much shorter proof using the admissibility of cut:

$$\frac{\dfrac{\overline{P \Rightarrow P}\ (ax) \quad \overline{Q \Rightarrow Q}\ (ax)}{P, Q \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}}) \quad \Gamma, P \otimes Q \Rightarrow R}{\Gamma, P, Q \Rightarrow R}\ (cut)$$

The admissibility of all the inverse rules can be shown in similar ways.

We give a counter-example to the invertibility of $\&_{\mathcal{L}}$, that is, we show that the following rule is not admissible in ILL:

$$\frac{\Gamma, P\&Q \Rightarrow R}{\Gamma, P \Rightarrow R}\ (\&_{\mathcal{L}}^{inv_1})$$

A simple counter-example is:

$$\frac{\overline{A\&B \Rightarrow A\&B}\ (ax)}{A \Rightarrow A\&B}\ (\&_{\mathcal{L}}^{inv_1})$$

Similar counter-examples can be provided for the other non-invertible rules. ■

It is possible that for the context splitting rules we could have defined a weak notion of invertibility. For example, for $(\otimes_{\mathcal{R}})$ we might have said that if $\Gamma \Rightarrow P \otimes Q$ is provable, then there exists a splitting of $\Gamma$ into $\Gamma_1$ and $\Gamma_2$ such that $\Gamma_1 \Rightarrow P$ and $\Gamma_2 \Rightarrow Q$ are both provable. However, the possibility of contraction prevents any such notion. The following example illustrates this.

The sequent $!A \Rightarrow A \otimes A$ is provable in ILL:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}(ax)}{!A \Rightarrow A}(D) \quad \dfrac{\dfrac{}{A \Rightarrow A}(ax)}{!A \Rightarrow A}(D)}{!A, !A \Rightarrow A \otimes A}(\otimes_{\mathcal{R}})}{!A \Rightarrow A \otimes A}(C)$$

However, neither the pair of sequents ($!A \Rightarrow A$ and $\Rightarrow A$) nor the pair of sequents ($\Rightarrow A$ and $!A \Rightarrow A$) are provable.

The proof of the admissibility of the inverse rules makes it clear why invertibility is related to permutability. The proof uses the interchangeability of the inference with the other rules of the calculus, the fact that the rule is permutable with all others. We get invertibility when we have permutability. We prove a general theorem for all sequent calculi.

**Theorem 2.1** *For sequent calculus $\mathcal{G}$, if rule R strictly permutes over all rules in $\mathcal{G}$, and the active formulae can be combined, using the connectives of the logic, to make the principal formula, then rule R is invertible.*

PROOF: Consider rule $R$ with principal formula $P$ and active formulae $\mathcal{P}_i$ in the $i$th premiss ($\mathcal{P}_i$ is a set):

$$\frac{S_1 \quad \ldots \quad S_n}{S} \; R$$

then this has inverses:

$$\frac{S}{S_1} \; R^{inv_1} \qquad \ldots \qquad \frac{S}{S_n} \; R^{inv_n}$$

Consider the $i$th such rule:

$$\frac{S}{S_i} \; R^{inv_i}$$

Consider any derivation of $S$ in $\mathcal{G}$. We show that we have a derivation of $S_i$. There are four cases to consider. When we refer to $P$, we refer to an occurrence of $P$ traceable to its occurrence in the root.

1. $P$ is the principal formula for some occurrence of rule $R$. Since $R$ strictly permutes over all the inference rules of $\mathcal{G}$, we can permute it to the root, hence we have derivation ending:

$$\frac{S_1 \quad \ldots \quad S_i \quad \ldots \quad S_n}{S} \; R$$

   Hence we have a derivation of $S_i$.

2. $P$ is never principal, but is the side formula of some leaf node (or nodes) of the derivation. By replacing $P$ by the $\mathcal{P}_i$ in each sequent in which it appears, we have a derivation of $S_i$.

3. $P$ is never principal in a logical rule, but is principal in a structural rule. Similar to previous case.

4. $P$ is never principal in a logical rule, but is principal in an axiom. This case only applies when $\mathcal{G}$ allows non-atomic axioms. In this case the rule for the top connective on the other side needs to be applied where the axiom was.

■

Having proved this theorem, we illustrate the proof with an example. Suppose that the following rule permutes over all others in G6.

$$\frac{\Gamma, P, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}})$$

Consider any derivation of the sequent $\Gamma, P \wedge Q \Rightarrow R$. Again, when we refer to an occurrence of a formula we mean an occurrence that can be traced its occurrence at the root. The cases are:

1. $P \wedge Q$ is principal for some inference. Then since $(\wedge_{\mathcal{L}})$ permutes over all rules of the calculus, it can be permuted to the root. Hence

$$\frac{\begin{array}{c}\vdots\\ \Gamma, P, Q \Rightarrow R\end{array}}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}})$$

   We have a proof of $\Gamma, P, Q \Rightarrow R$.

2. $P \wedge Q$ is never principal, and is the side formula of some leaf node. Then

$$\begin{array}{c}\overline{\Gamma', P \wedge Q, S \Rightarrow S}\ (ax)\\ \vdots\\ \Gamma, P \wedge Q \Rightarrow R\end{array} \quad \text{becomes} \quad \begin{array}{c}\overline{\Gamma', P, Q, S \Rightarrow S}\ (ax)\\ \vdots\\ \Gamma, P, Q \Rightarrow R\end{array}$$

   We have a proof of $\Gamma, P, Q \Rightarrow R$.

3. $P \wedge Q$ is never principal in a logical rule, but is principal in a structural rule. Consider weakening:

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Gamma' \Rightarrow S\end{array}}{\Gamma', P \wedge Q \Rightarrow S}\ (W)\\ \vdots\\ \Gamma, P \wedge Q \Rightarrow R\end{array} \quad \text{becomes} \quad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma' \Rightarrow S\end{array}}{\Gamma', Q \Rightarrow S}\ (W)}{\Gamma', P, Q \Rightarrow S}\ (W)\\ \vdots\\ \Gamma, P, Q \Rightarrow R\end{array}$$

We have a proof of $\Gamma, P, Q \Rightarrow R$. Consider contraction:

$$\dfrac{\Gamma', P \wedge Q, P \wedge Q \Rightarrow S}{\Gamma', P \wedge Q \Rightarrow S}\,(C) \quad \vdots \quad \Gamma, P \wedge Q \Rightarrow R \qquad \text{becomes} \qquad \dfrac{\dfrac{\Gamma', P, P, Q, Q \Rightarrow S}{\Gamma', P, P, Q \Rightarrow S}\,(C)}{\Gamma', P, Q \Rightarrow S}\,(C) \quad \vdots \quad \Gamma, P, Q \Rightarrow R$$

4. $P \wedge Q$ is never principal in a logical rule, but is principal in an axiom. Then:

$$\dfrac{}{\Gamma', P \wedge Q \Rightarrow P \wedge Q}\,(ax) \quad \vdots \quad \Gamma, P \wedge Q \Rightarrow R \qquad \text{becomes} \qquad \dfrac{\dfrac{}{\Gamma', P, Q \Rightarrow P}\,(ax) \quad \dfrac{}{\Gamma', P, Q \Rightarrow Q}\,(ax)}{\Gamma', P, Q \Rightarrow P \wedge Q}\,(\wedge_{\mathcal{R}}) \quad \vdots \quad \Gamma, P, Q \Rightarrow R$$

We have a proof of $\Gamma, P, Q \Rightarrow R$.

### 2.2.2 Permutability Table for ILL

In section 2.1.3 we gave definitions of permutability of inferences and rules. Table 2.5 gives the permutability of the rules of ILL, indicating whether two rules permute (and if so under what definition), or are never in permutation position, or do not permute.

Study of this table suggests that some inferences are more suitable for permuting backwards (toward the leaves) and others forwards to the root. Following [GP94] we call the rules suitable for backward permutation, $T_\uparrow$; those suitable for forward permutation, $T_\downarrow$. $T_\downarrow = \{(\otimes_{\mathcal{L}}), (\oplus_{\mathcal{L}}), (\multimap_{\mathcal{R}}), (\&_{\mathcal{R}}), (I_{\mathcal{L}})\}$. $T_\uparrow = \{(\multimap_{\mathcal{L}}), (\otimes_{\mathcal{R}}), (\&_{\mathcal{L}}), (\oplus_{\mathcal{R}}), (W), (D), (C)\}$. Notice that, as one would expect, the inference rules that are suitable for forward permutation are those that are invertible, and those suitable for backward permutation are the non-invertible rules. The only exception to this is contraction, which is invertible, but is moved backward since the more formulae there are in a sequent, the harder it is to control. Also note that $(P)$, also invertible, is not included in either of these sets. Study of the table doesn't suggest an obvious answer to how we should try and move this inference. In fact, we leave it as a pivot about which the structure of proofs revolve.

Having studied the permutation of inference rules in classical linear logic, Galmiche & Perrier define a normal form for sequent derivations. We give a version of this definition for two-sided ILL. The aim being to avoid redundancies in proofs, we first observe that cut elimination holds for ILL (see [Bie94]) and so we do not have to consider a system with cut. We also try to avoid weakening/contraction pairs,

| | | $I_1$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\otimes_{\mathcal{L}}$ | $\oplus_{\mathcal{L}}$ | $\multimap_{\mathcal{L}}$ | $\&_{\mathcal{L}}$ | $\otimes_{\mathcal{R}}$ | $\oplus_{\mathcal{R}}$ | $\multimap_{\mathcal{R}}$ | $\&_{\mathcal{R}}$ | $C$ | $W$ | $D$ | $P$ | $I_{\mathcal{L}}$ |
| $I_2$ | $\otimes_{\mathcal{L}}$ | p | p | x | p | x | p | p | p | p | p | p | x | p |
| | $\oplus_{\mathcal{L}}$ | i | i | x | x | x | x | i | i | i | x | x | x | i |
| | $\multimap_{\mathcal{L}}$ | p | p | p | p | p | p | p | p | p | p | p | x | p |
| | $\&_{\mathcal{L}}$ | p | p | p | p | p | p | p | p | p | p | p | x | p |
| | $\otimes_{\mathcal{R}}$ | p | p | p | p | n | n | n | n | p | p | p | n | p |
| | $\oplus_{\mathcal{R}}$ | p | p | p | p | n | n | n | n | p | p | p | n | p |
| | $\multimap_{\mathcal{R}}$ | p | p | x | p | n | n | n | n | p | p | p | n | p |
| | $\&_{\mathcal{R}}$ | i | i | x | x | n | n | n | n | i | x | x | n | i |
| | $C$ | p | p | x | p | x | p | p | p | p | p | p | p | p |
| | $W$ | p | p | p | p | p | p | p | p | p | p | p | p | p |
| | $D$ | p | p | p | p | p | p | p | p | p | p | p | p | p |
| | $P$ | n | n | n | n | n | n | n | n | p | p | x | n | n |
| | $I_{\mathcal{L}}$ | p | p | p | p | p | p | p | p | p | p | p | x | p |

Key:
p stands for strictly permutable
x stands for non-permutable
n stands for not possible
i stands for permutable (using invertibility)

Table 2.5: Permutability of inference rules in propositional ILL

such as:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow P}\,(ax)}{!P \Rightarrow P}\,(D)}{!P, !P \Rightarrow P}\,(W)}{!P \Rightarrow P}\,(C)$$

**Definition 2.9** *Proof* $\Pi$ *in* ILL *is* **under weakening and contraction reduction** *if for any instance of rule* $(C)$*, the active formulae are not principal formulae of an immediate descendant inference* $(W)$*.*

Note that following definition (from Galmiche & Perrier) of normal proof is unrelated to the notion of normal natural deduction used elsewhere in this thesis.

**Definition 2.10** *Proof* $\Pi$ *in* ILL *is* **normal** *if it is cut-free, under weakening and contraction reduction and:*

1. *any sequent of form* $!\Gamma \Rightarrow !P$ *is the conclusion of a* $(P)$
2. *else if sequent* $S$ *contains formulae introduced by an inference rule in* $T_{\downarrow}$ *then* $S$ *is the conclusion of an inference rule in* $T_{\downarrow}$

3. *else if sequent $S$ contains formulae introduced by an inference rule in $T_\uparrow \setminus \{(W), (C)\}$, then each premiss is either the conclusion of a $(P)$ or the active formula (if it is not atomic) is the principal formula in the immediate descendant.*

4. *else if sequent $S$ has a principal formula $!P$, then either*

   (a) *$S$ is the conclusion of a $(W)$ which is the result of a chain of weakenings from an axiom*

   (b) *$S$ is the conclusion of a $(C)$ and the immediate descendant is a $(D)$ introducing one of the active formulae of the $(C)$ or a chain of contractions from a context splitting rule.*

Later we compare the sequent proofs in this normal form with proofs in the calculus given in the next section.

## 2.3 Focusing Proofs

In this section we describe the notion of a 'focusing proof' introduced in [And92] and apply it to the two-sided sequent calculus for ILL. In his paper, Andreoli gives a single-sided focusing calculus for classical linear logic. Here we use the same ideas to get a focusing calculus for two-sided ILL. We compare this with the permutability table for ILL and with the definition of a normal sequent proof from Galmiche & Perrier.

The motivation for focusing proofs is the same as for many of the calculi mentioned in this thesis – to have a calculus that avoids finding proofs that are, in some sense, essentially the same. Andreoli's work is developed syntactically from the sequent calculus presentation of linear logic, rather than the semantic approach taken later in this thesis. Sequent proofs are studied, and redundancies, such as permutations and trivial loops, are identified. A calculus that (as far as possible) avoids these is given. The resulting calculus is one suitable for theorem proving – finding a proof efficiently. By taking a purely syntactic view of proof, and considering the focusing proofs as normal forms with respect to permutations, ILLF can also be seen as a proof enumeration calculus. This is the view taken by Andreoli in [And92]. However, focusing calculi lack the semantic rationale that proof enumeration calculi should have.

We take the following definitions from Andreoli ([And92]).

**Definition 2.11** *Two proofs are said to be* **P-equivalent** *if each can be transformed to the other by simple permutation of inference figures and elimination or introduction of weakening/contraction pairs.*

We give a calculus for ILL similar to that of Andreoli's for classical linear logic which finds only one proof in each P-equivalence class.

We give two definitions:

**Definition 2.12** *A connective is* **asynchronous** *if, when a principal formula with this connective as the top connective has been selected, there is only one applicable instance of an inference rule.*

**Definition 2.13** *A connective is* **synchronous** *if, when a principal formula with this connective as the top connective has been selected, there is more than one applicable instance of an inference rule.*

For two-sided ILL we need to distinguish between the positive and negative occurrences of connectives (their occurrence on the right and on the left). We find that the asynchronous connectives are: $\multimap^+$, $\&^+$, $\otimes^\perp$, $\oplus^\perp$. The synchronous connectives are: $\multimap^\perp$, $\&^\perp$, $\otimes^+$, $\oplus^+$. The negative occurrences will be principal formulae of left rules and the positive occurrences will be principal formulae of right rules. It is observed that the rules for the asynchronous connectives are invertible and in the set $T_\downarrow$ and that the rules for the synchronous connectives are not invertible and are in the set $T_\uparrow$. We haven't mentioned ! as it doesn't fit as neatly into this pattern and will be treated differently from the other connectives in the calculus we give.

We give a calculus which we will call ILLF. This has similar properties to Andreoli's focusing calculus for classical linear logic (called $\Sigma_3$). Firstly the problem of where to apply the structural rules is lessened. We add an extra field to the standard calculus in which we put only exponential formulae. Weakening can be permuted up towards the axioms, hence we can drop the weakening rule and in its place change the axiom rule so that any number of exponential formulae are allowed in the context. Contraction doesn't permute over context splitting rules, but if we duplicate all the exponential formulae in the new field at the application of one of these rules, then we will have duplicated the necessary formulae. Hence no explicit contraction rule is necessary. We may perform unnecessary contractions, but this is unproblematic with the new axiom rule. Note that there is a small cost to this – the possibility of dereliction of formulae that would not otherwise be in the context is introduced. Which rules can be applied is also restricted. We try to apply (backward) the asynchronous, invertible, rules first. To this end we split the context into further fields: a list of formulae and a multiset of synchronous formulae. The list places an (arbitrary) order on the way the asynchronous formulae are considered. We also have two kinds of goal (for synchronous and asynchronous goals). ILLF is displayed in Figure 2.1.

The calculus ILLF has four forms of sequent. These direct proof search by forcing asynchronous formulae (those with invertible rules) to be broken up first, and then by focusing on a formula (the active formula of a premiss is principal in the next backward inference) for as long as possible.

$$\frac{\Sigma;\Gamma\Uparrow L,P\Rightarrow Q\Uparrow}{\Sigma;\Gamma\Uparrow L\Rightarrow P\multimap Q\Uparrow}\ (\multimap_{\mathcal{R}})\qquad \frac{\Sigma;\Gamma\Uparrow L\Rightarrow P\Uparrow\quad \Sigma;\Gamma\Uparrow L\Rightarrow Q\Uparrow}{\Sigma;\Gamma\Uparrow L\Rightarrow P\&Q\Uparrow}\ (\&_{\mathcal{R}})$$

$$\frac{}{\Sigma;\Gamma\Uparrow L\Rightarrow\top\Uparrow}\ (\top_{\mathcal{R}})$$

$$\frac{\Sigma;\Gamma\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L\Rightarrow R\Uparrow}\ (\Uparrow_{\mathcal{R}})\quad \text{if } R \text{ not asynchronous}$$

$$\frac{\Sigma;\Gamma\Uparrow L,P,Q\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,P\otimes Q\Rightarrow R\Downarrow}\ (\otimes_{\mathcal{L}})$$

$$\frac{\Sigma;\Gamma\Uparrow L,P\Rightarrow R\Downarrow\quad \Sigma;\Gamma\Uparrow L,Q\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,P\oplus Q\Rightarrow R\Downarrow}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Sigma,P;\Gamma\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,!P\Rightarrow R\Downarrow}\ (S)\qquad \frac{\Sigma;\Gamma\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,I\Rightarrow R\Downarrow}\ (I_{\mathcal{L}})$$

$$\frac{}{\Sigma;\Gamma\Uparrow L,0\Rightarrow R\Downarrow}\ (0_{\mathcal{L}})$$

$$\frac{\Sigma;\Gamma,P\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,P\Rightarrow R\Downarrow}\ (Pop)\quad P \text{ not asynchronous}$$

$$\frac{\Sigma;\Gamma\Downarrow\Rightarrow R\Uparrow}{\Sigma;\Gamma\Uparrow\Rightarrow R\Downarrow}\ (\Downarrow_{\mathcal{R}})\qquad \frac{\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma;\Gamma,P\Uparrow\Rightarrow R\Downarrow}\ (Push)$$

$$\frac{\Sigma,P;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma,P;\Gamma\Uparrow\Rightarrow R\Downarrow}\ (D)$$

$$\frac{\Sigma;\Gamma\Downarrow\Rightarrow P\Uparrow\quad \Sigma;\Delta\Downarrow\Rightarrow Q\Uparrow}{\Sigma;\Gamma,\Delta\Downarrow\Rightarrow P\otimes Q\Uparrow}\ (\otimes_{\mathcal{R}})\qquad \frac{}{\Sigma;\Downarrow\Rightarrow I\Uparrow}\ (I_{\mathcal{R}})$$

$$\frac{\Sigma;\Gamma\Downarrow\Rightarrow P\Uparrow}{\Sigma;\Gamma\Downarrow\Rightarrow P\oplus Q\Uparrow}\ (\oplus_{\mathcal{R}_1})\qquad \frac{\Sigma;\Gamma\Downarrow\Rightarrow Q\Uparrow}{\Sigma;\Gamma\Downarrow\Rightarrow P\oplus Q\Uparrow}\ (\oplus_{\mathcal{R}_2})$$

$$\frac{\Sigma;\Uparrow\Rightarrow P\Uparrow}{\Sigma;\Downarrow\Rightarrow !P\Uparrow}\ (P)$$

$$\frac{\Sigma;\Gamma\Uparrow\Rightarrow R\Uparrow}{\Sigma;\Gamma\Downarrow\Rightarrow R\Uparrow}\ (\Downarrow_{\mathcal{L}_1})\quad R \text{ not synchronous}$$

$$\frac{}{\Sigma;\Downarrow A\Rightarrow A\Downarrow}\ (ax)\qquad \frac{\Sigma;\Gamma\Downarrow\Rightarrow P\Uparrow\quad \Sigma;\Delta\Downarrow Q\Rightarrow R\Downarrow}{\Sigma;\Gamma,\Delta\Downarrow P\multimap Q\Rightarrow R\Downarrow}\ (\multimap_{\mathcal{L}})$$

$$\frac{\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma;\Gamma\Downarrow P\&Q\Rightarrow R\Downarrow}\ (\&_{\mathcal{L}_1})\qquad \frac{\Sigma;\Gamma\Downarrow Q\Rightarrow R\Downarrow}{\Sigma;\Gamma\Downarrow P\&Q\Rightarrow R\Downarrow}\ (\&_{\mathcal{L}_2})$$

$$\frac{\Sigma;\Gamma\Uparrow P\Rightarrow R\Downarrow}{\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow}\ (\Downarrow_{\mathcal{L}_2})\quad P \text{ not synchronous}$$

**Figure 2.1:** The focusing calculus ILLF for Intuitionistic Linear Logic

Initial sequents have form $\Sigma; \Gamma \Uparrow L \Rightarrow R \Uparrow$. The only rules with such a sequent as the conclusion are those for asynchronous connectives on the right and $(\Uparrow_{\mathcal{R}})$, which is only applicable when the goal is not asynchronous. Hence the goal formula is forced to be broken up until not asynchronous, then the form of the sequent is changed to $\Sigma; \Gamma \Uparrow L \Rightarrow R \Downarrow$.

When the sequent has form $\Sigma; \Gamma \Uparrow L \Rightarrow R \Downarrow$ and the list, $L$, is non-empty, the only rules applicable are those for the asynchronous connectives on the left, $(Pop)$ and $(S)$. The latter two rules are only applicable with principal formulae which are not asynchronous. That is, all asynchronous formulae on the left are broken up in a fixed order and other formulae are put to one side to be dealt with later. When there are no more asynchronous formulae on the left (that is, when $L$ is the empty list) we reach the major choice point in the calculus. Using one of $(\Downarrow_{\mathcal{R}})$, $(Push)$ and $(D)$, a formula is selected and focused on.

If $(\Downarrow_{\mathcal{R}})$ is used, the goal is selected and sequent changed to form $\Sigma; \Gamma \Downarrow \Rightarrow R \Uparrow$. Only the rules for synchronous connectives on the right, $(P)$ and $(\Downarrow_{\mathcal{L}_1})$ are applicable to a sequent of this form. That is, the goal is broken up until a sequent with a non-synchronous right hand side is reached.

If $(S)$ or $(D)$ is used, a formula on the left is selected and the sequent changed to the form $\Sigma; \Gamma \Downarrow P \Rightarrow R \Downarrow$. Only the rules for synchronous formulae on the left, $(ax)$ and $(\Downarrow_{\mathcal{L}_2})$ are applicable to a sequent of this form. That is, the selected formula is broken up until the formula in the special position is not synchronous.

### 2.3.1 Soundness and Completeness

The calculus ILLF is the result of entirely syntactic observations and results on the permutability and invertibility of inference rules. To prove the required results, a lot of lemmas about the admissibility of various rules in ILLF are needed. This makes the full detail of the proof very long, although there is nothing too involved in these details. Here we state the lemmas, proving only one as an illustration of the standard techniques used in the proofs. We then prove the theorem which, once we have the lemmas, is routine.

We prove the result via the equivalence of both ILL and ILLF to an intermediate calculus, ILL$^{\Sigma}$. This calculus has two fields which absorb the structural rules of ILL. The calculus ILL$^{\Sigma}$ can be seen in Figure 2.2. Note that $\Sigma$ could have been given as a set, but for our purposes it is easier for it to be a multiset. (Intuitionistic) Linear Logic is often presented with the context split into non-linear (or classical) and linear field. A calculus similar to ILL$^{\Sigma}$ can be found in, for example, [HM94]. Treating linear and non-linear formulae separately is taken to its extremes in Girard's Logic of Unity ([Gir93]).

We prove the equivalence of ILL and ILL$^{\Sigma}$. This requires a couple of standard results.

$$\frac{}{\Sigma; A \Rightarrow A}\ (ax) \qquad \frac{}{\Sigma; \Gamma \Rightarrow \top}\ (\top_{\mathcal{R}}) \qquad \frac{}{\Sigma; \Gamma, 0 \Rightarrow R}\ (0_{\mathcal{L}})$$

$$\frac{}{\Sigma; \Rightarrow I}\ (I_{\mathcal{R}}) \qquad \frac{\Sigma; \Gamma \Rightarrow R}{\Sigma; \Gamma, I \Rightarrow R}\ (I_{\mathcal{L}})$$

$$\frac{\Sigma; \Gamma, P \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}}) \qquad \frac{\Sigma; \Gamma \Rightarrow P \quad \Sigma; \Delta, Q \Rightarrow R}{\Sigma; \Gamma, \Delta, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})$$

$$\frac{\Sigma; \Gamma \Rightarrow P \quad \Sigma; \Gamma \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \& Q}\ (\&_{\mathcal{R}})$$

$$\frac{\Sigma; \Gamma, P \Rightarrow R}{\Sigma; \Gamma, P \& Q \Rightarrow R}\ (\&_{\mathcal{L}_1}) \qquad \frac{\Sigma; \Gamma, Q \Rightarrow R}{\Sigma; \Gamma, P \& Q \Rightarrow R}\ (\&_{\mathcal{L}_2})$$

$$\frac{\Sigma; \Gamma \Rightarrow P \quad \Sigma; \Delta \Rightarrow Q}{\Sigma; \Gamma, \Delta \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}}) \qquad \frac{\Sigma; \Gamma, P, Q \Rightarrow R}{\Sigma; \Gamma, P \otimes Q \Rightarrow R}\ (\otimes_{\mathcal{L}})$$

$$\frac{\Sigma; \Gamma \Rightarrow P}{\Sigma; \Gamma \Rightarrow P \oplus Q}\ (\oplus_{\mathcal{R}_1}) \qquad \frac{\Sigma; \Gamma \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \oplus Q}\ (\oplus_{\mathcal{R}_2})$$

$$\frac{\Sigma; \Gamma, P \Rightarrow R \quad \Sigma; \Gamma, Q \Rightarrow R}{\Sigma; \Gamma, P \oplus Q \Rightarrow R}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Sigma, P; \Gamma \Rightarrow R}{\Sigma; \Gamma, !P \Rightarrow R}\ (S) \qquad \frac{\Sigma, P; \Gamma, P \Rightarrow R}{\Sigma, P; \Gamma \Rightarrow R}\ (D)$$

$$\frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P}\ (P)$$

**Figure 2.2:** The calculus ILL$^{\Sigma}$ for Intuitionistic Linear Logic

**Definition 2.14** *The* **height of a derivation** *is the number of nodes on the longest branch.*

**Lemma 2.1** *The following rules are admissible in* ILL$^{\Sigma}$*:*

$$\frac{\Sigma, P, P; \Gamma \Rightarrow R}{\Sigma, P; \Gamma \Rightarrow R}\ (C) \qquad \frac{\Sigma; \Gamma \Rightarrow P}{\Sigma, \Delta; \Gamma \Rightarrow P}\ (W*)$$

PROOF: The admissibility of both rules can be shown by standard induction arguments. ■

**Lemma 2.2** *The sequent* $!\Sigma, \Gamma \Rightarrow P$ *is provable in* ILL *iff the sequent* $\Sigma; \Gamma \Rightarrow P$ *is provable in* ILL$^{\Sigma}$.

PROOF: We illustrate the proof of this theorem for the $\multimap$, ! fragment of ILL. The extended proof is similar.

First we show that if $\Sigma; \Gamma \Rightarrow P$ is provable in ILL$^{\Sigma}$ then $!\Sigma, \Gamma \Rightarrow P$ is provable in ILL.

The proof is by induction on the height of derivations.

1. $$\frac{}{\Sigma; A \Rightarrow A}\ (ax)$$

   then

   $$\frac{\frac{}{A \Rightarrow A}\ (ax)}{!\Sigma, A \Rightarrow A}\ (W*)$$

2. In ILL$^{\Sigma}$ we have

   $$\frac{\Sigma; \Gamma, P \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}})$$

   by induction hypothesis we have:

   $$\frac{!\Sigma, \Gamma, P \Rightarrow Q}{!\Sigma, \Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}})$$

3. In ILL$^{\Sigma}$ we have

   $$\frac{\Sigma; \Gamma \Rightarrow P \quad \Sigma; \Delta, Q \Rightarrow R}{\Sigma; \Gamma, \Delta, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})$$

   by induction hypothesis we have:

   $$\frac{\frac{!\Sigma, \Gamma \Rightarrow P \quad !\Sigma, \Delta, Q \Rightarrow R}{!\Sigma, !\Sigma, \Gamma, \Delta, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})}{!\Sigma, \Gamma, \Delta, P \multimap Q \Rightarrow R}\ (C*)$$

4. In ILL$^{\Sigma}$ we have
$$\frac{\Sigma, P; \Gamma \Rightarrow R}{\Sigma; \Gamma, !P \Rightarrow R}\ (S)$$
by induction hypothesis we have:
$$!\Sigma, !P, \Gamma \Rightarrow R$$
as required.

5. In ILL$^{\Sigma}$ we have
$$\frac{\Sigma, P; \Gamma, P \Rightarrow R}{\Sigma, P; \Gamma \Rightarrow R}\ (D)$$
by induction hypothesis we have:
$$\cfrac{\cfrac{!\Sigma, !P, \Gamma, P \Rightarrow R}{!\Sigma, !P, !P, \Gamma \Rightarrow R}\ (D)}{!\Sigma, !P, \Gamma \Rightarrow R}\ (C)$$

6. In ILL$^{\Sigma}$ we have
$$\frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P}\ (P)$$
by induction hypothesis we have:
$$\frac{!\Sigma \Rightarrow P}{!\Sigma \Rightarrow !P}\ (P)$$

Now we show that if $!\Sigma, \Gamma \Rightarrow P$ (where $\Gamma$ contains no banged formulae) is provable in ILL then $\Sigma; \Gamma \Rightarrow P$ is provable in ILL$^{\Sigma}$.

The proof is by induction on the height of derivations.

1. In ILL we have
$$\frac{}{A \Rightarrow A}\ (ax)$$
then
$$\frac{}{; A \Rightarrow A}\ (ax)$$

2. In ILL we have
$$\frac{!\Sigma, \Gamma, P \Rightarrow Q}{!\Sigma, \Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}})$$
by induction hypothesis we have:
$$\frac{\Sigma; \Gamma, P \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}})$$

3. In ILL we have

$$\frac{!\Sigma_1, \Gamma \Rightarrow P \quad !\Sigma_2, \Delta, Q \Rightarrow R}{!\Sigma_1, !\Sigma_2, \Gamma, \Delta, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})$$

by induction hypothesis and Lemma 2.1 we have:

$$\frac{\dfrac{\Sigma_1; \Gamma \Rightarrow P}{\Sigma_1, \Sigma_2; \Gamma \Rightarrow P}\ (W*) \quad \dfrac{\Sigma_2; \Delta, Q \Rightarrow R}{\Sigma_1, \Sigma_2; \Delta, Q \Rightarrow R}\ (W*)}{\Sigma_1, \Sigma_2; \Gamma, \Delta, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})$$

4. In ILL we have

$$\frac{!\Sigma, \Gamma \Rightarrow R}{!\Sigma, !P, \Gamma \Rightarrow R}\ (W)$$

by induction hypothesis and Lemma 2.1 we have:

$$\frac{\Sigma; \Gamma \Rightarrow R}{\Sigma, P; \Gamma \Rightarrow R}\ (W)$$

5. In ILL we have

$$\frac{!\Sigma, !P, !P, \Gamma \Rightarrow R}{!\Sigma, !P, \Gamma \Rightarrow R}\ (C)$$

by induction hypothesis and Lemma 2.1 we have:

$$\frac{\Sigma, P, P; \Gamma \Rightarrow R}{\Sigma, P; \Gamma \Rightarrow R}\ (C)$$

6. In ILL we have

$$\frac{!\Sigma, P, \Gamma \Rightarrow R}{!\Sigma, !P, \Gamma \Rightarrow R}\ (D)$$

by induction hypothesis and Lemma 2.1 we have:

$$\frac{\dfrac{\Sigma; \Gamma, P \Rightarrow R}{\Sigma, P; \Gamma, P \Rightarrow R}\ (W)}{\Sigma, P; \Gamma \Rightarrow R}\ (D)$$

7. In ILL we have

$$\frac{!\Sigma \Rightarrow P}{!\Sigma \Rightarrow !P}\ (P)$$

by induction hypothesis we have:

$$\frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P}\ (P)$$

■

The following lemmas are needed in order to prove the equivalence of $\mathrm{ILL}^{\Sigma}$ and ILLF. Note that the rules given have an $\Uparrow$ in the succedent, but that in most cases the rule with this arrow reversed is also admissible. This is noted in the statement of the appropriate lemmas. The proofs are all by induction on the height of derivations (in fact we often simultaneously prove the admissibility of several rules with different positions of arrows). We illustrate the proofs by giving a restricted proof of the first lemma, but omit all other proofs.

**Lemma 2.3** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L \Rightarrow R \Uparrow}{\Sigma, \Delta; \Gamma \Uparrow L \Rightarrow R \Uparrow}\ (W*)$$

*(In fact, we prove this result for any legitimate combination of the arrows).*

PROOF: We illustrate the proof for the $\multimap$, ! fragment of the logic.

The proof is by induction on the height of derivations.

1. $(\multimap_{\mathcal{R}})$

$$\frac{\Sigma; \Gamma \Uparrow L, P \Rightarrow Q \Uparrow}{\Sigma; \Gamma \Uparrow L \Rightarrow P \multimap Q \Uparrow}\ (\multimap_{\mathcal{R}})$$

   by induction hypothesis we have:

$$\frac{\dfrac{\Sigma; \Gamma \Uparrow L, P \Rightarrow Q \Uparrow}{\Sigma, \Delta; \Gamma \Uparrow L, P \Rightarrow Q \Uparrow}\ (W*)}{\Sigma, \Delta; \Gamma \Uparrow L \Rightarrow P \multimap Q \Uparrow}\ (\multimap_{\mathcal{R}})$$

2. $(\Uparrow_{\mathcal{R}})$

$$\frac{\Sigma; \Gamma \Uparrow L \Rightarrow P \Downarrow}{\Sigma; \Gamma \Uparrow L \Rightarrow P \Uparrow}\ (\Uparrow_{\mathcal{R}}) \quad P \text{ not asynchronous}$$

   by induction hypothesis we have:

$$\frac{\dfrac{\Sigma; \Gamma \Uparrow L \Rightarrow P \Downarrow}{\Sigma, \Delta; \Gamma \Uparrow L \Rightarrow P \Downarrow}\ (W*)}{\Sigma, \Delta; \Gamma \Uparrow L \Rightarrow P \Uparrow}\ (\Uparrow_{\mathcal{R}}) \quad P \text{ not asynchronous}$$

3. $(S)$

$$\frac{\Sigma, P; \Gamma \Uparrow L \Rightarrow R \Downarrow}{\Sigma; \Gamma \Uparrow L, !P \Rightarrow R \Downarrow}\ (S)$$

   by induction hypothesis we have:

$$\frac{\dfrac{\Sigma, P; \Gamma \Uparrow L \Rightarrow R \Downarrow}{\Sigma, \Delta, P; \Gamma \Uparrow L \Rightarrow R \Downarrow}\ (W*)}{\Sigma, \Delta; \Gamma \Uparrow L, !P \Rightarrow R \Downarrow}\ (S)$$

4. $(Pop)$

$$\frac{\Sigma;\Gamma,P\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,P\Rightarrow R\Downarrow}\ (Pop)$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Gamma,P\Uparrow L\Rightarrow R\Downarrow}{\Sigma,\Delta;\Gamma,P\Uparrow L\Rightarrow R\Downarrow}\ (W*)}{\Sigma,\Delta;\Gamma\Uparrow L,P\Rightarrow R\Downarrow}\ (Pop)$$

5. $(\Downarrow_{\mathcal{R}})$

$$\frac{\Sigma;\Gamma\Downarrow\Rightarrow R\Uparrow}{\Sigma;\Gamma\Uparrow\Rightarrow R\Downarrow}\ (\Downarrow_{\mathcal{R}})$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Gamma\Downarrow\Rightarrow R\Uparrow}{\Sigma,\Delta;\Gamma\Downarrow\Rightarrow R\Uparrow}\ (W*)}{\Sigma,\Delta;\Gamma\Uparrow\Rightarrow R\Downarrow}\ (\Downarrow_{\mathcal{R}})$$

6. $(Push)$

$$\frac{\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma;\Gamma,P\Uparrow\Rightarrow R\Downarrow}\ (Push)$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma,\Delta;\Gamma\Downarrow P\Rightarrow R\Downarrow}\ (W*)}{\Sigma,\Delta;\Gamma,P\Uparrow\Rightarrow R\Downarrow}\ (Push)$$

7. $(D)$

$$\frac{\Sigma,P;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma,P;\Gamma\Uparrow\Rightarrow R\Downarrow}\ (D)$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma,P;\Gamma\Downarrow P\Rightarrow R\Downarrow}{\Sigma,\Delta,P;\Gamma\Downarrow P\Rightarrow R\Downarrow}\ (W*)}{\Sigma,\Delta,P;\Gamma\Uparrow\Rightarrow R\Downarrow}\ (D)$$

8. $(P)$

$$\frac{\Sigma;\Uparrow\Rightarrow P\Uparrow}{\Sigma;\Downarrow\Rightarrow !P\Uparrow}\ (P)$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Uparrow\Rightarrow P\Uparrow}{\Sigma,\Delta;\Uparrow\Rightarrow P\Uparrow}\ (W*)}{\Sigma,\Delta;\Downarrow\Rightarrow !P\Uparrow}\ (P)$$

9. $(\Downarrow_{\mathcal{L}_1})$

$$\frac{\Sigma;\Gamma\Uparrow\Rightarrow R\Uparrow}{\Sigma;\Gamma\Downarrow\Rightarrow R\Uparrow}\ (\Downarrow_{\mathcal{L}_1})$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Gamma\Uparrow\Rightarrow R\Uparrow}{\Sigma,\Delta;\Gamma\Uparrow\Rightarrow R\Uparrow}\ (W*)}{\Sigma,\Delta;\Gamma\Downarrow\Rightarrow R\Uparrow}\ (\Downarrow_{\mathcal{L}_1})$$

10. $(ax)$

$$\frac{}{\Sigma;\Downarrow A\Rightarrow A\Downarrow}\ (ax)$$

then

$$\frac{}{\Sigma,\Delta;\Downarrow A\Rightarrow A\Downarrow}\ (ax)$$

11. $(\multimap_{\mathcal{L}})$

$$\frac{\Sigma;\Gamma\Downarrow\Rightarrow P\Uparrow\quad \Sigma;\Psi\Downarrow Q\Rightarrow R\Downarrow}{\Sigma;\Gamma,\Psi\Downarrow P\multimap Q\Rightarrow R\Downarrow}\ (\multimap_{\mathcal{L}})$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Gamma\Downarrow\Rightarrow P\Uparrow}{\Sigma,\Delta;\Gamma\Downarrow\Rightarrow P\Uparrow}\ (W*)\quad \cfrac{\Sigma;\Psi\Downarrow Q\Rightarrow R\Downarrow}{\Sigma,\Delta;\Psi\Downarrow Q\Rightarrow R\Downarrow}\ (W*)}{\Sigma,\Delta;\Gamma,\Psi\Downarrow P\multimap Q\Rightarrow R\Downarrow}\ (\multimap_{\mathcal{L}})$$

12. $(\Downarrow_{\mathcal{L}_2})$

$$\frac{\Sigma;\Gamma\Uparrow P\Rightarrow R\Downarrow}{\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow}\ (\Downarrow_{\mathcal{L}_2})\quad P \text{ not synchronous}$$

by induction hypothesis we have:

$$\cfrac{\cfrac{\Sigma;\Gamma\Uparrow P\Rightarrow R\Downarrow}{\Sigma,\Delta;\Gamma\Uparrow P\Rightarrow R\Downarrow}\ (W*)}{\Sigma,\Delta;\Gamma\Downarrow P\Rightarrow R\Downarrow}\ (\Downarrow_{\mathcal{L}_2})\quad P \text{ not synchronous}$$

■

**Lemma 2.4** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma;\Gamma,P\Uparrow L,M\Rightarrow R\Uparrow}{\Sigma;\Gamma\Uparrow L,P,M\Rightarrow R\Uparrow}\quad P \textit{ not asynchronous}$$

*(Note that this lemma still holds with the succedent arrow reversed).*

**Lemma 2.5** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma;\Gamma\Uparrow L,P,Q,M\Rightarrow R\Uparrow}{\Sigma;\Gamma\Uparrow L,P\otimes Q,M\Rightarrow R\Uparrow}$$

*(Note that this lemma still holds with the succedent arrow reversed).*

**Lemma 2.6** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, P, M \Rightarrow R \Uparrow \quad \Sigma; \Gamma \Uparrow L, Q, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, P \oplus Q, M \Rightarrow R \Uparrow}$$

*(Note that this lemma still holds with the succedent arrow reversed).*

**Lemma 2.7** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma, P; \Gamma \Uparrow L, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, !P, M \Rightarrow R \Uparrow}$$

*(Note that this lemma still holds with the succedent arrow reversed).*

**Lemma 2.8** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, I, M \Rightarrow R \Uparrow}$$

*(Note that this lemma still holds with the succedent arrow reversed).*

**Lemma 2.9** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow M \Rightarrow P \Uparrow}{\Sigma; \Gamma \Uparrow L \Rightarrow P \Uparrow} \quad \textit{where } L \equiv M$$

*where $L \equiv M$ means that $L$ and $M$ are different lists of the same elements. That is, this is an exchange rule. (Note that this lemma also holds with the succedent arrow reversed).*

**Lemma 2.10** *Proving "$\Sigma; \Gamma, L \Rightarrow R$ implies $\Sigma; \Gamma \Uparrow L \Rightarrow R \Uparrow$ provable" is equivalent to proving "$\Sigma; L \Rightarrow R$ provable implies $\Sigma; \Uparrow L \Rightarrow R \Uparrow$".*

PROOF: "$\Rightarrow$" This direction is trivial, simply put $\Gamma = \phi$.

"$\Leftarrow$" Let $L'$ be any ordering of $\Gamma$. We can then prove $\Sigma; \Uparrow L, L' \Rightarrow R \Uparrow$. By Lemma 2.4 we can prove $\Sigma; \Gamma \Uparrow L \Rightarrow R \Uparrow$. ■

**Lemma 2.11** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, P \otimes Q, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, P, Q, M \Rightarrow R \Uparrow}$$

*(Note that this rule with the succedent arrow reversed is also admissible).*

**Lemma 2.12** *The following rules are admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, P \oplus Q, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, P, M \Rightarrow R \Uparrow} \qquad \frac{\Sigma; \Gamma \Uparrow L, P \oplus Q, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, Q, M \Rightarrow R \Uparrow}$$

*(Note that these rules with the succedent arrows reversed are also admissible).*

**Lemma 2.13** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, !P, M \Rightarrow R \Uparrow}{\Sigma, P; \Gamma \Uparrow L, M \Rightarrow R \Uparrow}$$

*(Note that this rule with the succedent arrow reversed is also admissible).*

**Lemma 2.14** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, I, M \Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow L, M \Rightarrow R \Uparrow}$$

*(Note that this rule with the succedent arrow reversed is also admissible).*

**Lemma 2.15** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L, P, M \Rightarrow R \Uparrow}{\Sigma; \Gamma, P \Uparrow L, M \Rightarrow R \Uparrow} \quad P \textit{ not asynchronous}$$

*(Note that this rule with the succedent arrow reversed is also admissible).*

**Lemma 2.16** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L \Rightarrow P \multimap Q \Uparrow}{\Sigma; \Gamma \Uparrow L, P \Rightarrow Q \Uparrow}$$

**Lemma 2.17** *The following rules are admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L \Rightarrow P \& Q \Uparrow}{\Sigma; \Gamma \Uparrow L \Rightarrow P \Uparrow} \qquad \frac{\Sigma; \Gamma \Uparrow L \Rightarrow P \& Q \Uparrow}{\Sigma; \Gamma \Uparrow L \Rightarrow Q \Uparrow}$$

**Lemma 2.18** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow L \Rightarrow P \Uparrow}{\Sigma; \Gamma \Uparrow L \Rightarrow P \Downarrow} \quad P \textit{ not asynchronous}$$

**Lemma 2.19** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma, P \Uparrow L \Rightarrow R \Uparrow}{\Sigma; \Gamma, P \& Q \Uparrow L \Rightarrow R \Uparrow}$$

*(Note that this rule with the other appropriate combination of arrows is also admissible).*

**Lemma 2.20** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma, Q \Uparrow L \Rightarrow R \Uparrow}{\Sigma; \Gamma, P \& Q \Uparrow L \Rightarrow R \Uparrow}$$

*(Note that this rule with the other appropriate combination of arrows is also admissible).*

**Lemma 2.21** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma_1 \Uparrow \Rightarrow P \Uparrow \quad \Sigma; \Gamma_2, Q \Uparrow L \Rightarrow R \Uparrow}{\Sigma; \Gamma_1, \Gamma_2, P \multimap Q \Uparrow L \Rightarrow R \Uparrow}$$

*(Note that this rule with the other appropriate combination of arrows is also admissible).*

**Lemma 2.22** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma, P; \Gamma, P \Uparrow L \Rightarrow R \Uparrow}{\Sigma, P; \Gamma \Uparrow L \Rightarrow R \Uparrow}$$

*(Note that this rule with the other appropriate combination of arrows is also admissible).*

**Lemma 2.23** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma_1 \Uparrow \Rightarrow P \Uparrow \quad \Sigma; \Gamma_2, \Uparrow Q \Rightarrow R \Downarrow}{\Sigma; \Gamma_1, \Gamma_2, P \multimap Q \Uparrow \Rightarrow R \Downarrow}$$

**Lemma 2.24** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow P \Rightarrow R \Downarrow}{\Sigma; \Gamma, P \& Q \Uparrow \Rightarrow R \Downarrow}$$

**Lemma 2.25** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow Q \Rightarrow R \Downarrow}{\Sigma; \Gamma, P \& Q \Uparrow \Rightarrow R \Downarrow}$$

**Lemma 2.26** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma, P; \Gamma \Uparrow L, P, M \Rightarrow R \Uparrow}{\Sigma, P; \Gamma \Uparrow L, M \Rightarrow R \Uparrow}$$

*(Note that this rule with the other appropriate combinations of arrows is also admissible).*

**Lemma 2.27** *The following rule is admissible in* ILLF*:*

$$\frac{\Sigma; \Gamma \Uparrow \Rightarrow R \Uparrow}{\Sigma; \Gamma \Downarrow \Rightarrow R \Uparrow}$$

**Lemma 2.28** *The following rule is admissible in* ILLF*:*

$$\frac{}{\Sigma; \Gamma \Uparrow L, 0, M \Rightarrow R \Uparrow}$$

Now we have all the results we need to complete the equivalence proof.

**Lemma 2.29** *Sequent* $\Sigma;\Gamma,L\Rightarrow P$ *is provable in* $\mathrm{ILL}^{\Sigma}$ *iff sequent* $\Sigma;\Gamma\Uparrow L\Rightarrow P\Uparrow$ *is provable in* ILLF.

PROOF: The proof is again by induction on the height of derivations.

"$\Leftarrow$". We need to show that:

- If $\Sigma;\Gamma\Uparrow L\Rightarrow R\Uparrow$ is provable in ILLF then $\Sigma;\Gamma,L\Rightarrow R$ is provable in $\mathrm{ILL}^{\Sigma}$.
- If $\Sigma;\Gamma\Uparrow L\Rightarrow R\Downarrow$ is provable in ILLF then $\Sigma;\Gamma,L\Rightarrow R$ is provable in $\mathrm{ILL}^{\Sigma}$.
- If $\Sigma;\Gamma\Downarrow P\Rightarrow R\Downarrow$ is provable in ILLF then $\Sigma;\Gamma,P\Rightarrow R$ is provable in $\mathrm{ILL}^{\Sigma}$.

For this direction we illustrate the proof for the $\multimap$, ! fragment of the logic.

1. In ILLF we have
$$\frac{\Sigma;\Gamma\Uparrow L,P\Rightarrow Q\Uparrow}{\Sigma;\Gamma\Uparrow L\Rightarrow P\multimap Q\Uparrow}\ (\multimap_{\mathcal{R}})$$
by induction hypothesis we have:
$$\frac{\Sigma;\Gamma,L,P\Rightarrow Q}{\Sigma;\Gamma,L\Rightarrow P\multimap Q}\ (\multimap_{\mathcal{R}})$$

2. In ILLF we have
$$\frac{\Sigma;\Gamma\Uparrow L\Rightarrow P\Downarrow}{\Sigma;\Gamma\Uparrow L\Rightarrow P\Uparrow}\ (\Uparrow_{\mathcal{R}})$$
by induction hypothesis we have:
$$\Sigma;\Gamma,L\Rightarrow P$$
as required.

3. In ILLF we have:
$$\frac{\Sigma,P;\Gamma\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,!P\Rightarrow R\Downarrow}\ (S)$$
by induction hypothesis we have:
$$\frac{\Sigma,P;\Gamma,L\Rightarrow R}{\Sigma;\Gamma,L,!P\Rightarrow R}\ (S)$$

4. In ILLF we have
$$\frac{\Sigma;\Gamma,P\Uparrow L\Rightarrow R\Downarrow}{\Sigma;\Gamma\Uparrow L,P\Rightarrow R\Downarrow}\ (Pop)\quad P\text{ not asynchronous}$$
by induction hypothesis we get:
$$\Sigma;\Gamma,L,P\Rightarrow R$$
as required.

5. In ILLF we have

$$\frac{\Sigma; \Gamma \Downarrow\Rightarrow R \Uparrow}{\Sigma; \Gamma \Uparrow\Rightarrow R \Downarrow} (\Downarrow_{\mathcal{R}})$$

by induction hypothesis we get:

$$\Sigma; \Gamma \Rightarrow R$$

as required.

6. In ILLF we have

$$\frac{\Sigma; \Gamma \Downarrow P \Rightarrow R \Downarrow}{\Sigma; \Gamma, P \Uparrow\Rightarrow R \Downarrow} (Push)$$

by induction hypothesis we get:

$$\Sigma; \Gamma, P \Rightarrow R$$

as required.

7. In ILLF we have

$$\frac{\Sigma, P; \Gamma \Downarrow P \Rightarrow R \Downarrow}{\Sigma, P; \Gamma \Uparrow\Rightarrow R \Downarrow} (D)$$

by induction hypothesis we get:

$$\frac{\Sigma, P; \Gamma, P \Rightarrow R}{\Sigma, P; \Gamma \Rightarrow R} (D)$$

8. In ILLF we have

$$\frac{\Sigma; \Uparrow\Rightarrow P \Uparrow}{\Sigma; \Downarrow\Rightarrow !P \Uparrow} (P)$$

by induction hypothesis we have:

$$\frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P} (P)$$

9. In ILLF we have

$$\frac{\Sigma; \Gamma \Uparrow\Rightarrow R \Uparrow}{\Sigma; \Gamma \Downarrow\Rightarrow R \Uparrow} (\Downarrow_{\mathcal{L}_1}) \quad R \text{ not synchronous}$$

by induction hypothesis we get:

$$\Sigma; \Gamma \Rightarrow R$$

10. In ILLF we have

$$\frac{}{\Sigma; \Downarrow A \Rightarrow A \Downarrow} (ax)$$

then

$$\frac{}{\Sigma; A \Rightarrow A} (ax)$$

11. In ILLF we have

$$\frac{\Sigma;\Gamma \Downarrow\Rightarrow P \Uparrow \quad \Sigma;\Delta \Downarrow Q \Rightarrow R \Downarrow}{\Sigma;\Gamma,\Delta \Downarrow P \multimap Q \Rightarrow R \Downarrow}\ (\multimap_{\mathcal{L}})$$

by induction hypothesis we have:

$$\frac{\Sigma;\Gamma \Rightarrow P \quad \Sigma;\Delta, Q \Rightarrow R}{\Sigma;\Gamma,\Delta, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})$$

12. In ILLF we have

$$\frac{\Sigma;\Gamma \Uparrow P \Rightarrow R \Downarrow}{\Sigma;\Gamma \Downarrow P \Rightarrow R \Downarrow}\ (\Downarrow_{\mathcal{L}}) \quad P \text{ not synchronous}$$

by induction hypothesis we have:

$$\Sigma;\Gamma, P \Rightarrow R$$

as required.

"⇒" For this direction we give the entire proof rather than a fragment of it, as this is the non-trivial part.

By Lemma 2.10 it is enough to show that $\Sigma; L \Rightarrow P$ provable in ILL$^{\Sigma}$ implies $\Sigma; \Uparrow L \Rightarrow P \Uparrow$ provable in ILLF.

1. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L, P \Rightarrow Q}{\Sigma; L \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}})$$

by induction hypothesis we have:

$$\frac{\Sigma; \Uparrow L, P \Rightarrow Q \Uparrow}{\Sigma; \Uparrow L \Rightarrow P \multimap Q \Uparrow}\ (\multimap_{\mathcal{R}})$$

2. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L \Rightarrow P \quad \Sigma; L \Rightarrow Q}{\Sigma; L \Rightarrow P \& Q}\ (\&_{\mathcal{R}})$$

by induction hypothesis we have

$$\frac{\Sigma; \Uparrow L \Rightarrow P \Uparrow \quad \Sigma; \Uparrow L \Rightarrow Q \Uparrow}{\Sigma; \Uparrow L \Rightarrow P \& Q \Uparrow}\ (\&_{\mathcal{R}})$$

3. In ILL$^{\Sigma}$ we have

$$\frac{}{\Sigma; L \Rightarrow \top}\ (\top_{\mathcal{R}})$$

in ILLF

$$\frac{}{\Sigma; \Uparrow L \Rightarrow \top \Uparrow}\ (\top_{\mathcal{R}})$$

4. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L, P, Q \Rightarrow R}{\Sigma; L, P \otimes Q \Rightarrow R}\ (\otimes_{\mathcal{L}})$$

by induction hypothesis and Lemma 2.5 we have:

$$\frac{\Sigma; \Uparrow L, P, Q \Rightarrow R \Uparrow}{\Sigma; \Uparrow L, P \otimes Q \Rightarrow R \Uparrow}$$

5. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L, P \Rightarrow R \quad \Sigma; L, Q \Rightarrow R}{\Sigma; L, P \oplus Q \Rightarrow R}\ (\oplus_{\mathcal{L}})$$

by induction hypothesis and Lemma 2.6 we have:

$$\frac{\Sigma; \Uparrow L, P \Rightarrow R \Uparrow \quad \Sigma; \Uparrow L, Q \Rightarrow R \Uparrow}{\Sigma; \Uparrow L, P \oplus Q \Rightarrow R \Uparrow}$$

6. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma, P; L \Rightarrow R}{\Sigma; L, !P \Rightarrow R}\ (S)$$

by induction hypothesis and Lemma 2.7 we have

$$\frac{\Sigma, P; \Uparrow L \Rightarrow R \Uparrow}{\Sigma; \Uparrow L, !P \Rightarrow R \Uparrow}$$

7. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L \Rightarrow R}{\Sigma; L, I \Rightarrow R}\ (I_{\mathcal{L}})$$

by induction hypothesis and Lemma 2.8 we have

$$\frac{\Sigma; \Uparrow L \Rightarrow R \Uparrow}{\Sigma; \Uparrow L, I \Rightarrow R \Uparrow}$$

8. In ILL$^{\Sigma}$ we have

$$\frac{}{\Sigma; L, 0 \Rightarrow R}\ (0_{\mathcal{L}})$$

by Lemma 2.28 we have:

$$\frac{}{\Sigma; \Uparrow L, 0 \Rightarrow R \Uparrow}$$

9. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L_1 \Rightarrow P \quad \Sigma; L_2 \Rightarrow Q}{\Sigma; L_1, L_2 \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}})$$

by induction hypothesis we have:

$$\dfrac{\dfrac{\dfrac{\Sigma; \Uparrow L_1 \Rightarrow P \Uparrow}{\Sigma'; L_1' \Uparrow\Rightarrow P \Uparrow}(1)}{\Sigma'; L_1' \Downarrow\Rightarrow P \Uparrow}(3) \quad \dfrac{\dfrac{\Sigma; \Uparrow L_2 \Rightarrow Q \Uparrow}{\Sigma'; L_2' \Uparrow\Rightarrow Q \Uparrow}(2)}{\Sigma'; L_2' \Downarrow\Rightarrow P \Uparrow}(4)}{\dfrac{\dfrac{\Sigma'; L_1', L_2' \Uparrow\Rightarrow P \otimes Q \Downarrow}{\Sigma; \Uparrow L_1, L_2 \Rightarrow P \otimes Q \Downarrow}(5)}{\Sigma; \Uparrow L_1, L_2 \Rightarrow P \otimes Q \Uparrow}(\Uparrow_{\mathcal{R}})}(\otimes_{\mathcal{R}})$$

Where $(1)$ a series of applications of Lemmas 2.3, 2.11-2.15, $(2)$ a series of applications of Lemmas 2.3, 2.11-2.15, $(3)$ Lemma 2.27, $(4)$ Lemma 2.27, $(5)$ a series of applications of $(Pop)$, $(\otimes_{\mathcal{L}})$, $(\oplus_{\mathcal{L}})$, $(I_{\mathcal{L}})$, $(S)$. Also we may need to build extra bits of proof for additive rules.

10. In ILL$^{\Sigma}$ we have

$$\dfrac{\Sigma, P; L, P \Rightarrow R}{\Sigma, P; L \Rightarrow R}(D)$$

by induction hypothesis and Lemma 2.25 we get:

$$\dfrac{\Sigma, P; \Uparrow L, P \Rightarrow R \Uparrow}{\Sigma, P; \Uparrow L \Rightarrow R \Uparrow}$$

11. In ILL$^{\Sigma}$ we have

$$\dfrac{}{\Sigma; \Rightarrow I}(I_{\mathcal{R}})$$

then

$$\dfrac{\dfrac{\dfrac{}{\Sigma; \Downarrow\Rightarrow I \Uparrow}(I_{\mathcal{R}})}{\Sigma; \Uparrow\Rightarrow I \Downarrow}(\Downarrow_{\mathcal{R}})}{\Sigma; \Uparrow\Rightarrow I \Uparrow}(\Uparrow_{\mathcal{R}})$$

12. In ILL$^{\Sigma}$ we have

$$\dfrac{\Sigma; L \Rightarrow P}{\Sigma; L \Rightarrow P \oplus Q}(\oplus_{\mathcal{R}_1})$$

by induction hypothesis we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Sigma; \Uparrow L \Rightarrow P \Uparrow}{\Sigma'; L' \Uparrow\Rightarrow P \Uparrow}(1)}{\Sigma'; L' \Downarrow\Rightarrow P \Uparrow}(2)}{\Sigma'; L' \Downarrow\Rightarrow P \oplus Q \Uparrow}(\oplus_{\mathcal{R}_1})}{\Sigma'; L' \Uparrow\Rightarrow P \oplus Q \Downarrow}(\Downarrow_{\mathcal{R}})}{\Sigma; \Uparrow L \Rightarrow P \oplus Q \Downarrow}(3)}{\Sigma; \Uparrow L \Rightarrow P \oplus Q \Uparrow}(\Uparrow_{\mathcal{R}})$$

where $(1)$ a series of application of Lemmas 11-15, $(2)$ Lemma 2.26, $(3)$ a series of applications of $(Pop)$, $(\otimes_{\mathcal{L}})$, $(\oplus_{\mathcal{L}})$, $(I_{\mathcal{L}})$, $(S)$. Also we may need to build extra bits of proof for additive rules

13. $(\oplus_{\mathcal{R}_2})$ similar to $(\oplus_{\mathcal{R}_1})$.

14. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P}\ (P)$$

by induction hypothesis we have

$$\cfrac{\cfrac{\cfrac{\Sigma; \Uparrow \Rightarrow P \Uparrow}{\Sigma; \Downarrow \Rightarrow !P \Uparrow}\ (P)}{\Sigma; \Uparrow \Rightarrow !P \Downarrow}\ (\Downarrow_{\mathcal{R}})}{\Sigma; \Uparrow \Rightarrow !P \Uparrow}\ (\Uparrow_{\mathcal{R}})$$

15. In ILL$^{\Sigma}$ we have

$$\frac{}{\Sigma; A \Rightarrow A}\ (ax)$$

by induction hypothesis we have

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\Sigma; \Downarrow A \Rightarrow A \Downarrow}\ (ax)}{\Sigma; A \Uparrow \Rightarrow A \Downarrow}\ (Push)}{\Sigma; \Uparrow A \Rightarrow A \Downarrow}\ (Pop)}{\Sigma; \Uparrow A \Rightarrow A \Uparrow}\ (\Uparrow_{\mathcal{R}})$$

16. In ILL$^{\Sigma}$ we have

$$\frac{\Sigma; L_1 \Rightarrow P \quad \Sigma; L_2, Q \Rightarrow R}{\Sigma; L_1, L_2, P \multimap Q \Rightarrow R}\ (\multimap_{\mathcal{L}})$$

by induction hypothesis we have

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Sigma; \Uparrow L_1 \Rightarrow P \Uparrow}{\Sigma'; L_1' \Uparrow\Rightarrow P \Uparrow}\ (4)}{\Sigma'; L_1' \Downarrow\Rightarrow P \Uparrow}\ (5) \quad \cfrac{\cfrac{\cfrac{\Sigma; \Uparrow L_2, Q \Rightarrow R \Uparrow}{\Sigma; \Uparrow L_2, Q, L_3 \Rightarrow S \Uparrow}\ (1)}{\Sigma'; L_2', L_3' \Uparrow Q \Rightarrow S \Uparrow}\ (2)}{\Sigma'; L_2', L_3' \Uparrow Q \Rightarrow S \Downarrow}\ (3)}{\Sigma'; L_1', L_2', L_3', P \multimap Q \Uparrow\Rightarrow S \Downarrow}\ (6)}{\Sigma; \Uparrow L_1, L_2, L_3, P \multimap Q \Rightarrow S \Downarrow}\ (7)}{\Sigma; \Uparrow L_1, L_2, L_3, P \multimap Q \Rightarrow S \Uparrow}\ (\Uparrow_{\mathcal{R}})}{\Sigma; \Uparrow L_1, L_2, P \multimap Q \Rightarrow R \Uparrow}\ (8)$$

where $(1)$ a series of application of Lemmas 2.16, 2.17, $(2)$ a series of applications of Lemmas 2.3, 2.11-2.15, $(3)$ Lemma 2.18, $(4)$ a series of applications of Lemmas 2.3, 2.11-2.15, $(5)$ Lemma 2.26, $(6)$ Lemma 2.23 $(7)$ a series of applications of $(Pop)$, $(\otimes_{\mathcal{L}})$, $(\oplus_{\mathcal{L}})$, $(I_{\mathcal{L}})$, $(S)$. Also we may need to build extra bits of proof for additive rules, $(8)$ a series of applications of $(\multimap_{\mathcal{R}})$, $(\&_{\mathcal{R}})$. Also we may need to build extra bits of proof for additive rules.

17. In ILL$^\Sigma$ we have
$$\frac{\Sigma; L_1, P \Rightarrow R}{\Sigma; L_1, P\&Q \Rightarrow R}\ (\&_{\mathcal{L}_1})$$
by induction hypothesis we have
$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Sigma; \Uparrow L_1, P \Rightarrow R \Uparrow}{\Sigma; \Uparrow L_1, P, L_2 \Rightarrow S \Uparrow}\ (1)}{\Sigma'; L_1', L_2' \Uparrow P \Rightarrow S \Uparrow}\ (2)}{\Sigma'; L_1', L_2' \Uparrow P \Rightarrow S \Downarrow}\ (3)}{\Sigma'; L_1', L_2', P\&Q \Uparrow\Rightarrow S \Downarrow}\ (4)}{\Sigma; \Uparrow L_1, L_2, P\&Q \Rightarrow S \Downarrow}\ (5)}{\Sigma; \Uparrow L_1, L_2, P\&Q \Rightarrow S \Uparrow}\ (\Uparrow_{\mathcal{R}})}{\Sigma; \Uparrow L_1, P\&Q \Rightarrow R \Uparrow}\ (6)$$
where $(1)$ a series of applications of Lemmas 2.16, 2.17, $(2)$ a series of applications of Lemmas 2.11-2.15, $(3)$ Lemma 2.18, $(4)$ Lemma 2.24, $(5)$ a series of applications of $(Pop)$, $(\otimes_{\mathcal{L}})$, $(\oplus_{\mathcal{L}})$, $(I_{\mathcal{L}})$, $(S)$. Also we may need to build extra bits of proof for additive rules, $(6)$ a series of applications of $(\multimap_{\mathcal{R}})$, $(\&_{\mathcal{R}})$. Also we may need to build extra bits of proof for additive rules.

18. $(\&_{\mathcal{L}_2})$. Similar to $(\&_{\mathcal{L}_1})$.

■

**Theorem 2.2** *The calculi* ILL *and* ILLF *are equivalent: the sequent* $!\Sigma, \Gamma, L \Rightarrow P$ *is provable in* ILL *iff the sequent* $\Sigma; \Gamma \Uparrow L \Rightarrow P$ *is provable in* ILLF. *Hence* ILLF *is sound and complete with respect to provability in* ILL.

PROOF: Immediate from Lemma 2.2 and Lemma 2.29. ■

## 2.4 ILLF and Permutations

The motivation for focusing calculi is the reduction of redundancy in proof search. Calculi good for delivering a yes/no answer to a query have as much determinism as possible, and on backtracking will not investigate an essentially similar path. A calculus where trivial permutation of inferences is not possible is good for this.

ILLF has a lot of determinism and avoids permutations. Occurrences of $(\multimap_{\mathcal{R}})$ and $(\&_{\mathcal{R}})$ are forced to occur as soon as possible, hence cannot be permuted. Occurrences of $(\otimes_{\mathcal{L}})$, $(\oplus_{\mathcal{L}})$, and $(I_{\mathcal{L}})$ are forced to occur together, and the list structure forces this treatment to be in fixed sequence, thus preventing (in a somewhat arbitrary manner) the permutation of these inferences with each other, as well as with other rules. The major choice point is where it is decided which synchronous formula to consider, or whether to derelict. Once a formula had been decided upon, it is principal, as are the active formulae in the premisses (unless atomic) – the formula is focused upon (further restriction could be placed on $(\Downarrow_{\mathcal{L}_2})$ to prevent its use with a focused atom). Promotion occurs as soon as possible. In fact in ILLF as presented, there is a choice between promotion and dereliction in ILLF. A side condition could be placed on $(D)$, restricting its application to when $(P)$ is not possible. Indeed it could be further restricted so that it is only applicable when the formulae in $\Gamma$ are atomic. Due to the separate field for the exponential formulae, we do not have to worry about permutations of $(C)$ and $(W)$.

ILLF finds proofs up to permutations. Compare ILLF with the table of permutation of inference rules (Table 2.5). It is observed that inferences which it says are permutable cannot occur in permutation position in ILLF (or the proofs of ILLF mapped into proofs in ILL). For each ordering of the list, the calculus finds only one proof in each P-equivalence class.

Finally we compare the proofs of ILLF with the definition of a normal sequent proof (Definition 2.10). As discussed above, because of the formulation of $(D)$, sequents of form $!\Gamma \Rightarrow !P$ might not be the result of promotion, but we could restrict ILLF so that they are. The other clauses of the definition are satisfied, or irrelevant, so it can be said that ILLF only finds proofs in normal form with respect to permutations.

## 2.5 Concluding Remarks

This chapter has studied ILL proofs in a purely syntactic way. Having investigated permutations of inferences and inference rules, as well as the invertibility of rules, we gave a calculus avoiding permutations. This calculus, ILLF, gives a reduced proof search space and hence is suitable for theorem proving. It could also be viewed as a proof enumeration calculus as it can be argued that normal proofs with respect to permutation of inferences in a sequent calculus are the proofs of interest. We prefer to have a semantic motivation for proof enumeration calculi. This results in the calculus SILL given in Chapter 6.

# Chapter 3

# A Permutation-free Sequent Calculus for Lax Logic

This chapter is a study in the application of permutation-free techniques. The methods that develop MJ (see Chapter 1, [Her95], [DP96], [DP98a]) are used to find a 'permutation-free' calculus for an intuitionistic modal logic. The results and their proofs are simple extensions of those for MJ, their inclusion here being to illustrate the wide applicability of permutation-free techniques and for reference purposes.

The logic we look at, now called Lax Logic, dates back to Curry ([Cur52a]). Interest in Lax Logic has recently been renewed. We give a very short introduction to Lax Logic and its applications, then build the machinery to develop the permutation-free calculus PFLAX. Much of the work contained in this chapter can also be found in [How98].

## 3.1 Lax Logic

Lax Logic is an intuitionistic modal logic with a single modality (○, *somehow*). This modality is unusual in that it has properties both of necessity and of possibility. The modality can be thought of as expressing correctness up to a constraint, abstracting away from the detail (hence the choice of name, Lax Logic). A formula $\circ P$ can be read as "for some constraint $c$, formula $P$ holds under $c$". The modality is axiomatised by three axioms:

$$\begin{array}{ll} \circ R: & S \supset \circ S \\ \circ M: & \circ \circ S \supset \circ S \\ \circ F: & (S \supset T) \supset (\circ S \supset \circ T) \end{array}$$

The logic can also be presented as a natural deduction calculus (displayed in sequent style in Figure 3.1) and as a sequent calculus (Figure 3.6). Lax logic has recently been investigated by Fairtlough, Mendler & Walton ([Men93], [FM94], [FM97], [FMW97], [FW97]) and by Benton, Bierman and de Paiva ([BBdP98]).

Curry ([Cur52a]) introduced the logic to illustrate cut-elimination in the presence of modalities. The logic was rediscovered by Mendler, who developed the logic for abstract reasoning about constraints in hardware verification ([Men93]). The timing constraints that need to be satisfied in a circuit can be abstracted away as instances of the modality and reasoned about separately from the logical analysis of the circuit. In [Men93], [FM94], [FM97], the proof theory and semantics of Lax Logic are developed, giving Gentzen calculi, natural deduction calculi and Kripke semantics for the logic as well as giving details of the logic's use as a tool for hardware verification.

Lax Logic has also been observed ([BBdP98]) as the type system for Moggi's computational lambda-calculus (see [Mog89]). In [BBdP98] the correspondence between the natural deduction presentation of Lax Logic (there called computational logic) and the computational lambda calculus is given, along with some proof theoretic results on the logic.

In [FMW97] the ability of Lax Logic to give an abstract expression of constraints is utilised to give a semantics to constraint logic programming languages. The constraints to be satisfied can be abstracted away as modalities and the query can be reasoned about logically. The constraints can then be analysed separately. The logic is used to give proofs of queries. These proofs give the constraints to be satisfied. The work in this chapter gives a calculus suitable searching for these proofs.

The calculi in this chapter are presented as first order, but we only give proofs of results for the propositional implicational and modal fragment (for brevity).

## 3.2 Natural Deduction

We give the natural deduction calculus for Lax Logic. This is taken directly from [BBdP98] (with quantifiers and falsum added) and can be seen in Figure 3.1.

We look at the normalisation steps. Again these are taken from [BBdP98], with the extra cases for $-$ and $\exists$ added. The reduction rules for the intuitionistic connectives are completely standard. We do not include them here, concentrating instead on those involving the modality. We give these reductions in tree style rather than in sequent style.

First the $\beta$-reduction:

–

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ P\end{array}}{\circ P}\,(\circ_{\mathcal{I}}) \qquad \begin{array}{c}[P]\\ \vdots\\ \circ Q\end{array}}{\circ Q}\,(\circ_{\varepsilon}) \qquad \rightsquigarrow \qquad \begin{array}{c}\vdots\\ P\\ \vdots\\ \circ Q\end{array}$$

Now we give the commuting conversions (or $c$-reductions) involving the modality:

–

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \circ P\end{matrix} \quad \begin{matrix}[P]\\ \vdots\\ \circ Q\end{matrix}}{\circ Q}\,(\circ_{\mathcal{E}}) \quad \begin{matrix}[Q]\\ \vdots\\ \circ R\end{matrix}}{\circ R}\,(\circ_{\mathcal{E}}) \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\\ \circ P\end{matrix} \quad \dfrac{\begin{matrix}[P]\\ \vdots\\ \circ Q\end{matrix} \quad \begin{matrix}[Q]\\ \vdots\\ \circ R\end{matrix}}{\circ R}\,(\circ_{\mathcal{E}})}{\circ R}\,(\circ_{\mathcal{E}})$$

–

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ P \vee Q\end{matrix} \quad \begin{matrix}[P]\\ \vdots\\ \circ R\end{matrix} \quad \begin{matrix}[Q]\\ \vdots\\ \circ R\end{matrix}}{\circ R}\,(\vee_{\mathcal{E}}) \quad \begin{matrix}[R]\\ \vdots\\ \circ S\end{matrix}}{\circ S}\,(\circ_{\mathcal{E}})$$

$$\rightsquigarrow \quad \dfrac{\begin{matrix}\vdots\\ P \vee Q\end{matrix} \quad \dfrac{\begin{matrix}[P]\\ \vdots\\ \circ R\end{matrix} \quad \begin{matrix}[R]\\ \vdots\\ \circ S\end{matrix}}{\circ S}\,(\circ_{\mathcal{E}}) \quad \dfrac{\begin{matrix}[Q]\\ \vdots\\ \circ R\end{matrix} \quad \begin{matrix}[R]\\ \vdots\\ \circ S\end{matrix}}{\circ S}\,(\circ_{\mathcal{E}})}{\circ S}\,(\vee_{\mathcal{E}})$$

$$\dfrac{}{\Gamma, P \vdash P}\,(ax) \qquad \dfrac{}{\Gamma \vdash \top}\,(\top) \qquad \dfrac{\Gamma \vdash -}{\Gamma \vdash P}\,(-)$$

$$\dfrac{\Gamma, P \vdash Q}{\Gamma \vdash P \supset Q}\,(\supset_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash P \supset Q \quad \Gamma \vdash P}{\Gamma \vdash Q}\,(\supset_{\mathcal{E}})$$

$$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\,(\wedge_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P}\,(\wedge_{\mathcal{E}_1}) \qquad \dfrac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}\,(\wedge_{\mathcal{E}_2})$$

$$\dfrac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\,(\vee_{\mathcal{I}_1}) \qquad \dfrac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\,(\vee_{\mathcal{I}_2})$$

$$\dfrac{\Gamma \vdash P \vee Q \quad \Gamma, P \vdash R \quad \Gamma, Q \vdash R}{\Gamma \vdash R}\,(\vee_{\mathcal{E}})$$

$$\dfrac{\Gamma \vdash P}{\Gamma \vdash \circ P}\,(\circ_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash \circ P \quad \Gamma, P \vdash \circ Q}{\Gamma \vdash \circ Q}\,(\circ_{\mathcal{E}})$$

$$\dfrac{\Gamma \vdash P[u/x]}{\Gamma \vdash \forall x.P}\,(\forall_{\mathcal{I}})\dagger \qquad \dfrac{\Gamma \vdash \forall x.P}{\Gamma \vdash P[t/x]}\,(\forall_{\mathcal{E}})$$

$$\dfrac{\Gamma \vdash P[t/x]}{\Gamma \vdash \exists x.P}\,(\exists_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash \exists x.P \quad \Gamma, P[u/x] \vdash R}{\Gamma \vdash R}\,(\exists_{\mathcal{E}})\dagger$$

† $u$ not free in $\Gamma, R$

**Figure 3.1:** NLL: Sequent style presentation of natural deduction for Lax Logic

–

$$\dfrac{\dfrac{-}{\circ P}\,(-_\varepsilon) \quad \begin{array}{c}[P]\\ \vdots\\ \circ Q\end{array}}{\circ Q}\,(\circ_\varepsilon) \quad \rightsquigarrow \quad \dfrac{-}{\circ Q}\,(-_\varepsilon)$$

–

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists x.P\end{array} \quad \begin{array}{c}[P[u/x]]\\ \vdots\\ \circ Q\end{array}}{\circ Q}\,(\exists_\varepsilon) \quad \begin{array}{c}[Q]\\ \vdots\\ \circ R\end{array}}{\circ R}\,(\circ_\varepsilon) \quad \rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \exists x.P\end{array} \quad \dfrac{\begin{array}{c}[P[u/x]]\\ \vdots\\ \circ Q\end{array} \quad \begin{array}{c}[Q]\\ \vdots\\ \circ R\end{array}}{\circ R}\,(\circ_\varepsilon)}{\circ R}\,(\exists_\varepsilon)$$

**Definition 3.1** *A natural deduction is said to be in* **($\beta, c$)-normal form** *when no $\beta$-reductions and no $c$-reductions are applicable.*

We give a presentation of a restricted version of natural deduction for Lax Logic. In this calculus, the only deductions are those that are in $(\beta, c)$-normal form. This calculus has two kinds of 'sequents', differentiated by their consequence relations, $\rhd$ and $\rhd\!\rhd$. Rules are applicable only when the premisses have a certain consequence relation. The conclusions have a fixed consequence relation. Thus those deductions that are valid are of a restricted form. This calculus, which we shall call NLAX, is given in Figure 3.2.

**Proposition 3.1** *The calculus* NLAX *only allows deductions to which no $\beta$-reductions and no $c$-reductions are applicable. Moreover, it allows all ($\beta, c$)-normal deductions.*

PROOF: By inspection one can see that deductions to which one could apply a reduction are not allowed in NLAX because they would involve a rule application with a premiss with the wrong consequence relation.

It can be seen that by use of the $(M)$ rule, all other deductions are possible. ■

## 3.3 Term Assignment

In this section we give a term assignment system for NLAX. In [Mog89] Moggi gave a $\lambda$-calculus, which he called the *computational $\lambda$-calculus*. This calculus naturally matches Lax Logic, as can be seen in Figure 3.3. The only exception to this match is our inclusion of the rules for $(-_\varepsilon)$ and the quantifiers. We leave these rules out of Figure 3.3. More about the computational $\lambda$-calculus and Lax Logic (there called computational logic) can be found in [BBdP98].

We give this term system again in a syntax we prefer – an abstract syntax with explicit constructors. We give a translation of Moggi's terms to ours, and then give

$$\frac{}{\Gamma, P \rhd P}\ (ax) \qquad \frac{\Gamma \rhd P}{\Gamma \rhd\!\!\rhd P}\ (M)$$

$$\frac{}{\Gamma \rhd\!\!\rhd \top}\ (\top) \qquad \frac{\Gamma \rhd -}{\Gamma \rhd\!\!\rhd P}\ (-_{\varepsilon})$$

$$\frac{\Gamma, P \rhd\!\!\rhd Q}{\Gamma \rhd\!\!\rhd P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \rhd P \supset Q \quad \Gamma \rhd\!\!\rhd P}{\Gamma \rhd Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd P \quad \Gamma \rhd\!\!\rhd Q}{\Gamma \rhd\!\!\rhd P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \rhd P \wedge Q}{\Gamma \rhd P}\ (\wedge_{\varepsilon_1}) \qquad \frac{\Gamma \rhd P \wedge Q}{\Gamma \rhd Q}\ (\wedge_{\varepsilon_2})$$

$$\frac{\Gamma \rhd\!\!\rhd P}{\Gamma \rhd\!\!\rhd P \vee Q}\ (\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \rhd\!\!\rhd Q}{\Gamma \rhd\!\!\rhd P \vee Q}\ (\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \rhd P \vee Q \quad \Gamma, P \rhd\!\!\rhd R \quad \Gamma, Q \rhd\!\!\rhd R}{\Gamma \rhd\!\!\rhd R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd P}{\Gamma \rhd\!\!\rhd \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \rhd \circ P \quad \Gamma, P \rhd\!\!\rhd \circ Q}{\Gamma \rhd\!\!\rhd \circ Q}\ (\circ_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd P[u/x]}{\Gamma \rhd\!\!\rhd \forall x.P}\ (\forall_{\mathcal{I}})\dagger \qquad \frac{\Gamma \rhd \forall x.P}{\Gamma \rhd P[t/x]}\ (\forall_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd P[t/x]}{\Gamma \rhd\!\!\rhd \exists x.P}\ (\exists_{\mathcal{I}}) \qquad \frac{\Gamma \rhd \exists x.P \quad \Gamma, P[u/x] \rhd\!\!\rhd R}{\Gamma \rhd\!\!\rhd R}\ (\exists_{\varepsilon})\dagger$$

† $u$ not free in $\Gamma, R$

**Figure 3.2:** NLAX: Sequent style presentation for normal natural deduction for Lax Logic

$$\frac{}{\Gamma, x : P \vdash x : P}\ (ax) \qquad \frac{}{\Gamma \vdash * : \top}\ (\top)$$

$$\frac{\Gamma, x : P \vdash e : Q}{\Gamma \vdash \lambda x.e : P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \supset Q \quad \Gamma \vdash f : P}{\Gamma \vdash e\ f : Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P \quad \Gamma \vdash f : Q}{\Gamma \vdash (e, f) : P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \wedge Q}{\Gamma \vdash fst(e) : P}\ (\wedge_{\varepsilon_1}) \qquad \frac{\Gamma \vdash e : P \wedge Q}{\Gamma \vdash snd(e) : Q}\ (\wedge_{\varepsilon_2})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash inl(e) : P \vee Q}\ (\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \vdash e : Q}{\Gamma \vdash inr(e) : P \vee Q}\ (\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \vdash e : P \vee Q \quad \Gamma, x : P \vdash f : R \quad \Gamma, y : Q \vdash g : R}{\Gamma \vdash case\ e\ of\ inl(x)\ \rightarrow\ f \mid inr(y)\ \rightarrow\ g : R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash val(e) : \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : \circ P \quad \Gamma, x : P \vdash f : \circ Q}{\Gamma \vdash let\ x\ \Leftarrow\ e\ in\ f : \circ Q}\ (\circ_{\varepsilon})$$

**Figure 3.3:** Sequent style presentation of natural deduction for Lax Logic, with Moggi's computational $\lambda$ terms.

yet another presentation of natural deduction for Lax Logic, this time annotated with proof terms in our prefered syntax, in Figure 3.4.

**Translation:** Moggi's terms $\rightsquigarrow$ proof terms in our preferred syntax

$$\begin{array}{lcl}
x & \rightsquigarrow & var(x) \\
* & \rightsquigarrow & * \\
\lambda x.e & \rightsquigarrow & \lambda x.e \\
e\ f & \rightsquigarrow & ap(e, f) \\
(e, f) & \rightsquigarrow & pr(e, f) \\
fst(e) & \rightsquigarrow & fst(e) \\
snd(e) & \rightsquigarrow & snd(e) \\
inl(e) & \rightsquigarrow & i(e) \\
inr(e) & \rightsquigarrow & j(e) \\
case\ e\ of\ inl(x)\ \rightarrow\ f \mid inr(y)\ \rightarrow\ g & \rightsquigarrow & wn(e, x.f, y.g) \\
val(e) & \rightsquigarrow & smhi(e) \\
let\ x\ \Leftarrow\ e\ in\ f & \rightsquigarrow & smhe(e, x.f)
\end{array}$$

We are interested in the 'real' proofs for Lax Logic – the normal natural deductions. We now restrict the terms that can be built, in order that they match our restricted natural deduction calculus NLAX, giving us proof objects. (That is, no reductions will be applicable at the term level; the term reductions match the $\beta$- and $c$-reductions for types given earlier). The proof terms come in two syntactic categories, **A** and **N**. V is the category of variables (proofs), U is the category of variables (individuals), and T the category of terms. The extra constructor $an(A)$ matches the $(M)$ rule of NLAX.

**A**::=

$$var(V) \mid ap(A, N) \mid fst(A) \mid snd(A) \mid apn(A, T)$$

**N**::=

$$* \mid efq(A) \mid an(A) \mid \lambda V.N \mid pr(N, N) \mid i(N) \mid j(N)$$
$$wn(A, V.N, V.N) \mid smhi(N) \mid smhe(A, V.N)$$
$$\lambda U.N \mid prq(T, N) \mid ee(A, U.V.N)$$

$$\frac{}{\Gamma, x : P \vdash var(x) : P}\ (ax)$$

$$\frac{}{\Gamma \vdash * : \top}\ (\top) \qquad \frac{\Gamma \vdash e : -}{\Gamma \vdash efq(e) : P}\ (-_{\varepsilon})$$

$$\frac{\Gamma, x : P \vdash e : Q}{\Gamma \vdash \lambda x.e : P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \supset Q \quad \Gamma \vdash f : P}{\Gamma \vdash ap(e, f) : Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P \quad \Gamma \vdash f : Q}{\Gamma \vdash pr(e, f) : P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \wedge Q}{\Gamma \vdash fst(e) : P}\ (\wedge_{\varepsilon_1}) \qquad \frac{\Gamma \vdash f : P \wedge Q}{\Gamma \vdash snd(f) : Q}\ (\wedge_{\varepsilon_2})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash i(e) : P \vee Q}\ (\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \vdash e : Q}{\Gamma \vdash j(e) : P \vee Q}\ (\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \vdash e : P \vee Q \quad \Gamma, x : P \vdash f : R \quad \Gamma, y : Q \vdash g : R}{\Gamma \vdash wn(e, x.f, y.g) : R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash smhi(e) : \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : \circ P \quad \Gamma, x : P \vdash f : \circ Q}{\Gamma \vdash smhe(e, x.f) : \circ Q}\ (\circ_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P[u/x]}{\Gamma \vdash \lambda u.e : \forall x.P}\ (\forall_{\mathcal{I}})\dagger \qquad \frac{\Gamma \vdash e : \forall x.P}{\Gamma \vdash apn(e, t) : P[t/x]}\ (\forall_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P[t/x]}{\Gamma \vdash prq(t, e) : \exists x.P}\ (\exists_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : \exists x.P \quad \Gamma, x : P[u/x] \vdash f : R}{\Gamma \vdash ee(e, u.x.f) : R}\ (\exists_{\varepsilon})\dagger$$

† $u$ not free in $\Gamma, R$

**Figure 3.4:** Sequent style presentation of natural deduction for Lax Logic

In Figure 3.5 we give one final presentation of a natural deduction calculus for Lax Logic, this time NLAX together with proof annotations.

## 3.4 Sequent Calculus

The stated aim of this chapter is to present a sequent calculus for Lax Logic whose proofs naturally correspond in a 1–1 way to normal natural deductions for Lax Logic – i.e. the proofs of NLAX. In this section we give such a sequent calculus, but first we remind the reader of the sequent calculus as presented in [FM97] and [BBdP98]. This can be seen in Figure 3.6.

In fact, our presentation is slightly different from both those cited. The calculus in [BBdP98] has no structural rules, that is, the contexts are sets. [FM97] have both weakening and contraction on both the left and the right, plus exchange. Here the only structural rule we consider (or need) is contraction on the left. The contexts in our presentation are labelled sets. We leave all discussion of cut until later.

We now present a new sequent calculus which we call PFLAX ('permutation-free' Lax Logic). Like MJ this calculus has two forms of judgement, $\Gamma \Rightarrow R$ and $\Gamma \xrightarrow{Q} R$. The calculus is displayed in Figure 3.7.

The stoup is a form of focusing: the formula in the stoup is always principal in the premiss unless it is a disjunction or a somehow formula. One might ask why we do not formulate the $(\circ_{\mathcal{L}})$ rule as follows

$$\frac{\Gamma \xrightarrow{P} \circ R}{\Gamma \xrightarrow{\circ P} \circ R}\ (\circ_{\mathcal{L}})$$

To answer this, we point out that the resulting calculus would not match normal natural deductions in the manner we would like. Also, consider proofs of the sequent $\circ\circ(P \wedge Q) \Rightarrow \circ(Q \wedge P)$.

## 3.5 Term Assignment for Sequent Calculus

We give a term assignment system for PFLAX. This we get by extending that given for intuitionistic logic in [Her95], [DP96], [DP98a]. The term calculus has two syntactic categories, **M** and **Ms**. V is the category of variables (proofs), U is the category of variables (individuals) and T is the category of terms .

**M**::=

$$* \mid (V; Ms) \mid \lambda V.M \mid pair(M, M) \mid inl(M) \mid inr(M)$$

$$smhr(M) \mid \lambda U.M \mid pairq(T, M)$$

$$\frac{}{\Gamma, x : P \rhd var(x) : P}\ (ax) \qquad \frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P}\ (M)$$

$$\frac{}{\Gamma \rhd\!\!\rhd * : \top}\ (\top) \qquad \frac{\Gamma \rhd A : -}{\Gamma \rhd\!\!\rhd efq(A) : P}\ (-_\varepsilon)$$

$$\frac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd \lambda x.N : P \supset Q}\ (\supset_\mathcal{I}) \qquad \frac{\Gamma \rhd A : P \supset Q \quad \Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd ap(A, N) : Q}\ (\supset_\varepsilon)$$

$$\frac{\Gamma \rhd\!\!\rhd N_1 : P \quad \Gamma \rhd\!\!\rhd N_2 : Q}{\Gamma \rhd\!\!\rhd pr(N_1, N_2) : P \wedge Q}\ (\wedge_\mathcal{I})$$

$$\frac{\Gamma \rhd A : P \wedge Q}{\Gamma \rhd fst(A) : P}\ (\wedge_{\varepsilon_1}) \qquad \frac{\Gamma \rhd A : P \wedge Q}{\Gamma \rhd snd(A) : Q}\ (\wedge_{\varepsilon_2})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd i(N) : P \vee Q}\ (\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd j(N) : P \vee Q}\ (\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \rhd A : P \vee Q \quad \Gamma, x_1 : P \rhd\!\!\rhd N_1 : R \quad \Gamma, x_2 : Q \rhd\!\!\rhd N_2 : R}{\Gamma \rhd\!\!\rhd wn(A, x_1.N_1, x_2.N_2) : R}\ (\vee_\varepsilon)$$

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd smhi(N) : \circ P}\ (\circ_\mathcal{I}) \qquad \frac{\Gamma \rhd A : \circ P \quad \Gamma, x : P \rhd\!\!\rhd N : \circ Q}{\Gamma \rhd\!\!\rhd smhe(A, x.N) : \circ Q}\ (\circ_\varepsilon)$$

$$\frac{\Gamma \rhd\!\!\rhd N : P[u/x]}{\Gamma \rhd\!\!\rhd \lambda u.N : \forall x.P}\ (\forall_\mathcal{I})\dagger \qquad \frac{\Gamma \rhd A : \forall x.P}{\Gamma \rhd apn(A, t) : P[t/x]}\ (\forall_\varepsilon)$$

$$\frac{\Gamma \rhd\!\!\rhd N : P[t/x]}{\Gamma \rhd\!\!\rhd prq(t, N) : \exists x.P}\ (\exists_\mathcal{I}) \qquad \frac{\Gamma \rhd A : \exists x.P \quad \Gamma, x : P[u/x] \rhd\!\!\rhd N : R}{\Gamma \rhd\!\!\rhd ee(A, u.x.N) : R}\ (\exists_\varepsilon)\dagger$$

† $u$ not free in $\Gamma, R$

**Figure 3.5:** NLAX with proof annotations

$$\frac{}{\Gamma, P \Rightarrow P}\ (ax) \qquad \frac{\Gamma, P, P \Rightarrow R}{\Gamma, P \Rightarrow R}\ (C)$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{\Gamma, - \Rightarrow P}\ (-)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma, Q \Rightarrow R}{\Gamma, P \supset Q \Rightarrow R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_2}) \qquad \frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma, P \vee Q \Rightarrow R}\ (\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow \circ P}\ (\circ_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \circ R}{\Gamma, \circ P \Rightarrow \circ R}\ (\circ_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P[y/x]}{\Gamma \Rightarrow \forall x.P}\ (\forall_{\mathcal{R}})\dagger \qquad \frac{\Gamma, P[t/x] \Rightarrow R}{\Gamma, \forall x.P \Rightarrow R}\ (\forall_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P[t/x]}{\Gamma \Rightarrow \exists x.P}\ (\exists_{\mathcal{R}}) \qquad \frac{\Gamma, P[y/x] \Rightarrow R}{\Gamma, \exists x.P \Rightarrow R}\ (\exists_{\mathcal{L}})\dagger$$

† $y$ not free in $\Gamma, R$

**Figure 3.6:** LAX: Sequent Calculus for Lax Logic

$$\frac{}{\Gamma \xrightarrow{P} P}\ (ax) \qquad \frac{\Gamma, P \xrightarrow{P} R}{\Gamma, P \Rightarrow R}\ (C)$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top) \qquad \frac{}{\Gamma \xrightarrow{\bot} P}\ (-)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \supset Q} R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{R}}) \qquad \frac{\Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{P \wedge Q} R}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \wedge Q} R}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_2}) \qquad \frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma \xrightarrow{P \vee Q} R}\ (\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow \circ P}\ (\circ_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \circ R}{\Gamma \xrightarrow{\circ P} \circ R}\ (\circ_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P[y/x]}{\Gamma \Rightarrow \forall x.P}\ (\forall_{\mathcal{R}})\dagger \qquad \frac{\Gamma \xrightarrow{P[t/x]} R}{\Gamma \xrightarrow{\forall x.P} R}\ (\forall_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P[t/x]}{\Gamma \Rightarrow \exists x.P}\ (\exists_{\mathcal{R}}) \qquad \frac{\Gamma, P[y/x] \Rightarrow R}{\Gamma \xrightarrow{\exists x.P} R}\ (\exists_{\mathcal{L}})\dagger$$

† $y$ not free in $\Gamma, R$

**Figure 3.7:** The Sequent Calculus PFLAX

**Ms**::=

$$[\,] \mid ae \mid (M :: Ms) \mid p(Ms) \mid q(Ms) \mid when(V.M, V.M)$$
$$smhl(V.M) \mid apq(T, Ms) \mid spl(U.V.M)$$

These terms can easily be typed by PFLAX, as seen in Figure 3.8.

## 3.6 Results

Having presented the calculi for Lax Logic, we now prove that they have the properties we claim. We prove soundness and adequacy for PFLAX, and the equivalence of the term calculi. These results prove the desired correspondence.

The full details of these proofs are rather repetitive: therefore we only give the proofs for the $\supset, \circ$ fragment of Lax Logic. The remainder of the calculus is intuitionistic logic as presented in [DP96]. The details of the proofs extended to the rest of the calculus can be found there.

We start by giving pairs of functions that define translations between the term assignment systems for natural deduction and sequent calculus.

**Sequent Calculus to Natural Deduction:**

$\theta$ : M $\rightarrow$ N

$$\theta(x; Ms) = \theta'(var(x), Ms)$$
$$\theta(\lambda x.M) = \lambda x.\theta(M)$$
$$\theta(smhr(M)) = smhi(\theta(M))$$

$\theta'$ : A $\times$ Ms $\rightarrow$ N

$$\theta'(A, [\,]) = an(A)$$
$$\theta'(A, (M :: Ms)) = \theta'(ap(A, \theta(M)), Ms)$$
$$\theta'(A, smhl(x.Ms)) = smhe(A, x.\theta(M))$$

$$\frac{}{\Gamma \xrightarrow{P} [\,] : P}\ (ax) \qquad \frac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\ (C)$$

$$\frac{}{\Gamma \Rightarrow * : \top}\ (\top) \qquad \frac{}{\Gamma \xrightarrow{\bot} ae : R}\ (-)$$

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} (M :: Ms) : R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma \Rightarrow M_2 : Q}{\Gamma \Rightarrow pair(M_1, M_2) : P \wedge Q}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{P \wedge Q} p(Ms) : R}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \wedge Q} q(Ms) : R}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow inl(M) : P \vee Q}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow M : Q}{\Gamma \Rightarrow inr(M) : P \vee Q}\ (\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, x_1 : P \Rightarrow M_1 : R \quad \Gamma, x_2 : Q \Rightarrow M_2 : R}{\Gamma \xrightarrow{P \vee Q} when(x_1.M_1, x_2.M_2) : R}\ (\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow smhr(M) : \circ P}\ (\circ_{\mathcal{R}}) \qquad \frac{\Gamma, x : P \Rightarrow M : \circ R}{\Gamma \xrightarrow{\circ P} smhl(x.M) : \circ R}\ (\circ_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P[u/x]}{\Gamma \Rightarrow \lambda u.M : \forall x.P}\ (\forall_{\mathcal{R}})\dagger \qquad \frac{\Gamma \xrightarrow{P[t/x]} Ms : R}{\Gamma \xrightarrow{\forall x.P} apq(t, Ms) : R}\ (\forall_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P[t/x]}{\Gamma \Rightarrow pairq(T, M) : \exists x.P}\ (\exists_{\mathcal{R}}) \qquad \frac{\Gamma, P[u/x] \Rightarrow M : R}{\Gamma \xrightarrow{\exists x.P} spl(u.x.M) : R}\ (\exists_{\mathcal{L}})\dagger$$

† $u$ not free in $\Gamma, R$

**Figure 3.8:** The Sequent Calculus PFLAX, with Term Assignment

**Natural Deduction to Sequent Calculus:**

$\psi$ : N $\rightarrow$ M

$$\psi(an(A)) = \psi'(A, [\,])$$
$$\psi(\lambda x.N) = \lambda x.\psi(N)$$
$$\psi(smhe(A, x.N)) = \psi'(A, smhl(x.\psi(N)))$$
$$\psi(smhi(N)) = smhr(\psi(N))$$

$\psi'$ : A $\times$ Ms $\rightarrow$ M

$$\psi'(var(x), Ms) = (x; Ms)$$
$$\psi'(ap(A, N), Ms) = \psi'(A, (\psi(N) :: Ms))$$

We prove two lemmas showing the equivalence of the term calculi.

**Lemma 3.1**

**i)** $\psi(\theta(M)) \;=\; M$

**ii)** $\psi(\theta'(A, Ms)) \;=\; \psi'(A, Ms)$

PROOF: The proof is by simultaneous induction on the structure of $M$ and $Ms$.

Case 1. The **M** term is $(x; Ms)$

$$\begin{array}{lcll} \psi(\theta(x; Ms)) & = & \psi(\theta'(var(x), Ms)) & \text{def } \theta \\ & = & \psi'(var(x), Ms) & \text{ind ii)} \\ & = & (x; Ms) & \text{def } \psi' \end{array}$$

Case 2.The **M** term is $\lambda x.M$

$$\begin{array}{lcll} \psi(\theta(\lambda x.M)) & = & \psi(\lambda x.\theta(M)) & \text{def } \theta \\ & = & \lambda x.\psi(\theta(M)) & \text{def } \psi \\ & = & \lambda x.M & \text{ind i)} \end{array}$$

Case 3. The **M** term is $smhr(M)$

$$\begin{array}{lcll} \psi(\theta(smhr(M))) & = & \psi(smhi(\theta(M))) & \text{def } \theta \\ & = & smhr(\psi(\theta(M))) & \text{def } \psi \\ & = & smhr(M) & \text{ind i)} \end{array}$$

Case 4.The **Ms** term is $[\,]$

$$\begin{array}{lcll} \psi(\theta'(A,[\,])) & = & \psi(an(A)) & \text{def } \theta \\ & = & \psi'(A,[\,]) & \text{def } \psi' \end{array}$$

Case 5.The **Ms** term is $(M :: Ms)$

$$\begin{array}{lcll} \psi(\theta'(A,(M :: Ms))) & = & \psi(\theta'(ap(A,\theta(M)),Ms)) & \text{def } \theta' \\ & = & \psi'(ap(A,\theta(M)),Ms) & \text{ind ii)} \\ & = & \psi'(A,(\psi(\theta(M)) :: Ms)) & \text{def } \psi' \\ & = & \psi'(A,(M :: Ms)) & \text{ind i)} \end{array}$$

Case 6. The **M** term is $smhl(x.M)$

$$\begin{array}{lcll} \psi(\theta'(A,smhl(x.M))) & = & \psi(smhe(A,x.\theta(M))) & \text{def } \theta' \\ & = & \psi'(A,smhl(x.\psi(\theta(M)))) & \text{def } \psi \\ & = & \psi'(A,smhl(x.M)) & \text{ind i)} \end{array}$$

■

**Lemma 3.2**

**i)** $\theta(\psi(N)) \;=\; N$

**ii)** $\theta(\psi'(A,Ms)) \;=\; \theta'(A,Ms)$

PROOF: By simultaneous induction on the structure of $N$ and $A$.

Case 1.The **N** term is $an(A)$

$$\begin{array}{lcll} \theta(\psi(an(A)) & = & \theta(\psi'(A,[\,])) & \text{def } \psi \\ & = & \theta'(A,[\,]) & \text{ind ii)} \\ & = & an(A) & \text{def } \theta' \end{array}$$

Case 2. The **N** term is $\lambda x.N$

$$\begin{array}{lcll} \theta(\psi(\lambda x.N)) & = & \theta(\lambda x.\psi(N)) & \text{def } \psi \\ & = & \lambda x.\theta(\psi(N)) & \text{def } \theta \\ & = & \lambda x.N & \text{ind i)} \end{array}$$

Case 3. The **N** term is $smhi(N)$

$$\begin{array}{lcll} \theta(\psi(smhi(N))) & = & \theta(smhr(\psi(N))) & \text{def } \psi \\ & = & smhi(\theta(\psi(N))) & \text{def } \theta \\ & = & smhi(N) & \text{ind i)} \end{array}$$

Case 4. The **N** term is $smhe(A, x.N)$

$$\begin{array}{rcll} \theta(\psi(smhe(A, x.N))) & = & \theta(\psi'(A, smhl(x.\psi(N)))) & \text{def } \psi \\ & = & \theta'(A, smhl(x.\psi(N))) & \text{ind ii)} \\ & = & smhe(A, x.\theta(\psi(N))) & \text{def } \theta' \\ & = & smhe(A, x.N) & \text{ind i)} \end{array}$$

Case 5. The **A** term is $var(x)$

$$\begin{array}{rcll} \theta(\psi'(var(x), Ms)) & = & \theta(x; Ms) & \text{def } \psi' \\ & = & \theta'(var(x), Ms) & \text{def } \theta \end{array}$$

Case 6. The **A** term is $ap(A, N)$

$$\begin{array}{rcll} \theta(\psi'(ap(A, N), Ms)) & = & \theta(\psi'(A, (\psi(N) :: Ms))) & \text{def } \psi' \\ & = & \theta'(A, (\psi(N) :: Ms)) & \text{ind ii)} \\ & = & \theta'(ap(A, \theta(\psi(N))), Ms) & \text{def } \theta' \\ & = & \theta'(ap(A, N), Ms) & \text{ind i)} \end{array}$$

■

We now prove soundness and adequacy theorems.

**Theorem 3.1** *(*SOUNDNESS*) The following rules are admissible:*

$$\frac{\Gamma \Rightarrow M : R}{\Gamma \rhd\!\!\rhd \theta(M) : R}\ i) \qquad \frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(A, Ms) : R}\ ii)$$

PROOF: By simultaneous induction on the structure of $M$ and $Ms$.

Case 1. The **M** term is $(x; Ms)$

We have a derivation ending in:

$$\frac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\ (C)$$

and we know that

$$\Gamma, x : P \rhd var(x) : P$$

is deducible.

So we have:

$$\frac{\Gamma, x : P \rhd var(x) : P \quad \Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(var(x), Ms) : R}\ ii)$$

and we know that

$$\theta'(var(x), Ms) = \theta(x; Ms)$$

Case 2. The **M** term is $\lambda x.M$

We have a derivation ending in

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \supset Q}\ (\supset_{\mathcal{R}})$$

whence

$$\frac{\dfrac{\Gamma, x : P \Rightarrow M : Q}{\Gamma, x : P \rhd\!\!\rhd \theta(M) : Q}\ i)}{\Gamma \rhd\!\!\rhd \lambda x.\theta(M) : P \supset Q}\ (\supset_{\mathcal{I}})$$

and we know that

$$\lambda x.\theta(M) = \theta(\lambda x.M)$$

Case 3. The **M** term is $smhr(M)$

We have a derivation ending as follows

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow smhr(M) : \circ P}\ (\circ_{\mathcal{R}})$$

whence

$$\frac{\dfrac{\Gamma \Rightarrow M : P}{\Gamma \rhd\!\!\rhd \theta(M) : P}\ i)}{\Gamma \rhd\!\!\rhd smhi(\theta(M)) : \circ P}\ (\circ_{\mathcal{I}})$$

and we know that

$$smhi(\theta(M)) = \theta(smhr(M))$$

Case 4. The **Ms** term is $[\,]$

We have a deduction and a derivation:

$$\Gamma \rhd A : P \qquad \frac{}{\Gamma \xrightarrow{P} [\,] : P}\ (ax)$$

From the deduction, we obtain:

$$\frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P}\ (M)$$

and since

$$an(A) = \theta'(A, [\,])$$

we have what we require.

Case 5. The **Ms** term is $(M :: Ms)$

We have a derivation ending in

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} (M :: Ms) : R} \ (\supset_{\mathcal{L}})$$

whence

$$\frac{\dfrac{\Gamma \rhd A : P \supset Q \quad \dfrac{\Gamma \Rightarrow M : P}{\Gamma \rhd\!\!\rhd \theta(M) : P}\ i)}{\Gamma \rhd ap(A, \theta(M)) : Q}\ (\supset_{\varepsilon}) \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(ap(A, \theta(M)), Ms) : R}\ ii)$$

and we know that

$$\theta'(ap(A, \theta(M)), Ms) = \theta'(A, (M :: Ms))$$

Case 6. The **Ms** term is $smhl(x.Ms)$

We have a derivation ending

$$\frac{\Gamma, x : P \Rightarrow M : \circ Q}{\Gamma \xrightarrow{\circ P} smhl(x.M) : \circ Q}\ (\circ_{\mathcal{L}})$$

whence

$$\frac{\Gamma \rhd A : \circ P \quad \dfrac{\Gamma, x : P \Rightarrow M : \circ Q}{\Gamma, x : P \rhd\!\!\rhd \theta(M) : \circ Q}\ i)}{\Gamma \rhd\!\!\rhd smhe(A, x.\theta(M)) : \circ Q}\ (\circ_{\varepsilon})$$

and we know that

$$smhe(A, x.\theta(M)) = \theta'(A, smhl(x.M))$$

■

**Theorem 3.2** *(*ADEQUACY*) The following rules are admissible:*

$$\frac{\Gamma \rhd\!\!\rhd N : R}{\Gamma \Rightarrow \psi(N) : R}\ i) \qquad \frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \Rightarrow \psi'(A, Ms) : R}\ ii)$$

PROOF: By simultaneous induction on the structure of $A$ and $N$.

Case 1. The **N** term is $an(A)$

We have a deduction ending

$$\frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P}\ (M)$$

We know that we can derive

$$\frac{}{\Gamma \xrightarrow{P} [\,] : P}\ (ax)$$

hence we have

$$\frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} [\,] : P}{\Gamma \Rightarrow \psi'(A, []) : P}\ ii)$$

We know that

$$\psi'(A, [\,]) = \psi(an(A))$$

Case 2. The **N** term is $\lambda x.N$

We have a deduction ending

$$\frac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd \lambda x.N : P \supset Q}\ (\supset_{\mathcal{I}})$$

whence

$$\frac{\dfrac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma, x : P \Rightarrow \psi(N) : Q}\ i)}{\Gamma \Rightarrow \lambda x.\psi(N) : P \supset Q}\ (\supset_{\mathcal{R}})$$

and we know that

$$\lambda x.\psi(N) = \psi(\lambda x.N)$$

Case 3. The **N** term is $smhe(A, x.N)$

We have a deduction ending in

$$\frac{\Gamma \rhd A : \circ P \quad \Gamma, x : P \rhd\!\!\rhd N : \circ Q}{\Gamma \rhd\!\!\rhd smhe(A, x.N) : \circ Q}\ (\circ_{\mathcal{E}})$$

whence

$$\frac{\Gamma \rhd A : \circ P \quad \dfrac{\dfrac{\Gamma, x : P \rhd\!\!\rhd N : \circ Q}{\Gamma, x : P \Rightarrow \psi(N) : \circ Q}\ i)}{\Gamma \xrightarrow{\circ P} smhl(x.\psi(N)) : \circ Q}\ (\circ_{\mathcal{L}})}{\Gamma \Rightarrow \psi'(A, smhl(x.\psi(N))) : \circ Q}\ ii)$$

and we know that

$$\psi'(A, smhl(x.\psi(N))) = \psi(smhe(A, x.N))$$

Case 4. The **N** term is $smhi(N)$

We have a deduction ending in

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd smhi(N) : \circ P}\ (\circ_{\mathcal{I}})$$

whence

$$\frac{\dfrac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \Rightarrow \psi(N) : P}\ i)}{\Gamma \Rightarrow smhr(\psi(N)) : \circ P}\ (\circ_{\mathcal{R}})$$

and we know that

$$smhr(\psi(N)) = \psi(smhi(N))$$

Case 5. The **A** term is $var(x)$

We can extend to

$$\frac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\ (C)$$

and since

$$(x; Ms) = \psi'(var(x), Ms)$$

we have the result.

Case 6. The **A** term is $ap(A, N)$

We have a deduction ending in

$$\frac{\Gamma \rhd A : P \supset Q \quad \Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd ap(A, N) : Q}\ (\supset_{\mathcal{E}})$$

whence

$$\frac{\Gamma \rhd A : P \supset Q \qquad \dfrac{\dfrac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \Rightarrow \psi(N) : P}\ i) \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} (\psi(N) :: Ms) : R}\ (\supset_{\mathcal{L}})}{\Gamma \Rightarrow \psi'(A, (\psi(N) :: Ms)) : R}\ ii)$$

and we know that

$$\psi'(A, (\psi(N) :: Ms)) = \psi'(ap(A, N), Ms)$$

■

**Theorem 3.3** *The normal natural deductions of Lax Logic (the proofs of* NLAX*) are in 1–1 correspondence to the proofs of* PFLAX.

PROOF: Immediate from Theorems 1 and 2 and Lemmas 1 and 2. ■

**Corollary 3.1** *The calculus* PFLAX *is sound and complete with respect to provability in the logic.*

## 3.7 Cut Elimination

Now we move on to a study of cut in PFLAX. In the usual sequent calculus, cut may be formulated as follows:

$$\frac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow Q}{\Gamma \Rightarrow Q}\ (cut)$$

In PFLAX, the two judgement forms lead to the following four cut rules:.

$$\frac{\Gamma \xrightarrow{Q} P \quad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{Q} R}\ (cut_1) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{Q} R}{\Gamma \xrightarrow{Q} R}\ (cut_2)$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{P} R}{\Gamma \Rightarrow R}\ (cut_3) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow R}{\Gamma \Rightarrow R}\ (cut_4)$$

These have associated terms:

**M**::=

$$cut_1^P(Ms, Ms) \mid cut_2^P(M, V.Ms)$$

**Ms**::=

$$cut_3^P(M, Ms) \mid cut_4^P(M, V.M)$$

We can give the cut rules again annotated by the proof terms:

$$\frac{\Gamma \xrightarrow{Q} Ms_1 : P \quad \Gamma \xrightarrow{P} Ms_2 : R}{\Gamma \xrightarrow{Q} cut_1^P(Ms_1, Ms_2) : R}\ (cut_1)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma, x : P \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{Q} cut_2^P(M, x.Ms) : R}\ (cut_2)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \Rightarrow cut_3^P(M, Ms) : R}\ (cut_3)$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma, x : P \Rightarrow M_2 : R}{\Gamma \Rightarrow cut_4^P(M_1, x.M_2) : R}\ (cut_4)$$

We call PFLAX extended with the four cut rules PFLAX$^{cut}$. We give reduction rules for PFLAX$^{cut}$. As in the previous section, we restrict ourselves to the $\supset, \circ$ fragment of the logic, in order to prevent repetition of results that can be found elsewhere ([DP96]). Here we give reductions without terms, together with the associated term reductions.

Case 1. $cut_1^P([\,], Ms) \rightsquigarrow Ms$

$$\frac{\dfrac{}{\Gamma \xrightarrow{P} P}\,(ax) \qquad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{P} R}\,(cut_1) \qquad \rightsquigarrow \qquad \Gamma \xrightarrow{P} R$$

Case 2. $cut_1^P((M :: Ms_1), Ms_2) \rightsquigarrow (M :: cut_1^P(Ms_1, Ms_2))$

$$\frac{\dfrac{\Gamma \Rightarrow S \quad \Gamma \xrightarrow{T} P}{\Gamma \xrightarrow{S \supset T} P}\,(\supset_{\mathcal{L}}) \qquad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{S \supset T} R}\,(cut_1)$$

$$\rightsquigarrow \qquad \frac{\Gamma \Rightarrow S \qquad \dfrac{\Gamma \xrightarrow{T} P \quad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{T} R}\,(cut_1)}{\Gamma \xrightarrow{S \supset T} R}\,(\supset_{\mathcal{L}})$$

Case 3. $cut_1^{\circ P}(smhl(x.M), Ms) \rightsquigarrow smhl(x.cut_3^{\circ P}(M, Ms))$

$$\frac{\dfrac{\Gamma, S \Rightarrow \circ P}{\Gamma \xrightarrow{\circ S} \circ P}\,(\circ_{\mathcal{L}}) \qquad \Gamma \xrightarrow{\circ P} \circ R}{\Gamma \xrightarrow{\circ S} \circ R}\,(cut_1) \qquad \rightsquigarrow \qquad \frac{\dfrac{\Gamma, S \Rightarrow \circ P \quad \Gamma \xrightarrow{\circ P} \circ R}{\Gamma, S \Rightarrow \circ R}\,(cut_3)}{\Gamma \xrightarrow{\circ S} \circ R}\,(\circ_{\mathcal{L}})$$

Case 4. $cut_2^P(M, x.[\,]) \rightsquigarrow [\,]$

$$\frac{\Gamma \Rightarrow P \qquad \dfrac{}{\Gamma, P \xrightarrow{R} R}\,(ax)}{\Gamma \xrightarrow{R} R}\,(cut_2) \qquad \rightsquigarrow \qquad \frac{}{\Gamma \xrightarrow{R} R}\,(ax)$$

Case 5. $cut_2^P(M_1, x.(M_2 :: Ms)) \rightsquigarrow ((cut_4^P(M_1, x.M_2)) :: (cut_2^P(M_1, x.Ms)))$

$$\frac{\Gamma \Rightarrow P \qquad \dfrac{\Gamma, P \Rightarrow S \quad \Gamma, P \xrightarrow{T} R}{\Gamma, P \xrightarrow{S \supset T} R}\,(\supset_{\mathcal{L}})}{\Gamma \xrightarrow{S \supset T} R}\,(cut_2)$$

$$\rightsquigarrow \qquad \frac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow S}{\Gamma \Rightarrow S}\,(cut_4) \qquad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{T} R}{\Gamma \xrightarrow{T} R}\,(cut_2)}{\Gamma \xrightarrow{S \supset T} R}\,(\supset_{\mathcal{L}})$$

Case 6. $cut_2^P(M_1, x_1.smhl(x_2.M_2)) \rightsquigarrow smhl(x_2.cut_4^P(M_1, x_1.M_2))$

$$\frac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P, S \Rightarrow \circ R}{\Gamma, P \xrightarrow{\circ S} \circ R}(\circ_{\mathcal{L}})}{\Gamma \xrightarrow{\circ S} \circ R}(cut_2) \quad \rightsquigarrow \quad \frac{\dfrac{\dfrac{\Gamma \Rightarrow P}{\Gamma, S \Rightarrow P}(W) \quad \Gamma, P, S \Rightarrow \circ R}{\Gamma, S \Rightarrow \circ R}(cut_4)}{\Gamma \xrightarrow{\circ S} \circ R}(\circ_{\mathcal{L}})$$

Case 7. $cut_3^P((x; Ms_1), Ms_2) \rightsquigarrow (x; cut_1^P(Ms_1, Ms_2))$

$$\frac{\dfrac{\Gamma, S \xrightarrow{S} P}{\Gamma, S \Rightarrow P}(C) \quad \Gamma, S \xrightarrow{P} R}{\Gamma, S \Rightarrow R}(cut_3)$$

$$\rightsquigarrow \quad \frac{\dfrac{\Gamma, S \xrightarrow{S} P \quad \Gamma, S \xrightarrow{P} R}{\Gamma, S \xrightarrow{S} R}(cut_3)}{\Gamma, S \Rightarrow R}(C)$$

Case 8. $cut_3^{P \supset Q}(\lambda x.M_1, (M_2 :: Ms)) \rightsquigarrow cut_3^Q(cut_4^P(M_2, x.M_1), Ms)$

$$\frac{\dfrac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}(\supset_{\mathcal{R}}) \quad \dfrac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \supset Q} R}(\supset_{\mathcal{L}})}{\Gamma \Rightarrow R}(cut_3)$$

$$\rightsquigarrow \quad \frac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow Q}{\Gamma \Rightarrow Q}(cut_4) \quad \Gamma \xrightarrow{Q} R}{\Gamma \Rightarrow R}(cut_3)$$

Case 9. $cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \rightsquigarrow cut_4^P(M_1, x.M_2)$

$$\frac{\dfrac{\Gamma \Rightarrow P}{\Gamma \Rightarrow \circ P}(\circ_{\mathcal{R}}) \quad \dfrac{\Gamma, P \Rightarrow \circ R}{\Gamma \xrightarrow{\circ P} \circ R}(\circ_{\mathcal{L}})}{\Gamma \Rightarrow \circ R}(cut_3) \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow \circ R}{\Gamma \Rightarrow \circ R}(cut_4)$$

Case 10. $cut_3^P(M, [\,]) \rightsquigarrow M$

$$\frac{\Gamma \Rightarrow P \quad \dfrac{}{\Gamma \xrightarrow{P} P}(ax)}{\Gamma \Rightarrow P}(cut_3) \quad \rightsquigarrow \quad \Gamma \Rightarrow P$$

Case 11. $cut_4^P(M, x_1.(x_2; Ms)) \rightsquigarrow (x_2; cut_2^P(M, x_1.Ms))$

$$\frac{\Gamma, S \Rightarrow P \quad \dfrac{\Gamma, P, S \xrightarrow{S} R}{\Gamma, P, S \Rightarrow R}(C)}{\Gamma, S \Rightarrow R}(cut_4) \quad \rightsquigarrow \quad \frac{\dfrac{\Gamma, S \Rightarrow P \quad \Gamma, S, P \xrightarrow{S} R}{\Gamma, S \xrightarrow{S} R}(cut_2)}{\Gamma, S \Rightarrow R}(C)$$

Case 12. $cut_4^P(M, x.(x; Ms)) \rightsquigarrow cut_3^P(M, cut_2^P(M, x.Ms))$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P \xrightarrow{P} R}{\Gamma, P \Rightarrow R}\,(C)}{\Gamma \Rightarrow R}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{P} R}{\Gamma \xrightarrow{P} R}\,(cut_2)}{\Gamma \Rightarrow R}\,(cut_3)$$

Case 13. $cut_4^P(M_1, x_1.\lambda x_2.M_2) \rightsquigarrow \lambda x_2.cut_4^P(M_1, x_1.M_2)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P, S \Rightarrow T}{\Gamma, P \Rightarrow S \supset T}\,(\supset_{\mathcal{R}})}{\Gamma \Rightarrow S \supset T}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow P}{\Gamma, S \Rightarrow P}\,(W) \quad \Gamma, S, P \Rightarrow T}{\Gamma, S \Rightarrow T}\,(cut_4)}{\Gamma \Rightarrow S \supset T}\,(\supset_{\mathcal{R}})$$

Case 14. $cut_4^P(M_1, x.smhr(M_2)) \rightsquigarrow smhr(cut_4^P(M_1, x.M_2))$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P \Rightarrow S}{\Gamma, P \Rightarrow \circ S}\,(\circ_{\mathcal{R}})}{\Gamma \Rightarrow \circ S}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow S}{\Gamma \Rightarrow S}\,(cut_4)}{\Gamma \Rightarrow \circ S}\,(\circ_{\mathcal{R}})$$

Notice that we used the following lemma:

**Lemma 3.3 (**WEAKENING) *The following rules are admissible in* PFLAX*:*

$$\dfrac{\Gamma \Rightarrow R}{\Gamma, P \Rightarrow R}\,(W) \qquad \dfrac{\Gamma \xrightarrow{Q} R}{\Gamma, P \xrightarrow{Q} R}\,(W)$$

PROOF: Induction on the height of derivations. ■

We summarise the term reductions:

1. $cut_1^P([\,], Ms) \rightsquigarrow Ms$
2. $cut_1^P((M :: Ms_1), Ms_2) \rightsquigarrow (M :: cut_1^P(Ms_1, Ms_2))$
3. $cut_1^{\circ P}(smhl(x.M), Ms) \rightsquigarrow smhl(x.cut_3^{\circ P}(M, Ms))$
4. $cut_2^P(M, x.[\,]) \rightsquigarrow [\,]$
5. $cut_2^P(M_1, x.(M_2 :: Ms)) \rightsquigarrow ((cut_4^P(M_1, x.M_2)) :: (cut_2^P(M_1, x.Ms)))$
6. $cut_2^P(M_1, x_1.smhl(x_2.M_2)) \rightsquigarrow smhl(x_2.cut_4^P(M_1, x_1.M_2))$
7. $cut_3^P((x; Ms_1), Ms_2) \rightsquigarrow (x; cut_1^P(Ms_1, Ms_2))$
8. $cut_3^{P \supset Q}(\lambda x.M_1, (M_2 :: Ms)) \rightsquigarrow cut_3^Q(cut_4^P(M_2, x.M_1), Ms)$
9. $cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \rightsquigarrow cut_4^P(M_1, x.M_2)$

10. $cut_3^P(M, [\,]) \rightsquigarrow M$

11. $cut_4^P(M, x_1.(x_2; Ms)) \rightsquigarrow (x_2; cut_2^P(M, x_1.Ms))$

12. $cut_4^P(M, x.(x; Ms)) \rightsquigarrow cut_3^P(M, cut_2^P(M, x.Ms))$

13. $cut_4^P(M_1, x_1.\lambda x_2.M_2) \rightsquigarrow \lambda x_2.cut_4^P(M_1, x_1.M_2)$

14. $cut_4^P(M_1, x.smhr(M_2)) \rightsquigarrow smhr(cut_4^P(M_1, x.M_2))$

**Definition 3.2** *A* **simple cut** *instance is an instance of cut with cut-free premisses.*

**Definition 3.3** *The* **size** *of a formula is the number of connectives in that formula plus one.*

**Definition 3.4** *The* **weight** *of a simple cut instance is the quadruple:*

$$(|A|, cutno., h_1, h_2)$$

*where:*

- $|A|$ *is the size of the cut formula.*
- $cutno.$ *is the kind of the cut (i.e. 1, 2, 3, 4)*
- $h_1$ *is the height of the derivation of the right premiss*
- $h_2$ *is the height of the derivation of the left premiss*

*we make the convention that* $cut_1 = cut_3 < cut_2 = cut_4$.

*The quadruple is lexicographically ordered from the left.*

**Lemma 3.4** *The weights defined in Definition 3.4 are well-ordered.*

We now prove the theorem.

**Theorem 3.4** *(*WEAK CUT ELIMINATION*) The rules* $(cut_1), (cut_2), (cut_3), (cut_4)$ *are admissible in* PFLAX.

PROOF: We give a weak cut reduction strategy:

- pick any simple cut instance and reduce
- recursively reduce any simple cut instances in the result

By induction on the weight of the cut instance, and induction on the number of simple cut instances, this strategy terminates.

This can easily be seen by inspection. ■

## 3.8 Strong Normalisation

In this section we prove that the cut reduction system strongly normalises, giving another proof of cut elimination for PFLAX.

We prove strong normalisation using the recursive path-order from term rewriting ([Der82], see also [BN98]). This is attractive since it is purely syntactic; reasoning is about the structure of the terms themselves rather than about a mapping of terms into tuples of natural numbers. More on proving normalisation using term rewriting can be found in [Sel98].

Again we restrict ourselves to the $\supset, \circ$ fragment of Lax Logic to avoid repetition.

### 3.8.1 Termination Using the Recursive Path-Order

We define two strict partial orders, one on term constructors (or operators), $>$, and one on terms, $\succ$. This second strict partial order, the recursive path-order, is defined in terms of the first. Given that $>$ has some simple properties (transitivity, irreflexivity, well-foundedness – all true by definition), the recursive path-order theorem tells us that $\succ$ is well-founded; that is, there is no infinite decreasing sequence $\alpha_1 \succ \alpha_2 \succ ....$ Finally we show for any reduction $\xi \rightsquigarrow \xi'$, that $\xi \succ \xi'$. By the well foundedness of $\succ$, every reduction sequence terminates; the cut reduction rules are strongly normalising.

**Definition 3.5** *The* **recursive path-order** *is defined as follows.*

*Let F be a set of operators, $f, g \in F$. Let T(F) be the set of terms over F and an infinite set of variables, $s, t \in T(F)$. We also write terms as $f(s_1, ...s_n)$, where $f(s_1, ..., s_n)$ is built from operator $f$ applied to terms $s_1, ..., s_n$.*

*Let $>$ be a strict partial order on F. Then $\succ$ is defined recursively on T(F) as follows:*

$$s = f(s_1, ..., s_m) \succ g(t_1, ..., t_n) = t$$

*iff*

$i)$ $s_i \succeq t$ *for some* $i \in \{1, ..., m\}$

**or** $ii)$ $f > g$ *and* $s \succ t_j$ *for every* $j \in \{1, ..., n\}$

**or** $iii)$ $f = g$ *and* $[s_1, ..., s_m] \succ\succ [t_1, ..., t_n]$

We have used the following abbreviations: $\succeq$ for $\succ$ or equivalent up to permutation of subterms; $\succ\succ$ for the extension of $\succ$ to finite multisets.

**Definition 3.6** *A relation $\sqsupset$ on set $K$ (with $\kappa_1, \kappa_2, ... \in K$) is* **well-founded** *iff there is no infinite decreasing sequence $\kappa_1 \sqsupset \kappa_2 \sqsupset ....$*

**Theorem 3.5 (RECURSIVE PATH-ORDER THEOREM)** *If $>$ is well founded, then $\succ$ is well-founded.*

PROOF: See [Der82], [BN98]. ■

## 3.8.2 Strong Normalisation for PFLAX

We apply the recursive path-order to the term assignment system of PFLAX.

The operators are the term constructors of PFLAX; that is, the constructors $;$, $\lambda$, $::$, $[\,]$, $smhl$, $smhr$, together with those for cut. The cut constructors are in fact an infinite family of constructors parameterised by the formulae of Lax Logic, i.e. the constructors are $cut_i^P$ where $P$ ranges over the formulae of Lax Logic.

$$Op = \{cut_i^P \mid i \in \{1,2,3,4\},\ P \text{ a formula}\} \cup \{;\,,\lambda, ::, [\,], smhl, smhr\}$$

The terms over $Op$ contain the proof terms of PFLAX$^{cut}$.

If we write $f(s_1, ..., s_n)$, $f$ is the top term constructor and $s_1, ..., s_n$ are the immediate subterms.

We define a strict partial order on term constructors:

- if $P$ and $Q$ are formulae then $P > Q$ if $Q$ is a subformula of $P$ (i.e. $>$ has the subterm property)
- $cut_i^P > cut_j^Q$ if $P > Q$, $i, j \in \{1,2,3,4\}$
- $cut_4^P, cut_2^P > cut_3^P, cut_1^P$
- we put $cut_1^P = cut_3^P$ and $cut_2^P = cut_4^P$ (so in fact we have two cut operators $cut_H$ and $cut_M$)
- $cut_i^P \quad > \quad ;\,,\lambda, ::, [\,], smhl, smhr$
- $;\,,\lambda, ::, [\,], smhl, smhr$ are ordered equally.

**Lemma 3.5** *The ordering $>$ on Op is transitive, irreflexive and well-founded.*

PROOF: Transitivity and irreflexivity obvious.

We have an infinite number of term constructors, so it is possible that we could have an infinite decreasing sequence:

$$cut_{i_1}^P > cut_{i_2}^Q > ....$$

As either the cut suffix or the size of the cut formula must decrease, the length of the sequence is bounded (by twice the size of $P$). ■

**Corollary 3.2** *$\succ$ is well founded for the terms of* PFLAX.

PROOF: By the recursive path-ordering theorem. ■

We also need the following lemma.

**Lemma 3.6** *For each cut reduction $\alpha \rightsquigarrow \alpha'$, $\alpha \succ \alpha'$ holds.*

PROOF: We analyse each of the fourteen cases. In each case we give an argument that for every pair of terms of the form involved, the relation holds.

Case 1.

$cut_1^P([\,], Ms) \succ Ms$

since $Ms \succeq Ms$

Case 2.

$cut_1^P((M :: Ms_1), Ms_2) \succ (M :: cut_1^P(Ms_1, Ms_2))$

since $cut_1^P >::$ and

$cut_1^P((M :: Ms_1), Ms_2) \succ M$

since $(M :: Ms_1) \succeq M$

$cut_1^P((M :: Ms_1), Ms_2) \succ cut_1^P(Ms_1, Ms_2)$

since $cut_1^P = cut_1^P$ and

$[(M :: Ms_1), Ms_2] \succ\!\!\succ [Ms_1, Ms_2]$

Case 3.

$cut_1^{\circ P}(smhl(x.M), Ms) \succ smhl(x.cut_3^{\circ P}(M, Ms))$

since $cut_1^{\circ P} > smhl$ and

$cut_1^{\circ P}(smhl(x.M), Ms) \succ cut_3^{\circ P}(M, Ms)$

since $cut_1^{\circ P} = cut_3^{\circ P}$ and

$[smhl(x.M), Ms] \succ\!\!\succ [M, Ms]$

Case 4.

$cut_2^P(M, x.[\,]) \succ [\,]$

since $[\,] \succeq [\,]$

Case 5.

$cut_2^P(M_1, x.((M_2 :: Ms))) \succ ((cut_4^P(M_1, x.M_2)) :: cut_2^P(M_1, x.Ms))$

since $cut_2^P >::$ and

$cut_2^P(M_1, x.(M_2 :: Ms)) \succ cut_4^P(M_1, x.M_2)$

since $cut_2^P = cut_4^P$ and

$[M_1, (M_2 :: Ms)] \succ\!\succ [M_1, M_2]$

$cut_2^P(M_1, x.(M_2 :: Ms)) \succ cut_2^P(M_1, x.Ms)$

since $cut_2^P = cut_2^P$ and

$[M_1, (M_2 :: Ms)] \succ\!\succ [M_1, M_2]$

Case 6.

$cut_2^P(M_1, x_1.smhl(x_2.M_2)) \succ smhl(x_2.cut_4^P(M_1, x_1.M_2))$

since $cut_2^P > smhl$ and

$cut_2^P(M_1, x_1.smhl(x_2.M_2)) \succ cut_4^P(M_1, x_1.M_2)$

since $cut_2^P = cut_4^P$ and

$[M_1, smhl(x_2.M_2)] \succ\!\succ [M_1, M_2]$

Case 7.

$cut_3^P((x; Ms_1), Ms_2) \succ (x; cut_1^P(Ms_1, Ms_2))$

since $cut_3^P >;$ and

$cut_3^P((x; Ms_1), Ms_2) \succ cut_1^P(Ms_1, Ms_2)$

since $cut_3^P = cut_1^P$ and

$[(x; Ms_1), Ms_2] \succ\!\succ [Ms_1, Ms_2]$

Case 8.

$cut_3^{P\supset Q}(\lambda x.M_1, (M_2 :: Ms)) \succ cut_3^Q(cut_4^P(M_2, x.M_1), Ms)$

since $cut_3^{P\supset Q} > cut_3^Q$ and

$cut_3^{P\supset Q}(\lambda x.M_1, (M_2 :: Ms)) \succ cut_4^P(M_2, x.M_1)$

since $cut_3^{P\supset Q} > cut_4^P$ and

$cut_3^{P\supset Q}(\lambda x.M_1, (M_2 :: Ms)) \succ M_2$

since $(M_2 :: Ms) \succeq M_2$

$cut_3^{P \supset Q}(\lambda x.M_1, (M_2 :: Ms)) \succ M_1$

since $\lambda x.M_1 \succeq M_1$

$cut_3^{P \supset Q}(\lambda x.M_1, (M_2 :: Ms)) \succ Ms$

since $(M_2 :: Ms) \succeq Ms$

Case 9.

$cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \succ cut_4^P(M_1, x.M_2)$

since $cut_3^{\circ P} > cut_4^P$ and

$cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \succ M_1$

since $smhr(M_1) \succeq M_1$

$cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \succ M_2$

since $smhl(x.M_2) \succeq M_2$

Case 10.

$cut_3^P(M, [\,]) \succ M$

since $M \succeq M$

Case 11.

$cut_4^P(M, x_1.(x_2; Ms)) \succ (x_2; cut_2^P(M, x_1.Ms))$

since $cut_4^P >;$ and

$cut_4^P(M, x_1.(x_2; Ms)) \succ cut_2^P(M, x_1.Ms)$

since $cut_4^P = cut_2^P$ and

$[M, (x_2; Ms)] \succ\!\succ [M, Ms]$

Case 12.

$cut_2^P(M, x.(x; Ms)) \succ cut_3^P(M, cut_2^P(M, x.Ms))$

since $cut_2^P > cut_3^P$ and

$cut_2^P(M, x.(x; Ms)) \succ M$

since $M \succeq M$

$cut_2^P(M, x.(x; Ms)) \succ cut_2^P(M, x.Ms)$

since $cut_2^P = cut_2^P$ and

$[M, (x; Ms)] \succ\!\succ [M, Ms]$

Case 13.

$cut_4^P(M_1, x_1.\lambda x_2.M_2) \succ \lambda x_2.cut_4^P(M_1, x_2.M_2)$

since $cut_4^P > \lambda$ and

$cut_4^P(M_1, x_1.\lambda x_2.M_2) \succ cut_4^P(M_1, x_1.M_2)$

since $cut_4^P = cut_4^P$ and

$[M_1, \lambda x_2.M_2] \succ\!\!\succ [M_1, M_2]$

Case 14.

$cut_4^P(M_1, x.smhr(M_2)) \succ smhr(cut_4^P(M_1, x.M_2))$

since $cut_4^P > smhr$ and

$cut_4^P(M_1, x.smhr(M_2)) \succ cut_4^P(M_1, x.M_2)$

since $cut_4^P = cut_4^P$ and

$[M_1, smhr(M_2)] \succ\!\!\succ [M_1, M_2]$

■

**Theorem 3.6** *The cut reduction system for* PFLAX *strongly normalises.*

PROOF: Immediate from Corollary 3.2, Lemma 3.4, Lemma 3.5 and Theorem 3.5.
■

## 3.9 Lax Logic and Constraint Logic Programming

In [FMW97] and [Wal97], quantified Lax Logic is used to give a logical analysis of constraint logic programming. Lax Logic is used to separate the logical analysis of provability and the analysis of constraints. Here we summarise their approach.

Constraint logic programs consist of clauses, CLP clauses, which are closed formulae of the form:

$$\forall x_1 ... x_n . S \supset H$$

where $H$ is an atom $A(x_1, ..., x_n)$ and $S$ is a formula according to the following grammar:

**S**::=

$$\top \mid A \mid S \vee S \mid S \wedge S \mid \exists V.S$$

These clauses can contain constraints. An example of a constraint logic program clause is

$$\forall s.s \geq 5 \supset A(s)$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top_{\mathcal{I}}) \qquad \frac{}{\Gamma \Rightarrow \circ\top}\ (\circ\top_{\mathcal{I}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \Rightarrow \circ P \quad \Gamma \Rightarrow \circ Q}{\Gamma \Rightarrow \circ(P \wedge Q)}\ (\circ\wedge_{\mathcal{I}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \Rightarrow \circ P}{\Gamma \Rightarrow \circ(P \vee Q)}\ (\circ\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \Rightarrow \circ Q}{\Gamma \Rightarrow \circ(P \vee Q)}\ (\circ\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \Rightarrow P[t/x]}{\Gamma \Rightarrow \exists x.P}\ (\exists_{\mathcal{I}}) \qquad \frac{\Gamma \Rightarrow \circ P[t/x]}{\Gamma \Rightarrow \circ\exists x.P}\ (\circ\exists_{\mathcal{I}})$$

$$\frac{\Gamma, P \supset \circ A \Rightarrow \circ P}{\Gamma, P \supset \circ A \Rightarrow \circ A}\ (\supset \circ_{\varepsilon})$$

$$\frac{\Gamma, P \supset A \Rightarrow P}{\Gamma, P \supset A \Rightarrow A}\ (\supset_{\varepsilon_1})$$

$$\frac{\Gamma, P \supset A \Rightarrow \circ P}{\Gamma, P \supset A \Rightarrow \circ A}\ (\supset \circ_{\varepsilon_2})$$

**Figure 3.9:** Proof search calculus for LLP

Queries (goal formulae) are also formulae of **S**. Queries contain no constraints.

Lax Logic is used to separate the constraints from the logical parts of the programs. This is done by a simple procedure: replace all occurrences of constraints in $S$ by $\top$ and modalise the head. For example:

$$\forall s.s \geq 5 \supset A(s) \quad \text{becomes} \quad \forall s.\top \supset \circ A(s)$$

The constraint can be encoded as a special kind of lambda term.

The result of this abstraction is called a Lax Logic program clause (LLP clause). These have the form:

$$\forall x_1 ... x_n.S \supset \circ H$$

where $S$ and $H$ are as for constraint logic program clauses (except that no constraints are allowed in $S$). Note the constraint program clauses and Lax Logic program clauses are part of the same logic (quantified Lax Logic) and so programs with LLP clauses and constraint-free CLP clauses can be reasoned about together.

If we want to answer a query $Q$ from a program containing LLP clauses, then we try to prove formula $\circ Q$, meaning that $Q$ is proved up to the satisfaction of some, as yet unspecified, constraints. This is done using the natural deduction calculus given in Figure 3.9.

For any query, we get one or many proofs from the program by using the LLP calculus. This gives us different solutions up to the satisfaction of constraints. What these constraints are differs for each proof. Using the proof term system for the

LLP calculus, together with the lambda term (in a different system) encoding the abstracted constraints, the actual constraints to be satisfied can be calculated and then solved using suitable machinery.

For every query, we are interested in the proofs of this query. As has been discussed in the introduction, permutation-free calculi, such as PFLAX, are peculiarly well suited for the enumeration of all proofs. PFLAX has an advantage over the LLP calculus given, in that it generates exactly the normal forms of proofs, whereas the LLP calculus involves transformation of proof terms to normal form. The drawback is that PFLAX, even for the fragment of Lax Logic used for constraint logic programming, does not allow goal directed proof search. However, despite there being no obvious correspondence between LLP and PFLAX, we consider PFLAX to be a suitable calculus for proof search in the context of constraint logic programming.

## 3.10 Conclusion

In this chapter we have presented a Gentzen system for Lax Logic whose proofs naturally correspond in a 1–1 way to the normal natural deductions. This calculus is syntax-directed and hence suitable for proof enumeration. The search space for PFLAX is smaller than that for the usual Lax Logic sequent calculus. In the following chapter this calculus is used as the basis for a theorem proving calculus.

Lax Logic gives a proof-theoretic approach to constraint logic programming (see [FMW97], [Wal97]). The modality can be used to abstract away the constraints, separating the logical and constraint parts of the analysis. Lax Logic is used to prove the modal formula. Permutation-free calculi are natural extensions to logic programming thought of as backward proof search on hereditary Harrop formulae; the work in this chapter provides an extension of the setting for constraint logic programming founded upon Lax Logic. PFLAX is also useful since the proofs it generates are in normal form, unlike the LLP calculus for constraint logic programming.

# Chapter 4

# Loop-Checking Using a History Mechanism

This chapter is an investigation of one technique for propositional theorem proving – the use of a 'history' to prevent looping. We develop a new history mechanism and apply it to several calculi, utilising work from the first three chapters of this thesis. The resulting calculi with loop checking are proved to be sound and complete. Although it seems intuitively obvious that the history calculi are complete, the proofs of this are surprisingly involved.

Backward proof search in the usual formulations of many non-classical propositional sequent calculi is non-terminating. Backward application of the rules can easily produce the same sequent again and again. A simple example in the G3 calculus for propositional intuitionistic logic is (with $A$ atomic):

$$\dfrac{\dfrac{\overset{\vdots}{(A\wedge A)\supset A \Rightarrow A} \quad \overset{\vdots}{(A\wedge A)\supset A\Rightarrow A}}{(A\wedge A)\supset A\Rightarrow A\wedge A}\,(\wedge_{\mathcal{R}}) \qquad \dfrac{}{A,(A\wedge A)\supset A\Rightarrow A}\,(ax)}{(A\wedge A)\supset A\Rightarrow A}\,(\supset_{\mathcal{L}})$$

Here the sequent $(A \wedge A) \supset A \Rightarrow A$ may continue to occur in the proof tree for this sequent.

There are several approaches to decision procedures for logics whose usual sequent formulations are not decision procedures themselves. One can attempt to find a sequent formulation of the logic that terminates when used for backward proof search. An example of this is the contraction-free calculus G4 for propositional intuitionistic logic, originating with Vorob'ev ([Vor52], [Vor58]), and rediscovered and expounded by Dyckhoff ([Dyc92]) and by Hudelmaier ([Hud93]). These contraction-free calculi are not easily discovered (indeed may not be possible), and so other methods can be useful. Another approach is to place conditions on the sequent calculus to ensure termination of search. It is elegant to be able to build the content of these conditions into the sequent calculus itself. This is how we develop calculi for theorem proving in this section. The technique for doing this is quite general and can be applied to many calculi. We apply it to the intuitionistic sequent calculi G3 and MJ, as well as to some modal logics: S4, Lax Logic and intuitionistic S4.

## 4.1 History Mechanisms

In order to ensure termination of backward proof search, we need to check that the same sequent (modulo number of occurrences of formulae of the same type) does not appear again on a branch. In the example above we easily see that there is a loop: we need a mechanical way to detect such loops.

One way to do this is to add a *history* to a sequent. The history is the set of all sequents to have occurred so far on a branch of a proof tree. After each backwards inference the new sequent (without its history) is checked to see whether it is a member of this set. If it is we have looping and backtrack. If not the new history is the extension of the old history by the old sequent (without the history component), and we try to prove the new sequent, and so on. Unfortunately, this method is space inefficient as it requires long lists of sequents to be stored by the computer, and all of this list has to be checked at each stage. When the sequents are stored, far more information than necessary is kept. Efficiency would be improved by cutting down the amount of storage and checking to the bare minimum needed to prevent looping.

The basis of the reduced history is the realisation (as in [HSZ96]) that one need only store goal formulae in order to loop-check. The contexts of the sequents in this section are multisets rather than sets of labelled formulae. For most of the calculi dealt with in this chapter, the context cannot decrease; once a formula is in the context it will be in the context of all sequents above it in the proof tree. We say that the calculus has *increasing context*. For two sequents to be the same they need to have the same context (up to multiple occurrences of formulae of the same type). Therefore we may empty the history every time the context is (properly) extended. All we need store in the history are goal formulae. If we have a sequent whose goal is already in the history, then we have the same goal and the same context as another sequent, that is, a loop.

We describe two slightly different approaches to doing this. There is the straightforward extension of the calculus described in [HSZ96] (which we call the 'Swiss history'; more on this loop-checking method can be found in [Heu98]). There is also related work on histories for intuitionistic logic by Gabbay in [Gab91]. The other approach involves storing slightly more formulae in the history, but which for some calculi detects loops more quickly. This we describe as the 'Scottish history' ([How96], [How97]); it can in many cases be more efficient than the Swiss method.

$$\frac{}{\Gamma, P \Rightarrow P; \mathcal{H}}\ (ax) \qquad \frac{}{\Gamma, - \Rightarrow P; \mathcal{H}}\ (-)$$

$$\frac{\Gamma, P \Rightarrow Q; \phi}{\Gamma \Rightarrow P \supset Q; \mathcal{H}}\ (\supset_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma \qquad \frac{\Gamma \Rightarrow Q; \mathcal{H}}{\Gamma \Rightarrow P \supset Q; \mathcal{H}}\ (\supset_{\mathcal{R}2}) \quad \text{if } P \in \Gamma$$

$$\frac{\Gamma, P \Rightarrow -; \phi}{\Gamma \Rightarrow \neg P; \mathcal{H}}\ (\neg_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma \qquad \frac{\Gamma \Rightarrow -; \mathcal{H}}{\Gamma \Rightarrow \neg P; \mathcal{H}}\ (\neg_{\mathcal{R}2}) \quad \text{if } P \in \Gamma$$

$$\frac{\Gamma, P \supset Q \Rightarrow P; (D, \mathcal{H}) \quad \Gamma, P \supset Q, Q \Rightarrow D; \phi}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}}\ (\supset_{\mathcal{L}}) \quad \text{if } D \notin \mathcal{H}, \text{ and } Q \notin \Gamma$$

$$\frac{\Gamma, \neg P \Rightarrow P; (D, \mathcal{H})}{\Gamma, \neg P \Rightarrow D; \mathcal{H}}\ (\neg_{\mathcal{L}}) \quad \text{if } D \notin \mathcal{H}, \text{ and } - \notin \Gamma$$

$$\frac{\Gamma \Rightarrow P; \mathcal{H} \quad \Gamma \Rightarrow Q; \mathcal{H}}{\Gamma \Rightarrow P \wedge Q; \mathcal{H}}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma, P \wedge Q, P \Rightarrow D; \phi}{\Gamma, P \wedge Q \Rightarrow D; \mathcal{H}}\ (\wedge_{\mathcal{L}_1}) \quad \text{if } P \notin \Gamma \qquad \frac{\Gamma, P \wedge Q, Q \Rightarrow D; \phi}{\Gamma, P \wedge Q \Rightarrow D; \mathcal{H}}\ (\wedge_{\mathcal{L}_2}) \quad \text{if } Q \notin \Gamma$$

$$\frac{\Gamma \Rightarrow P; \mathcal{H}}{\Gamma \Rightarrow P \vee Q; \mathcal{H}}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q; \mathcal{H}}{\Gamma \Rightarrow P \vee Q; \mathcal{H}}\ (\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \vee Q, P \Rightarrow D; \phi \quad \Gamma, P \vee Q, Q \Rightarrow D; \phi}{\Gamma, P \vee Q \Rightarrow D; \mathcal{H}}\ (\vee_{\mathcal{L}}) \quad \text{if } P, Q \notin \Gamma$$

$D$ is either an atom, $-$ or a disjunction.
When the history has been extended we have parenthesised $(D, \mathcal{H})$ for emphasis.

**Figure 4.1:** G3$^{Hist}$ in the Swiss-style

### 4.1.1 The Swiss History

In this section we describe the application of the Swiss history to the G3 calculus for propositional intuitionistic logic. We should first point out that the calculus we describe as Swiss is different from the one in [HSZ96]. We are trying to focus on the history mechanism and hence have not included the subsumption checks that the calculus in [HSZ96] uses. It has also been extended to cover disjunction.

The Swiss-style calculus G3$^{Hist}$ is displayed in Figure 4.1. Let us make some general points about it (which will apply to the Scottish G3$^{Hist}$ too). We give explicit rules for negation (which are just special cases of the rules for implication) for the sake of completeness of connectives. There are two rules for $(\supset_{\mathcal{R}})$. These correspond to the two cases where the new formula, $P$, is or is not in the context. As noted above, this is very important for history mechanism. Also notice that the number of formulae in the history is at most equal to the length of the formula we check for provability.

The loop checking works in a similar way to that of $\mathsf{IPC}^{RP}{}_{\wedge,\rightarrow}{}^{SU}$ in [HSZ96]. A sequent is matched against the conclusions of right rules until the goal formula is either a propositional variable, falsum, or a disjunction (note that disjunction is not covered in [HSZ96], and requires special treatment). This has been ensured by the restriction on goal formulae given in the calculus (although the calculus would still be terminating without this restriction, it gives a much more efficient implementation). A formula from the context is then picked and matched against the left rules of the calculus. The history mechanism applies to prevent looping in the $(\supset_{\mathcal{L}})$ rule (and similarly in the $(\neg_{\mathcal{L}})$ rule). The left premiss of the rule has the same context as the conclusion, but the goal is, in general, different. If the goal, $D$, of the conclusion is not in the history, $\mathcal{H}$, we store $D$ in the history and continue backward proof search on the left premiss. Alternatively, $D$ might already be in $\mathcal{H}$. In this case there is a loop, and so this branch is not pursued. We backtrack and look for a proof in a different way.

There are other places where the rules are restricted to prevent looping. The left rules have side conditions to ensure that the context is increasing. For the $(\supset_{\mathcal{R}})$ rule (which attempts to extend the context) there are two cases corresponding to when the context is and when it is not extended. Something similar is happening in the left rules. Take $(\vee_{\mathcal{L}})$ as an example. In both premisses of the rule a formula may be added to context. If both contexts really are extended, then we can continue building the proof tree. If one or both contexts are not extended then the sequent, $S$, with the non-extended context, will be the same as some sequent at a lesser height in the proof tree – there is a loop (which we describe as a trivial loop). This is easy to see: since the context and the goal of $S$ are the same as that of the conclusion, the conclusion is the same as the premiss $S$.

What does a history sequent say? What, in logical terms, is the meaning of a sequent with a history field? Take, for example, the G3$^{Hist}$ sequent $S = \Gamma \Rightarrow R; \mathcal{H}$. This says that for every proof of $S$, if $P \in \mathcal{H}$, then no sequent of the form $\Gamma \Rightarrow P; \mathcal{H}'$ appears in the proof tree of $S$.

We now prove the equivalence theorems. This is done in two stages. First we prove the equivalence of G3 and G3 with goals of the left rules restricted to atoms, $-$ and disjunctions (a calculus we shall refer to as G3$^D$). Then we prove the equivalence of G3$^D$ and G3$^{Hist}$ .

**Definition 4.1** *The* **size of a proof tree** *is equal to the number of nodes in it.*

We need the following lemmas:

**Lemma 4.1 (**WEAKENING) *The following rule is admissible in both* G3 *and* G3$^D$*:*

$$\frac{\Gamma \Rightarrow R}{\Gamma, P \Rightarrow R}\ (W)$$

PROOF: By induction on the height of the derivation of the premiss. ■

**Lemma 4.2** *If sequent $S$ is provable in* G3*, then $S$ is provable in* G3 *with the axioms and the goal of $(-)$ restricted to atomic formulae (and $-$).*

PROOF: By breaking up the axiom formulae and induction on the size of the goal. ■

We prove the following result using the permutation properties of the calculus as studied in Chapter 2 (see Table 2.1).

**Proposition 4.1** *The calculi* G3 *and* G3$^D$ *are equivalent. That is, a sequent is provable in* G3 *iff it is provable in* G3$^D$.

PROOF: It is trivial that if sequent $S$ is provable in G3$^D$ then it is provable in G3. We show the converse.

We show that if the goal is an implication or a conjunction, and the next inference is $(\vee_\mathcal{L})$ and in both premisses the goal is principal, then the rules permute. i.e.

$$\dfrac{\dfrac{\Gamma, S \vee T, S, P \Rightarrow Q}{\Gamma, S \vee T, S \Rightarrow P \supset Q}\,(\supset_\mathcal{R}) \quad \dfrac{\Gamma, S \vee T, T, P \Rightarrow Q}{\Gamma, S \vee T, T \Rightarrow P \supset Q}\,(\supset_\mathcal{R})}{\Gamma, S \vee T \Rightarrow P \supset Q}\,(\vee_\mathcal{L})$$

permutes to:

$$\dfrac{\dfrac{\Gamma, S \vee T, S, P \Rightarrow Q \quad \Gamma, S \vee T, T, P \Rightarrow Q}{\Gamma, S \vee T, P \Rightarrow Q}\,(\vee_\mathcal{L})}{\Gamma, S \vee T \Rightarrow P \supset Q}\,(\supset_\mathcal{R})$$

and (where $\Gamma' = \Gamma, S \vee T$)

$$\dfrac{\dfrac{\Gamma', S \Rightarrow P \quad \Gamma', S \Rightarrow Q}{\Gamma', S \Rightarrow P \wedge Q}\,(\wedge_\mathcal{R}) \quad \dfrac{\Gamma', T \Rightarrow P \quad \Gamma', T \Rightarrow Q}{\Gamma', T \Rightarrow P \wedge Q}\,(\wedge_\mathcal{R})}{\Gamma, S \vee T \Rightarrow P \wedge Q}\,(\vee_\mathcal{L})$$

permutes to:

$$\dfrac{\dfrac{\Gamma', S \Rightarrow P \quad \Gamma', T \Rightarrow P}{\Gamma' \Rightarrow P}\,(\vee_\mathcal{L}) \quad \dfrac{\Gamma', S \Rightarrow Q \quad \Gamma', T \Rightarrow Q}{\Gamma' \Rightarrow Q}\,(\vee_\mathcal{L})}{\Gamma, S \vee T \Rightarrow P \wedge Q}\,(\wedge_\mathcal{R})$$

We proceed by induction on the height of derivations.

Consider a G3 inference which is not a G3$^D$ inference. This must be an instance of a left rule with an implicational or conjunctive goal. By the induction hypothesis we have G3$^D$ proofs of the premiss(es). Hence the premisses with implicational or conjunctive goals have these goals as the principal formula. From Table 2.1 and the permutations given above, we see that we can permute these inferences with the left rule we are looking at. The result follows by induction on the size of the goal. ■

**Lemma 4.3 (**CONTRACTION) *The following rule is admissible in* G3$^{Hist}$ *:*

$$\dfrac{\Gamma, P, P \Rightarrow R; \mathcal{H}}{\Gamma, P \Rightarrow R; \mathcal{H}}\,(C)$$

PROOF: By induction on the height of the derivations of the premisses. ■

**Theorem 4.1** *The calculi* G3 *and* G3$^{Hist}$ *are equivalent. That is, sequent* $\Gamma \Rightarrow G$ *is provable in* G3 *iff* $\Gamma \Rightarrow G; \phi$ *is provable in* G3$^{Hist}$.

PROOF: From Proposition 4.1 we know that it is enough to show that G3$^{D}$ is equivalent to G3$^{Hist}$.

It is trivial that any sequent provable in G3$^{Hist}$ is provable in G3$^{D}$. (Simply drop the history part of the sequent and use contraction above instances of $(\supset_{\mathcal{R}2})$). We prove the converse.

Take any proof tree for sequent $S$ in G3$^{D}$. By definition this proof tree is finite. That is, all branches of the tree end with an occurrence of $(ax)$ or $(-)$, with all branches having a finite number of nodes (there is also no infinite branching at any node). Using a proof tree for a sequent $S$ in G3$^{D}$ we construct a proof tree for the sequent $S; \phi$ in G3$^{Hist}$. Essentially we take a G3$^{D}$ proof tree and give a recipe for 'snipping out' the loops: removing the sequents that form the loop. Or, looking at it in another way we shall show that failure due to the history mechanism only occurs when there is a loop.

Take any G3$^{D}$ proof tree with $n > 0$ nodes. We take this proof tree and use the following construction to give a G3$^{Hist}$ proof tree.

The following construction takes a G3$^{D}$ proof tree and builds a G3$^{Hist}$ proof tree from the root up. For simplicity we ignore negation, although this can easily be added. In this construction we use 'hybrid trees'. A hybrid tree is a fragment of G3$^{Hist}$ proof tree with all branches that do not have $(ax)$ or $(-)$ leaves ending with G3$^{D}$ proof trees. These G3$^{D}$ proof trees have roots which can be obtained by backwards application of a G3$^{D}$ rule to the top history sequent (ignoring its history). We analyse each case of a topmost history sequent with non-history premiss(es) resulting from application of rule $(R)$ in the sequent tree.

- The root of the G3$^{D}$ tree. We change (non-history) sequent $S$ to history sequent $S; \phi$.
- $(R)$ is one of $(ax)$, $(-)$, $(\wedge_{\mathcal{R}})$, $(\vee_{\mathcal{R}_1})$, $(\vee_{\mathcal{R}_2})$, i.e. a rule which in G3$^{Hist}$ has no side conditions. The premiss(es) are changed by adding the appropriate history. They become the history sequents obtained by applying (backwards) the G3$^{Hist}$ rule to the original conclusion.

  For example, if the situation we are analysing is:

  $$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q; \mathcal{H}} \; (\wedge_{\mathcal{R}})$$

  then we change this part of the hybrid tree to:

  $$\frac{\Gamma \Rightarrow P; \mathcal{H} \quad \Gamma \Rightarrow Q; \mathcal{H}}{\Gamma \Rightarrow P \wedge Q; \mathcal{H}} \; (\wedge_{\mathcal{R}})$$

We have an extended G3$^{Hist}$ proof tree fragment with G3$^{D}$ proof tree(s) as premiss(es).

- $(R)$ is $(\supset_{\mathcal{R}})$. We simply add a history as appropriate to the version of the rule, depending on the context. We use contraction when the context is not properly extended.

- $(R)$ is $(\wedge_{\mathcal{L}_1})$. If the side conditions of the history rule $(\wedge_{\mathcal{L}_1})$ are satisfied, we simply add the appropriate history to the premiss. Else, we have:

$$\cfrac{\cfrac{\vdots}{\Gamma, P \wedge Q, P \Rightarrow D}\ (inf)}{\Gamma, P \wedge Q \Rightarrow D; \mathcal{H}}\ (\wedge_{\mathcal{L}_1})$$

where $P \in \Gamma$. From the point of view of looping, both the premiss and conclusion are the same. This is a loop that we describe as trivial. The new hybrid tree is simply the old one with the premiss obtained by contraction:

$$\cfrac{\cfrac{\vdots}{\Gamma, P \wedge Q, P \Rightarrow D; \mathcal{H}}\ (inf)}{\Gamma, P \wedge Q \Rightarrow D; \mathcal{H}}\ (C)$$

- $(R)$ is $(\wedge_{\mathcal{L}_2})$ is treated analogously to $(\wedge_{\mathcal{L}_1})$.

- $(R)$ is $(\vee_{\mathcal{L}})$. Similar to $(\wedge_{\mathcal{L}_1})$. If the side conditions are satisfied, then we simply add the appropriate histories. Else, we have:

$$\frac{\Gamma, P \vee Q, P \Rightarrow D \quad \Gamma, P \vee Q, Q \Rightarrow D}{\Gamma, P \vee Q \Rightarrow D; \mathcal{H}}\ (\vee_{\mathcal{L}})$$

if $P \in \Gamma$ then the left premiss and the conclusion are the same and there is a trivial loop. In this case the new hybrid tree is obtained by removing the completed subtree above the right premiss and obtaining the left premiss by contraction:

$$\frac{\Gamma, P \vee Q, P \Rightarrow D; \mathcal{H}}{\Gamma, P \vee Q \Rightarrow D; \mathcal{H}}\ (C)$$

Similarly if $Q \in \Gamma$. (If $P, Q \in \Gamma$, then we have a choice of which branch to remove).

- $(R)$ is $(\supset_{\mathcal{L}})$. If the side conditions are satisfied then we simply add the appropriate histories. Else, we have:

$$\frac{\Gamma, P \supset Q \Rightarrow P \quad \Gamma, P \supset Q, Q \Rightarrow D}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}}\ (\supset_{\mathcal{L}})$$

If $Q \in \Gamma$, then for the purposes of looping the right premiss and the conclusion are the same. The new hybrid tree is obtained by removing the subtree deriving the left premiss, and obtaining conclusion from the right premiss by contraction.

We now need to consider what happens if $D \in \mathcal{H}$. This is where the history mechanism prevents looping. If the history condition is not met, we know that below the conclusion the hybrid tree has the form:

$$\dfrac{\dfrac{\Gamma, P \supset Q \Rightarrow P \quad \Gamma, P \supset Q, Q \Rightarrow D}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}}\,(\supset_{\mathcal{L}}) \atop {\vdots \atop \Gamma, P \supset Q \Rightarrow S; (D, \mathcal{H}')} \qquad \vdots}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}'}\,(\supset_{\mathcal{L}})$$

where $D \in \mathcal{H}$ and $\mathcal{H} \supseteq \mathcal{H}'$. The history is not reset at any point in this fragment.

This can easily be seen to contain the loop which is the reason for the side conditions not being met. The new hybrid tree is obtained by removing from the previous hybrid tree all the sequents from, but not including, the sequent $\Gamma, P \supset Q \Rightarrow D; \mathcal{H}'$ up to and including $\Gamma, P \supset Q \Rightarrow D; \mathcal{H}$. (We may need some contractions). We can now apply (backwards) $(\supset_{\mathcal{L}})$ to the first of these sequents. Either the side conditions will be satisfied, or $Q \in \Gamma$: in either case we know how to proceed.

As has been noted, G3 proof trees are finite and at every stage in this construction, the number of nodes of the hybrid tree without a history strictly decreases. Therefore the construction is terminating. As every branch in the G3 tree ends in an application of $(ax)$ or $(-)$, the history tree we construct is a proof tree. ■

G3$^{Hist}$ (Swiss) is a calculus with a history mechanism for propositional intuitionistic logic. It is sound and complete. We claimed earlier that this calculus gives a decision procedure for propositional intuitionistic logic. We prove that backwards proof search in G3$^{Hist}$ in the Swiss style terminates.

**Theorem 4.2** *Backwards proof search in the Swiss calculus* G3$^{Hist}$ *is terminating.*

PROOF: We associate with every sequent $\Gamma \Rightarrow R; \mathcal{H}$ a triple of natural numbers:

$$W = (k - n, k - m, r)$$

where $k$ is the number of elements in the *set* of subformulae of $(\Gamma, R)$; $n$ is the number of elements in the *set* of elements of $\Gamma$; $m$ is the number of elements in $\mathcal{H}$ and $r$ is the size of goal formula $R$. (Notice that although $\Gamma$ is a multiset, we count its elements as a set). These triples are lexicographically ordered from the left.

By inspection we see that $W$ is lower for the premisses of every inference rule than for the conclusion. Consider as an example, $(\supset_{\mathcal{L}})$:

$$\dfrac{\Gamma, P \supset Q \Rightarrow P; (D, \mathcal{H}) \quad \Gamma, P \supset Q, Q \Rightarrow D; \phi}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}}\,(\supset_{\mathcal{L}}) \quad \text{if } D \notin \mathcal{H} \text{ and } Q \notin \Gamma$$

The conclusion has $W = (k - n, k - m, r)$. The left premiss has $W' = (k' - n, k' - (m+1), r')$ (where $k' \leq k$). Therefore $W' < W$. The right premiss has $W'' = (k - (n+1), k, r)$. Therefore $W'' < W$. The weights of both premisses are less than the conclusion.

Hence backwards proof search is terminating. ■

**Note 4.1** *The proof of Theorem 4.2 makes explicit a theme that can be seem throughout this chapter. We have given the calculi using multisets, in order that they match usual presentations, and the way that they are implemented in Prolog. However, it is often more natural to think of contexts as sets. In particular, the idea of increasing context is one that is based on a view of the context as a set. In fact, as the theory has shown, we can look at multisets, and we only need the set view when we need to map proofs to tuples of natural numbers to get a termination argument as in Theorem 4.2. We could rework the entire section using sets, or we could give a collection of propositions about treating multisets as sets and the effect of this on these calculi. However, we do not do this, instead restricting ourself to this note.*

## 4.2 Scottish History and G3

In this section we discuss the ‘Scottish’ history mechanism as applied to G3. This calculus takes a slightly different approach to the ‘Swiss’ calculus. Again we call the calculus G3$^{Hist}$. The calculus adds to the history at several points, rather than just one (as is the case for the Swiss history) so has to store a larger set. It also checks for looping more often than the Swiss history, so proof trees do not have to be so large. The Scottish calculus G3$^{Hist}$ can be seen in Figure 4.2.

We said earlier that when using a history mechanism to prevent looping it would be good to cut down the amount of storage and checking to a bare minimum. This was done in the Swiss G3$^{Hist}$ – the history mechanism operates in one place and one place only and other restrictions for loop prevention involve no storage. However, it is not clear that this is the best or most attractive approach. There is a tradeoff between these advantages and the obvious disadvantage of not looking for loops very often. We find loops more quickly if we look for them at more points. That is, we might continue building a proof tree needlessly when a loop might have already been spotted. The Scottish G3$^{Hist}$ has larger histories. This allows us to check for loops in more places, and in certain situations this is advantageous.

As in the Swiss history, when attempting to prove a sequent, right rules are applied first, breaking up a formula until it is atomic, falsum or a disjunction, and only then can left rules be applied. Looping due to context extensions is prevented in the same way. The difference between the two calculi is in the way that the history mechanism works.

Whereas the Swiss calculus only places formulae in the history that have been the goal of the conclusions of a $(\supset_{\mathcal{L}})$ (or $(\neg_{\mathcal{L}})$), the Scottish calculus keeps a complete

$$\frac{}{\Gamma, P \Rightarrow P; \mathcal{H}}\ (ax) \qquad \frac{}{\Gamma, - \Rightarrow P; \mathcal{H}}\ (-)$$

$$\frac{\Gamma, P \Rightarrow Q; \{Q\}}{\Gamma \Rightarrow P \supset Q; \mathcal{H}}\ (\supset_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma$$

$$\frac{\Gamma \Rightarrow Q; (Q, \mathcal{H})}{\Gamma \Rightarrow P \supset Q; \mathcal{H}}\ (\supset_{\mathcal{R}2}) \quad \text{if} P \in \Gamma \text{ and } Q \notin \mathcal{H}$$

$$\frac{\Gamma, P \Rightarrow -; \{-\}}{\Gamma \Rightarrow \neg P; \mathcal{H}}\ (\neg_{\mathcal{R}1}) \quad \text{if} P \notin \Gamma$$

$$\frac{\Gamma \Rightarrow -; (-, \mathcal{H})}{\Gamma \Rightarrow \neg P; \mathcal{H}}\ (\neg_{\mathcal{R}2}) \quad \text{if} P \in \Gamma \text{ and } - \notin \mathcal{H}$$

$$\frac{\Gamma, P \supset Q \Rightarrow P; (P, \mathcal{H}) \quad \Gamma, P \supset Q, Q \Rightarrow D; \{D\}}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}}\ (\supset_{\mathcal{L}}) \quad \text{if } P \notin \mathcal{H} \text{ and } Q \notin \Gamma$$

$$\frac{\Gamma, \neg P \Rightarrow P; (P, \mathcal{H})}{\Gamma, \neg P \Rightarrow D; \mathcal{H}}\ (\neg_{\mathcal{L}}) \quad \text{if } P \notin \mathcal{H} \text{ and } - \notin \Gamma$$

$$\frac{\Gamma \Rightarrow P; (P, \mathcal{H}) \quad \Gamma \Rightarrow Q; (Q, \mathcal{H})}{\Gamma \Rightarrow P \wedge Q; \mathcal{H}}\ (\wedge_{\mathcal{R}}) \quad \text{if } P, Q \notin \mathcal{H}$$

$$\frac{\Gamma, P \wedge Q, P \Rightarrow D; \{D\}}{\Gamma, P \wedge Q \Rightarrow D; \mathcal{H}}\ (\wedge_{\mathcal{L}_1}) \quad \text{if } P \notin \Gamma$$

$$\frac{\Gamma, P \wedge Q, Q \Rightarrow D; \{D\}}{\Gamma, P \wedge Q \Rightarrow D; \mathcal{H}}\ (\wedge_{\mathcal{L}_2}) \quad \text{if } Q \notin \Gamma$$

$$\frac{\Gamma \Rightarrow P; (P, \mathcal{H})}{\Gamma \Rightarrow P \vee Q; \mathcal{H}}\ (\vee_{\mathcal{R}_1}) \quad \text{if } P \notin \mathcal{H} \qquad \frac{\Gamma \Rightarrow Q; (Q, \mathcal{H})}{\Gamma \Rightarrow P \vee Q; \mathcal{H}}\ (\vee_{\mathcal{R}_2}) \quad \text{if } Q \notin \mathcal{H}$$

$$\frac{\Gamma, P \vee Q, P \Rightarrow D; \{D\} \quad \Gamma, P \vee Q, Q \Rightarrow D; \{D\}}{\Gamma, P \vee Q \Rightarrow D; \mathcal{H}}\ (\vee_{\mathcal{L}}) \quad \text{if } P, Q \notin \Gamma$$

$D$ is either an atom, $-$ or a disjunction.
When the history has been extended, we have parenthesised $(P, \mathcal{H})$ for emphasis.

**Figure 4.2:** G3$^{Hist}$ in the Scottish style

record of goal formulae between context extensions. At each of the places where the history might be extended, the new goal is checked against the history. If it is in the history, then there is a loop. The heart of the difference between the two calculi is that in the Swiss calculus loop checking is done when a formula leaves the goal, whereas in the Scottish calculus it is done when it becomes the goal.

We prove the same theorems for the Scottish G3$^{Hist}$ as for the Swiss G3$^{Hist}$ . The proofs are very similar to those for the Swiss G3$^{Hist}$ and are omitted because of their length and repetitiveness.

**Theorem 4.3** *The calculi* G3 *and* G3$^{Hist}$ *are equivalent. That is, sequent* $\Gamma \Rightarrow G$ *is provable in* G3 *iff* $\Gamma \Rightarrow G; \{G\}$ *is provable in* G3$^{Hist}$ .

PROOF: Similar to the proof of Theorem 4.1. ■

**Theorem 4.4** *Backwards proof search in the calculus* G3$^{Hist}$ *is terminating.*

PROOF: Similar to the proof of Theorem 4.2. ■

### 4.2.1 Comparison of the Two Calculi

Because of the way that the Swiss history works, loop detection is delayed. Let us illustrate this with an example. Consider the sequent:

$$A, B, (A \supset B \supset C) \supset C \Rightarrow A \supset B \supset C$$

In the Swiss G3$^{Hist}$ (where $\Gamma = A, B, (A \supset B \supset C) \supset C$ and $G = A \supset B \supset C$) we get the following:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow G; \{C\} \quad \dfrac{}{\Gamma, C \Rightarrow C; \phi}\,(ax)}{\Gamma \Rightarrow C; \phi}\,(\supset_{\mathcal{L}})}{\Gamma \Rightarrow B \supset C; \phi}\,(\supset_{\mathcal{R}2})}{\Gamma \Rightarrow G; \phi}\,(\supset_{\mathcal{R}2})$$

We have to go through all the inference steps again (in the branch above the left premiss) before the loop is detected – even though we can clearly see the loop. However, in the Scottish calculus we get:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow G; \{G, C, B \supset C, G\} \quad \dfrac{}{\Gamma, C \Rightarrow C; \{C\}}\,(ax)}{\Gamma \Rightarrow C; \{C, B \supset C, G\}}\,(\supset_{\mathcal{L}})}{\Gamma \Rightarrow B \supset C; \{B \supset C, G\}}\,(\supset_{\mathcal{R}2})}{\Gamma \Rightarrow G; \{G\}}\,(\supset_{\mathcal{R}2})$$

The topmost inference, $(\supset_{\mathcal{L}})$, is not valid, since the left premiss has goal formula, $G$, which is already in the history. That is, the loop is detected, and is detected lower in the proof tree than in the Swiss style calculus.

Spotting the loop as it occurs is not only theoretically more attractive, but could also prevent a lot of costly extra computation.

The two calculi both have their good points. The Swiss calculus is efficient from the point of view that its history mechanism requires little storage and checking. The Scottish calculus is efficient in that it detects loops as they occur, avoiding unnecessary computation. The Swiss calculus needs less space for each sequent, but more for the entire proof tree.

The question is whether or not in general an overhead in storage and checking of the history (which should not be too great due to regular resetting) is preferable to the larger proof trees which are the result of delaying checking. The approach we take to this question is to look at empirical results in the form of timings for theorem proving in implementations of the calculi. Note that as the two calculi are rather similar it is more than likely that any optimisation that can be applied to one can also be applied to the other.

Results for the implementations of the G3$^{Hist}$ calculi can be found in section 4.4.

## 4.3 Histories and MJ

So far we have used the history mechanisms with G3 to give decision procedures for intuitionistic logic. We can, however, improve on these decision procedures by using a different base calculus. The calculus MJ has all the features (such as increasing context) which make it suitable for the history mechanisms to be applied to. MJ has fewer derivations than G3 and has focusing, therefore when searching for a proof, there are fewer possible proofs to check on backtracking. Hence the decision as to whether or not a formula is provable in intuitionistic logic ought to be made quicker. This is the approach taken in [How96], [How97]. The calculi MJ$^{Hist}$ in the Swiss style and MJ$^{Hist}$ in the Scottish style can be seen in Figures 4.3 and 4.4 respectively.

We can prove similar theorems for MJ$^{Hist}$ as for G3$^{Hist}$. The proofs are similar; some of these proofs can be found in detail in [How96].

**Proposition 4.2** *The calculi* MJ *and* MJ$^D$ *(*MJ *with the goal of* $(C)$ *restricted to atoms, falsum or disjunctions) are equivalent. That is, sequent* $S$ *is provable in* MJ *iff it is provable in* MJ$^D$*.*

PROOF: Similar to proof of Proposition 4.1. ■

**Theorem 4.5** *The Swiss calculus* MJ$^{Hist}$ *is equivalent to* MJ*.*

PROOF: Similar to proof of Theorem 4.1. ■

**Theorem 4.6** *Backwards proof search in the Swiss* MJ$^{Hist}$ *is terminating.*

PROOF: Similar to proof of Theorem 4.2. ■

$$\frac{}{\Gamma \xrightarrow{D} D;\mathcal{H}}\ (ax) \qquad \frac{}{\Gamma \xrightarrow{\perp} D;\mathcal{H}}\ (-) \qquad \frac{\Gamma, P \xrightarrow{P} D;\mathcal{H}}{\Gamma, P \Rightarrow D;\mathcal{H}}\ (C)$$

$$\frac{\Gamma, P \Rightarrow Q;\phi}{\Gamma \Rightarrow P \supset Q;\mathcal{H}}\ (\supset_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma \qquad \frac{\Gamma \Rightarrow Q;\mathcal{H}}{\Gamma \Rightarrow P \supset Q;\mathcal{H}}\ (\supset_{\mathcal{R}2}) \quad \text{if } P \in \Gamma$$

$$\frac{\Gamma, P \Rightarrow -;\phi}{\Gamma \Rightarrow \neg P;\mathcal{H}}\ (\neg_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma \qquad \frac{\Gamma \Rightarrow -;\mathcal{H}}{\Gamma \Rightarrow \neg P;\mathcal{H}}\ (\neg_{\mathcal{R}2}) \quad \text{if } P \in \Gamma$$

$$\frac{\Gamma \Rightarrow P;(D,\mathcal{H}) \quad \Gamma \xrightarrow{Q} D;\mathcal{H}}{\Gamma \xrightarrow{P \supset Q} D;\mathcal{H}}\ (\supset_{\mathcal{L}}) \quad \text{if } D \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P;(D,\mathcal{H})}{\Gamma \xrightarrow{\neg P} D;\mathcal{H}}\ (\neg_{\mathcal{L}}) \quad \text{if } D \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P;\mathcal{H} \quad \Gamma \Rightarrow Q;\mathcal{H}}{\Gamma \Rightarrow P \wedge Q;\mathcal{H}}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma \xrightarrow{P} D;\mathcal{H}}{\Gamma \xrightarrow{P \wedge Q} D;\mathcal{H}}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} D;\mathcal{H}}{\Gamma \xrightarrow{P \wedge Q} D;\mathcal{H}}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P;\mathcal{H}}{\Gamma \Rightarrow P \vee Q;\mathcal{H}}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q;\mathcal{H}}{\Gamma \Rightarrow P \vee Q;\mathcal{H}}\ (\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \Rightarrow D;\phi \quad \Gamma, Q \Rightarrow D;\phi}{\Gamma \xrightarrow{P \vee Q} D;\mathcal{H}}\ (\vee_{\mathcal{L}}) \quad \text{if } P \notin \Gamma \text{ and } Q \notin \Gamma$$

$D$ is either a propositional variable, $-$ or a disjunction.
When the history has been extended we have parenthesised $(D,\mathcal{H})$ for emphasis.

**Figure 4.3:** The Calculus MJ$^{Hist}$ in the Swiss style.

$$\frac{}{\Gamma \xrightarrow{D} D;\mathcal{H}}\,(ax) \qquad \frac{}{\Gamma \xrightarrow{\perp} D;\mathcal{H}}\,(-) \qquad \frac{\Gamma, P \xrightarrow{P} D;\mathcal{H}}{\Gamma, P \Rightarrow D;\mathcal{H}}\,(C)$$

$$\frac{\Gamma, P \Rightarrow Q; \{Q\}}{\Gamma \Rightarrow P \supset Q;\mathcal{H}}\,(\supset_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma$$

$$\frac{\Gamma \Rightarrow Q; (Q,\mathcal{H})}{\Gamma \Rightarrow P \supset Q;\mathcal{H}}\,(\supset_{\mathcal{R}2}) \quad \text{if } P \in \Gamma \text{ and } Q \notin \mathcal{H}$$

$$\frac{\Gamma, P \Rightarrow -; \{-\}}{\Gamma \Rightarrow \neg P;\mathcal{H}}\,(\neg_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma$$

$$\frac{\Gamma \Rightarrow -; (-,\mathcal{H})}{\Gamma \Rightarrow \neg P;\mathcal{H}}\,(\neg_{\mathcal{R}2}) \quad \text{if } P \in \Gamma \text{ and } - \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P; (P,\mathcal{H}) \quad \Gamma \xrightarrow{Q} D;\mathcal{H}}{\Gamma \xrightarrow{P \supset Q} D;\mathcal{H}}\,(\supset_{\mathcal{L}}) \quad \text{if } P \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P; (P,\mathcal{H})}{\Gamma \xrightarrow{\neg P} D;\mathcal{H}}\,(\neg_{\mathcal{L}}) \quad \text{if } P \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P; (P,\mathcal{H}) \quad \Gamma \Rightarrow Q; (Q,\mathcal{H})}{\Gamma \Rightarrow P \wedge Q;\mathcal{H}}\,(\wedge_{\mathcal{R}}) \quad \text{if } P \notin \mathcal{H} \text{ and } Q \notin \mathcal{H}$$

$$\frac{\Gamma \xrightarrow{P} D;\mathcal{H}}{\Gamma \xrightarrow{P \wedge Q} D;\mathcal{H}}\,(\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} D;\mathcal{H}}{\Gamma \xrightarrow{P \wedge Q} D;\mathcal{H}}\,(\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P; (P,\mathcal{H})}{\Gamma \Rightarrow P \vee Q;\mathcal{H}}\,(\vee_{\mathcal{R}_1}) \quad \text{if } P \notin \mathcal{H} \qquad \frac{\Gamma \Rightarrow Q; (Q,\mathcal{H})}{\Gamma \Rightarrow P \vee Q;\mathcal{H}}\,(\vee_{\mathcal{R}_2}) \quad \text{if } Q \notin \mathcal{H}$$

$$\frac{\Gamma, P \Rightarrow D; \{D\} \quad \Gamma, Q \Rightarrow D; \{D\}}{\Gamma \xrightarrow{P \vee Q} D;\mathcal{H}}\,(\vee_{\mathcal{L}}) \quad \text{if } P \notin \Gamma \text{ and } Q \notin \Gamma$$

$D$ is either a propositional variable, $-$ or a disjunction.
Where the history has been extended we have parenthesised $(P,\mathcal{H})$ for emphasis.

**Figure 4.4:** The Calculus MJ$^{Hist}$ in the Scottish style

**Theorem 4.7** *The Scottish calculus* $\mathrm{MJ}^{Hist}$ *is equivalent to* MJ.

PROOF: Similar to proof of Theorem 4.1. ■

**Theorem 4.8** *Backwards proof search in the Scottish* $\mathrm{MJ}^{Hist}$ *is terminating.*

PROOF: Similar to proof of Theorem 4.2. ■

### 4.3.1 Propositional Theorem Proving

We have described four calculi that are decision procedures for propositional intuitionistic logic. Another calculus which is a decision procedure for propositional intuitionistic logic is the contraction-free calculus G4. (This calculus can be found in the appendix).

We have already given a discussion of why we think that the Scottish history applied to propositional intuitionistic logic is theoretically more attractive than the Swiss history. We also said that we would like to compare implementations of the calculi to add experimental evidence to the theoretical argument. We also compare with an implementation of G4.

The calculi were all naïvely implemented in Prolog. By a naïve implementation we mean one that follows as closely as possible the unintelligent searching through the proof trees as generated by the sequent calculi presented. We describe this for $\mathrm{MJ}^{Hist}$.

Our implementations of the calculi are syntax directed. A sequent $\Gamma \Rightarrow P; \phi$ (for the Swiss calculus), or $\Gamma \Rightarrow P; \{P\}$ (for the Scottish calculus) is passed to the theorem prover. For a sequent with an empty stoup, the next inference is determined by the goal. If the goal is an implication, negation or conjunction, then the appropriate right rule is applied. If an instance of one of these rules fails, then we have to backtrack (as no other rule is applicable). If the goal is a propositional variable, falsum or a disjunction, the contraction rule is applied, selecting a formula and placing it in the stoup. If a contraction fails, another contraction is attempted, placing a different formula in the stoup. If the goal is a propositional variable or falsum, and contraction has failed for all possible stoup formulae, then we backtrack. If the goal is a disjunction and contraction has failed for all possible stoup formulae, then we may apply disjunction on the right. If this fails we have to backtrack. For a sequent with a stoup formula, the next inference is determined by the stoup formula. The next inference must be an instance of the appropriate rule on the left. If such an inference fails then we have to backtrack. Note that in $(\supset_{\mathcal{L}})$ we check the right branch, the one with the stoup formula, first. We get failure if at any point no rule instance can be applied. We give an example of failure owing to the history:

$$\frac{}{\Gamma, P \Rightarrow P \supset Q; \{P, Q\}} \; (\supset_{\mathcal{R}2})$$

fails as $P \in \{\Gamma, P\}$ and $Q \in \{P, Q\}$; the side conditions are not satisfied. Owing to the condition on $(C)$, no other rule instances are applicable to this sequent and we must backtrack. We can describe a similar process for the procedures for G3$^{Hist}$ and G4.

Knowledge of the invertibility of inference rules can be useful when implementing theorem provers. Although we have not used such knowledge here, we still think it is useful to give the following lemma.

**Lemma 4.4** *The following rules of* MJ$^{Hist}$ *(both Swiss and Scottish) are invertible:* $(\supset_{\mathcal{R}1})$, $(\supset_{\mathcal{R}2})$, $(\neg_{\mathcal{R}1})$, $(\neg_{\mathcal{R}2})$, $(\supset_{\mathcal{L}})$, $(\neg_{\mathcal{L}})$, $(\wedge_{\mathcal{R}})$, $(\vee_{\mathcal{L}})$. *The rules* $(\wedge_{\mathcal{L}_1})$, $(\wedge_{\mathcal{L}_2})$, $(\vee_{\mathcal{R}_1})$, $(\vee_{\mathcal{R}_2})$, $(C)$ *are not invertible.*

PROOF: The invertibilities are proved by some easy inductions. We can give simple counterexamples to the invertibility of the other rules. ■

## 4.4 Results

We tested our implementations of the Swiss and Scottish G3$^{Hist}$, the Swiss and Scottish MJ$^{Hist}$ and G4 on a set of benchmarks for propositional intuitionistic logic ([Dyc97]) and on the example formulae from [How97]. The example sets may be found in Appendix B. As we have already said, the implementations of these calculi are naïve. Much more efficient implementations are imaginable, and many better implementations of G4 exist. The purpose of these implementations is for them to be simple and in the same style in order that we can make a meaningful comparison of the calculi.

The results are displayed in Table 4.1 and Table 4.2. The benchmark formulae are all parameterised by natural number $n$. The entries in the table represent the largest $n$ for which the formula was decided in a particular calculus in less than 10 seconds of processor time (the larger the entry, the better the prover has performed). The timings in the second table are simply average timings (to two significant figures, with a cut off at 100000ms) for proving the formulae (the smaller the entry, the better the prover has performed). The Prolog code was run using SISCTUS Prolog2.1 on a Sun SPARCStation 10.

We can make several comparisons: we can compare the history provers with the contraction free prover; we can compare G3 and MJ as base calculi for applying a history mechanism to; we can compare the two forms of history mechanism.

The G4 decision procedure takes a different approach from that of the history provers. Therefore the implementation, though we have attempted to write it in the same style, is significantly different from the implementations of the history provers. Comparison is hard and uncertain. We therefore do not want to say anything definite based on the timings given. However, the results might indicate that

| | $G3^{Hist}$ Sc. | $G3^{Hist}$ Sw. | $MJ^{Hist}$ Sc. | $MJ^{Hist}$ Sw. | G4 |
|---|---|---|---|---|---|
| `de_bruijn_p` | 3 | 2 | 7 | 6 | 10 |
| `de_bruijn_n` | 1 | 1 | 2 | 2 | 1 |
| `ph_p` | 3 | 3 | 3 | 3 | 4 |
| `ph_n` | 1 | 1 | 2 | 2 | 1 |
| `con_p` | 50 | 23 | 54 | 85 | 130 |
| `con_n` | 5 | 2 | 5 | 6 | 2 |
| `schwicht_p` | 5 | 5 | 12 | 11 | 157 |
| `schwicht_n` | 2 | 1 | 102 | 89 | 38 |
| `kk_p` | 2 | 1 | 330 | 404 | 4 |
| `kk_n` | 0 | 0 | 5 | 5 | 1 |
| `equiv_p` | 3 | 2 | 13 | 13 | 4 |
| `equiv_n` | 4 | 4 | 1191 | 1219 | 3 |

Table 4.1: Results for Theorem Provers (largest parameter giving proof in less than 10sec)

G4 is generally a faster decision procedure, but that for certain classes of problem, the history provers can be comparable or even quicker.

The comparison between G3 and MJ as the base calculus for the history mechanism seems quite straightforward. In all cases the Swiss $MJ^{Hist}$ is better than the Swiss $G3^{Hist}$, and the Scottish $MJ^{Hist}$ is better than the Scottish $G3^{Hist}$. This is to be expected as MJ search space is a restriction of G3 search space. We conclude that MJ is a better calculus than G3 for basing a history mechanism propositional theorem prover on.

Our experimental results show that with both MJ and G3 as a base calculus, the Swiss and Scottish calculi give similar results for most examples. However, as expected, there are some examples where the Swiss mechanism is a little better, and others where the Scottish mechanism considerably outperforms (by several orders of magnitude) the Swiss mechanism. We conclude that for propositional intuitionistic logic, the Scottish mechanism seems to be the better approach to loop detection. However, G4 seems to give the best decision procedure.

Of course, if one is interested in finding loops, or a certain class of proofs rather than in decision procedures, then the history calculi are very useful and G4 is not.

## 4.5 Histories and Modal Logic

So far we have discussed history mechanisms only with respect to propositional intuitionistic logic. However, their use is possible for other logics, such as modal logics. Indeed, as contraction-free calculi for modal logics are either not known or

| Eg | Uni. | P/U | G3$^{Hist}$ Sw. | G3$^{Hist}$ Sc | MJ$^{Hist}$ Sw. | MJ$^{Hist}$ Sc. | G4 |
|---|---|---|---|---|---|---|---|
| 1. | | P | 49 | 56 | 14 | 18 | 7 |
| 2. | | P | 3900 | 4500 | 1400 | 1700 | 260 |
| 3. | | U | 1800 | 1800 | 170 | 160 | 61 |
| 4. | | P | 0.7 | 0.8 | 0.2 | 0.2 | 0.5 |
| 5. | | P | 0.3 | 0.4 | 0.1 | 0.1 | 0.2 |
| 6. | | P | 3.4 | 2.7 | 0.6 | 0.8 | 0.5 |
| 7. | | P | 77 | 57 | 11 | 14 | 13 |
| 8. | | P | 1.2 | 1.1 | 0.5 | 0.5 | 0.4 |
| 9. | | P | NR | NR | 4.3 | 4.3 | 1500 |
| 10. | | U | 1.1 | 1.0 | 0.4 | 0.5 | 0.7 |
| 11. | | U | NR | 61 | 24 | 10 | NR |
| 12. | | P | 1.4 | 1.7 | 0.7 | 1.0 | 0.9 |
| 13. | | U | 47 | 6.3 | 4.5 | 3.2 | 3.9 |
| 14. | | P | 6.8 | 4.8 | 3.5 | 2.7 | 1.5 |
| 15. | | P | 79 | 38 | 50 | 57 | 11 |
| 16. | 3 | P | 6400 | 6500 | 800 | 960 | 1100 |
| 17. | 2 | P | 46000 | 46000 | 7500 | 8500 | 3300 |
| 18. | 4 | P | 63 | 41 | 63 | 8.5 | 13 |
| 18. | 5 | P | 120 | 71 | 150 | 15 | 24 |
| 19. | 2 | P | 52000 | 2500 | 7.8 | 8.1 | 13 |
| 19. | 3 | P | NR | NR | 18000 | 27 | 260 |
| 20. | 2 | P | 17 | 17 | 1.1 | 2.1 | 2.4 |
| 20. | 4 | P | 970 | 950 | 5.3 | 6.6 | 33 |
| 21. | 2 | U | 290 | 260 | 8.6 | 10 | 12 |
| 21. | 3 | U | 1500 | 1500 | 27 | 33 | 37 |
| 22. | 2 | P | 3200 | 190 | 370 | 22 | 8.0 |
| 22. | 3 | P | NR | 11000 | 12000 | 510 | 20 |
| 23. | 2 | P | NR | NR | 35 | 45 | 140 |
| 23. | 3 | P | NR | NR | 2200 | 1400 | 8900 |
| 24. | 2 | U | NR | NR | 49 | 31 | NR |
| 25. | 2 | P | NR | NR | 11000 | 20 | 29 |
| 25. | 4 | P | NR | NR | NR | 370 | 18000 |
| 26. | 2 | P | NR | NR | 3.4 | 5.8 | 5.6 |
| 26. | 5 | P | NR | NR | 17 | 30 | 40 |
| 27. | 2 | P | 380 | 110 | 10000 | 47 | 9.3 |

Key:
Uni.:size of the universe the formula has been instantiated over; P: provable; U: unprovable; NR: no result in less 100000ms)

Table 4.2: Results and Timings (averages in milliseconds)

$$\frac{}{\Sigma|\Pi, A, \neg A; \mathcal{H}}\ (ax)$$

$$\frac{}{\Sigma|\Pi, \top; \mathcal{H}}\ (\top) \qquad \frac{\Sigma|\Pi; \mathcal{H}}{\Sigma|\Pi, -; \mathcal{H}}\ (-)$$

$$\frac{\Sigma|\Pi, P, Q; \mathcal{H}}{\Sigma|\Pi, P \vee Q; \mathcal{H}}\ (\vee) \qquad \frac{\Sigma|\Pi, P; \mathcal{H} \quad \Sigma|\Pi, Q; \mathcal{H}}{\Sigma|\Pi, P \wedge Q; \mathcal{H}}\ (\wedge)$$

$$\frac{\Sigma, \Diamond P|\Pi, P; \phi}{\Sigma|\Pi, \Diamond P; \mathcal{H}}\ (\Diamond_1) \quad \text{if } \Diamond P \notin \Sigma \qquad \frac{\Sigma|\Pi, P; \mathcal{H}}{\Sigma|\Pi, \Diamond P; \mathcal{H}}\ (\Diamond_2) \quad \text{if } \Diamond P \in \Sigma$$

$$\frac{\Sigma|\Sigma, P; \mathcal{H}, \Pi_2, P}{\Sigma|\Pi_1, \Box\Pi_2, \Box P; \mathcal{H}}\ (\Box) \quad \text{if } P \notin \mathcal{H}$$

**Figure 4.5:** S4$^{Hist}$ in the Swiss style

are complicated, history mechanisms are of more interest here than for intuitionistic logic. The Heuerding *et al.* paper ([HSZ96]) is mainly about loop checking for modal logics. In this section we discuss the application of histories to some modal logics: S4, intuitionistic S4 and Lax Logic. We know of no contraction-free calculi for intuitionistic S4 or Lax Logic (although [AF96] contains an unsuccessful attempt at developing one). Hudelmaier has given a contraction-free calculus for S4 ([Hud96]). However, this calculus is complicated and hard to understand, motivating other approaches to theorem proving in S4, such as the one from [HSZ96] discussed here.

As S4 is a modal logic with classical logic underlying it, we do not need a calculus which deals with all the connectives, but simply one which can deal with formulae in negation normal form. We give the calculus for S4$^{Hist}$ from [HSZ96] (where it is called S4$^{\mathcal{SU}}$), [Heu98] (where it is called S4$^{\mathcal{S},3}$) in Figure 4.5. Sequents are one-sided and of the form $\Sigma|\Pi; \mathcal{H}$. $\Sigma$ is a set of formulae of the form $\Diamond P$. $\Pi$ is a set of formulae in negation normal form. $\mathcal{H}$ is a set of formula.

**Definition 4.2** *A formula is said to be in* **negation normal form** *if it contains no occurrences of* $\supset$*, the only negated subformulae are atoms and the formula contains no repeated instances of a modality (no* $\Box\Box$ *and no* $\Diamond\Diamond$*).*

Since the base calculus is classical logic, no loop checking is needed for this. All we need to consider for looping are the modalities. This is fortunate, since generally speaking this calculus does not have the fundamental requirement that the context is increasing. What it does have is an increasing context of $\Diamond$ formulae. As noted in [HSZ96], this is enough to allow loop checking with a history.

In the previous section we identified two different approaches to loop detection in intuitionistic propositional logic. The obvious thing to do next is to see if the same distinction can be drawn for the modal logic.

We reiterate the difference between the Swiss and Scottish method for intuitionistic logic, with reference to the $(\supset_{\mathcal{L}})$ rule in the G3$^{Hist}$ calculi. First the Swiss $(\supset_{\mathcal{L}})$:

$$\frac{\Gamma, P \supset Q \Rightarrow P; (D, \mathcal{H}) \quad \Gamma, P \supset Q, Q \Rightarrow D; \phi}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}} \; (\supset_{\mathcal{L}}) \quad \text{if } D \notin \mathcal{H} \text{ and } Q \notin \Gamma$$

And the Scottish $(\supset_{\mathcal{L}})$:

$$\frac{\Gamma, P \supset Q \Rightarrow P; (P, \mathcal{H}) \quad \Gamma, P \supset Q, Q \Rightarrow D; \{D\}}{\Gamma, P \supset Q \Rightarrow D; \mathcal{H}} \; (\supset_{\mathcal{L}}) \quad \text{if } P \notin \mathcal{H} \text{ and } Q \notin \Gamma$$

The Swiss calculus checks that the goal of the *conclusion* is not in the history and if not, adds this formula to the history. The Scottish calculus checks that the goal of the left *premiss* is not in the history and if not, adds this goal to the history. For intuitionistic logic this makes a significant difference to where a loop is detected.

Now look at the $(\Box)$ inference of S4$^{Hist}$ . To illustrate the point we will look at a sequent with only one boxed formula in it:

$$\frac{\Sigma | \Sigma, P; \mathcal{H}, P}{\Sigma | \Pi, \Box P; \mathcal{H}} \; (\Box) \quad \text{if } P \notin \mathcal{H}$$

An alternative rule would have been:

$$\frac{\Sigma | \Sigma, P; \mathcal{H}, \Box P}{\Sigma | \Pi, \Box P; \mathcal{H}} \; (\Box) \quad \text{if } \Box P \notin \mathcal{H}$$

In terms of checking against the history and adding to it, these two rules are analogous to those given above for intuitionistic logic. But here it is easy to see that these rules will have exactly the same effect. The difference between checking the premiss and conclusion formula is simply a box. The addition of more boxed formulae to sequents makes no difference to this.

We see that the two slightly different approaches that were taken for intuitionistic logic merge into one for S4.

In the rest of this section we illustrate the wide applicability of history mechanisms by applying them to two more logics. Both are intuitionistic modal logics: intuitionistic S4 and Lax Logic.

### 4.5.1 Histories and Lax Logic

In this section we briefly present a history calculus which is a decision procedure for propositional Lax Logic, as presented in Chapter 3.

Lax Logic extends usual calculi for intuitionistic logic by two rules, one for the modality on the left and one for the modality on the right. The calculus we use here as the basis for the history calculus is PFLAX (see Figure 3.7). Essentially, no extra

$$\frac{}{\Gamma \xrightarrow{P} P; \mathcal{H}} (ax) \qquad \frac{\Gamma, P \xrightarrow{P} D; \mathcal{H}}{\Gamma, P \Rightarrow D; \mathcal{H}} (C)$$

$$\frac{\Gamma, P \Rightarrow Q; \{Q\}}{\Gamma \Rightarrow P \supset Q; \mathcal{H}} (\supset_{\mathcal{R}1}) \quad \text{if } P \notin \Gamma$$

$$\frac{\Gamma \Rightarrow Q; (Q, \mathcal{H})}{\Gamma \Rightarrow P \supset Q; \mathcal{H}} (\supset_{\mathcal{R}2}) \quad \text{if } P \in \Gamma \text{ and } Q \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P; (P, \mathcal{H}) \quad \Gamma \xrightarrow{Q} D; \mathcal{H}}{\Gamma \xrightarrow{P \supset Q} D : \mathcal{H}} (\supset_{\mathcal{L}}) \quad \text{if} P \notin \mathcal{H}$$

$$\frac{\Gamma \Rightarrow P; (P, \mathcal{H})}{\Gamma \Rightarrow \circ P; \mathcal{H}} (\circ_{\mathcal{R}}) \quad \text{if } P \notin \mathcal{H}$$

$$\frac{\Gamma, P \Rightarrow \circ R; \{\circ R\}}{\Gamma \xrightarrow{\circ P} \circ R; \mathcal{H}} (\circ_{\mathcal{L}}) \quad \text{if } P \notin \Gamma$$

$D$ is either an atom or a modal formula.
Where the history has been extended we have parenthesised $(P, \mathcal{H})$ for emphasis.

**Figure 4.6:** The calculus PFLAX$^{Hist}$ (Scottish)

work has to be done for this calculus. It has all necessary features, such as increasing context, for use with history mechanisms. We simply take the history mechanism for intuitionistic logic (in either the Swiss or Scottish style, we only present one in the Scottish style) and apply it without change, only noting that there are formulae with a modality – this presents no difficulties. The calculus PFLAX$^{Hist}$ restricted to the connectives $\supset$ and $\circ$ is presented in Figure 4.6.

We can again prove all the usual theorems about soundness, completeness and termination.

### 4.5.2 Histories and IS4

Intuitionistic S4 (IS4) is a modal logic with a modality like that of S4, but built on intuitionistic logic rather than classical logic. The two sided single succedent calculus with a single modality that we deal with here can be found in the appendix. More details on IS4 can be found in [BdP96] and [Sim94].

As for S4, we are faced with an immediate problem – the context is not increasing (owing to the $(\Box_{\mathcal{R}})$ rule). For S4 this wasn't problematic as we only needed to check for looping owing to the modalities – the propositional classical logic needs no history. The modal context was increasing: hence we could easily use our histories. Now that the modal logic is based on intuitionistic logic, we have to consider loops in the base calculus, as well as ones owing to the modality (which can be dealt with since we still have an increasing modal context).

We do loop checking in this calculus by using two histories – one to deal with the modalities (like that for S4 above) and one for intuitionistic propositional logic (again, as above). We formulate the calculus in order to prove formulae with no repeated modalities. (Sequents can be preprocessed to such a form since $\Box\Box P \equiv \Box P$. Notice that for proof enumeration such a preprocessing would not be allowed as it would identify non-equivalent proofs. However, for theorem proving this is a valid step). We display IS4$^{Hist}$ (in a Scottish style) in Figures 4.7 and 4.8. Because of the two history mechanisms we have a lot of, but not unmanageably many, rules. Sequents have form $\Box\Gamma, \Delta \Rightarrow P; \mathcal{H}_1; \mathcal{H}_2$, where none of the formulae in $\Delta$ are boxed, $\mathcal{H}_1$ is the modal history and $\mathcal{H}_2$ is the intuitionistic history.

The calculus uses the Scottish style of history for the intuitionistic component. We could easily have used the Swiss style instead. As already discussed, there is only one approach to the modal looping. We prove the soundness, completeness and termination of this calculus.

**Theorem 4.9** *The calculi* IS4 *and* IS4$^{Hist}$ *are equivalent. That is, sequent $S$ is provable in* IS4 *iff it is provable in* IS4$^{Hist}$.

PROOF: Soundness is trivial. The completeness is similar to the other proofs. To see this, one simply has to note that the two histories work independently, with the modality history taking precedence. Between $(\Box_{\mathcal{R}})$ inferences the second history is much like the intuitionistic history. That the first history is much like the S4 history is also obvious. Building a proof tree can be done as in Theorem 4.1. ■

**Theorem 4.10** *Backwards proof search in the calculus* IS4$^{Hist}$ *is terminating.*

PROOF: The proof is similar of that of Theorem 4.2. We associate with every sequent $\Box\Gamma, \Delta \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2$ a quadruple of natural numbers

$$W = (k - m, k - l_1, k - n, k - l_2)$$

where $k$ is the number of elements of the set of subformulae of $\Box\Gamma, \Delta, R$; $m$ is number of elements in the set of formulae of $\Box\Gamma$; $n$ is the number of elements in $\Delta$ when considered as a set; $l_1$ is the number of elements in $\mathcal{H}_1$; $l_2$ is the number of elements in $\mathcal{H}_2$. The quadruples are ordered lexicographically from the left.

By inspection we see that for every inference, the premisses have lower $W$ than the conclusion. Hence backwards proof search is terminating. ■

We can easily formulate a two-sided classical S4 calculus similar to the IS4 calculus we have given. We simply allow multiple succedents and adjust the rules accordingly. What effect will this have on the histories? Basing the calculus on classical logic immediately means that we do not need the second history – loop-checking is not needed for classical logic. We still need to keep track of the boxed formulae and this is done by noting all the boxed formulae in the succedent when performing $(\Box_{\mathcal{R}})$. That is, we end up with a two-sided calculus S4$^{Hist}$.

$$\frac{}{\Box\Gamma, \Delta, P \Rightarrow P; \mathcal{H}_1; \mathcal{H}_2}\ (ax) \qquad \frac{}{\Box\Gamma, \Delta, - \Rightarrow P; \mathcal{H}_1; \mathcal{H}_2}\ (-)$$

$$\frac{\Box\Gamma, \Delta, N \Rightarrow Q; \mathcal{H}_1; \{Q\}}{\Box\Gamma, \Delta \Rightarrow N \supset Q; \mathcal{H}_1; \mathcal{H}_2}\ (\supset_{\mathcal{R}1}) \quad \text{if } N \notin \Delta$$

$$\frac{\Box\Gamma, M, \Delta \Rightarrow Q; \phi; \{Q\}}{\Box\Gamma, \Delta \Rightarrow M \supset Q; \mathcal{H}_1; \mathcal{H}_2}\ (\supset_{\mathcal{R}2}) \quad \text{if } M \notin \Box\Gamma$$

$$\frac{\Box\Gamma, \Delta \Rightarrow Q; \mathcal{H}_1; (Q, \mathcal{H}_2)}{\Box\Gamma, \Delta \Rightarrow N \supset Q; \mathcal{H}_1; \mathcal{H}_2}\ (\supset_{\mathcal{R}3}) \quad \text{if } N \in \Delta \text{ and } Q \notin \mathcal{H}_2$$

$$\frac{\Box\Gamma, \Delta \Rightarrow Q; \mathcal{H}_1; (Q, \mathcal{H}_2)}{\Box\Gamma, \Delta \Rightarrow M \supset Q; \mathcal{H}_1; \mathcal{H}_2}\ (\supset_{\mathcal{R}4}) \quad \text{if } M \in \Gamma \text{ and } Q \notin \mathcal{H}_2$$

$$\frac{\Box\Gamma, \Delta, P \supset N \Rightarrow P; \mathcal{H}_1; (P, \mathcal{H}_2) \quad \Box\Gamma, \Delta, P \supset N, N \Rightarrow R; \mathcal{H}_1; \{R\}}{\Box\Gamma, \Delta, P \supset N \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2}\ (\supset_{\mathcal{L}1})\dagger$$

$$\frac{\Box\Gamma, \Delta, P \supset M \Rightarrow P; \mathcal{H}_1; (P, \mathcal{H}_2) \quad \Box\Gamma, M, \Delta, P \supset M \Rightarrow R; \phi; \{R\}}{\Box\Gamma, \Delta, P \supset M \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2}\ (\supset_{\mathcal{L}2})\ddagger$$

$$\frac{\Box\Gamma, \Delta \Rightarrow P : \mathcal{H}_1; (P, \mathcal{H}_2) \quad \Box\Gamma, \Delta \Rightarrow Q; \mathcal{H}_1; (Q, \mathcal{H}_2)}{\Box\Gamma, \Delta \Rightarrow P \wedge Q; \mathcal{H}_1; \mathcal{H}_2}\ (\wedge_{\mathcal{R}}) \quad \text{if } P, Q \notin \mathcal{H}_2$$

$$\frac{\Box\Gamma, \Delta, N \wedge Q, N \Rightarrow R; \mathcal{H}_1; \{R\}}{\Box\Gamma, \Delta, N \wedge Q \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2}\ (\wedge_{\mathcal{L}1}) \quad \text{if} N \notin \Delta$$

$$\frac{\Box\Gamma, \Delta, P \wedge N, N \Rightarrow R; \mathcal{H}_1; \{R\}}{\Box\Gamma, \Delta, P \wedge N \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2}\ (\wedge_{\mathcal{L}2}) \quad \text{if} N \notin \Delta$$

$$\frac{\Box\Gamma, M, \Delta, M \wedge Q \Rightarrow R; \phi; \{R\}}{\Box\Gamma, \Delta, M \wedge Q \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2}\ (\wedge_{\mathcal{L}3}) \quad \text{if} M \notin \Box\Gamma$$

$$\frac{\Box\Gamma, M, \Delta, P \wedge M \Rightarrow R; \phi; \{R\}}{\Box\Gamma, \Delta, P \wedge M \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2}\ (\wedge_{\mathcal{L}4}) \quad \text{if} M \notin \Box\Gamma$$

† if $P \notin \mathcal{H}_2$ and $N \notin \Delta$.
‡ if $P \notin \mathcal{H}_2$ and $M \notin \Delta$.
All boxed formulae in the context are in $\Box\Gamma$
$M$ is a modal formula, $N$ is a non-modal formula. $P, Q, R$ can be either.
Where the history is extended we have parenthesised $(P, \mathcal{H})$ for emphasis.

**Figure 4.7:** IS4$^{Hist}$ : axioms and rules for $-$, $\supset$ and $\wedge$

$$\frac{\Box\Gamma, \Delta \Rightarrow P; \mathcal{H}_1; (P, \mathcal{H}_2)}{\Box\Gamma, \Delta \Rightarrow P \vee Q; \mathcal{H}_1; \mathcal{H}_2} \; (\vee_{\mathcal{R}_1}) \quad \text{if } P \notin \mathcal{H}_2$$

$$\frac{\Box\Gamma, \Delta \Rightarrow Q; \mathcal{H}_1; (Q, \mathcal{H}_2)}{\Box\Gamma, \Delta \Rightarrow P \vee Q; \mathcal{H}_1; \mathcal{H}_2} \; (\vee_{\mathcal{R}_2}) \quad \text{if } Q \notin \mathcal{H}_2$$

$$\frac{\Box\Gamma, \Delta, N_1 \vee N_2, N_1 \Rightarrow R; \mathcal{H}_1; \{R\} \quad \Box\Gamma, \Delta, N_1 \vee N_2, N_2 \Rightarrow R; \mathcal{H}_1; \{R\}}{\Box\Gamma, \Delta, N_1 \vee N_2 \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2} \; (\vee_{\mathcal{L}1})\dagger$$

$$\frac{\Box\Gamma, M, \Delta, M \vee N \Rightarrow R; \phi; \{R\} \quad \Box\Gamma, \Delta, M \vee N, N \Rightarrow R; \mathcal{H}_1; \{R\}}{\Box\Gamma, \Delta, M \vee N \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2} \; (\vee_{\mathcal{L}2})\ddagger$$

$$\frac{\Box\Gamma, \Delta, N \vee M, N \Rightarrow R; \mathcal{H}_1; \{R\} \quad \Box\Gamma, M, \Delta, N \vee M \Rightarrow R; \phi; \{R\}}{\Box\Gamma, \Delta, N \vee M \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2} \; (\vee_{\mathcal{L}3})\flat$$

$$\frac{\Box\Gamma, M_1, \Delta, M_1 \vee M_2 \Rightarrow R; \phi; \{R\} \quad \Box\Gamma, M_2, \Delta, M_1 \vee M_2 \Rightarrow R; \phi; \{R\}}{\Box\Gamma, \Delta, M_1 \vee M_2 \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2} \; (\vee_{\mathcal{L}4})\sharp$$

$$\frac{\Box\Gamma \Rightarrow P; (P, \mathcal{H}_1); \{P\}}{\Box\Gamma, \Delta \Rightarrow \Box P; \mathcal{H}_1; \mathcal{H}_2} \; (\Box_{\mathcal{R}}) \quad \text{if } P \notin \mathcal{H}_1$$

$$\frac{\Box\Gamma, \Delta, \Box P, P \Rightarrow R; \mathcal{H}_1; \{R\}}{\Box\Gamma, \Delta, \Box P \Rightarrow R; \mathcal{H}_1; \mathcal{H}_2} \; (\Box_{\mathcal{L}}) \quad \text{if } P \notin \Delta$$

† if $N_1, N_2 \notin \Delta$
‡ if $M \notin \Box\Gamma$ and $N \notin \Delta$
♭ if $M \notin \Box\Gamma$ and $N \notin \Delta$
♯ if $M_1, M_2 \notin \Box\Gamma$
All boxed formulae in the context are in $\Box\Gamma$
$M$ is a modal formula. $N$ is a non-modal formula. $P, Q, R$ can be either.
Where the history has been extended we have parenthesised $(P, \mathcal{H})$ for emphasis.

**Figure 4.8:** The calculus IS4$^{Hist}$ : rules for $\vee$, $\Box$

## 4.6 Conclusion

In this chapter we have investigated the use of calculi with history mechanisms as decision procedures for a variety of logics. We have given history calculi for intuitionistic logic, S4, IS4, and Lax Logic. We have proved the soundness, completeness and termination of these calculi. We have compared our approach with that of Heuerding *et al* in [HSZ96]. We have given a theoretical discussion of this and have also performed a practical comparison of the calculi for intuitionistic logic using Prolog implementations of the calculi. We conclude that the Scottish mechanism gives a better decision procedure for intuitionistic logic, but that for this logic, G4 gives as good or better a decision procedure. For classical S4, the approaches coincide. We have also illustrated the wide applicability and flexibility of the technique by applying it to IS4 and Lax Logic.

# Chapter 5

# Embedding MJ in Intuitionistic Linear Logic

Girard's original paper on linear logic, [Gir87], gives an embedding of intuitionistic logic into linear logic, for formulae, sequents and for proofs. The translation, known as the Girard embedding (the $(.)^0$ embedding of [Gir87]), was claimed to be correct and faithful. A detailed proof wasn't provided in that paper – this was supplied by Schellinx in [Sch91]. [Sch94], [TS96] show how the Girard embedding of proofs induces a sequent calculus for implicational intuitionistic logic – this Gentzen system is known as IU. We know of no satisfactory semantic justification for the form of IU. In this chapter we discuss IU and the Girard embedding and give a new embedding (defined by two functions) of a fragment of intuitionistic logic into (Intuitionistic) Linear Logic. This embedding induces (and was designed to induce) a fragment of the sequent calculus MJ. This calculus is syntactically similar to IU, but has in addition a semantic justification – its proofs correspond naturally in a 1–1 way to the normal natural deductions of intuitionistic logic. In fact, for reasons discussed below, the largest fragment of intuitionistic logic that we have a satisfactory solution for is hereditary Harrop logic.

## 5.1 The Girard Embedding

Before we discuss this particular embedding, we give a definition of what we mean by an embedding of one logic into another.

**Definition 5.1** *An* **embedding** *of logic $L_1$ into logic $L_2$ is a function, $f$, interpreting formulae of $L_1$ into formulae of $L_2$ such that for every formula $P$ of $L_1$, $\vdash_{L_1} P$ iff $\vdash_{L_2} f(P)$.*

We should note that this definition is given in terms simply of provability – later in the chapter we ask for a little more.

The Girard embedding has been quite widely discussed and can be found in, for example, [Gir87]. We present the embedding below:

$$\begin{array}{lcll}
A^g & = & A & \text{where } A \text{ is atomic} \\
-^g & = & 0 & \\
(P \supset Q)^g & = & !P^g \multimap Q^g & \\
(P \wedge Q)^g & = & P^g \& Q^g & \\
(P \vee Q)^g & = & !P^g \oplus !Q^g & \\
(\forall x.P)^g & = & \forall x.P^g & \\
(\exists x.P)^g & = & \exists x.!P^g &
\end{array}$$

To embed sequents we interpret $\Gamma \Rightarrow R$ as $!\Gamma^g \Rightarrow R^g$ and (with $P$ the next principal formula) $\Gamma, P \Rightarrow R$ as $!\Gamma^g, P^g \Rightarrow R^g$. Hence if the last step in a proof of $\Gamma, P \supset Q \Rightarrow R$ is $(\supset_{\mathcal{L}})$, then in the translation we would have:

$$\frac{\dfrac{!\Gamma^g \Rightarrow P^g}{!\Gamma^g \Rightarrow !P^g}\,(P) \qquad !\Gamma^g, Q^g \Rightarrow R^g}{!\Gamma^g, !P^g \multimap Q^g \Rightarrow R^g}\,(\multimap_{\mathcal{L}})$$

We know that the embedding is correct for provability. The following theorem can be found in [Sch91]:

**Theorem 5.1** IL $\vdash \Gamma \Rightarrow P$ *iff* CLL $\vdash !\Gamma^g \Rightarrow P^g$.

PROOF: The proof can be found in [Sch91]. ■

In [Gir87] a translation of proofs is also given, showing how to interpret a (natural deduction) proof into a linear sequent calculus proof. It is translations of proofs that we are really interested in in this chapter. It should also be pointed out that embedding logics into linear logic results in the system of Unified Logic in [Gir93]. This calculus, by building embeddings of classical logic and intuitionistic logic into linear logic, has the connectives of all three logics and allows them to interact. We mention connections between Unified Logic and IU later in the chapter.

## 5.2 Induced Calculi and IU

We look at the fragment of ILL generated by the grammar of the embedding of intuitionistic logic into linear logic. From the proofs in this fragment, we find a sequent calculus for intuitionistic logic. We say that the calculus is *induced* by the embedding. The proofs in the restricted grammar take a certain form and the inference steps correspond to certain rules of intuitionistic logic. We make the notion of induced calculus precise.

**Definition 5.2** *A sequent calculus $G_1$ for logic $L_1$ is* **induced** *by embedding e of logic $L_1$ into logic $L_2$ (with sequent calculus $G_2$) if for every sequent $S$ of $L_1$:*

$$\frac{}{\Gamma; A \Rightarrow A}\ (ax) \qquad \frac{\Gamma, P; P \Rightarrow R}{\Gamma, P; - \Rightarrow R}\ (C)$$

$$\frac{\Gamma, P; \Theta \Rightarrow Q}{\Gamma; \Theta \Rightarrow P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma; - \Rightarrow P \quad \Gamma; Q \Rightarrow R}{\Gamma; P \supset Q \Rightarrow R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma; \Theta \Rightarrow P \quad \Gamma; \Theta \Rightarrow Q}{\Gamma; \Theta \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma; P \Rightarrow R}{\Gamma; P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma; Q \Rightarrow R}{\Gamma; P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma; - \Rightarrow P[u/x]}{\Gamma; - \Rightarrow \forall x P}\ (\forall_{\mathcal{R}})\dagger \qquad \frac{\Gamma; P[t/x] \Rightarrow R}{\Gamma; \forall x P \Rightarrow R}\ (\forall_{\mathcal{L}})$$

$\Theta$ stands for either empty or a single formula.
$-$ stands for empty.
† $u$ not free in $\Gamma$.

**Figure 5.1:** The sequent calculus IU for a fragment of intuitionistic logic

– *there is a bijection between proofs of sequent $S$ in $G_1$ and proofs of sequent $e(S)$ in $G_2$*

The Gentzen calculus IU induced by the Girard embedding for the $\supset, \wedge, \forall$ fragment of intuitionistic logic is displayed in Figure 5.1.

**Proposition 5.1** IU$\vdash \Gamma; \Theta \Rightarrow R$ *iff* IL$\vdash \Gamma, \Theta \Rightarrow R$

PROOF: See [TS96]. ■

We have not treated disjunction, bottom or the existential quantifier in IU. This is because with these connectives the calculus loses the attractive feature of the focused formula on the left. For example, the following could occur in ILL:

$$\frac{\dfrac{!\Gamma, !P_1, Q \Rightarrow R \quad !\Gamma, !P_2, Q \Rightarrow R}{!\Gamma, !P_1 \oplus !P_2, Q \Rightarrow R}\ (\oplus_{\mathcal{L}})}{!\Gamma, !(!P_1 \oplus !P_2), Q \Rightarrow R}\ (D)$$

Or even more illustrative:

$$\frac{}{!\Gamma, \Delta, 0 \Rightarrow R}\ (0)$$

These correspond to intuitionistic proofs with many focused formulae.

We know of no treatment of IU and the Girard embedding that explicitly mentions disjunction or falsum, although it is hinted in [Sch94] that there is a correspondence between the induced calculus for the whole of intuitionistic logic and the intuitionistic fragment of Girard's Logic of Unity ([Gir93]). Schellinx says that we find "the neutral fragment of intuitionistic implicational logic as it appears in Girard's system of Unified Logic" ([Sch94], pg. 50). The neutral intuitionistic fragment of Unified Logic is the fragment of the logic with connectives $\supset, \wedge, \forall$, those of IU. However, beyond this fragment, the interpretations of the intuitionistic connectives is more complicated, and simply taking the fragment of Unified Logic for intuitionistic logic gives an unattractive calculus (losing the single formula focusing). It does not give MJ.

## 5.3 Inducing MJ

The $\supset, \wedge, \forall$ fragment MJ is similar to the calculus IU. Its form is that of IU, but with the restriction that the $\Theta$s of Figure 5.1 are empty. MJ also has satisfactory rules for disjunction, falsum and the existential quantifier. We would like to find an embedding of intuitionistic logic into ILL which induces this calculus. This seems to be hard to achieve using a single mapping. Instead we use two mappings: a positive one which applies to formulae on the right; and a negative one which applies to formulae on the left. Unfortunately, we have been unable to find an embedding of disjunction (on the left), falsum and the existential quantifier (on the left) that works as we would like, and so these have been left out. We give this embedding:

| | | | |
|---|---|---|---|
| $A^+$ | = | $A$ | where A is atomic |
| $A^\perp$ | = | $A$ | where A is atomic |
| $(P \supset Q)^+$ | = | $!P^\perp \multimap !Q^+$ | |
| $(P \supset Q)^\perp$ | = | $!P^+ \multimap Q^\perp$ | |
| $(P \wedge Q)^+$ | = | $!P^+ \& !Q^+$ | |
| $(P \wedge Q)^\perp$ | = | $P^\perp \& Q^\perp$ | |
| $(\forall x.P)^+$ | = | $\forall x.!P^+$ | |
| $(\forall x.P)^\perp$ | = | $\forall x.P^\perp$ | |

We should also note the following extensions:

| | | |
|---|---|---|
| $(P \vee Q)^+$ | = | $!P^+ \oplus !Q^+$ |
| $-^+$ | = | $0$ |
| $(\exists x.P)^+$ | = | $\exists x.!P^+$ |

We embed sequents into the ILL calculus with split context, the system ILL$^\Sigma$ (Figure 2.2).

The intuitionistic sequent $\Sigma \Rightarrow R$ is interpreted as the sequent $\Sigma^\perp; - \Rightarrow R^+$ in ILL$^\Sigma$. The MJ sequent $\Sigma \xrightarrow{P} R$ is interpreted as the sequent $\Sigma^\perp; P^\perp \Rightarrow R^+$ in ILL$^\Sigma$.

The sequent $\Sigma \Rightarrow R$ of intuitionistic logic, is interpreted as $\Sigma^\perp; - \Rightarrow R^+$. Every proof of this ILL$^\Sigma$ sequent, when viewed as an intuitionistic proof, is an MJ proof. Moreover, all MJ proofs can be found in this way.

We now try to explain why we have chosen this embedding. When we translate a formula on the right, say $P \supset Q$, to a linear logic formula $!P \multimap Q$, the $(\multimap_{\mathcal{R}})$ rule can be applied straight away, independently of whether there is a stoup (unbanged) formula on the left. This does not match MJ. One fix is to translate to $!P \multimap !Q$ instead. In order to get to a sequent that is the translation of an intuitionistic logic sequent, we have to unbang the goal – to do this we must have no unbanged formulae in the context. The negative formula is still banged and is moved to the context. However, we then find that the translation of implication on the left loses the notion of a privileged formula. Hence the two translations. One for the left, to retain the notion of a privileged formula, one for the right to ensure the rule can only be applied when we require.

The two rules where IU differs from MJ are $(\supset_{\mathcal{R}})$ and $(\wedge_{\mathcal{R}})$. The new embedding induces the MJ rules. For example (using Lemma 5.1 below):

$$\dfrac{\dfrac{\Sigma^{\perp}, P^{\perp}; - \Rightarrow Q^{+}}{\Sigma^{\perp}, P^{\perp}; - \Rightarrow !Q^{+}}\ (P)}{\Sigma^{\perp}; - \Rightarrow !P^{\perp} \multimap !Q^{+}}\ (\multimap_{\mathcal{R}})$$

Obviously if there was a stoup formula, the linear context would be non-empty and so we would not be able to perform the promotion (see Lemma 5.1).

The presentation of ILL that we use to prove results about the embedding is ILL$^{\Sigma}$. This can be seen in Figure 2.2. We prove that the embedding is correct and faithful. Note that for presentational purposes we write MJ sequents differently from normal: we write $\Sigma; - \Rightarrow R$ instead of $\Sigma \Rightarrow R$ and we write $\Sigma; P \Rightarrow R$ instead of $\Sigma \xrightarrow{P} R$.

**Theorem 5.2** *The embedding given above is correct for proofs. That is, for every proof in* MJ *of* $\Sigma; \Theta \Rightarrow P$ *there is a proof in* ILL$^{\Sigma}$ *of* $\Sigma^{\perp}; \Theta^{\perp} \Rightarrow P^{+}$.

PROOF: The proof is by an easy induction on the height of derivations. We shall illustrate it for just one case, the others being very similar.

The last inference is $(\supset_{\mathcal{L}})$. We have:

$$\dfrac{\begin{matrix}\vdots \\ \Sigma; - \Rightarrow P\end{matrix} \quad \begin{matrix}\vdots \\ \Sigma; Q \Rightarrow R\end{matrix}}{\Sigma; P \supset Q \Rightarrow R}\ (\supset_{\mathcal{L}})$$

So by the induction hypothesis we have proofs in ILL$^{\Sigma}$ of:

$$\begin{matrix}\vdots \\ \Sigma^{\perp}; - \Rightarrow P^{+}\end{matrix} \qquad \begin{matrix}\vdots \\ \Sigma^{\perp}; Q^{\perp} \Rightarrow R^{+}\end{matrix}$$

And hence we have a proof:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \Sigma^{\perp}; - \Rightarrow P^{+}\end{matrix}}{\Sigma^{\perp}; - \Rightarrow !P^{+}}\ (P) \quad \begin{matrix}\vdots \\ \Sigma^{\perp}; Q^{\perp} \Rightarrow R^{+}\end{matrix}}{\Sigma^{\perp}; !P^{+} \multimap Q^{\perp} \Rightarrow R^{+}}\ (\multimap_{\mathcal{L}})$$

■

We need two lemmas:

**Lemma 5.1** *If sequent* $\Sigma^{\perp}; \Theta^{\perp} \Rightarrow !R^{+}$ *is provable in* ILL$^{\Sigma}$ *then* $\Theta^{\perp} = \phi$.

PROOF: Induction on the height of derivations. ■

**Lemma 5.2** *If sequent* $\Sigma^{\perp}; \Theta^{\perp} \Rightarrow R^{+}$ *is provable in* ILL$^{\Sigma}$ *then* $\Theta^{\perp}$ *has zero or one elements.*

PROOF: Induction on the height of derivations. ■

**Theorem 5.3** *The embedding given above is faithful for proofs. That is, for every proof in* ILL *of* $\Sigma^{\perp}; \Theta^{\perp} \Rightarrow P^{+}$ *there is a proof in* MJ *of* $\Sigma; \Theta \Rightarrow P$.

PROOF: We prove the result by induction on the height of derivations.

1. Case: the last inference is an instance of $(\multimap_{\mathcal{R}})$; we have the following:

$$\frac{\Sigma^{\perp}, P^{\perp}; \Theta^{\perp} \Rightarrow !Q^{+}}{\Sigma^{\perp}; \Theta^{\perp} \Rightarrow !P^{\perp} \multimap !Q^{+}} \ (\multimap_{\mathcal{R}})$$

   By Lemma 5.1 $\Theta^{\perp}$ is empty and the next inference in therefore $(P)$. By the induction hypothesis we have an MJ proof ending in:

$$\frac{\Sigma, P; - \Rightarrow Q}{\Sigma; - \Rightarrow P \supset Q} \ (\supset_{\mathcal{R}})$$

2. Case: the last inference is an instance of $(\multimap_{\mathcal{L}})$; we have the following:

$$\frac{\Sigma^{\perp}; \Theta_1^{\perp} \Rightarrow !P^{+} \quad \Sigma^{\perp}; Q^{\perp}, \Theta_2^{\perp} \Rightarrow R^{+}}{\Sigma^{\perp}; !P^{+} \multimap Q^{\perp}, \Theta_1^{\perp}, \Theta_2^{\perp} \Rightarrow R^{+}} \ (\multimap_{\mathcal{L}})$$

   By Lemma 5.1 and Lemma 5.2, $\Theta_1^{\perp}$ and $\Theta_2^{\perp}$ are empty and the left premiss must result from $(P)$. By the induction hypothesis we have an MJ proof ending in:

$$\frac{\Sigma; - \Rightarrow P \quad \Sigma; Q \Rightarrow R}{\Sigma; P \multimap Q \Rightarrow R} \ (\supset_{\mathcal{L}})$$

3. Case: the last rule is an instance of $(\&_{\mathcal{R}})$; we have the following:

$$\frac{\Sigma^{\perp}; \Theta^{\perp} \Rightarrow !P^{+} \quad \Sigma^{\perp}; \Theta^{\perp} \Rightarrow !Q^{+}}{\Sigma^{\perp}; \Theta^{\perp} \Rightarrow !P^{+} \& !Q^{+}} \ (\&_{\mathcal{R}})$$

   By Lemma 5.1, $\Theta^{\perp}$ is empty and therefore both premisses are the result of $(P)$. By the induction hypothesis we have an MJ proof ending in:

$$\frac{\Sigma; - \Rightarrow P \quad \Sigma; - \Rightarrow Q}{\Sigma; - \Rightarrow P \wedge Q} \ (\wedge_{\mathcal{R}})$$

4. Case: the last inference is an instance of $(\&_{\mathcal{L}1})$. We have the following:

$$\frac{\Sigma^\perp; P^\perp, \Theta^\perp \Rightarrow R^+}{\Sigma^\perp; P^\perp \& Q^\perp, \Theta^\perp \Rightarrow R^+}\ (\&_{\mathcal{L}1})$$

   By Lemma 5.2, $\Theta^\perp$ is empty. By the induction hypothesis we have an MJ proof ending as follows:

$$\frac{\Sigma; P \Rightarrow R}{\Sigma; P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}1})$$

5. The case for $(\&_{\mathcal{L}2})$ is similar to $(\&_{\mathcal{L}_1})$.

6. Case: the last inference is an instance of $(D)$. We have the following:

$$\frac{\Sigma^\perp, P^\perp; \Theta^\perp, P^\perp \Rightarrow R^+}{\Sigma^\perp, P^\perp; \Theta^\perp \Rightarrow R^+}\ (D)$$

   By Lemma 5.2, $\Theta^\perp$ is empty. By the induction hypothesis we have an MJ proof ending:

$$\frac{\Sigma, P; P \Rightarrow R}{\Sigma, P; - \Rightarrow R}\ (C)$$

7. Case: the last inference is an instance of $(ax)$. We have the following:

$$\frac{}{\Sigma^\perp; P^\perp \Rightarrow P^+}\ (ax)$$

   We have an MJ proof:

$$\frac{}{\Sigma; P \Rightarrow P}\ (ax)$$

8. Case: the last inference is an instance of $(\forall_{\mathcal{R}})$. We have the following:

$$\frac{\Sigma^\perp; \Theta^\perp \Rightarrow P[y/x]^+}{\Sigma^\perp; \Theta^\perp \Rightarrow \forall x.P^+}\ (\forall_{\mathcal{R}})$$

   (with $y$ not free in $\Sigma^\perp, \Theta^\perp$). By Lemma 5.1, $\Theta^\perp$ is empty. By the induction hypothesis we have an MJ proof ending in:

$$\frac{\Sigma; - \Rightarrow P[y/x]}{\Sigma; - \Rightarrow \forall x P}\ (\forall_{\mathcal{R}})$$

   (with $y$ not free in $\Sigma$).

9. Case: the last inference is an instance of $(\forall_{\mathcal{L}})$. We have the following:

$$\frac{\Sigma^\perp; P[t/x]^\perp, \Theta^\perp \Rightarrow R^+}{\Sigma^\perp; \forall x.P^\perp, \Theta^\perp \Rightarrow R^+}\ (\forall_{\mathcal{L}})$$

   By Lemma 5.2, $\Theta^\perp$ is empty. By the induction hypothesis we have an MJ proof ending in:

$$\frac{\Sigma; P[t/x] \Rightarrow R}{\Sigma; \forall x.P \Rightarrow R}\ (\forall_{\mathcal{L}})$$

10. We can also add the cases for disjunction and the existential quantifier on the right.

■

The proofs of the two theorems above show that a fragment of MJ is indeed the calculus induced by the new embedding. There is therefore an isomorphism between the proofs in the fragment of ILL described above (that is, ILL over the grammar of the embedding) and the proofs in the fragment of MJ. The $\supset, \wedge, \forall$ fragment of MJ lives inside ILL. An obvious corollary of the above theorems is that the new embedding is correct and faithful for provability.

The embedding given above uses two translations: one for occurrences on the left and one for occurrences on the right. That the embedding requires this is, perhaps, not surprising, given the lack of symmetry in intuitionistic logic between the left and the righthand sides of the consequence relation and the symmetry that is observed in CLL. Note that embeddings using a positive and a negative translation have been used by several people when embedding calculi in linear logic. See for example, [Tro92], [HM94], [HP94].

As noted several times above, we have only given the embedding for the $\supset, \wedge, \forall$ fragment of intuitionistic logic. This is because our interest is in the induced calculi and these are unattractive outside of this fragment. We leave it as an open problem how to embed disjunction on the left, falsum and the existential quantifier on the left in order to induce MJ. We are not optimistic that a solution can be found.

The problems with some of the connectives result from trying to embed intuitionistic logic into unrestricted ILL. If we restricted the fragment we were looking at by, for example, only looking at sequents with one unbanged formula on the left, then we could embed to get the result required. However, in this case we are simply making the ILL calculus closer to the intuitionistic calculus.

Notice that the fragment of MJ we can induce by the new embedding is enough to cover hereditary Harrop formulae. That is, we can reason about this fragment of intuitionistic logic (important from the logic programming perspective) inside ILL. As noted in [Har94], hereditary Harrop logic is in some natural sense the largest well behaved fragment of intuitionistic logic (for example with respect to goal-directed proof search), and so we are not surprised that this is the largest fragment that can easily be embedded to give MJ. Harland and Pym have also embedded hereditary Harrop formulae into linear logic using a two function, positive and negative, embedding (see [HP94]). Their embedding into ILL doesn't induce a uniform proof calculus for hereditary Harrop formulae. If, however, the embedding is into a uniform proof calculus for linear logic, then the calculus induced will be a uniform proof calculus.

Embedding intuitionistic logic into linear logic has also been investigated (with different motivation) by Negri in [Neg95]. Also by Lincoln, Scedrov & Shankar ([LSS93]). Danos, Joinet and Schellinx have written extensively on embedding logics into linear logics. As well as Schellinx's thesis ([Sch94]), embeddings in intuitionistic logic into linear logic are given in [DJS95], [Sch92].

# Chapter 6

# A Sequent Calculus for Intuitionistic Linear Logic

In this chapter, the ideas behind the MJ calculus for intuitionistic logic are applied to Intuitionistic Linear Logic (ILL). We develop a Gentzen-system, SILL, for ILL whose derivations can be translated in a 1–1 way to the normal natural deductions for ILL. We prove some properties of SILL and discuss possible alternative systems. We also discuss SILL in relation to linear logic programming languages, paying particular attention to Lolli.

## 6.1 Natural Deduction

The primary natural deduction system we consider is that of Benton, Bierman, de Paiva and Hyland ([BBdPH92], [BBdPH93b], [BBdPH93a], [Bie94]). This can be seen in Figure 1.6. We are interested in deductions in normal form and we give the beta-reductions and commuting conversions from [Bie94] in order to define normal natural deductions for ILL.

With the promotion rule, the discharged assumptions are written as $[\![!P_1 \ldots !P_n]\!]$. This means that all assumptions are of the form $!P_i$ and that they are all discharged at $(P)$.

First beta-reductions:

1. Linear implication:

$$\dfrac{\dfrac{\begin{matrix}[P]\\ \vdots \\ Q\end{matrix}}{P \multimap Q}\,(\multimap_{\mathcal{I}}) \qquad \begin{matrix}\vdots \\ P\end{matrix}}{Q}\,(\multimap_{\mathcal{E}}) \qquad \rightsquigarrow_{\beta} \qquad \begin{matrix}\vdots \\ P \\ \vdots \\ Q\end{matrix}$$

2. $I$:

$$\dfrac{\dfrac{}{I}\,(I_{\mathcal{I}}) \qquad \begin{matrix}\vdots\\ P\end{matrix}}{P}\,(I_{\varepsilon}) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots\\ P\end{matrix}$$

3. Tensor:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ P\end{matrix} \quad \begin{matrix}\vdots\\ Q\end{matrix}}{P \otimes Q}\,(\otimes_{\mathcal{I}}) \qquad \begin{matrix}[P][Q]\\ \vdots\\ R\end{matrix}}{R}\,(\otimes_{\varepsilon}) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots & \vdots\\ P & Q\\ \multicolumn{2}{c}{\vdots}\\ \multicolumn{2}{c}{R}\end{matrix}$$

4. With:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ P\end{matrix} \quad \begin{matrix}\vdots\\ Q\end{matrix}}{P \& Q}\,(\&_{\mathcal{I}})}{P}\,(\&_{\varepsilon_1}) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots\\ P\end{matrix}$$

Also:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ P\end{matrix} \quad \begin{matrix}\vdots\\ Q\end{matrix}}{P \& Q}\,(\&_{\mathcal{I}})}{Q}\,(\&_{\varepsilon_2}) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots\\ Q\end{matrix}$$

5. Plus:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ P\end{matrix}}{P \oplus Q}\,(\oplus_{\mathcal{I}_1}) \qquad \begin{matrix}[P]\\ \vdots\\ R\end{matrix} \quad \begin{matrix}[Q]\\ \vdots\\ R\end{matrix}}{R}\,(\oplus_{\varepsilon}) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots\\ P\\ \vdots\\ R\end{matrix}$$

Also:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ Q\end{matrix}}{P \oplus Q}\,(\oplus_{\mathcal{I}_2}) \qquad \begin{matrix}[P]\\ \vdots\\ R\end{matrix} \quad \begin{matrix}[Q]\\ \vdots\\ R\end{matrix}}{R}\,(\oplus_{\varepsilon}) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots\\ Q\\ \vdots\\ R\end{matrix}$$

6. Ofcourse, promotion with dereliction:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ !P_1\end{matrix} \quad .... \quad \begin{matrix}\vdots\\ !P_n\end{matrix} \qquad \begin{matrix}[\![!P_1....!P_n]\!]\\ \vdots\\ Q\end{matrix}}{!Q}\,(P)}{Q}\,(D) \quad \leadsto_{\beta} \quad \begin{matrix}\vdots & & \vdots\\ !P_1 & .... & !P_n\\ & \vdots & \\ & Q & \end{matrix}$$

7. Ofcourse, promotion with weakening:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ !P_1\end{matrix} \quad .... \quad \begin{matrix}\vdots\\ !P_n\end{matrix} \qquad \begin{matrix}[\![!P_1....!P_n]\!]\\ \vdots\\ Q\end{matrix}}{!Q}\,(P) \qquad \begin{matrix}\vdots\\ R\end{matrix}}{R}\,(W) \quad \leadsto_{\beta} \quad \dfrac{\begin{matrix}\vdots\\ !P_1\end{matrix} \quad .... \quad \begin{matrix}\vdots\\ !P_n\end{matrix} \quad \begin{matrix}\vdots\\ R\end{matrix}}{R}\,(W*)$$

Note that the minor premisses of the promotion rule are ordered left to right, and therefore that the minor premisses of $(W*)$ are also ordered left to right. This also applies in the next reduction.

8. Ofcourse, promotion with contraction:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ !P_1\end{matrix} \quad \dots \quad \begin{matrix}\vdots\\ !P_n\end{matrix} \qquad \begin{matrix}[\![!P_1 \dots !P_n]\!]\\ \vdots \\ Q\end{matrix}}{!Q}\,(P) \qquad \begin{matrix}[!Q][!Q]\\ \vdots\\ R\end{matrix}}{R}\,(C)$$

$\leadsto_\beta$

$$\dfrac{\begin{matrix}\vdots\\ !P_1\end{matrix} \quad \dots \quad \begin{matrix}\vdots\\ !P_n\end{matrix} \qquad \begin{matrix}\dfrac{[!P_1]\dots[!P_n] \qquad \begin{matrix}[\![!P_1 \dots !P_n]\!]\\ \vdots \\ Q\end{matrix}}{!Q}\,(P) \qquad \dfrac{[!P_1]\dots[!P_n] \qquad \begin{matrix}[\![!P_1 \dots !P_n]\!]\\ \vdots \\ Q\end{matrix}}{!Q}\,(P) \\ \vdots \\ R\end{matrix}}{R}\,(C*)$$

Notice that the last two reductions involve the use of $(C*)$ and $(W*)$ — inference rules which are combinations of multiple uses of the ordinary $(C)$ and $(W)$ rules. Ordering these occurrences makes them a shorthand for $n$ rule applications, rather than new admissible rules in their own right (although the admissibility of such a rule could be shown).

Next, the commuting conversions. Here, the ellipsis *s* is for one of $(\otimes_\varepsilon)$, $(I_\varepsilon)$, $(\oplus_\varepsilon)$, $(0_\varepsilon)$, $(W)$, $(C)$, and *r* is for one of *s* or one of $(\multimap_\varepsilon)$, $(\&_{\varepsilon_1})$, $(\&_{\varepsilon_2})$, $(D)$. Note that all the commutations are presented with two premiss rules. The changes where the rule is in fact a single (or multiple) premiss rule are obvious.

9. Commutation of $(\otimes_\varepsilon)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ P\otimes Q\end{matrix} \qquad \begin{matrix}[P][Q]\\ \vdots\\ R\end{matrix}}{R}\,(\otimes_\varepsilon) \qquad \vdots}{S}\,r \quad \leadsto_c \quad \dfrac{\begin{matrix}\vdots\\ P\otimes Q\end{matrix} \qquad \dfrac{\begin{matrix}[P][Q]\\ \vdots\\ R\end{matrix} \qquad \vdots}{S}\,r}{S}\,(\otimes_\varepsilon)$$

10. Commutation of $(I_\varepsilon)$:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ I\end{matrix} \quad \begin{matrix}\vdots\\ P\end{matrix}}{P}\,(I_\varepsilon) \qquad \vdots}{S}\,r \quad \leadsto_c \quad \dfrac{\begin{matrix}\vdots\\ I\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ P\end{matrix} \quad \vdots}{S}\,r}{S}\,(I_\varepsilon)$$

11. Commutation of $(0_\varepsilon)$:

$$\dfrac{\dfrac{\vdots \quad \overset{\vdots}{0}}{Q}\,(0_\varepsilon) \qquad \overset{\Delta}{\vdots}}{R}\,r \quad \rightsquigarrow_c \quad \dfrac{\vdots \quad \overset{\Delta}{\vdots} \quad \overset{\vdots}{0}}{R}\,(0_\varepsilon)$$

12. Commutation of $(\oplus_\varepsilon)$:

$$\dfrac{\dfrac{\overset{\Gamma}{\vdots} \atop P \oplus Q \qquad \overset{\Delta, P}{\vdots} \atop R \qquad \overset{\Delta, Q}{\vdots} \atop R}{R}\,(\oplus_\varepsilon) \qquad \vdots}{S}\,r \quad \rightsquigarrow_c \quad \dfrac{\overset{\Gamma}{\vdots} \atop P \oplus Q \qquad \dfrac{\overset{\Delta, P}{\vdots} \atop R \quad \vdots}{S}\,r \qquad \dfrac{\overset{\Delta, Q}{\vdots} \atop R \quad \vdots}{S}\,r}{S}\,(\oplus_\varepsilon)$$

13. Commutation of $(W)$:

$$\dfrac{\dfrac{\overset{\vdots}{!Q} \quad \overset{\vdots}{R}}{R}\,(W) \qquad \vdots}{S}\,r \quad \rightsquigarrow_c \quad \dfrac{\overset{\vdots}{!Q} \qquad \dfrac{\overset{\vdots}{R} \quad \vdots}{S}\,r}{S}\,(W)$$

14. Commutation of $(C)$:

$$\dfrac{\dfrac{\overset{\vdots}{!Q} \qquad \overset{[!Q][!Q]}{\vdots} \atop R}{R}\,(C) \qquad \vdots}{S}\,r \quad \rightsquigarrow_c \quad \dfrac{\overset{\vdots}{!Q} \qquad \dfrac{\overset{[!Q][!Q]}{\vdots} \atop R \quad \vdots}{S}\,r}{S}\,(C)$$

15. Commutation of an $s$-rule with $(P)$:

$$\dfrac{\overset{\vdots}{!P_1} \quad .... \quad \dfrac{\overset{\vdots}{R} \quad \overset{\vdots}{!P_i}}{!P_i}\,s \quad .... \quad \overset{\vdots}{!P_n} \qquad \overset{[\![!P_1, ...., !P_n]\!]}{\vdots} \atop Q}{!Q}\,(P)$$

$\rightsquigarrow_c$

$$\dfrac{\overset{\vdots}{R} \qquad \dfrac{\overset{\vdots}{!P_1} \quad .... \quad \overset{\vdots}{!P_n} \qquad \overset{[\![!P_1 .... !P_n]\!]}{\vdots} \atop Q}{!Q}\,(P)}{!Q}\,s$$

16. Commutation of $(P)$ with $(P)$:

$$\dfrac{\overset{\vdots}{!P_1} \quad .... \quad \dfrac{\overset{\vdots}{!R_1} \quad .... \quad \overset{\vdots}{!R_m} \qquad \overset{[\![!R_1 .... !R_m]\!]}{\vdots} \atop P_i}{!P_i}\,(P) \quad .... \quad \overset{\vdots}{!P_n} \qquad \overset{[\![!P_1 .... !P_n]\!]}{\vdots} \atop Q}{!Q}\,(P)$$

$\leadsto_c$

$$\dfrac{\begin{matrix}\vdots \\ !P_1\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ !R_1\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ !R_m\end{matrix} \quad \dots \quad \begin{matrix}\vdots \\ !P_n\end{matrix} \qquad \begin{matrix} \dfrac{[\![!R_1 \dots !R_m]\!] \qquad \begin{matrix}[\![!R_1 \dots !R_m]\!] \\ \vdots \\ P_i\end{matrix}}{[\![!P_1 \dots !P_{i\perp 1}]\!] \; !P_i \; [\![!P_{i+1} \dots !P_n]\!]}\,(P) \\ \vdots \\ Q\end{matrix}}{!Q}\,(P)$$

We finally have some commutations associated with with the $(0_\varepsilon)$ and $(\top_\mathcal{I})$ rules.

17. Commutation of $(0_\varepsilon)$:

$$\dfrac{\begin{matrix}\vdots \\ 0\end{matrix} \quad \begin{matrix}\Gamma_1 \\ \vdots \\ P_1\end{matrix} \quad \dots \quad \begin{matrix}\Gamma_n \\ \vdots \\ P_n\end{matrix}}{Q}\,(0_\varepsilon) \quad \leadsto_c \quad \dfrac{\begin{matrix}\vdots \\ 0\end{matrix} \quad \Gamma_1 \quad \dots \quad \Gamma_n}{Q}\,(0_\varepsilon)$$

18. Commutation of $(\top_\mathcal{I})$:

$$\dfrac{\begin{matrix}\Gamma_1 \\ \vdots \\ P_1\end{matrix} \quad \dots \quad \begin{matrix}\Gamma_n \\ \vdots \\ P_n\end{matrix}}{\top}\,(\top_\mathcal{I}) \quad \leadsto_c \quad \dfrac{\Gamma_1 \quad \dots \quad \Gamma_n}{\top}\,(\top_\mathcal{I})$$

**Definition 6.1** *A deduction in* NILL *is said to be in* **($\beta$, $c$)-normal form** *if no $\beta$-reductions and no $c$-reductions can be applied to it.*

### 6.1.1 Rewriting to normal form

It would be preferable if every natural deduction rewrote to a unique normal form. That is, that the above normalisation process was confluent. Unfortunately the above normalisation procedure is not confluent. Firstly, consider the beta-reductions for promotion with weakening and promotion with contraction (reductions 7. and 8.) As noted above, these reductions involve multiple applications of weakening/contraction. As the usual formulations of these rules have single formulae being weakened/contracted, we are left with a choice: either we make the multiple rules primitive, or we have them simply as a shorthand for multiple applications of the single rule.

We choose the latter option. This means that the $(W*)$ and $(C*)$ rules have ordered premisses. Now consider promotion. We again have a choice. Are the minor premisses ordered or not?

On the one hand, if they are, then there are proofs which are not equivalent, yet are the same up to the ordering of the minor premisses. On the other hand, if they are not ordered, then we can perform the reductions in many ways.

In fact we stipulate that the minor premisses of promotion are ordered (left to right as written). Later in this chapter, the existence of an order will make the calculus easier to handle.

However, this is not the only place where we have a problem with confluence. Consider the commutation of an $s$-rule with promotion (reduction 15.) Even with an order on the minor premisses of promotion, this reduction is not confluent. For example:

$$\dfrac{\dfrac{B_1 \otimes B_2 \quad !A_1}{!A_1}(\otimes_\varepsilon) \qquad \dfrac{B_3 \otimes B_4 \quad !A_2}{!A_2}(\otimes_\varepsilon) \qquad \begin{matrix}[\![!A_1, !A_2]\!] \\ \vdots \\ C\end{matrix}}{!C}(P)$$

$\leadsto$

$$\dfrac{B_1 \otimes B_2 \qquad \dfrac{B_3 \otimes B_4 \qquad \dfrac{!A_1 \quad !A_2 \qquad \begin{matrix}[\![!A_1, !A_2]\!] \\ \vdots \\ C\end{matrix}}{!C}(P)}{!C}(\otimes_\varepsilon)}{!C}(\otimes_\varepsilon)$$

or

$\leadsto$

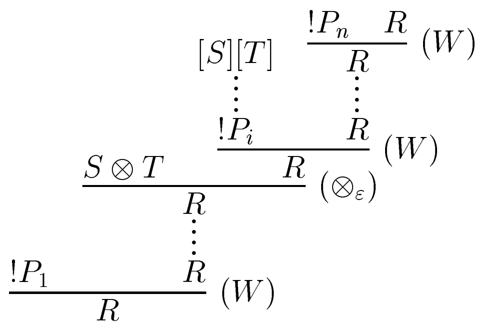
$$\dfrac{B_3 \otimes B_4 \qquad \dfrac{B_1 \otimes B_2 \qquad \dfrac{!A_1 \quad !A_2 \qquad \begin{matrix}[\![!A_1, !A_2]\!] \\ \vdots \\ C\end{matrix}}{!C}(P)}{!C}(\otimes_\varepsilon)}{!C}(\otimes_\varepsilon)$$

The following example illustrates the non-confluence introduced by the interaction of $\beta$- and c-reductions:

$$\dfrac{\dfrac{!P_1 \quad \dots \quad \dfrac{S \otimes T \qquad \begin{matrix}[S][T] \\ \vdots \\ !P_i\end{matrix}}{!P_i}(\otimes_\varepsilon) \quad \dots \quad !P_n \qquad \begin{matrix}[\![!P_1 \dots !P_n]\!] \\ \vdots \\ Q\end{matrix}}{!Q}(P) \qquad R}{R}(W)$$

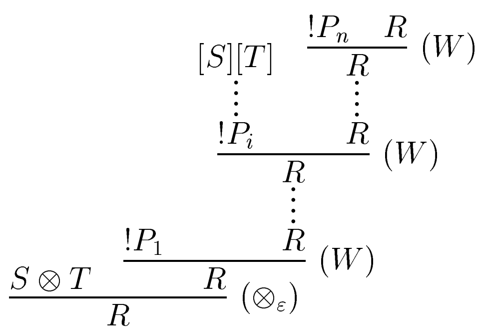
We can reduce in two ways: either first perform the $\beta$-reduction then the commuting conversion, or first perform the commuting conversion then then $\beta$-reduction.

$\leadsto$



or

$\leadsto$



It is known that if we consider only $\beta$-reductions then normalisation is confluent and strongly normalising. See [Bie94] and [Ben95].

What can we say for $(\beta, c)$-reduction? The above examples show that confluence does not hold. We can, of course, give a strategy for normalising non-normal deductions which would give a unique normal form. For example, pick any top most non-normal inference and recursively normalise.

What can be said is that all proofs in $(\beta, c)$-normal form are *irreducible*. A proof is irreducible if no normalisation steps can be applied to it. The $(\beta, c)$-normal proofs are all the irreducible proofs.

These problems with normal proofs suggest a more involved notion of normal form for ILL, as discussed in section 6.3.

## 6.2 Term Assignment for Normal Natural Deductions

This section details a term assignment system whose terms are in 1–1 correspondence with NILL deductions in $(\beta,\ c)$-normal form. We also give a sequent-style natural deduction calculus allowing only deductions in normal form. This deduction system exactly types the proof terms. This calculus has two judgement forms in order to restrict the deductions to those in $(\beta, c)$-normal form.

The term assignment has two syntactic categories, **A** and **N**. The normal proofs are given by the **N** terms. These are displayed below (where V is the category of variables):

**A**::=

$$var(V) \mid ap(A, N) \mid der(A) \mid withe1(A) \mid withe2(A)$$

**N**::=

$$* \mid ie(A, N) \mid tene(A, V.V.N) \mid weak(A, N) \mid cont(A, V.V.N) \mid withi(N, N) \mid$$

$$plusi1(N) \mid plusi2(N) \mid pluse(A, V.N, V.N) \mid tr(\{var(V), ..., var(V)\})$$

$$an(A) \mid \lambda V.N \mid teni(N, N) \mid prom(\vec{A}, \vec{V}.N) \mid fal(A, \{var(V), ..., var(V)\})$$

The full calculus with term assignments, which we call NNILL, is presented in sequent style in Figure 6.1.

### 6.2.1 Justification of the Restrictions

We now go through each of the $\beta$-reductions and commuting conversions and show that none of them can be performed in the calculus presented in the previous section.

1. This is not applicable since the conclusion of $(\multimap_{\mathcal{I}})$ is an **N** term whereas the left premiss of $(\multimap_{\varepsilon})$ has to be an **A** term.

2. This is not applicable since the conclusion of $(I_{\mathcal{I}})$ is an **N** term whereas the left premiss of $(I_{\varepsilon})$ has to be an **A** term.

3. This is not applicable since the conclusion of $(\otimes_{\mathcal{I}})$ is an **N** term whereas the left premiss of $(\otimes_{\varepsilon})$ has to be an **A** term.

4. This is not applicable since the conclusion of $(\&_{\mathcal{I}})$ is an **N** term whereas the premiss of $(\&_{\varepsilon})$ has to be an **A** term.

5. This is not applicable since the conclusion of $(\oplus_{\mathcal{I}})$ is an **N** term whereas the leftmost premiss of $(\oplus_{\varepsilon})$ has to be an **A** term.

6. This is not applicable since the conclusion of $(P)$ is an **N** term whereas the premiss of $(D)$ has to be an **A** term.

7. This is not applicable since the conclusion of $(P)$ is an **N** term whereas the left premiss of $(W)$ has to be an **A** term.

8. This is not applicable since the conclusion of $(P)$ ia an **N** term whereas the left premiss of $(C)$ has to be an **A** term.

9. This is not applicable since the conclusion of $(\otimes_{\varepsilon})$ is an **N** term whereas the left premiss of any of the $r$-rules has to be an **A** term.

$$\frac{}{x : P \rhd var(x) : P}\ (ax)$$

$$\frac{\Gamma \rhd A : P \multimap Q \quad \Delta \rhd\!\!\rhd N : P}{\Gamma, \Delta \rhd ap(A, N) : Q}\ (\multimap_{\varepsilon}) \qquad \frac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd \lambda x.N : P \multimap Q}\ (\multimap_{\mathcal{I}})$$

$$\frac{}{\phi \rhd\!\!\rhd * : I}\ (I_{\mathcal{I}}) \qquad \frac{\Gamma \rhd A : I \quad \Delta \rhd\!\!\rhd N : P}{\Gamma, \Delta \rhd\!\!\rhd ie(A, N) : P}\ (I_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd N_1 : P \quad \Delta \rhd\!\!\rhd N_2 : Q}{\Gamma, \Delta \rhd\!\!\rhd teni(N_1, N_2) : P \otimes Q}\ (\otimes_{\mathcal{I}})$$

$$\frac{\Gamma \rhd A : P \otimes Q \quad \Delta, x_1 : P, x_2 : Q \rhd\!\!\rhd N : R}{\Gamma, \Delta \rhd\!\!\rhd tene(A, x_1.x_2.N) : R}\ (\otimes_{\varepsilon})$$

$$\frac{\Gamma \rhd A :!P \quad \Delta \rhd\!\!\rhd N : Q}{\Gamma, \Delta \rhd\!\!\rhd weak(A, N) : Q}\ (W) \qquad \frac{\Gamma \rhd A :!P \quad \Delta, x_1 :!P, x_2 :!P \rhd\!\!\rhd N : Q}{\Gamma, \Delta \rhd\!\!\rhd cont(A, x_1.x_2.N) : Q}\ (C)$$

$$\frac{\Gamma \rhd A :!P}{\Gamma \rhd der(A) : P}\ (D) \qquad \frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P}\ (M)$$

$$\frac{\Delta_1 \rhd A_1 :!P_1 \quad .... \quad \Delta_n \rhd A_n :!P_n \quad x_1 :!P_1, ...., x_n :!P_n \rhd\!\!\rhd N : Q}{\Delta_1, ...., \Delta_n \rhd\!\!\rhd prom(\vec{A}, \vec{x}\ .N) :!Q}\ (P)$$

$$\frac{\Gamma \rhd\!\!\rhd N_1 : P \quad \Gamma \rhd\!\!\rhd N_2 : Q}{\Gamma \rhd\!\!\rhd withi(N_1, N_2) : P \& Q}\ (\&_{\mathcal{I}})$$

$$\frac{\Gamma \rhd A : P \& Q}{\Gamma \rhd withe1(A) : P}\ (\&_{\varepsilon_1}) \qquad \frac{\Gamma \rhd A : P \& Q}{\Gamma \rhd withe2(A) : Q}\ (\&_{\varepsilon_2})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd plusi1(N) : P \oplus Q}\ (\oplus_{\mathcal{I}_1}) \qquad \frac{\Gamma \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd plusi2(N) : P \oplus Q}\ (\oplus_{\mathcal{I}_2})$$

$$\frac{\Gamma \rhd A : P \oplus Q \quad \Delta, x_1 : P \rhd\!\!\rhd N_1 : R \quad \Delta, x_2 : Q \rhd\!\!\rhd N_2 : R}{\Gamma, \Delta \rhd\!\!\rhd pluse(A, x_1.N_1, x_2.N_2) : R}\ (\oplus_{\varepsilon})$$

$$\frac{P_1 \rhd var(x_1) : P_1 \quad .... \quad P_n \rhd var(x_n) : P_n}{P_1, ..., P_n \rhd\!\!\rhd tr(\{var(x_1), ..., var(x_n)\}) : \top}\ (\top_{\mathcal{I}})$$

$$\frac{P_1 \rhd var(x_1) : P_1 \quad .... \quad P_n \rhd var(x_n) : P_n \quad \Delta \rhd A : 0}{\Delta, P_1, ..., P_n \rhd\!\!\rhd fal(A, \{var(x_1), ..., var(x_n)\}) : Q}\ (0_{\varepsilon})$$

**Figure 6.1:** NNILL: Sequent style natural deduction calculus for ILL, giving normal natural deductions, together with term assignments.

10. This is not applicable since the conclusion of $(I_\varepsilon)$ is an **N** term whereas the left premiss of any of the $r$-rules has to be an **A** term.

11. This is not applicable since the conclusion of $(0_\varepsilon)$ is an **N** term whereas the left premiss of any of the $r$-rules has to be an **A** term.

12. This is not applicable since the conclusion of $(\oplus_\varepsilon)$ is an **N** term whereas the left premiss of any of the $r$-rules has to be an **A** term.

13. This is not applicable since the conclusion of $(W)$ is an **N** term whereas the left premiss of any of the $r$-rules has to be an **A** term.

14. This is not applicable since the conclusion of $(C)$ is an **N** term whereas the left premiss of any of the $r$-rules has to be an **A** term.

15. This is not applicable since the conclusion of any $s$-rule is an **N** term whereas the minor premisses of $(P)$ have to be **A** terms.

16. This is not applicable since the conclusion of $(P)$ is an **N** term whereas the minor premisses of $(P)$ have to be **A** terms.

17. This is not applicable since all the minor premisses of $(0_\varepsilon)$ must be instances of $(ax)$.

18. This is not applicable since all the premisses of $(\top_\mathcal{I})$ must be instances of $(ax)$.

Hence none of the reductions and commutations are applicable. Due to the $(M)$ rule, every other combination of inferences that was possible before is still possible. Therefore the calculus does, as claimed, capture exactly the ($\beta$, $c$)-normal natural deductions of ILL.

**Proposition 6.1** *The calculus* NNILL *generates exactly the (*$\beta$*,* $c$*)-normal natural deductions of* ILL*.*

### 6.2.2 Multiple Field Version of Natural Deduction

It should be noted that natural deduction for ILL might be presented with the assumptions split into two fields. One field contains linear assumptions which have to be discharged exactly once. The other contains non-linear (that is, banged) assumptions packets – as in the usual natural deduction formulations for intuitionistic logic. The rules then have to be adapted to take this into account and weakening and contraction can be replaced by a single structural rule. We might find this an attractive approach as it ties in with other work on linear logic and logic programming. See for example the calculus ILL$^\Sigma$ in Figure 2.2 and the discussion in section 6.7.

## 6.3 Alternative Natural Deduction and Term Systems

We have given the results so far for one presentation of natural deduction for ILL. There are, however, several others in the literature, some of which are discussed here.

### 6.3.1 Logical Constants

We have given the system as formulated in [Bie94]. This formulation of natural deduction has multiple premiss rules for $(\top_{\mathcal{I}})$ and $(0_\varepsilon)$ along with some reduction rules. We could have replaced these rules with the following:

$$\frac{}{x_1 : P_1, ..., x_n : P_n \rhd\!\rhd tr(\{x_1, ..., x_n\}) : \top}\ (\top_{\mathcal{I}})$$

$$\frac{\Delta \rhd A : 0}{x_1 : P_1, ..., x_n : P_n, \Delta \rhd\!\rhd fal(\{x_1, ..., x_n\}, A) : R}\ (0_\varepsilon)$$

NNILL with these rules (or NILL with similar rules) remains closed under substitution.

### 6.3.2 Promotion

Early formulations of natural deduction used the following apparently simpler introduction rule for !:

$$\frac{!\Gamma \Rightarrow Q}{!\Gamma \Rightarrow !Q}\ (P)$$

This is the rule to be found in [Avr88], [Abr93], [Wad92], originally in [Tro92], [Val92] and [RdRR97]. Unfortunately, natural deduction with this rule is not closed under substitution. This is a fairly fundamental property from a computational point of view, and so another formulation is desirable. The system we have already described above is closed under substitution, as is the system NAT in [LM92] (this system is similar to the one we discuss, in particular, it has the same rule for promotion).

The promotion rule for ILL suggested so far is still a rather strange looking rule. It is an introduction rule, yet looks more like an elimination rule. It has the form taken in order to make the possibility of substitution explicit – the rule can be thought of as a promotion in the style rejected above, together with $n$ substitutions. As noted above, it has to be decided whether the premisses of promotion are ordered or not. We are unsatisfied with our answer to this question. This motivates attempts to find another way of looking at the promotion rule in natural deduction.

$$\frac{\Xi \vdash Q}{\Xi \vdash !Q}\ (P)$$

$$\frac{\Gamma \vdash !P}{< \Gamma >_{!P} \vdash !P}\ (<>_{\mathcal{I}}) \qquad \frac{< \Gamma >_{!Q}, \Delta \vdash P}{\Gamma, \Delta \vdash P}\ (<>_{\varepsilon})$$

$$\frac{\Gamma, M, M \vdash R}{\Gamma, M \vdash R}\ (C)$$

$\Xi$ consists of bracket formulae and ! formulae.
$M$ consists of bracket formulae and ! formulae.

**Figure 6.2:** MBILL: natural deduction rules involving the $<>$ bracket

In [Tro95] another approach to natural deduction for ILL is developed. The promotion rule is given as follows:

$$\begin{array}{ccc} \mathcal{D}_1 & & \mathcal{D}_n \\ \vdots & & \vdots \\ [\![ !R_1 & \ldots & !R_n ]\!] \\ & \vdots\ \mathcal{D} & \\ & \dfrac{Q}{!Q}\ (P) & \end{array}$$

Here $[\![ !R_1 ... !R_n ]\!]$ is a complete list of open assumptions in deduction $\mathcal{D}$ and where deductions $\mathcal{D}_i$ may be substituted for the $!R_i$. That is, this rule is much like the one above in that it makes the possibility of substitution explicit. However, this approach is hard to extend into the additives.

Yet another approach to natural deduction for ILL comes from Mints. In [Min95] a natural deduction system is presented which avoids the use of an elimination-like promotion rule by using an explicit notation for these substitution-like aspects of promotion. This also orders the occurrences in a way that doesn't happen in the original system. The new rules can be seen in Figure 6.2 (the rest of the calculus is as before).

We have given the contraction rule as presented by Mints, although we could use the one given earlier. The judgement $< \Gamma >_{!P}, \Delta \vdash Q$ can be read as "$Q$ is deducible from $\Delta$ and $!P$; also, $!P$ is deducible from $\Gamma$". Notice that the system as presented is not closed under substitution, but that by restricting the condition on promotion so that all assumptions are bracketed, the system becomes closed under substitution.

Whereas in the natural deduction system presented in Figure 1.6, all the substitutions occur as part of the promotion rule, here they occur individually before the promotion. Although we use the Benton *et al.* system we could easily have used that of Mints instead. Indeed we find some of its features more attractive than the one we use, but are unhappy about the use of the brackets as some sort of logical connective – we do not feel that we understand it properly. Also, the commuting conversions for the bracket elimination rule are not obvious. It appears to commute with everything, including itself, in either direction. Then in what order do these eliminations occur?

### 6.3.3 Tensor Elimination

One of the most unsatisfactory features of normal natural deductions for ILL are chains of tensor eliminations. For example:

$$\dfrac{\Gamma_2 \rhd C \otimes D \quad \dfrac{\Gamma_1 \rhd A \otimes B \quad \Delta, A, B, C, D \rhd\!\!\rhd E}{\Gamma_1, \Delta, C, D \rhd\!\!\rhd E}\ (\otimes_\varepsilon)}{\Gamma_1, \Gamma_2, \Delta \rhd\!\!\rhd E}\ (\otimes_\varepsilon)$$

and

$$\dfrac{\Gamma_1 \rhd A \otimes B \quad \dfrac{\Gamma_2 \rhd C \otimes D \quad \Delta, A, B, C, D \rhd\!\!\rhd E}{\Gamma_2, \Delta, A, B \rhd\!\!\rhd E}\ (\otimes_\varepsilon)}{\Gamma_1, \Gamma_2, \Delta \rhd\!\!\rhd E}\ (\otimes_\varepsilon)$$

would seem to the same, yet are different normal natural deductions. A solution would be to have an $n$ premiss tensor elimination rule. Such an approach is outlined by Mints in [Min97], [Min98]. The new tensor elimination rule is:

$$\dfrac{\Gamma_1 \rhd P_1 \otimes Q_1 \quad \ldots \quad \Gamma_n \rhd P_n \otimes Q_n \quad \Delta, P_1, \ldots, P_n, Q_1, \ldots, Q_n \rhd\!\!\rhd R}{\Gamma_1, \ldots, \Gamma_n, \Delta \rhd\!\!\rhd R}\ (\otimes_\varepsilon)$$

Extra normalisation steps would then have to be added to bring tensor eliminations together into such a rule. One might also add extra rules to commute tensor eliminations with other elimination rules. Such a natural deduction would greatly improve the denotational power of the natural deduction system, bringing it much closer to expressing the equalities we would like. However, in this thesis we work with the usual system for normal natural deductions.

## 6.4 Sequent Calculus

We now describe a calculus in the style of MJ for ILL. We call this calculus SILL (for 'Stouped' Intuitionistic Linear Logic). We do not describe SILL as 'permutation-free' since the study of the permutability of the inference rules of ILL conducted in Chapter 2 shows that many derivations that would be seem to be equivalent are not identified by SILL. Most obvious amongst these are those to do with $(\otimes_{\mathcal{L}})$ – these permutations correspond to ones that it would appear natural to identify even in natural deduction, but are not identified under usual formulations of normal form for natural deductions (see discussion in the previous section). SILL does have the property that its proofs can be translated in a 1–1 with normal natural deductions for ILL, the proofs of NNILL. The sequent calculus SILL can be seen in Figure 6.3.

This calculus has three forms of judgement. There are the usual sequents with no privileged formula, there are sequents with a single stoup which behave much like stoup sequents for MJ. Finally there are sequents with a multiple stoup, of the form $\Gamma \xrightarrow{[!\Psi][\Delta]}\!\!\!\gg R$. This is the form of judgement reflecting the structure of the promotion rule for natural deduction. The *multistoup* contains two lists, one of banged

$$\frac{}{\phi \xrightarrow{P} P}\ (ax) \qquad \frac{\Gamma \stackrel{[!\Phi,!P][\Psi]}{\longrightarrow\!\!\!\!\gg} R}{\Gamma \stackrel{[!\Phi][!P,\Psi]}{\longrightarrow\!\!\!\!\gg} R}\ (tog)$$

$$\frac{\Gamma \stackrel{[\ ][\Delta]}{\longrightarrow\!\!\!\!\gg} R}{\Gamma, \Delta \Rightarrow R}\ (sel*) \qquad \frac{\Gamma \xrightarrow{Q} R}{\Gamma, Q \Rightarrow R}\ (sel)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow P \quad \Delta \stackrel{[!\Phi][Q,\Psi]}{\longrightarrow\!\!\!\!\gg} R}{\Gamma, \Delta \stackrel{[!\Phi][P\multimap Q,\Psi]}{\longrightarrow\!\!\!\!\gg} R}\ (\multimap_{\mathcal{L}}*) \qquad \frac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{Q} R}{\Gamma, \Delta \xrightarrow{P\multimap Q} R}\ (\multimap_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Delta \Rightarrow Q}{\Gamma, \Delta \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}}) \qquad \frac{\Gamma, P, Q \Rightarrow R}{\Gamma \xrightarrow{P\otimes Q} R}\ (\otimes_{\mathcal{L}})$$

$$\frac{}{\phi \Rightarrow I}\ (I_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P}{\Gamma \xrightarrow{I} P}\ (I_{\mathcal{L}})$$

$$\frac{}{\Gamma \Rightarrow \top}\ (\top_{\mathcal{R}}) \qquad \frac{}{\Gamma \xrightarrow{0} P}\ (0_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \oplus Q}\ (\oplus_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \oplus Q}\ (\oplus_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma \xrightarrow{P\oplus Q} R}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P\&Q}\ (\&_{\mathcal{R}}) \qquad \frac{\Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{P\&Q} R}\ (\&_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P\&Q} R}\ (\&_{\mathcal{L}_2})$$

$$\frac{\Gamma \stackrel{[!\Phi][P,\Psi]}{\longrightarrow\!\!\!\!\gg} R}{\Gamma \stackrel{[!\Phi][P\&Q,\Psi]}{\longrightarrow\!\!\!\!\gg} R}\ (\&_{\mathcal{L}_1}*) \qquad \frac{\Gamma \stackrel{[!\Phi][Q,\Psi]}{\longrightarrow\!\!\!\!\gg} R}{\Gamma \stackrel{[!\Phi][P\&Q,\Psi]}{\longrightarrow\!\!\!\!\gg} R}\ (\&_{\mathcal{L}_2}*)$$

$$\frac{\Gamma \Rightarrow R}{\Gamma \xrightarrow{!P} R}\ (W) \qquad \frac{\Gamma, !P, !P \Rightarrow R}{\Gamma \xrightarrow{!P} R}\ (C)$$

$$\frac{\Gamma \stackrel{[!\Phi][P,\Psi]}{\longrightarrow\!\!\!\!\gg} R}{\Gamma \stackrel{[!\Phi][!P,\Psi]}{\longrightarrow\!\!\!\!\gg} R}\ (D*) \qquad \frac{\Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{!P} R}\ (D) \qquad \frac{!\Psi \Rightarrow P}{\phi \stackrel{[!\Psi][\ ]}{\longrightarrow\!\!\!\!\gg} !P}\ (P)$$

**Figure 6.3:** Sequent calculus SILL for Intuitionistic Linear Logic

formulae, and one of formulae of a certain form (formulae built from any banged formulae using !, $\multimap$, $\&$, where the formula $P$ in $P \multimap Q$ can be of any form). In backwards proof search, a multiset of formulae is selected and ordered and becomes the second list in the multistoup of the premiss. The first formula is then principal. If it is banged it can be derelicted or appended to the first list. If the formula is an implication or a with, then the appropriate rules may be applied, the result remaining principal. There are no rules for formulae with other top connectives in the multistoup – that is to say, they should not be there. Each formula in turn is decomposed until the formula at the head of the second list is a bang formula. Each of these compositions corresponds to a minor premiss of promotion in normal natural deduction. When the second list is empty and the context is empty, a promotion is possible (and is the only applicable rule). Notice that, since this is the only way of leaving a multistoup, we should only perform $(sel*)$ when the goal is banged. We should point out that we have yet to mention cut. Cut is eliminable in ILL and will be discussed in section 6.6.

### 6.4.1 Term Assignment

We also give a term assignment system. There are again different kinds of term corresponding to the different judgement forms of the calculus, that is, there are three kinds of proof terms. Again the terms are typed by the sequents of SILL. We give the proof terms below (V is the category of variables).

**M::=**

$$(\overrightarrow{V}; Mss^i) \mid (V; Ms) \mid \lambda V.M \mid tenr(M, M) \mid * \mid tr(\{V, ..., V\})$$

$$plusr1(M) \mid plusr2(M) \mid withr(M, M)$$

**Ms::=**

$$[\,] \mid (M :: Ms) \mid tenl(V.V.M) \mid il(M) \mid fal(\{V, ..., V\}) \mid plusl(V.M, V.M)$$

$$withl1(Ms) \mid withl2(Ms) \mid w(M) \mid c(V.V.M) \mid d(Ms)$$

We need to explain the notation for the following **Mss**$^i$ terms. The **Mss**$^i$ have been written with a superscript. These superscripts are natural numbers and form part of the detail of the proof term. They are included to ensure that the terms are built in a specific order, as the sequents are.

**Mss**$^i$**::=**

$$(tog(Mss^{i+1}))^i \mid (M :: Mss^i)^i \mid (withl1(Mss^i))^i \mid (withl2(Mss^i))^i$$

$$(d(Mss^i))^i \mid (p(\overrightarrow{V}.M))^{n+1}$$

SILL together with its term system can be seen in Figures 6.4, 6.5 and 6.6.

$$\frac{\Gamma \xrightarrow{[\,][\Delta]}\!\!\gg Mss^1 : R}{\Gamma, \vec{x} : \Delta \Rightarrow (\vec{x}; Mss^1) : R}\ (sel*) \qquad \frac{\Gamma \xrightarrow{Q} Ms : R}{\Gamma, x : Q \Rightarrow (x; Ms) : R}\ (sel)$$

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \multimap Q}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Delta \Rightarrow M_2 : Q}{\Gamma, \Delta \Rightarrow tenr(M_1, M_2) : P \otimes Q}\ (\otimes_{\mathcal{R}})$$

$$\frac{}{\phi \Rightarrow * : I}\ (I_{\mathcal{R}}) \qquad \frac{}{\{x_i\} : \Gamma \Rightarrow tr(\{x_i\}) : \top}\ (\top_{\mathcal{R}})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow plusr1(M) : P \oplus Q}\ (\oplus_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow M : Q}{\Gamma \Rightarrow plusr2(M) : P \oplus Q}\ (\oplus_{\mathcal{R}_2})$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma \Rightarrow M_2 : Q}{\Gamma \Rightarrow withr(M_1, M_2) : P \& Q}\ (\&_{\mathcal{R}})$$

**Figure 6.4:** SILL with proof-term annotations: **M** terms.

$$\frac{}{\phi \xrightarrow{P} [\,] : P}\ (ax)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Delta \xrightarrow{Q} Ms : R}{\Gamma, \Delta \xrightarrow{P \multimap Q} (M :: Ms) : R}\ (\multimap_{\mathcal{L}})$$

$$\frac{\Gamma, x_1 : P, x_2 : Q \Rightarrow M : R}{\Gamma \xrightarrow{P \otimes Q} tenl(x_1.x_2.M) : R}\ (\otimes_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \xrightarrow{I} il(M) : P}\ (I_{\mathcal{L}}) \qquad \frac{}{\{x_i\} : \Gamma \xrightarrow{0} fal(\{x_i\}) : P}\ (0_{\mathcal{L}})$$

$$\frac{\Gamma, x_1 : P \Rightarrow M_1 : R \quad \Gamma, x_2 : Q \Rightarrow M_2 : R}{\Gamma \xrightarrow{P \oplus Q} plusl(x_1.M_1, x_2.M_2) : R}\ (\oplus_{\mathcal{L}})$$

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{P \& Q} withl1(Ms) : R}\ (\&_{\mathcal{L}_1}) \qquad \frac{\Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \& Q} withl2(Ms) : R}\ (\&_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow M : R}{\Gamma \xrightarrow{!P} w(M) : R}\ (W) \qquad \frac{\Gamma, x_1 :!P, x_2 :!P \Rightarrow M : R}{\Gamma \xrightarrow{!P} c(x_1.x_2.M) : R}\ (C)$$

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{!P} d(Ms) : R}\ (D)$$

**Figure 6.5:** SILL with proof-term annotation: **Ms** terms.

$$\frac{\Gamma \xrightarrow{[!\Phi,!P][\Psi]} Mss^{i+1} : R}{\Gamma \xrightarrow{[!\Phi][!P,\Psi]} (tog(Mss^{i+1}))^i : R}\ (tog)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Delta \xrightarrow{[!\Phi][Q,\Psi]} Mss^i : R}{\Gamma, \Delta \xrightarrow{[!\Phi][P\multimap Q,\Psi]} (M :: Mss^i)^i : R}\ (\multimap_{\mathcal{L}}*)$$

$$\frac{\Gamma \xrightarrow{[!\Phi][P,\Psi]} Mss^i : R}{\Gamma \xrightarrow{[!\Phi][P\&Q,\Psi]} (withl1(Mss^i))^i : R}\ (\&_{\mathcal{L}_1}*)$$

$$\frac{\Gamma \xrightarrow{[!\Phi][Q,\Psi]} Mss^i : R}{\Gamma \xrightarrow{[!\Phi][P\&Q,\Psi]} (withl2(Mss^i))^i : R}\ (\&_{\mathcal{L}_2}*)$$

$$\frac{\Gamma \xrightarrow{[!\Phi][P,\Psi]} Mss^i : R}{\Gamma \xrightarrow{[!\Phi][!P,\Psi]} (d(Mss^i))^i : R}\ (D*)$$

$$\frac{\{x_i\} :!\Psi \Rightarrow M : P}{\phi \xrightarrow{[!\Psi][\ ]} (p(\overrightarrow{x}.M))^{n+1} :!P}\ (P)\dagger$$

† n is the number of elements in $\{x_i\}$.

**Figure 6.6:** SILL with proof-term annotation: **Mss**$^i$ Terms.

## 6.5 The Correspondence Between Natural Deduction and Sequent Calculus for ILL

We have given a calculus for normal natural deductions, along with a term assignment system for this calculus. We have also given a sequent calculus for ILL, which restricts the sequent derivations which can be found in backward proof search in the calculus. Again we have given a term assignment system for this calculus. We claim that the sequent derivations in SILL naturally correspond to the normal natural deductions (the deductions of NNILL) in a 1–1 way. In order to prove this we give mappings from proof terms to proof terms in both directions, hence we have an isomorphism between proof terms.

**Sequent Calculus → Natural Deduction:**

$\theta$:**M** $\longrightarrow$ **N**

$$\theta(\overrightarrow{x}; Mss^1) = \theta'(\overrightarrow{var(x)}, Mss^1)$$

$$\theta((x; Ms)) = \theta'(var(x), Ms)$$

$$\theta(\lambda x.M) = \lambda x.\theta(M)$$

$$\theta(tenr(M_1, M_2)) = teni(\theta(M_1), \theta(M_2))$$

$$\theta(*) = *$$

$$\theta(tr(\{x_i\})) = tr(\{var(x_i)\})$$

$$\theta(withr(M_1, M_2)) = withi(\theta(M_1), \theta(M_2))$$

$$\theta(plusr1(M)) = plusi1(\theta(M))$$

$$\theta(plusr2(M)) = plusi2(\theta(M))$$

$\theta'$ : **A** $\times$ **Ms** $\longrightarrow$ **N**

$$\theta'(A, [\,]) = an(A)$$

$$\theta'(A, (M :: Ms)) = \theta'(ap(A, \theta(M)), Ms)$$

$$\theta'(A, tenl(x_1.x_2.M)) = tene(A, x_1.x_2.\theta(M))$$

$$\theta'(A, il(M)) = ie(A, \theta(M))$$

$$\theta'(A, fal(\{x_1, ..., x_n\})) = fal(A, \{var(x_1), ..., var(x_n)\})$$

$$\theta'(A, plusl(x_1.M_1, x_2.M_2)) = pluse(A, x_1.\theta(M_1), x_2.\theta(M_2))$$

$$\theta'(A, withl1(Ms)) = \theta'(withe1(A), Ms)$$

$\theta'(A, withl2(Ms)) = \theta'(withe2(A), Ms)$

$\theta'(A, w(M)) = weak(A, \theta(M))$

$\theta'(A, c(x_1.x_2.M)) = cont(A, x_1.x_2.\theta(M))$

$\theta'(A, d(Ms)) = \theta'(der(A), Ms)$

$\theta'' : \overrightarrow{\mathbf{A}} \times \mathbf{Mss}^i \longrightarrow \mathbf{N}$

$\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i)$
$= \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1})$

$\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (M :: Mss^i)^i)$
$= \theta''([A_1, ..., ap(A_i, \theta(M)), var(x_{i+1}), ..., var(x_n)], Mss^i)$

$\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i)$
$= \theta''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$

$\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i)$
$= \theta''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$

$\theta''([A_1, , ..., A_i, var(x_{i+1}), ..., var(x_n)], (d(Mss^i))^i)$
$= \theta''([A_1, ...der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$

$\theta''([A_1, ...., A_n], (p(\overrightarrow{x}.M))^{n+1}) = prom(\overrightarrow{A}, \overrightarrow{x}.\theta(M))$

**Natural Deduction $\longrightarrow$ Sequent Calculus**

$\psi$:**N** $\longrightarrow$ **M**

$\psi(\lambda x.N) = \lambda x.\psi(N)$

$\psi(teni(N_1, N_2)) = tenr(\psi(N_1), \psi(N_2))$

$\psi(*) = *$

$\psi(ie(A, N)) = \psi'(A, il(\psi(N)))$

$\psi(tene(A, x_1.x_2.N)) = \psi'(A, tenl(x_1.x_2.\psi(N)))$

$\psi(cont(A, x_1.x_2.N)) = \psi'(A, c(x_1.x_2.\psi(N)))$

$\psi(weak(A, N) = \psi'(A, w(\psi(N)))$

$\psi(prom(\overrightarrow{A}, \overrightarrow{x}.N)) = \psi''([A_1, ..., A_n], (p(\overrightarrow{x}.\psi(N)))^{n+1})$

$\psi(an(A)) = \psi'(A, [\,])$

$$\psi(fal(A, \{var(x_i)\})) = \psi'(A, fal(\{x_i\}))$$
$$\psi(withi(N_1, N_2)) = withr(\psi(N_1), \psi(N_2))$$
$$\psi(plusi1(N)) = plusr1(\psi(N))$$
$$\psi(plusi2(N)) = plusl2(\psi(N))$$
$$\psi(pluse(A, x_1.N_1, x_2.N_2)) = \psi'(A, plusl(x_1.\psi(N_1), x_2.\psi(N_2)))$$
$$\psi(tr(\{var(x_1), ..., var(x_n)\})) = tr(\{x_1, ..., x_n\})$$

$\psi'$:**A** $\times$ **Ms** $\longrightarrow$ **M**

$$\psi'(var(x), Ms) = (x; Ms)$$
$$\psi'(ap(A, N), Ms) = \psi'(A, (\psi(N) :: Ms))$$
$$\psi'(withe1(A), Ms) = \psi'(A, withl1(Ms))$$
$$\psi'(withe2(A), Ms) = \psi'(A, withl2(Ms))$$
$$\psi'(der(A), Ms) = \psi'(A, d(Ms))$$

$\psi''$ : $\overrightarrow{\mathbf{A}} \times \mathbf{Mss}^i \longrightarrow \mathbf{M}$

$$\psi''([var(x_1), ..., var(x_n)], Mss^1) = (\overrightarrow{x}; Mss^1)$$
$$\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1})$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i)$$
$$\psi''([A_1, ..., ap(A_i, N), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (\psi(N) :: Mss^i)^i)$$
$$\psi''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i)$$
$$\psi''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i)$$
$$\psi''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (d(Mss^i))^i)$$

In the following two lemmas we prove, using the translations above, that the two systems are isomorphic.

**Lemma 6.1**

**i)** *For all terms* $M$*,* $\psi(\theta(M)) = M$

**ii)** *Also, for all terms* $Ms$ *and* $A$*,*
$\psi(\theta'(A, Ms)) = \psi'(A, Ms)$.

**iii)** *Also, for all terms* $Mss^i$ *and* $A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)$*,*
$\psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^i)) =$
$\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^i)$

PROOF: By simultaneous structural induction on $M$, $Ms$, $Mss^i$.

1. The **M** term has form $(x; Ms)$

$$\begin{array}{rcll} \psi(\theta((x; Ms))) & = & \psi(\theta'(var(x), Ms)) & \text{def } \theta \\ & = & \psi'(var(x), Ms) & \text{ind ii)} \\ & = & (x; Ms) & \text{def } \psi' \end{array}$$

2. The **M** term has form $(\overrightarrow{x}; Mss^1)$

$$\begin{array}{rcll} \psi(\theta(\overrightarrow{x}; Mss^1)) & = & \psi(\theta''(\overrightarrow{var(x)}, Mss^1)) & \text{def } \theta \\ & = & \psi''(\overrightarrow{var(x)}, Mss^1) & \text{ind iii)} \\ & = & (\overrightarrow{x}; Mss^1) & \text{def } \psi'' \end{array}$$

3. The **M** term has form $\lambda x.M$

$$\begin{array}{rcll} \psi(\theta(\lambda x.M)) & = & \psi(\lambda x.\theta(M)) & \text{def } \theta \\ & = & \lambda x.\psi(\theta(M)) & \text{def } \psi \\ & = & \lambda x.M & \text{ind i)} \end{array}$$

4. The **M** term has form $tenr(M_1, M_2)$

$$\begin{array}{rcll} \psi(\theta(tenr(M_1, M_2))) & = & \psi(teni(\theta(M_1), \theta(M_2))) & \text{def } \theta \\ & = & tenr(\psi(\theta(M_1)), \psi(\theta(M_2))) & \text{def } \psi \\ & = & tenr(M_1, M_2) & \text{ind i)} \end{array}$$

5. The **M** term has form $*$

$$\begin{array}{rcll} \psi(\theta(*)) & = & \psi(*) & \text{def } \theta \\ & = & * & \text{def } \psi \end{array}$$

6. The **M** term has form $tr(\{x_1, ..., x_n\})$

$$\begin{array}{rcll} \psi(\theta(tr(\{x_1, ..., x_n\}))) & = & \psi(tr(\{var(x_1), ..., var(x_n)\})) & \text{def } \theta \\ & = & tr(\{x_1, ..., x_n\}) & \text{def } \phi \end{array}$$

7. The **M** term has form $plusr1(M)$

$$\begin{array}{rcll} \psi(\theta(plusr1(M))) & = & \psi(plusi1(\theta(M))) & \text{def } \theta \\ & = & plusr1(\psi(\theta(M))) & \text{def } \psi \\ & = & plusr1(M) & \text{ind i)} \end{array}$$

8. The **M** term has form $plusr2(M)$

$$\begin{array}{rcll} \psi(\theta(plusr2(M))) & = & \psi(plusi2(\theta(M))) & \text{def } \theta \\ & = & plusr2(\psi(\theta(M))) & \text{def } \psi \\ & = & plusr2(M) & \text{ind i)} \end{array}$$

9. The **M** term has form $withr(M_1, M_2)$

$$\begin{array}{rcll} \psi(\theta(withr(M_1, M_2))) & = & \psi(withi(\theta(M_1), \theta(M_2))) & \text{def } \theta \\ & = & withr(\psi(\theta(M_1)), \psi(\theta(M_2))) & \text{def } \psi \\ & = & withr(M_1, M_2) & \text{ind i)} \end{array}$$

10. The **Ms** term has form $[\,]$

$$\begin{array}{rcll} \psi(\theta'(A, [\,])) & = & \psi(an(A)) & \text{def } \theta' \\ & = & \psi'(A, [\,]) & \text{def } \psi \end{array}$$

11. The **Ms** term has form $(M :: Ms)$

$$\begin{array}{rcll} \psi(\theta'(A, (M :: Ms))) & = & \psi(\theta'(ap(A, \theta(M)), Ms)) & \text{def } \theta' \\ & = & \psi'(ap(A, \theta(M)), Ms) & \text{ind ii)} \\ & = & \psi'(A, (\psi(\theta(M)) :: Ms)) & \text{def } \psi' \\ & = & \psi'(A, (M :: Ms)) & \text{ind i)} \end{array}$$

12. The **Ms** term has form $tenl(x_1.x_2.M)$

$$\begin{array}{rcll} \psi(\theta'(A, tenl(x_1.x_2.M))) & = & \psi(tene(A, x_1.x_2.\theta(M))) & \text{def } \theta' \\ & = & \psi'(A, tenl(x_1.x_2.\psi(\theta(M)))) & \text{def } \psi \\ & = & \psi'(A, tenl(x_1.x_2.M)) & \text{ind i)} \end{array}$$

13. The **Ms** term has form $il(M)$

$$\begin{array}{rcll} \psi(\theta'(A, il(M))) & = & \psi(ie(A, \theta(M))) & \text{def } \theta' \\ & = & \psi'(A, il(\psi(\theta(M)))) & \text{def } \psi \\ & = & \psi'(A, il(M)) & \text{ind i)} \end{array}$$

14. The **Ms** term has form $fal(\{x_i\})$

$$\begin{array}{rcll} \psi(\theta'(A, fal(\{x_i\}))) & = & \psi(fal(A, \{var(x_i)\})) & \text{def } \theta' \\ & = & \psi'(A, fal(\{x_i\})) & \text{def } \psi \end{array}$$

15. The **Ms** term has form $plusl(x_1.M_1, x_2.M_2)$

$$\begin{array}{lll} \psi(\theta'(A, plusl(x_1.M_1, x_2.M_2))) & & \\ = & \psi(pluse(A, x_1.\theta(M_1), x_2.\theta(M_2))) & \text{def } \theta' \\ = & \psi'(A, plusl(x_1.\psi(\theta(M_1))), x_2.\psi(\theta(M_2))) & \text{def } \psi \\ = & \psi'(A, plusl(x_1.M_1, x_2.M_2)) & \text{ind i)} \end{array}$$

16. The **Ms** term has form $withl1(Ms)$

$$\begin{array}{llll} \psi(\theta'(A, withl1(Ms))) & = & \psi(\theta'(withe1(A), Ms)) & \text{def } \theta' \\ & = & \psi'(withe1(A), Ms) & \text{ind ii)} \\ & = & \psi'(A, withl1(Ms)) & \text{def } \psi' \end{array}$$

17. The **Ms** term has form $withl2(Ms)$

$$\begin{array}{llll} \psi(\theta'(A, withl2(Ms))) & = & \psi(\theta'(withe2(A), Ms)) & \text{def } \theta' \\ & = & \psi'(withe2(A), Ms) & \text{ind ii)} \\ & = & \psi'(A, withl2(Ms)) & \text{def } \psi' \end{array}$$

18. The **Ms** term has form $w(M)$

$$\begin{array}{llll} \psi(\theta'(A, w(M))) & = & \psi(weak(A, \theta(M))) & \text{def } \theta' \\ & = & \psi'(A, w(\psi(\theta(M)))) & \text{def } \psi \\ & = & \psi'(A, w(M)) & \text{ind i)} \end{array}$$

19. The **Ms** term has form $c(x_1.x_2.M)$

$$\begin{array}{llll} \psi(\theta'(A, c(x_1.x_2.M))) & = & \psi(cont(A, x_1.x_2.\theta(M))) & \text{def } \theta' \\ & = & \psi'(A, c(x_1.x_2.\psi(\theta(M)))) & \text{def } \psi \\ & = & \psi'(A, c(x_1.x_2.M)) & \text{ind i)} \end{array}$$

20. The **Ms** term has form $d(Ms)$

$$\begin{array}{llll} \psi(\theta'(A, d(Ms))) & = & \psi(\theta'(der(A), Ms)) & \text{def } \theta' \\ & = & \psi'(der(A), Ms) & \text{ind ii)} \\ & = & \psi'(A, d(Ms)) & \text{def } \psi' \end{array}$$

21. The **Mss**$^i$ term has form $(tog(Mss^{i+1}))^i$

$$\begin{array}{lll} \psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i)) & & \\ = & \psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1})) & \text{def } \theta'' \\ = & \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1}) & \text{ind iii)} \\ = & \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i) & \text{def } \psi'' \end{array}$$

22. The **Mss**$^i$ term has form $(M :: Mss^i)^i$

$$\psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (M :: Mss^i)^i))$$
$$= \psi(\theta''([A_1, ..., ap(A_i, \theta(M)), var(x_{i+1}), ..., var(x_n)], Mss^i)) \quad \text{def } \theta''$$
$$= \psi''([A_1, ..., ap(A_i, \theta(M)), var(x_{i+1}), ..., var(x_n)], Mss^i) \quad \text{ind iii)}$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (\psi(\theta(M)) :: Mss^i)^i) \quad \text{def } \psi''$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (M :: Mss^i)^i) \quad \text{ind i)}$$

23. The **Mss**$^i$ term has form $(withl1(Mss^i))^i$

$$\psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i))$$
$$= \psi(\theta''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)) \quad \text{def } \theta''$$
$$= \psi''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) \quad \text{ind iii)}$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i) \quad \text{def } \psi''$$

24. The **Mss**$^i$ term has form $(withl2(Mss^i))^i$

$$\psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i))$$
$$= \psi(\theta''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)) \quad \text{def } \theta''$$
$$= \psi''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) \quad \text{ind iii)}$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i) \quad \text{def } \psi''$$

25. The **Mss**$^i$ term has form $(d(Mss^i))^i$

$$\psi(\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (d(Mss^i))^i))$$
$$= \psi(\theta''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)) \quad \text{def } \theta''$$
$$= \psi''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) \quad \text{ind iii)}$$
$$= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (der(Mss^i))^i) \quad \text{def } \psi''$$

26. The **Mss**$^i$ term has form $(p(\overrightarrow{x}.M))^{n+1}$

$$\psi(\theta''([A_1, ..., A_n], (p(\overrightarrow{x}.M))^{n+1}))$$
$$= \psi(prom([A_1, ..., A_n], \overrightarrow{x}.\theta(M))) \quad \text{def } \theta''$$
$$= \psi''([A_1, .., A_n], (p(\overrightarrow{x}, \psi(\theta(M))))^{n+1}) \quad \text{def } \psi$$
$$= \psi''([A_1, .., A_n], (p(\overrightarrow{x}, M))^{n+1}) \quad \text{ind i)}$$

■

**Lemma 6.2**

**i)** *For all terms $N$, $\theta(\psi(N)) = N$.*

**ii)** *Also, for all terms $Ms$ and $A$,*
$\theta(\psi'(A, Ms)) = \theta'(A, Ms)$

**iii)** *Also for all terms $Mss^i$ and $A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)$,*
$\theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^i)) =$
$\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^i)$

PROOF: By simultaneous structural induction on $N$ and $A$.

1. The **N** term has form $*$

$$\begin{array}{lcll} \theta(\psi(*)) & = & \theta(*) & \text{def } \psi \\ & = & * & \text{def } \theta \end{array}$$

2. The **N** term has form $ie(A, N)$

$$\begin{array}{lcll} \theta(\psi(ie(A,N))) & = & \theta(\psi'(A, il(\psi(N)))) & \text{def } \psi \\ & = & \theta'(A, il(\psi(N))) & \text{ind ii)} \\ & = & ie(A, \theta(\psi(N))) & \text{def } \theta' \\ & = & ie(A, N) & \text{ind i)} \end{array}$$

3. The **N** term has form $tene(A, x_1.x_2.N)$

$$\begin{array}{lcll} \theta(\psi(tene(A, x_1.x_2.N))) & = & \theta(\psi'(A, tenl(x_1.x_2.\psi(N)))) & \text{def } \psi \\ & = & \theta'(A, tenl(x_1.x_2.\psi(N))) & \text{ind ii)} \\ & = & tene(A, x_1.x_2.\theta(\psi(N))) & \text{def } \theta' \\ & = & tene(A, x_1.x_2.N) & \text{ind i)} \end{array}$$

4. The **N** term has form $weak(A, N)$

$$\begin{array}{lcll} \theta(\psi(weak(A,N))) & = & \theta(\psi'(A, w(\psi(N)))) & \text{def } \psi \\ & = & \theta'(A, w(\psi(N))) & \text{ind ii)} \\ & = & weak(A, \theta(\psi(N))) & \text{def } \theta' \\ & = & weak(A, N) & \text{ind i)} \end{array}$$

5. The **N** term has form $cont(A, x_1.x_2.N)$

$$\begin{array}{lcll} \theta(\psi(cont(A, x_1.x_2.N))) & = & \theta(\psi'(A, c(x_1.x_2.\psi(N)))) & \text{def } \psi \\ & = & \theta'(A, c(x_1.x_2.\psi(N))) & \text{ind ii)} \\ & = & cont(A, x_1.x_2.\theta(\psi(N))) & \text{def } \theta' \\ & = & cont(A, x_1.x_2.N) & \text{ind i)} \end{array}$$

6. The **N** term has form $withi(N_1, N_2)$

$$\begin{array}{lcll} \theta(\psi(withi(N_1, N_2))) & = & \theta(withr(\psi(N_1), \psi(N_2))) & \text{def } \psi \\ & = & withi(\theta(\psi(N_1)), \theta(\psi(N_2))) & \text{def } \theta \\ & = & withi(N_1, N_2) & \text{ind i)} \end{array}$$

7. The **N** term has form $plusi1(N)$

$$\begin{array}{rcll} \theta(\psi(plusi1(N))) & = & \theta(plusr1(\psi(N))) & \text{def } \psi \\ & = & plusi1(\theta(\psi(N))) & \text{def } \theta \\ & = & plusi1(N) & \text{ind i)} \end{array}$$

8. The **N** term has form $plusi2(N)$

$$\begin{array}{rcll} \theta(\psi(plusi2(N))) & = & \theta(plusr2(\psi(N))) & \text{def } \psi \\ & = & plusi2(\theta(\psi(N))) & \text{def } \theta \\ & = & plusi2(N) & \text{ind i)} \end{array}$$

9. The **N** term has form $pluse(A, x_1.N_1, x_2.N_2)$

$$\begin{array}{rll} \multicolumn{3}{l}{\theta(\psi(pluse(A, x_1.N_1, x_2.N_2)))} \\ = & \theta(\psi'(A, plusl(x_1.\psi(N_1), x_2.\psi(N_2)))) & \text{def } \psi \\ = & \theta'(A, plusl(x_1.\psi(N_1), x_2.\psi(N_2))) & \text{ind ii)} \\ = & pluse(A, x_1.\theta(\psi(N_1))), x_2.\theta(\psi(N_2))) & \text{def } \theta' \\ = & pluse(A, x_1.N_1, x_2.N_2) & \text{ind i)} \end{array}$$

10. The **N** term has form $an(A)$

$$\begin{array}{rcll} \theta(\psi(an(A))) & = & \theta(\psi'(A, [\,])) & \text{def } \psi \\ & = & \theta'(A, [\,]) & \text{ind ii)} \\ & = & an(A) & \text{def } \theta' \end{array}$$

11. The **N** term has form $\lambda x.N$

$$\begin{array}{rcll} \theta(\psi(\lambda x.N)) & = & \theta(\lambda x.\psi(N)) & \text{def } \psi \\ & = & \lambda x.\theta(\psi(N)) & \text{def } \theta \\ & = & \lambda x.N & \text{ind i)} \end{array}$$

12. The **N** term has form $teni(N_1, N_2)$

$$\begin{array}{rcll} \theta(\psi(teni(N_1, N_2))) & = & \theta(tenr(\psi(N_1), \psi(N_2))) & \text{def } \psi \\ & = & teni(\theta(\psi(N_1)), \theta(\psi(N_2))) & \text{def } \theta \\ & = & teni(N_1, N_2) & \text{ind i)} \end{array}$$

13. The **N** term has form $prom(\overrightarrow{A}, \overrightarrow{x}.N)$

$$\begin{array}{rll} \multicolumn{3}{l}{\theta(\psi(prom(\overrightarrow{A}, \overrightarrow{x}.N)))} \\ = & \theta(\psi''([A_1, ..., A_n], (p(\overrightarrow{x}.\psi(N)))^{n+1})) & \text{def } \psi \\ = & \theta''([A_1, ..., A_n], (p(\overrightarrow{x}.\psi(N)))^{n+1}) & \text{ind iii)} \\ = & prom(\overrightarrow{A}, \overrightarrow{x}.\theta(\psi(N))) & \text{def } \theta'' \\ = & prom(\overrightarrow{A}, \overrightarrow{x}.N) & \text{ind i)} \end{array}$$

14. The **N** term has form $tr(\{var(x_i)\})$

$$\begin{array}{rcll} \theta(\psi(tr(var(\{x_i\})))) & = & \theta(tr(\{x_i\})) & \text{def } \psi \\ & = & tr(\{var(x_i)\}) & \text{def } \theta \end{array}$$

15. The **N** term has form $fal(A, \{var(x_i)\})$

$$\begin{array}{rcll} \theta(\psi(fal(A, \{var(x_i)\}))) & = & \theta(\psi'(A, fal(\{x_i\}))) & \text{def } \psi \\ & = & \theta'(A, fal(\{x_i\})) & \text{ind ii)} \\ & = & fal(A, \{var(x_i)\}) & \text{def } \theta' \end{array}$$

16. The **A** term has form $var(x)$

$$\begin{array}{rcll} \theta(\psi'(var(x), Ms)) & = & \theta((x; Ms)) & \text{def } \psi' \\ & = & \theta'(var(x), Ms) & \text{def } \theta \end{array}$$

17. The **A** term has form $ap(A, N)$

$$\begin{array}{rcll} \theta(\psi'(ap(A, N), Ms)) & = & \theta(\psi'(A, (\psi(N) :: Ms))) & \text{def } \psi' \\ & = & \theta'(A, (\psi(N) :: Ms)) & \text{ind ii)} \\ & = & \theta'(ap(A, \theta(\psi(N))), Ms) & \text{def } \theta' \\ & = & \theta'(ap(A, N), Ms) & \text{ind i)} \end{array}$$

18. The **A** term has form $der(A)$

$$\begin{array}{rcll} \theta(\psi'(der(A), Ms)) & = & \theta(\psi'(A, d(Ms))) & \text{def } \psi' \\ & = & \theta'(A, d(Ms)) & \text{ind ii)} \\ & = & \theta'(der(A), Ms) & \text{def } \theta' \end{array}$$

19. The **A** term has form $withe1(A)$

$$\begin{array}{rcll} \theta(\psi'(withe1(A), Ms)) & = & \theta(\psi'(A, withl1(Ms))) & \text{def } \psi' \\ & = & \theta'(A, withl1(Ms)) & \text{ind ii)} \\ & = & \theta'(withe1(A), Ms) & \text{def } \theta' \end{array}$$

20. The **A** term has form $withe2(A)$

$$\begin{array}{rcll} \theta(\psi'(withe2(A), Ms)) & = & \theta(\psi'(A, withl2(Ms))) & \text{def } \psi' \\ & = & \theta'(A, withl2(Ms)) & \text{ind ii)} \\ & = & \theta'(withe2(A), Ms) & \text{def } \theta' \end{array}$$

21. The $\overrightarrow{\mathbf{A}}$ term has form $\overrightarrow{var(x)}$

$$\begin{array}{rcll} \theta(\psi''(\overrightarrow{var(x)}, Mss^1)) & = & \theta(\overrightarrow{x}; Mss^1) & \text{def } \psi'' \\ & = & \theta''(\overrightarrow{var(x)}, Mss^1) & \text{def } \theta'' \end{array}$$

22. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)]$

$$\begin{array}{lll} & \theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1})) & \\ = & \theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i)) & \text{def } \psi'' \\ = & \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i) & \text{ind iii)} \\ = & \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1}) & \text{def } \theta'' \end{array}$$

23. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., ap(A_i, N), var(x_{i+1}), ..., var(x_n)]$

$$\begin{array}{lll} & \theta(\psi''([A_1, ..., ap(A_i, N), var(x_{i+1}), ..., var(x_n)], Mss^i)) & \\ = & \theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (\psi(N) :: Mss^i)^i)) & \text{def } \psi'' \\ = & \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (\psi(N) :: Mss^i)^i) & \text{ind iii)} \\ = & \theta''([A_1, ..., ap(A_i, \theta(\psi(N))), var(x_{i+1}), ..., var(x_n)], Mss^i) & \text{def } \theta'' \\ = & \theta''([A_1, ..., ap(A_i, N), var(x_{i+1}), ..., var(x_n)], Mss^i) & \text{ind i)} \end{array}$$

24. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)]$

$$\begin{array}{lll} & \theta(\psi''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)) & \\ = & \theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i)) & \text{def } \psi'' \\ = & \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i) & \text{ind iii)} \\ = & \theta''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) & \text{def } \theta'' \end{array}$$

25. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)]$

$$\begin{array}{lll} & \theta(\psi''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)) & \\ = & \theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i)) & \text{def } \psi'' \\ = & \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i) & \text{ind iii)} \\ = & \theta''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) & \text{def } \theta'' \end{array}$$

26. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)]$

$$\begin{array}{lll} & \theta(\psi''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)) & \\ = & \theta(\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (d(Mss^i))^i)) & \text{def } \psi'' \\ = & \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (d(Mss^i))^i) & \text{ind iii)} \\ = & \theta''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) & \text{def } \theta'' \end{array}$$

■

**Theorem 6.1** *The deductions of* NNILL *are in 1–1 correspondence with the sequent derivations given by the sequent calculus* SILL.

PROOF: Follows immediately from Lemma 6.1 and Lemma 6.2. ■

**Theorem 6.2 (**SOUNDNESS) *The following rules are admissible:*

$$\frac{\Gamma \Rightarrow M : R}{\Gamma \rhd\!\!\rhd \theta(M) : R}\text{ i)} \qquad \frac{\Delta \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma, \Delta \rhd\!\!\rhd \theta'(A, Ms) : R}\text{ ii)}$$

$$\frac{\mathcal{D}_1 \quad ... \quad \mathcal{D}_n \quad \Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma, \Delta_1, ..., \Delta_n \rhd\!\!\rhd \theta''([A_1, ..., A_i, var(x_{i+1}), ...var(x_n)], Mss^i) : R}\text{ iii)}$$

*where the* $\mathcal{D}_1, ..., \mathcal{D}_n$ *are (in order):*

$\Delta_1 \rhd A_1 :!P_1$

$\vdots$

$\Delta_{i\perp 1} \rhd A_{i\perp 1} :!P_{i\perp 1}$

$\Delta_i \rhd A_i : P_i$

$\Delta_{i+1} \rhd var(x_{i+1}) : P_{i+1}$

$\vdots$

$\Delta_n \rhd var(x_n) : P_n$

PROOF: By simultaneous structural induction on $M$, $Ms$ and $Mss^i$.

1. The **M** term has form $(\overrightarrow{x}; Mss^1)$

   We have a derivation ending in:

$$\frac{\Gamma \overset{[\ ][P_1,...,P_n]}{\ggg} Mss^1 : R}{\Gamma, P_1, ..., P_n \Rightarrow (\overrightarrow{x}; Mss^1) : R}\ (sel*)$$

   and we know that for all $i$

$$P_i \rhd var(x_i) : P_i$$

   is deducible.

   So we have:

$$\frac{P_1 \rhd var(x_1) : P_1 \quad ... \quad P_n \rhd var(x_n) : P_n \quad \Gamma \overset{[\ ][P_1,...,P_n]}{\ggg} Mss^1 : R}{\Gamma, P_1, ..., P_n \rhd\!\!\rhd \theta''(\overrightarrow{var(x)}, Mss^1) : R}\ iii)$$

   and we know that

$$\theta''(\overrightarrow{var(x)}, Mss^1) = \theta(\overrightarrow{x}; Mss^1)$$

2. The **M** term has form $(x; Ms)$

   We have a derivation ending in:

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma, P \Rightarrow (x; Ms) : R}\ (sel)$$

   and we know that

$$P \rhd var(x) : P$$

   is deducible.

   So we have

$$\frac{P \rhd var(x) : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma, P \rhd\!\!\rhd \theta'(var(x), Ms) : R}\ ii)$$

   and we know that

$$\theta'(var(x), Ms) = (x; Ms)$$

3. The **M** term has form $\lambda x.M$

   We have a derivation ending in:

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \multimap Q}\ (\multimap_{\mathcal{R}})$$

   whence

$$\frac{\dfrac{\Gamma, x : P \Rightarrow M : Q}{\Gamma, x : P \rhd\!\!\rhd \theta(M) : Q}\ \text{i)}}{\Gamma \rhd\!\!\rhd \lambda x.\theta(M) : P \multimap Q}\ (\multimap_{\mathcal{I}})$$

   and we know that

$$\lambda x.\theta(M) = \theta(\lambda x.M)$$

4. The **M** term has form $tenr(M_1, M_2)$

   We have a derivation ending in:

$$\frac{\Gamma_1 \Rightarrow M_1 : P \quad \Gamma_2 \Rightarrow M_2 : Q}{\Gamma_1, \Gamma_2 \Rightarrow tenr(M_1, M_2) : P \otimes Q}\ (\otimes_{\mathcal{R}})$$

   whence

$$\frac{\dfrac{\Gamma_1 \Rightarrow M_1 : P}{\Gamma_1 \rhd\!\!\rhd \theta(M_1) : P}\ \text{i)} \quad \dfrac{\Gamma_2 \Rightarrow M_2 : Q}{\Gamma_2 \rhd\!\!\rhd \theta M_2 : Q}\ \text{i)}}{\Gamma_1, \Gamma_2 \rhd\!\!\rhd teni(\theta(M_1), \theta(M_2)) : P \otimes Q}\ (\otimes_{\mathcal{I}})$$

   and we know that

$$teni(\theta(M_1), \theta(M_2)) = \theta(tenr(M_1, M_2))$$

5. The **M** term has form $*$

   We have the following derivation:

$$\frac{}{\Rightarrow * : I}\ (I_{\mathcal{R}})$$

   we also have the following deduction:

$$\frac{}{\rhd\!\rhd * : I}\ (I_{\mathcal{I}})$$

   giving the result.

6. The **M** term has form $tr(\{x_i\})$

   We have the following derivation:

$$\frac{}{x_1 : P_1, ..., x_n : P_n \Rightarrow tr(\{x_i\}) : \top}\ (\top_{\mathcal{R}})$$

   we also have the following deduction:

$$\frac{P_1 \rhd var(x_1) : P_1 \quad ... \quad P_n \rhd var(x_n) : P_n}{P_1, ..., P_n \rhd\!\rhd tr(\{var(x_i)\})}\ (\top_{\mathcal{I}})$$

   and we know that

$$tr(\{var(x_i)\}) = \theta(tr(\{x_i\}))$$

7. The **M** term has form $plusr1(M)$

   We have a derivation ending in:

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow plusr1(M) : P \oplus Q}\ (\oplus_{\mathcal{R}_1})$$

   whence

$$\frac{\dfrac{\Gamma \Rightarrow M : P}{\Gamma \rhd\!\rhd \theta(M) : P}\ \text{i)}}{\Gamma \rhd\!\rhd plusi1(\theta(M)) : P \oplus Q}\ (\oplus_{\mathcal{I}_1})$$

   and we know that

$$plusi1(\theta(M)) = \theta(plusr1(M))$$

8. The **M** term has form $plusr2(M)$

   We have a derivation ending in:

$$\frac{\Gamma \Rightarrow M : Q}{\Gamma \Rightarrow plusr2(M) : P \oplus Q}\ (\oplus_{\mathcal{R}_2})$$

whence

$$\dfrac{\dfrac{\Gamma \Rightarrow M : Q}{\Gamma \rhd\!\!\rhd \theta(M) : Q}\ \text{i)}}{\Gamma \rhd\!\!\rhd plusi2(\theta(M)) : P \oplus Q}\ (\oplus_{\mathcal{I}_2})$$

and we know that

$$plusi2(\theta(M)) = \theta(plusr2(M))$$

9. The **M** term has form $withr(M_1, M_2)$

   We have a derivation ending in:

$$\dfrac{\Gamma \Rightarrow M_1 : P \quad \Gamma \Rightarrow M_2 : Q}{\Gamma \Rightarrow withr(M_1, M_2) : P\&Q}\ (\&_{\mathcal{R}})$$

   whence

$$\dfrac{\dfrac{\Gamma \Rightarrow M_1 : P}{\Gamma \rhd\!\!\rhd \theta(M_1) : P}\ \text{i)} \quad \dfrac{\Gamma \Rightarrow M_2 : Q}{\Gamma \rhd\!\!\rhd \theta(M_2) : Q}\ \text{i)}}{\Gamma \rhd\!\!\rhd withi(\theta(M_1), \theta(M_2)) : P\&Q}\ (\&_{\mathcal{I}})$$

   and we know that

$$withi(\theta(M_1), \theta(M_2)) = \theta(withr(M_1, M_2))$$

10. The **Ms** term has form $[\,]$

    We have a derivation

$$\dfrac{}{\phi \xrightarrow{P} [\,] : P}\ (ax)$$

    and we can deduce

$$\dfrac{\Delta \rhd A : P}{\Delta \rhd\!\!\rhd an(A) : P}\ (M)$$

    and we know that

$$an(A) = \theta'(A, [\,])$$

11. The **Ms** term has form $(M :: Ms)$

    We have a derivation ending in:

$$\dfrac{\Gamma_1 \Rightarrow M : P \quad \Gamma_2 \xrightarrow{Q} Ms : R}{\Gamma_1, \Gamma_2 \xrightarrow{P \multimap Q} (M :: Ms) : R}\ (\multimap_{\mathcal{L}})$$

    whence

$$\dfrac{\dfrac{\Delta \rhd A : P \multimap Q \quad \dfrac{\Gamma_1 \Rightarrow M : P}{\Gamma_1 \rhd\!\!\rhd \theta(M) : P}\ \text{i)}}{\Gamma_1, \Delta \rhd ap(A, \theta(M)) : Q}\ (\multimap_{\varepsilon}) \quad \Gamma_2 \xrightarrow{Q} Ms : R}{\Gamma_1, \Gamma_2, \Delta \rhd\!\!\rhd \theta'(ap(A, \theta(M)), Ms) : R}\ \text{ii)}$$

and we know that

$$\theta'(ap(A,\theta(M)), Ms) = \theta'(A,(M :: Ms))$$

12. The **Ms** term has form $tenl(x_1.x_2.M)$

    We have a derivation ending in:

$$\frac{\Gamma, x_1 : P, x_2 : Q \Rightarrow M : R}{\Gamma \xrightarrow{P\otimes Q} timl(x_1.x_2.M) : R}\ (\otimes_{\mathcal{L}})$$

    whence

$$\frac{\Delta \rhd A : P \otimes Q \quad \dfrac{\Gamma, x_1 : P, x_2 : Q \Rightarrow M : R}{\Gamma, x_1 : P, x_2 : Q \rhd\!\!\rhd \theta(M) : R}\ \text{i)}}{\Gamma, \Delta \rhd\!\!\rhd tene(A, x_1.x_2.\theta(M)) : R}\ (\otimes_{\varepsilon})$$

    and we know that

$$tene(A, x_1.x_2.\theta(M)) = \theta'(A, timl(x_1.x_2.M))$$

13. The **Ms** term has form $il(M)$

    We have a derivation ending in:

$$\frac{\Gamma \Rightarrow M : R}{\Gamma \xrightarrow{I} il(M) : R}\ (I_{\mathcal{L}})$$

    whence

$$\frac{\Delta \rhd A : I \quad \dfrac{\Gamma \Rightarrow M : R}{\Gamma \rhd\!\!\rhd \theta(M) : R}\ \text{i)}}{\Gamma, \Delta \rhd\!\!\rhd ie(A, \theta(M)) : R}\ (I_{\varepsilon})$$

    and we know that

$$ie(A, \theta(M)) = \theta'(A, il(M))$$

14. The **Ms** term has form $fal(\{x_i\})$

    We have the following derivation:

$$\frac{}{x_1 : P_1, ..., x_n : P_n \xrightarrow{0} fal(\{x_i\}) : R}\ (0_{\mathcal{L}})$$

    We also have the following deduction:

$$\frac{P_1 \rhd var(x_1) : P_1 \quad ... \quad P_n \rhd var(x_n) : P_n \quad \Delta \rhd A : 0}{\Delta, P_1, ..., P_n \rhd\!\!\rhd fal(A, \{var(x_1), ..., var(x_n)\}) : R}\ (0_{\varepsilon})$$

    and we know that

$$fal(A, \{var(x_i)\}) = \theta'(A, fal(\{x_i\}))$$

15. The **Ms** term has form $plusl(x_1.M_1, x_2.M_2)$

    We have a derivation ending in

$$\frac{\Gamma, x_1 : P \Rightarrow M_1 : R \quad \Gamma, x_2 : Q \Rightarrow M_2 : R}{\Gamma \stackrel{P\oplus Q}{\longrightarrow} plusl(x_1.M_1, x_2.M_2) : R}\ (\oplus_{\mathcal{L}})$$

    whence

$$\frac{\Delta \rhd A : P \oplus Q \quad \dfrac{\Gamma, x_1 : P \Rightarrow M_1 : R}{\Gamma, x_1 : P \rhd\!\!\rhd \theta(M_1) : R}\ \text{i)} \quad \dfrac{\Gamma, x_2 : Q \Rightarrow M_2 : R}{\Gamma, x_2 : Q \rhd\!\!\rhd \theta(M_2) : R}\ \text{i)}}{\Gamma, \Delta \rhd\!\!\rhd pluse(A, x_1.\theta(M_1), x_2.\theta(M_2)) : R}\ (\oplus_{\varepsilon})$$

    and we know that

$$pluse(A, x_1.\theta(M_1), x_2.\theta(M_2)) = \theta'(A, plusl(x_1.M_1, x_2.M_2))$$

16. The **Ms** term has form $withl1(Ms)$

    We have a derivation ending in:

$$\frac{\Gamma \stackrel{P}{\longrightarrow} Ms : R}{\Gamma \stackrel{P\&Q}{\longrightarrow} withl1(Ms) : R}\ (\&_{\mathcal{L}_1})$$

    whence

$$\frac{\dfrac{\Delta \rhd A : P\&Q}{\Delta \rhd withe1(A) : P}\ (\&_{\varepsilon_1}) \quad \Gamma \stackrel{P}{\longrightarrow} Ms : R}{\Gamma, \Delta \rhd\!\!\rhd \theta'(withe1(A), Ms) : R}\ \text{ii)}$$

    and we know that

$$\theta'(withe1(A), Ms) = \theta'(A, withl1(Ms))$$

17. The **Ms** term has form $withl2(Ms)$

    We have a derivation ending in:

$$\frac{\Gamma \stackrel{Q}{\longrightarrow} Ms : R}{\Gamma \stackrel{P\&Q}{\longrightarrow} withl2(Ms) : R}\ (\&_{\mathcal{L}_2})$$

    whence

$$\frac{\dfrac{\Delta \rhd A : P\&Q}{\Delta \rhd withe2(A) : Q}\ (\&_{\varepsilon_2}) \quad \Gamma \stackrel{Q}{\longrightarrow} Ms : R}{\Gamma, \Delta \rhd\!\!\rhd \theta'(withe2(A), Ms) : R}\ \text{ii)}$$

    and we know that

$$\theta'(withe2(A), Ms) = \theta'(A, withl2(Ms))$$

18. The **Ms** term has form $w(M)$

We have a derivation ending in:

$$\frac{\Gamma \Rightarrow M : R}{\Gamma \xrightarrow{!P} w(M) : R}\ (W)$$

whence

$$\frac{\Delta \rhd A :!P \quad \dfrac{\Gamma \Rightarrow M : R}{\Gamma \rhd\!\!\rhd \theta(M) : R}\ \text{i)}}{\Gamma, \Delta \rhd\!\!\rhd weak(A, \theta(M)) : R}\ (W)$$

and we know that

$$weak(A, \theta(M)) = \theta'(A, w(M))$$

19. The **Ms** term has form $c(x_1.x_2.M)$

We have a derivation ending in

$$\frac{\Gamma, x_1 :!P, x_2 :!P \Rightarrow M : R}{\Gamma \xrightarrow{!P} c(x_1.x_2.M) : R}\ (C)$$

whence

$$\frac{\Delta \rhd A :!P \quad \dfrac{\Gamma, x_1 :!P, x_2 :!P \Rightarrow M : R}{\Gamma, x_1 :!P, x_2 :!P \rhd\!\!\rhd \theta(M) : R}\ \text{i)}}{\Gamma, \Delta \rhd\!\!\rhd cont(A, x_1.x_2.\theta(M)) : R}\ (C)$$

and we know that

$$cont(A, x_1.x_2.\theta(M)) = \theta'(A, c(x_1.x_2.M))$$

20. The **Ms** term has form $d(Ms)$

We have a derivation ending in

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{!P} d(Ms) : R}\ (D)$$

whence

$$\frac{\dfrac{\Delta \rhd A :!P}{\Delta \rhd der(A) : P}\ (D) \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma, \Delta \rhd\!\!\rhd \theta'(der(A), Ms) : R}\ \text{ii)}$$

and we know that

$$\theta'(der(A), Ms) = \theta'(A, d(Ms))$$

Mainly for reasons of typography, for the following **Mss**$^i$ cases we leave out the details of the left premisses unless absolutely necessary. We replace them with the ellipsis $\overrightarrow{lp}$.

21. The **Mss**$^i$ term has form $(tog(Mss^{i+1}))^i$

    We have a derivation ending in

$$\frac{\Gamma \xrightarrow{[!P_1,...,!P_i][P_{i+1},...,P_n]} Mss^{i+1} : R}{\Gamma \xrightarrow{[!P_1,...,!P_{i-1}][!P_i,..,P_n]} (tog(Mss^{i+1}))^i : R}\ (tog)$$

    whence

$$\frac{\overrightarrow{lp} \quad \Gamma \xrightarrow{[!P_1,...,!P_i][P_{i+1},...,P_n]} Mss^{i+1} : R}{\Gamma, \Delta_1, ..., \Delta_n \rhd\!\!\rhd \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1}) : R}\ \text{iii)}$$

    and we know that

$$\theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1})$$
$$= \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i)$$

22. The **Mss**$^i$ term has form $(M :: Mss^i)^i$

    We have a derivation ending in

$$\frac{\Gamma_1 \Rightarrow M : Q \quad \Gamma_2 \xrightarrow{[!P_1,...,!P_{i-1}][P_i,...P_n]} Mss^i : R}{\Gamma_1, \Gamma_2 \xrightarrow{[!P_1,..,!P_{i-1}][Q\multimap P_i,...,P_n]} (M :: Mss^i)^i : R}\ (\multimap_{\mathcal{L}}*)$$

    whence

$$\frac{\overrightarrow{lp} \quad \dfrac{\Delta_i \rhd A_i : Q\multimap P_i \quad \dfrac{\Gamma_1 \Rightarrow M : Q}{\Gamma_1 \rhd\!\!\rhd \theta(M) : Q}\ \text{i)}}{\Gamma_1, \Delta_i \rhd ap(A_i, \theta(M)) : P_i}\ (\multimap_\varepsilon) \quad \Gamma_2 \xrightarrow{[!P_1,...,!P_{i-1}][P_i,...P_n]} Mss^i : R}{\Gamma_1, \Gamma_2, \Delta_1, .., \Delta_n \rhd\!\!\rhd \theta''([A_1, ..., ap(A_i, \theta(M)), var(x_{i+1}), .., var(x_n)], Mss^i) : R}\ \text{iii)}$$

    and we know that

$$\theta''([A_1, ..., ap(A_i, \theta(M)), var(x_{i+1}), .., var(x_n)], Mss^i)$$
$$= \theta''([A_1, ..., A_i, var(x_{i+1}), .., var(x_n)], (M :: Ms^i)^i)$$

23. The **Mss**$^i$ term has form $(withl1(Mss^i))^i$

    We have a derivation ending in

$$\frac{\Gamma \xrightarrow{[!P_1,...,!P_{i-1}][P_i,...,P_n]} Mss^i : R}{\Gamma \xrightarrow{[!P_1,...,!P_{i-1}][P_i\&Q,..,P_n]} (withl1(Mss^i))^i : R}\ (\&_{\mathcal{L}_1}*)$$

whence

$$\dfrac{\overrightarrow{lp} \quad \dfrac{\Delta_i \rhd A_i : P_i \& Q}{\Delta_i \rhd withe1(A_i) : P_i}\ (\&_{\varepsilon_1}) \quad \Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma, \Delta_1, ..., \Delta_n \rhd\!\!\rhd \theta''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) : R} \text{ iii)}$$

and we know that

$$\theta''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl1(Mss^i))^i)$$

24. The **Mss**$^i$ term has form $(withl2(Mss^i))^i$

We have a derivation ending in

$$\dfrac{\Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma \overset{[!P_1,...,!P_{i-1}][P_i \& Q,..,P_n]}{\ggg} (withl2(Mss^i))^i : R}\ (\&_{\mathcal{L}_2}*)$$

whence

$$\dfrac{\overrightarrow{lp} \quad \dfrac{\Delta_i \rhd A_i : Q \& P_i}{\Delta_i \rhd withe2(A_i) : P_i}\ (\&_{\varepsilon_2}) \quad \Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma, \Delta_1, ..., \Delta_n \rhd\!\!\rhd \theta''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) : R} \text{ iii)}$$

and we know that

$$\theta''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (withl2(Mss^i))^i)$$

25. The **Mss**$^i$ term has form $(d(Mss^i))^i$

We have a derivation ending in

$$\dfrac{\Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma \overset{[!P_1,..,!Pi\bot 1][!P_i,...,P_n]}{\ggg} (d(Mss^i))^i : R}\ (D*)$$

whence

$$\dfrac{\overrightarrow{lp} \quad \dfrac{\Delta_i \rhd A_i : !P_i}{\Delta_i \rhd der(A_i) : P_i}\ (D) \quad \Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma, \Delta_1, ..., \Delta_n \rhd\!\!\rhd \theta''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i) : R} \text{ iii)}$$

and we know that

$$\theta''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$
$$= \theta''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (d(Mss^i))^i)$$

26. The **Mss**$^i$ term has form $(p(\overrightarrow{x}.M))^{n+1}$

    We have a derivation ending in

$$\frac{x_1 :!P_1, ..., x_n :!P_n \Rightarrow M : R}{\phi \overset{[!P_1,...,!P_n][\,]}{\ggg} (p(\overrightarrow{x}.M))^{n+1} :!R}\ (P)$$

    whence

$$\frac{\overrightarrow{lp} \quad \dfrac{x_1 :!P_1, ..., x_n :!P_n \Rightarrow M : R}{x_1 :!P_1, ..., x_n :!P_n \rhd\!\rhd \theta(M) : R}\ \text{i)}}{\Delta_1, ..., \Delta_n \rhd\!\rhd prom(\overrightarrow{A}, \overrightarrow{x}.\theta(M)) :!R}\ (P)$$

    and we know that

$$prom(\overrightarrow{A}, \overrightarrow{x}.\theta(M)) = \theta''(\overrightarrow{A}, (p(\overrightarrow{x}.M))^{n+1})$$

■

**Theorem 6.3 (**ADEQUACY) *The following rules are admissible:*

$$\frac{\Gamma \rhd\!\rhd N : R}{\Gamma \Rightarrow \psi(N) : R}\ \text{i)} \qquad \frac{\Delta \rhd A : P \quad \Gamma \overset{P}{\longrightarrow} Ms : R}{\Gamma, \Delta \Rightarrow \psi'(A, Ms) : R}\ \text{ii)}$$

$$\frac{\mathcal{D}_1 \quad ... \quad \mathcal{D}_n \quad \Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma, \Delta_1, ..., \Delta_n \Rightarrow \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^i) : R}\ \text{iii)}$$

*where the* $\mathcal{D}_1, ..., \mathcal{D}_n$ *are (in order):*

$\Delta_1 \rhd A_1 :!P_1$

$\vdots$

$\Delta_{i\perp 1} \rhd A_{i\perp 1} :!P_{i\perp 1}$

$\Delta_i \rhd A_i : P_i$

$\Delta_{i+1} \rhd var(x_{i+1}) : P_{i+1}$

$\vdots$

$\Delta_n \rhd var(x_n) : P_n$

PROOF:

1. The **N** term has form $\lambda x.N$

    We have a deduction ending in

$$\frac{\Gamma, x : P \rhd\!\rhd N : Q}{\Gamma \rhd\!\rhd \lambda x.N : P \multimap Q}\ (\multimap_{\mathcal{I}})$$

whence

$$\dfrac{\dfrac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma, x : P \Rightarrow \psi(N) : Q}\ \text{i)}}{\Gamma \Rightarrow \lambda x.\psi(N) : P \multimap Q}\ (\multimap_{\mathcal{R}})$$

and we know that

$$\lambda x.\psi(N) = \psi(\lambda x.N)$$

2. The **N** term has form $teni(N_1, N_2)$

   We have a deduction ending in

$$\dfrac{\Gamma_1 \rhd\!\!\rhd N_1 : P \quad \Gamma_2 \rhd\!\!\rhd N_2 : Q}{\Gamma_1, \Gamma_2 \rhd\!\!\rhd teni(N_1, N_2) : P \otimes Q}\ (\otimes_{\mathcal{I}})$$

   whence

$$\dfrac{\dfrac{\Gamma_1 \rhd\!\!\rhd N_1 : P}{\Gamma_1 \Rightarrow \psi(N_1) : P}\ \text{i)} \quad \dfrac{\Gamma_2 \rhd\!\!\rhd N_2 : Q}{\Gamma_2 \Rightarrow \psi(N_2) : Q}\ \text{i)}}{\Gamma_1, \Gamma_2 \Rightarrow tenr(\psi(N_1), \psi(N_2)) : P \otimes Q}\ (\otimes_{\mathcal{R}})$$

   and we know that

$$tenr(\psi(N_1), \psi(N_2)) = \psi(teni(N_1, N_2))$$

3. The **N** term has form $*$

   We have a deduction

$$\dfrac{}{\rhd\!\!\rhd * : I}\ (I_{\mathcal{I}})$$

   and we know that we have a derivation

$$\dfrac{}{\Rightarrow * : I}\ (I_{\mathcal{R}})$$

   Hence result.

4. The **N** term has form $ie(A, N)$

   We have a deduction ending in

$$\dfrac{\Gamma_1 \rhd A : I \quad \Gamma_2 \rhd\!\!\rhd N : R}{\Gamma_1, \Gamma_2 \rhd\!\!\rhd ie(A, N) : R}\ (I_{\varepsilon})$$

   whence

$$\dfrac{\Gamma_1 \rhd A : I \quad \dfrac{\dfrac{\Gamma_2 \rhd\!\!\rhd N : R}{\Gamma_2 \Rightarrow \psi(N) : R}\ \text{i)}}{\Gamma_2 \xrightarrow{I} il(\psi(N)) : R}\ (I_{\mathcal{L}})}{\Gamma_1, \Gamma_2 \Rightarrow \psi'(A, il(\psi(N))) : R}\ \text{ii)}$$

   and we know that

$$\psi'(A, il(\psi(N))) = \psi(ie(A, N))$$

5. The **N** term has form $tene(A, x_1.x_2.N)$

   We have a deduction ending in

$$\frac{\Gamma_1 \rhd A : P \otimes Q \quad \Gamma_2, x_1 : P, x_2 : Q \rhd\!\!\rhd N : R}{\Gamma_1, \Gamma_2 \rhd\!\!\rhd tene(A, x_1.x_2.N) : R}\ (\otimes_\varepsilon)$$

   whence

$$\frac{\Gamma_1 \rhd A : P \otimes Q \quad \dfrac{\dfrac{\Gamma_2, x_1 : P, x_2 : Q \rhd\!\!\rhd N : R}{\Gamma_2, x_1 : P, x_2 : Q \Rightarrow \psi(N) : R}\ \text{i)}}{\Gamma_2 \xrightarrow{P \otimes Q} tenl(x_1.x_2.\psi(N)) : R}\ (\otimes_\mathcal{L})}{\Gamma_1, \Gamma_2 \Rightarrow \psi'(A, tene(x_1.x_2.\psi(N))) : R}\ \text{ii)}$$

   and we know that

$$\psi'(A, tenl(x_1.x_2.\psi(N))) = \psi(tene(A, x_1.x_2.N))$$

6. The **N** term has form $cont(A, x_1.x_2.N)$

   We have a deduction ending in

$$\frac{\Gamma_1 \rhd A :!P \quad \Gamma_2, x_1 :!P, x_2 :!P \rhd\!\!\rhd N : R}{\Gamma_1, \Gamma_2 \rhd\!\!\rhd cont(A, x_1.x_2.N) : R}\ (C)$$

   whence

$$\frac{\Gamma_1 \rhd A :!P \quad \dfrac{\dfrac{\Gamma_2, x_1 :!P, x_2 :!P \rhd\!\!\rhd N : R}{\Gamma_2, x_1 :!P, x_2 :!P \Rightarrow \psi(N) : R}\ \text{i)}}{\Gamma_2 \xrightarrow{!P} c(x_1.x_2.\psi(N)) : R}\ (C)}{\Gamma_1, \Gamma_2 \Rightarrow \psi'(A, c(x_1.x_2.\psi(N))) : R}\ \text{ii)}$$

   and we know that

$$\psi'(A, c(x_1.x_2.\psi(N))) = \psi(cont(A, x_1.x_2.N))$$

7. The **N** term has form $weak(A, N)$

   We have a deduction ending in

$$\frac{\Gamma_1 \rhd A :!P \quad \Gamma_2 \rhd\!\!\rhd N : R}{\Gamma_1, \Gamma_2 \rhd\!\!\rhd weak(A, N) : R}\ (W)$$

   whence

$$\frac{\Gamma_1 \rhd A :!P \quad \dfrac{\dfrac{\Gamma_2 \rhd\!\!\rhd N : R}{\Gamma_2 \Rightarrow \psi(N) : R}\ \text{i)}}{\Gamma_2 \xrightarrow{!P} w(\psi(N)) : R}\ (W)}{\Gamma_1, \Gamma_2 \Rightarrow \psi'(A, w(\psi(N))) : R}\ \text{ii)}$$

   and we know that

$$\psi'(A, w(\psi(N))) = \psi(weak(A, N))$$

8. The **N** term has form $prom(\overrightarrow{A}, \overrightarrow{x}.N)$

   We have a deduction ending in

$$\frac{\Delta_1 \rhd A_1 :!P_1 \quad ... \quad \Delta_n \rhd A_n :!P_n \quad x_1 :!P_1, ..., x_n :!P_n \rhd\!\!\rhd N : R}{\Delta_1, ..., \Delta_n \rhd\!\!\rhd prom(\overrightarrow{A}, \overrightarrow{x}.N) :!R} \; (P)$$

   whence

$$\frac{\Delta_1 \rhd A_1 :!P_1 \quad ... \quad \Delta_n \rhd A_n :!P_n \quad \dfrac{\dfrac{x_1 :!P_1, ..., x_n :!P_n \rhd\!\!\rhd N : R}{x_1 :!P_1, ..., x_n :!P_n \Rightarrow \psi(N) : R} \text{ i)}}{\overset{[!P_1,...,!P_n][\,]}{\longrightarrow} (p(\overrightarrow{x}.\psi(N)))^{n+1} :!R} \; (P)}{\Delta_1, ..., \Delta_n \Rightarrow \psi''(\overrightarrow{A}, (p(\overrightarrow{x}.\psi(N)))^{n+1}) :!R} \text{ iii)}$$

   and we know that

$$\psi''(\overrightarrow{A}, (p(\overrightarrow{x}.\psi(N)))^{n+1}) = \psi(prom(\overrightarrow{A}, \overrightarrow{x}.N))$$

9. The **N** term has form $an(A)$

   We have a deduction ending in

$$\frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P} \; (M)$$

   and we have the following derivation:

$$\frac{}{\phi \overset{P}{\longrightarrow} [\,] : P} \; (ax)$$

   whence

$$\frac{\Gamma \rhd A : P \quad \dfrac{}{\psi \overset{P}{\longrightarrow} [\,] : P} \; (ax)}{\Gamma \Rightarrow \psi'(A, [\,]) : P} \text{ ii)}$$

   and we know that

$$\psi'(A, [\,]) = \psi(an(A))$$

10. The **N** term has form $fal(A, \{var(x_i)\})$

   We have a deduction ending in

$$\frac{P_1 \rhd var(x_1) : P_1 \quad ... \quad P_n \rhd var(x_n) : P_n \quad \Delta \rhd A : 0}{\Delta, P_1, ..., P_n \rhd\!\!\rhd fal(A, \{var(x_i)\}) : R} \; (0_\varepsilon)$$

   whence

$$\frac{\Delta \rhd A : 0 \quad \dfrac{}{x_1 : P_1, ..., x_n : P_n \overset{0}{\longrightarrow} fal(\{x_i\}) : R} \; (0_\mathcal{L})}{\Delta, P_1, ..., P_n \Rightarrow \psi'(A, , fal(\{x_i\})) : R} \text{ ii)}$$

   and we know that

$$\psi'(A, , fal(\{x_i\})) = \psi(fal(A, \{var(x_i)\}))$$

11. The **N** term has form $withi(N_1, N_2)$

    We have a deduction ending in

$$\frac{\Gamma \rhd\!\!\rhd N_1 : P \quad \Gamma \rhd\!\!\rhd N_2 : Q}{\Gamma \rhd\!\!\rhd withi(N_1, N_2) : P\&Q}\ (\&_{\mathcal{I}})$$

    whence

$$\frac{\dfrac{\Gamma \rhd\!\!\rhd N_1 : P}{\Gamma \Rightarrow \psi(N_1) : P}\ \text{i)} \quad \dfrac{\Gamma \rhd\!\!\rhd N_2 : Q}{\Gamma \Rightarrow \psi(N_2) : Q}\ \text{i)}}{\Gamma \Rightarrow withr(\psi(N_1), \psi(N_2)) : P\&Q}\ (\&_{\mathcal{R}})$$

    and we know that

$$withr(\psi(N_1), \psi(N_2)) = \psi(withi(N_1, N_2))$$

12. The **N** term has form $plusi1(N)$

    We have a deduction ending in

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd plusi1(N) : P \oplus Q}\ (\oplus_{\mathcal{I}_1})$$

    whence

$$\frac{\dfrac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \Rightarrow \psi(N) : P}\ \text{i)}}{\Gamma \Rightarrow plusr1(\psi(N)) : P \oplus Q}\ (\oplus_{\mathcal{R}_1})$$

    and we know that

$$plusr1(\psi(N)) = \psi(plusi1(N))$$

13. The **N** term has form $plusi2(N)$

    We have a deduction ending in

$$\frac{\Gamma \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd plusi2(N) : P \oplus Q}\ (\oplus_{\mathcal{I}_2})$$

    whence

$$\frac{\dfrac{\Gamma \rhd\!\!\rhd N : Q}{\Gamma \Rightarrow \psi(N) : Q}\ \text{i)}}{\Gamma \Rightarrow plusr2(\psi(N)) : P \oplus Q}\ (\oplus_{\mathcal{R}_2})$$

    and we know that

$$plusr2(\psi(N)) = \psi(plusi2(N))$$

14. The **N** term has form $pluse(A, x_1.N_1, x_2.N_2)$

We have a deduction ending in

$$\frac{\Gamma \rhd A : P \oplus Q \quad \Delta, x_1 : P \rhd\!\!\rhd N_1 : R \quad \Delta, x_2 : P \rhd\!\!\rhd N_2 : R}{\Gamma, \Delta \rhd\!\!\rhd pluse(A, x_1.N_1, x_2.N_2) : R} \; (\oplus_\varepsilon)$$

whence

$$\frac{\Gamma \rhd A : P \oplus Q \qquad \dfrac{\dfrac{\Delta, x_1 : P \rhd\!\!\rhd N_1 : R}{\Delta, x_1 : P \Rightarrow \psi(N_1) : R}\text{ i)} \quad \dfrac{\Delta, x_2 : Q \rhd\!\!\rhd N_2 : R}{\Delta, x_2 : Q \Rightarrow \psi(N_2) : R}\text{ i)}}{\Delta \xrightarrow{P \oplus Q} plusl(x_1.\psi(N_1), x_2.\psi(N_2)) : R} \; (\oplus_\mathcal{L})}{\Gamma, \Delta \Rightarrow \psi'(A, plusl(x_1.\psi(N_1), x_2.\psi(N_2))) : R} \text{ ii)}$$

and we know that

$$\psi'(A, plusl(x_1.\psi(N_1), x_2.\psi(N_2))) = \psi(pluse(A, x_1.N_1, x_2.N_2))$$

15. The **N** term has form $tr(\{var(x_i)\})$

We have a deduction

$$\frac{P_1 \rhd var(x_1) : P_1 \quad \ldots \quad P_n \rhd var(x_n) : P_n}{P_1, \ldots, P_n \rhd\!\!\rhd tr(\{var(x_i)\}) : \top} \; (\top_\mathcal{I})$$

we also have

$$\frac{}{x_1 : P_1, \ldots, x_n : P_n \Rightarrow tr(\{x_i\}) : \top} \; (\top_\mathcal{R})$$

and we know that

$$tr(\{x_i\}) = \psi(tr(\{var(x_i)\}))$$

16. The **A** term has form $var(x)$

We have deduction

$$\frac{}{x : P \rhd var(x) : P} \; (ax)$$

and we find the following:

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R} \; (sel)$$

and we know that

$$(x; Ms) = \psi'(var(x), Ms)$$

17. The **A** term has form $ap(A, N)$

We have a deduction ending in

$$\frac{\Delta_1 \rhd A : P \multimap Q \quad \Delta_2 \rhd\!\rhd N : P}{\Delta_1, \Delta_2 \rhd ap(A, N) : Q} \ (\multimap_\varepsilon)$$

whence

$$\frac{\Delta_1 \rhd A : P \multimap Q \qquad \dfrac{\dfrac{\Delta_2 \rhd\!\rhd N : P}{\Delta_2 \Rightarrow \psi(N) : P}\ \text{i)} \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma, \Delta_2 \xrightarrow{P \multimap Q} (\psi(N) :: Ms) : R}\ (\multimap_{\mathcal{L}})}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \psi'(A, (\psi(N) :: Ms)) : R}\ \text{ii)}$$

and we know that

$$\psi'(A, (\psi(N) :: Ms)) = \psi'(ap(A, N), Ms)$$

18. The **A** term has form $withe1(A)$

We have a deduction ending in

$$\frac{\Delta \rhd A : P\&Q}{\Delta \rhd withe1(A) : P} \ (\&_{\varepsilon_1})$$

whence

$$\frac{\Delta \rhd A : P\&Q \quad \dfrac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{P\&Q} withl1(Ms) : R}\ (\&_{\mathcal{L}_1})}{\Gamma, \Delta \Rightarrow \psi'(A, withl1(Ms)) : R}\ \text{ii)}$$

and we know that

$$\psi'(A, withl1(Ms)) = \psi'(withe1(A), Ms)$$

19. The **A** term has form $withe2(A)$

We have a deduction ending in

$$\frac{\Delta \rhd A : P\&Q}{\Delta \rhd withe2(A) : Q} \ (\&_{\varepsilon_1})$$

whence

$$\frac{\Delta \rhd A : P\&Q \quad \dfrac{\Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P\&Q} withl2(Ms) : R}\ (\&_{\mathcal{L}_2})}{\Gamma, \Delta \Rightarrow \psi'(A, withl2(Ms)) : R}\ \text{ii)}$$

and we know that

$$\psi'(A, withl2(Ms)) = \psi'(withe2(A), Ms)$$

20. The **A** term has form $der(A)$

    We have a deduction ending in

$$\frac{\Delta \rhd A :!P}{\Delta \rhd der(A) : P}\ (D)$$

    whence

$$\dfrac{\Delta \rhd A :!P \quad \dfrac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{!P} d(Ms) : R}\ (D)}{\Gamma, \Delta \Rightarrow \psi'(A, d(Ms)) : R}\ \text{ii)}$$

    and we know that

$$\psi'(A, d(Ms)) = \psi'(der(A), Ms)$$

    In the following we again frequently uses the ellipsis $\overrightarrow{lp}$ instead of spelling out all the detail of the left premisses.

21. The $\overrightarrow{\mathbf{A}}$ term has form $\overrightarrow{var(x)}$

    We can find the following derivation:

$$\frac{\Gamma \xrightarrow{[\ ][P_1,...,P_n]}\!\!\!\!\gg Mss^1 : R}{\Gamma, P_1, ..., P_n \Rightarrow (\overrightarrow{x}; Mss^1) : R}\ (sel*)$$

    and we know that

$$(\overrightarrow{x}; Mss^1) = \psi''(\overrightarrow{var(x)}, Mss^1)$$

22. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)]$

    We have derivation ending in

$$\frac{\Gamma \xrightarrow{[!P_1,...,!P_i][P_{i+1},...,P_n]}\!\!\!\!\gg Mss^{i+1} : R}{\Gamma \xrightarrow{[!P_1,...,!P_{i-1}][!P_i,...,P_n]}\!\!\!\!\gg (tog(Mss^{i+1}))^i : R}\ (tog)$$

    whence

$$\frac{\overrightarrow{lp} \quad \Gamma \xrightarrow{[!P_1,...,!P_i][P_{i+1},...,P_n]}\!\!\!\!\gg Mss^{i+1} : R}{\Gamma, \Delta_1, ..., \Delta_n \Rightarrow \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1}) : R}\ \text{iii)}$$

    and we know that

$$\begin{aligned} &\psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], Mss^{i+1}) \\ &= \psi''([A_1, ..., A_i, var(x_{i+1}), ..., var(x_n)], (tog(Mss^{i+1}))^i) \end{aligned}$$

23. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., ap(A_i, N), var(x_{i+1}), ..., var(x_n)]$

The $i$th deduction ends in

$$\frac{\Delta_i \rhd A_i : P \multimap P_i \quad \Delta_i' \rhd\!\rhd N : P}{\Delta_i, \Delta_i' \rhd ap(A_i, N) : P_i} \ (\multimap_\varepsilon)$$

whence

$$\frac{\overrightarrow{lp} \qquad \dfrac{\dfrac{\Delta_i' \rhd\!\rhd N : P}{\Delta_i' \Rightarrow \psi(N) : P}\ \text{i)} \qquad \Gamma \xrightarrow{[!P_1,...,!P_{i-1}][P_i,...,P_n]} Mss^i : R}{\Gamma \xrightarrow{[!P_1,..,!P_{i-1}][P \multimap P_i,...,P_n]} (\psi(N) :: Mss^i)^i : R}\ (\multimap_{\mathcal{L}}*)}{\Delta_1, ..., \Delta_i, \Delta_i', ..., \Delta_n \Rightarrow \psi''(\overrightarrow{A}, (\psi(N) :: Mss^i)^i) : R}\ \text{iii)}$$

and we know that

$$\psi''(\overrightarrow{A}, (\psi(N) :: Mss^i)^i) = \psi''([A_1, ..., ap(A_i, N), var(x_{i+1}), ..., var(x_n)], Mss^i)$$

24. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)]$

The $i$th deduction ends in

$$\frac{\Delta_i \rhd A_i : P_i \& Q}{\Delta_i \rhd withe1(A_i) : P_i} \ (\&_{\varepsilon_1})$$

whence

$$\frac{\overrightarrow{lp} \quad \Delta_i \rhd A_i : P_i \& Q \quad \dfrac{\Gamma \xrightarrow{[!P_1,...,!P_{i-1}][P_i,...,P_n]} Mss^i : R}{\Gamma \xrightarrow{[!P_1,..,!P_{i-1}][P_i \& Q,...,P_n]} (withl1(Mss^i))^i : R}\ (\&_{\mathcal{L}_1}*)}{\Delta_1 ..., \Delta_n \Rightarrow \psi''(\overrightarrow{A}, (withl1(Mss^i))^i : R}\ \text{iii)}$$

and we know that

$$\psi''(\overrightarrow{A}, (withl1(Mss^i))^i = \psi''([A_1, ..., withe1(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$

25. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)]$

The $i$th deduction ends in

$$\frac{\Delta_i \rhd A_i : Q \& P_i}{\Delta_i \rhd withe2(A_i) : P_i} \ (\&_{\varepsilon_2})$$

whence

$$\frac{\overrightarrow{lp} \quad \Delta_i \rhd A_i : Q \& P_i \quad \dfrac{\Gamma \xrightarrow{[!P_1,...,!P_{i-1}][P_i,...,P_n]} Mss^i : R}{\Gamma \xrightarrow{[!P_1,..,!P_{i-1}][Q \& P_i,...,P_n]} (withl2(Mss^i))^i : R}\ (\&_{\mathcal{L}_2}*)}{\Delta_1 ..., \Delta_n \Rightarrow \psi''(\overrightarrow{A}, (withl2(Mss^i))^i : R}\ \text{iii)}$$

and we know that

$$\psi''(\overrightarrow{A}, (withl2(Mss^i))^i = \psi''([A_1, ..., withe2(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$

26. The $\overrightarrow{\mathbf{A}}$ term has form $[A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)]$

The $i$th deduction ends in

$$\frac{\Delta_i \rhd A_i :!P_i}{\Delta_i \rhd der(A_i) : P_i}\ (D)$$

whence

$$\frac{\overrightarrow{lp} \quad \Delta_i \rhd A_i :!P_i \quad \dfrac{\Gamma \overset{[!P_1,...,!P_{i-1}][P_i,...,P_n]}{\ggg} Mss^i : R}{\Gamma \overset{[!P_1,..,!P_{i-1}][!P_i,...,P_n]}{\ggg} (d(Mss^i))^i : R}\ (D*)}{\Delta_1, ..., \Delta_n \Rightarrow \psi''(\overrightarrow{A}, (d(Mss^i))^i : R}\ \text{iii)}$$

and we know that

$$\psi''(\overrightarrow{A}, (d(Mss^i))^i = \psi''([A_1, ..., der(A_i), var(x_{i+1}), ..., var(x_n)], Mss^i)$$

■

## 6.6 Cut Elimination

In this section we discuss cut elimination for SILL. We give the (complicated) cut rules for SILL$^{cut}$ (SILL with these rules) and then a simple cut elimination argument for SILL$^{cut}$.

There are ten cut rules for SILL. We show all of these in Figure 6.7. In these rules we have some notation for multicuts: $(!P)^i$ stands for $i$ occurrences of formula $!P$ and $(!\Theta)^i$ stands for $i$ occurrences of multiset $!\Theta$.

In the next section we give reduction rules for the occurrences of cut, before giving a cut-elimination procedure and further discussion on cut for SILL and its elimination.

### 6.6.1 Cut Reductions

We give reduction rules for the occurrences of the ten cut rules. First, we try to clarify some of the notation used.

$$\frac{\Gamma \xrightarrow{Q} P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{Q} R}\ (cut_1) \qquad \frac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{Q} R}{\Gamma, \Delta \xrightarrow{Q} R}\ (cut_2)$$

$$\frac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \Rightarrow R}\ (cut_3) \qquad \frac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow R}{\Gamma, \Delta \Rightarrow R}\ (cut_4)$$

$$\frac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{[!\Phi][\Psi]}\!\!\!\gg R}{\Gamma, \Delta \xrightarrow{[!\Phi][\Psi]}\!\!\!\gg R}\ (cut_5) \qquad \frac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{[\ ][\Xi,P]}\!\!\!\gg R}{\Gamma, \Delta, \Xi \Rightarrow R}\ (cut_6)$$

$$\frac{\xrightarrow{[!\Theta][\ ]}\!\!\!\gg !P \quad \Delta, (!P)^i \xrightarrow{!P} R}{\Delta, (!\Theta)^{i+1} \Rightarrow R}\ (cut_7) \qquad \frac{\xrightarrow{[!\Theta][\ ]}\!\!\!\gg !P \quad \Delta, (!P)^i \Rightarrow R}{\Delta, (!\Theta)^i \Rightarrow R}\ (cut_8)$$

$$\frac{\xrightarrow{[!\Theta][\ ]}\!\!\!\gg !P \quad \Delta, (!P)^i \xrightarrow{Q} R}{\Delta, (!\Theta)^i \xrightarrow{Q} R}\ (cut_9)$$

$$\frac{\xrightarrow{[!\Theta][\ ]}\!\!\!\gg !P \quad \Delta, (!P)^i \xrightarrow{[!\Phi][\Psi]}\!\!\!\gg R}{\Delta, (!\Theta)^i \xrightarrow{[!\Phi][\Psi]}\!\!\!\gg R}\ (cut_{10})$$

**Figure 6.7:** Cut rules for SILL

Consider a promotion, written:

$$\cfrac{\cfrac{!Q_1, ..., !Q_n \Rightarrow P}{\xrightarrow{[!Q_1,...,!Q_n][\ ]}\!\!\!\gg !P}\ (P)}{\cfrac{\vdots}{\cfrac{\Gamma_1, ..., \Gamma_n \xrightarrow{[\ ][S_1,...,S_n]}\!\!\!\gg !P}{\Gamma_1, ..., \Gamma_n, S_1, ..., S_n \Rightarrow !P}\ (sel*)}}$$

The $\Gamma_i$ are the context formulae for the decomposition of $S_i$ to $!Q_i$. This section of proof can then be extracted:

$$\cfrac{\cfrac{}{\phi \xrightarrow{!Q_i} !Q_i}\ (ax)}{\cfrac{\vdots}{\cfrac{\Gamma_i \xrightarrow{S_i} !Q_i}{\Gamma_i, S_i \Rightarrow !Q_i}\ (sel)}}$$

When this extraction forms part of a reduction, we simply write the conclusion.

We often write $[\Xi, P]$. This stands for a list whose elements are those of the multiset $\Xi$ and the element $P$, with $P$ occurring at an unspecified position in the list. We write $[\Xi, (!Q_1, ..., !Q_n)]$ for a list of the elements of $\Xi$ with the list $[!Q_1, ..., !Q_n]$ occurring as a sublist in some position.

Now we give the reduction rules.

1. $(cut_1)$ Analysis by cases on the left premiss:

   (a) $(ax)$

$$\dfrac{\dfrac{}{\phi \xrightarrow{P} P}\,(ax) \qquad \Delta \xrightarrow{P} R}{\Delta \xrightarrow{P} R}\,(cut_1)$$

   reduces to:

$$\Delta \xrightarrow{P} R$$

   (b) $(\multimap_{\mathcal{L}})$

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow S \quad \Gamma_2 \xrightarrow{T} P}{\Gamma_1, \Gamma_2 \xrightarrow{S \multimap T} P}\,(\multimap_{\mathcal{L}}) \qquad \Delta \xrightarrow{P} R}{\Gamma_1, \Gamma_2, \Delta \xrightarrow{S \multimap T} R}\,(cut_1)$$

   reduces to:

$$\dfrac{\Gamma_1 \Rightarrow S \qquad \dfrac{\Gamma_2 \xrightarrow{T} P \quad \Delta \xrightarrow{P} R}{\Gamma_2, \Delta \xrightarrow{T} R}\,(cut_1)}{\Gamma_1, \Gamma_2, \Delta \xrightarrow{S \multimap T} R}\,(\multimap_{\mathcal{L}})$$

   (c) $(\otimes_{\mathcal{L}})$

$$\dfrac{\dfrac{\Gamma, S, T \Rightarrow P}{\Gamma \xrightarrow{S \otimes T} P}\,(\otimes_{\mathcal{L}}) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{S \otimes T} R}\,(cut_1)$$

   reduces to:

$$\dfrac{\dfrac{\Gamma, S, T \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta, S, T \Rightarrow R}\,(cut_3)}{\Gamma, \Delta \xrightarrow{S \otimes T} R}\,(\otimes_{\mathcal{L}})$$

   (d) $(I_{\mathcal{L}})$

$$\dfrac{\dfrac{\Gamma \Rightarrow P}{\Gamma \xrightarrow{I} P}\,(I_{\mathcal{L}}) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{I} R}\,(cut_1)$$

   reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \Rightarrow R}\,(cut_3)}{\Gamma, \Delta \xrightarrow{I} R}\,(I_{\mathcal{L}})$$

   (e) $(0_{\mathcal{L}})$

$$\dfrac{\dfrac{}{\Gamma \xrightarrow{0} P}\,(0_{\mathcal{L}}) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{0} R}\,(cut_1)$$

reduces to:

$$\dfrac{}{\Gamma, \Delta \xrightarrow{0} R}\,(0_{\mathcal{L}})$$

(f) $(\oplus_{\mathcal{L}})$

$$\dfrac{\dfrac{\Gamma, S \Rightarrow P \quad \Gamma, T \Rightarrow P}{\Gamma \xrightarrow{S\oplus T} P}\,(\oplus_{\mathcal{L}}) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{S\oplus T} R}\,(cut_1)$$

reduces to:

$$\dfrac{\dfrac{\Gamma, S \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta, S \Rightarrow R}\,(cut_3) \qquad \dfrac{\Gamma, T \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta, T \Rightarrow R}\,(cut_3)}{\Gamma, \Delta \xrightarrow{S\oplus T} R}\,(\oplus_{\mathcal{L}})$$

(g) $(\&_{\mathcal{L}_1})$

$$\dfrac{\dfrac{\Gamma \xrightarrow{S} P}{\Gamma \xrightarrow{S\&T} P}\,(\&_{\mathcal{L}_1}) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{S\&T} R}\,(cut_1)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \xrightarrow{S} P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{S} R}\,(cut_1)}{\Gamma, \Delta \xrightarrow{S\&T} R}\,(\&_{\mathcal{L}_1})$$

(h) $(\&_{\mathcal{L}_2})$ Similar to above.

(i) $(W)$

$$\dfrac{\dfrac{\Gamma \Rightarrow P}{\Gamma \xrightarrow{!S} P}\,(W) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{!S} R}\,(cut_1)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \Rightarrow R}\,(cut_3)}{\Gamma, \Delta \xrightarrow{!S} R}\,(W)$$

(j) $(C)$

$$\dfrac{\dfrac{\Gamma, !S, !S \Rightarrow P}{\Gamma \xrightarrow{!S} P}\,(C) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{!S} R}\,(cut_1)$$

reduces to:

$$\dfrac{\dfrac{\Gamma, !S, !S \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta, !S, !S \Rightarrow R}\,(cut_3)}{\Gamma, \Delta \xrightarrow{!S} R}\,(C)$$

(k) $(D)$

$$\dfrac{\dfrac{\Gamma \xrightarrow{S} P}{\Gamma \xrightarrow{!S} P}\,(D) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{!S} R}\,(cut_1)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \xrightarrow{S} P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{S} R}\,(cut_1)}{\Gamma, \Delta \xrightarrow{!S} R}\,(D)$$

2. $(cut_2)$ Analysis by cases on the right premiss.

(a) $(ax)$ Not possible.

(b) $(\multimap_{\mathcal{L}})$

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Delta_1, P \Rightarrow S \quad \Delta_2 \xrightarrow{T} R}{\Delta_1, \Delta_2, P \xrightarrow{S \multimap T} R}\,(\multimap_{\mathcal{L}})}{\Gamma, \Delta_1, \Delta_2 \xrightarrow{S \multimap T} R}\,(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta_1, P \Rightarrow S}{\Gamma, \Delta_1 \Rightarrow S}\,(cut_4) \qquad \Delta_2 \xrightarrow{T} R}{\Gamma, \Delta_1, \Delta_2 \xrightarrow{S \multimap T} R}\,(\multimap_{\mathcal{L}})$$

or

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Delta_1 \Rightarrow S \quad \Delta_2, P \xrightarrow{T} R}{\Delta_1, \Delta_2, P \xrightarrow{S \multimap T} R}\,(\multimap_{\mathcal{L}})}{\Gamma, \Delta_1, \Delta_2 \xrightarrow{S \multimap T} R}\,(cut_2)$$

reduces to:

$$\dfrac{\Delta_1 \Rightarrow S \qquad \dfrac{\Gamma \Rightarrow P \quad \Delta_2, P \xrightarrow{T} R}{\Gamma, \Delta_2 \xrightarrow{T} R}\,(cut_2)}{\Gamma, \Delta_1, \Delta_2 \xrightarrow{S \multimap T} R}\,(\multimap_{\mathcal{L}})$$

(c) $(\otimes_{\mathcal{L}})$

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Delta, P, S, T \Rightarrow R}{\Delta, P \xrightarrow{S \otimes T} R}\,(\otimes_{\mathcal{L}})}{\Gamma, \Delta \xrightarrow{S \otimes T} R}\,(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P, S, T \Rightarrow R}{\Gamma, \Delta, S, T \Rightarrow R}\,(cut_4)}{\Gamma, \Delta \xrightarrow{S \otimes T} R}\,(\otimes_{\mathcal{L}})$$

(d) $(I_{\mathcal{L}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \Rightarrow R}{\Delta, P \xrightarrow{I} R}\,(I_{\mathcal{L}})}{\Gamma, \Delta \xrightarrow{I} R}\,(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow R}{\Gamma, \Delta \Rightarrow R}\,(cut_4)}{\Gamma, \Delta \xrightarrow{I} R}\,(I_{\mathcal{L}})$$

(e) $(0_{\mathcal{L}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{}{\Delta, P \xrightarrow{0} R}\,(0_{\mathcal{L}})}{\Gamma, \Delta \xrightarrow{0} R}\,(cut_2)$$

reduces to:

$$\dfrac{}{\Gamma, \Delta \xrightarrow{0} R}\,(0_{\mathcal{L}})$$

(f) $(\oplus_{\mathcal{L}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P, S \Rightarrow R \quad \Delta, P, T \Rightarrow R}{\Delta, P \xrightarrow{S \oplus T} R}\,(\oplus_{\mathcal{L}})}{\Gamma, \Delta \xrightarrow{S \oplus T} R}\,(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P, S \Rightarrow R}{\Gamma, \Delta, S \Rightarrow R}\,(cut_4) \quad \dfrac{\Gamma \Rightarrow P \quad \Delta, P, T \Rightarrow R}{\Gamma, \Delta, T \Rightarrow R}\,(cut_4)}{\Gamma, \Delta \xrightarrow{S \oplus T} R}\,(\oplus_{\mathcal{L}})$$

(g) $(\&_{\mathcal{L}_1})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \xrightarrow{S} R}{\Delta, P \xrightarrow{S \& T} R}\,(\&_{\mathcal{L}_1})}{\Gamma, \Delta \xrightarrow{S \& T} R}\,(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{S} R}{\Gamma, \Delta \xrightarrow{S} R}\,(cut_2)}{\Gamma, \Delta \xrightarrow{S \& T} R}\,(\&_{\mathcal{L}_1})$$

(h) $(\&_{\mathcal{L}_2})$ Similar to above.

(i) $(W)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \Rightarrow R}{\Delta, P \xrightarrow{!S} R}\,(W)}{\Gamma, \Delta \xrightarrow{!S} R}\,(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow R}{\Gamma, \Delta \Rightarrow R}\;(cut_4)}{\Gamma, \Delta \xrightarrow{!S} R}\;(W)$$

(j) $(C)$

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Delta, P, !S, !S \Rightarrow R}{\Delta, P \xrightarrow{!S} R}\;(C)}{\Gamma, \Delta \xrightarrow{!S} R}\;(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P, !S, !S \Rightarrow R}{\Gamma, \Delta, !S, !S \Rightarrow R}\;(cut_4)}{\Gamma, \Delta \xrightarrow{!S} R}\;(C)$$

(k) $(D)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \xrightarrow{S} R}{\Delta, P \xrightarrow{!S} R}\;(D)}{\Gamma, \Delta \xrightarrow{!S} R}\;(cut_2)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{S} R}{\Gamma, \Delta \xrightarrow{S} R}\;(cut_2)}{\Gamma, \Delta \xrightarrow{!S} R}\;(D)$$

3. $(cut_3)$ Analysis by cases on the left premiss.

(a) $(sel)$

$$\dfrac{\dfrac{\Gamma \xrightarrow{S} P}{\Gamma, S \Rightarrow P}\;(sel) \qquad \Delta \xrightarrow{P} R}{\Gamma, \Delta, S \Rightarrow R}\;(cut_3)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \xrightarrow{S} P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \xrightarrow{S} R}\;(cut_1)}{\Gamma, \Delta, S \Rightarrow R}\;(sel)$$

(b) $(sel*)$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{!Q_1, ..., !Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\longrightarrow\!\!\!\!\gg} !P}\;(P) \\ \vdots \\ \Gamma_1, ..., \Gamma_n \overset{[\,][S_1,...,S_n]}{\longrightarrow\!\!\!\!\gg} !P\end{array}}{\Gamma_1, ..., \Gamma_n, S_1, ..., S_n \Rightarrow !P}\;(sel*) \qquad \Delta \xrightarrow{!P} R}{\Gamma_1, ..., \Gamma_n, \Delta, S_1, ..., S_n \Rightarrow R}\;(cut_3)$$

reduces to:

$$\cfrac{\Gamma_n, S_n \Rightarrow !Q_n \qquad \cfrac{\cfrac{!Q_1, ..., !Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\;]}{\longrightarrow\!\!\gg} !P}\,(P) \qquad \Delta \xrightarrow{!P} R}{\Delta, !Q_1, ..., !Q_n \Rightarrow R}\,(cut_7)}{\Delta, !Q_1, ..., !Q_{n\perp 1}, \Gamma_n, S_n \Rightarrow R}\,(cut_4)$$

$$\vdots\; cut_4 s$$

$$\Gamma_1, ..., \Gamma_n, \Delta, S_1, ..., S_n \Rightarrow R$$

(c) $(\multimap_{\mathcal{R}})$

$$\cfrac{\cfrac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \multimap Q}\,(\multimap_{\mathcal{R}}) \qquad \cfrac{\Delta_1 \Rightarrow P \quad \Delta_2 \xrightarrow{Q} R}{\Delta_1, \Delta_2 \xrightarrow{P \perp\!\circ Q} R}\,(\multimap_{\mathcal{L}})}{\Gamma, \Delta_1, \Delta_2 \Rightarrow R}\,(cut_3)$$

reduces to:

$$\cfrac{\Delta_1 \Rightarrow P \qquad \cfrac{\Gamma, P \Rightarrow Q \quad \Delta_2 \xrightarrow{Q} R}{\Gamma, \Delta_2, P \Rightarrow R}\,(cut_3)}{\Gamma, \Delta_1, \Delta_2 \Rightarrow R}\,(cut_4)$$

(d) $(\otimes_{\mathcal{R}})$

$$\cfrac{\cfrac{\Gamma_1 \Rightarrow P \quad \Gamma_2 \Rightarrow Q}{\Gamma_1, \Gamma_2 \Rightarrow P \otimes Q}\,(\otimes_{\mathcal{R}}) \qquad \cfrac{\Delta, P, Q \Rightarrow R}{\Delta \xrightarrow{P \otimes Q} R}\,(\otimes_{\mathcal{L}})}{\Gamma_1, \Gamma_2, \Delta \Rightarrow R}\,(cut_3)$$

reduces to:

$$\cfrac{\Gamma_1 \Rightarrow P \qquad \cfrac{\Gamma_2 \Rightarrow Q \quad \Delta, P, Q \Rightarrow R}{\Gamma_2, \Delta, P \Rightarrow R}\,(cut_4)}{\Gamma_1, \Gamma_2, \Delta \Rightarrow R}\,(cut_4)$$

(e) $(I_{\mathcal{R}})$

$$\cfrac{\cfrac{}{\Rightarrow I}\,(I_{\mathcal{R}}) \qquad \cfrac{\Delta \Rightarrow R}{\Delta \xrightarrow{I} R}\,(I_{\mathcal{L}})}{\Delta \Rightarrow R}\,(cut_3)$$

reduces to:

$$\Delta \Rightarrow R$$

(f) $(\top_{\mathcal{R}})$ Not possible.

(g) $(\oplus_{\mathcal{R}_1})$

$$\cfrac{\cfrac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \oplus Q}\,(\oplus_{\mathcal{R}_1}) \qquad \cfrac{\Delta, P \Rightarrow R \quad \Delta, Q \Rightarrow R}{\Delta \xrightarrow{P \oplus Q} R}\,(\oplus_{\mathcal{L}})}{\Gamma, \Delta \Rightarrow R}\,(cut_3)$$

reduces to:

$$\cfrac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow R}{\Gamma, \Delta \Rightarrow R}\,(cut_4)$$

(h) $(\oplus_{\mathcal{R}_2})$ Similar to above.

(i) $(\&_{\mathcal{R}})$

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P\&Q}\,(\&_{\mathcal{R}}) \quad \dfrac{\Delta \xrightarrow{P} R}{\Delta \xrightarrow{P\&Q} R}\,(\&_{\mathcal{L}_1})}{\Gamma, \Delta \Rightarrow R}\,(cut_3)$$

reduces to:

$$\dfrac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \Rightarrow R}\,(cut_3)$$

4. $(cut_4)$ Analysis by cases on the right premiss.

(a) $(sel)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \xrightarrow{S} R}{\Delta, P, S \Rightarrow R}\,(sel)}{\Gamma, \Delta, S \Rightarrow R}\,(cut_4)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{S} R}{\Gamma, \Delta \xrightarrow{S} R}\,(cut_2)}{\Gamma, \Delta, S \Rightarrow R}\,(sel)$$

or

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta \xrightarrow{P} R}{\Delta, P \Rightarrow R}\,(sel)}{\Gamma, \Delta \Rightarrow R}\,(cut_4)$$

reduces to:

$$\dfrac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Gamma, \Delta \Rightarrow R}\,(cut_3)$$

(b) $(sel*)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \xrightarrow{[\,][\Xi]} R}{\Delta, \Xi, P \Rightarrow R}\,(sel*)}{\Gamma, \Delta, \Xi \Rightarrow R}\,(cut_4)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{[\,][\Xi]} R}{\Gamma, \Delta \xrightarrow{[\,][\Xi]} R}\,(cut_5)}{\Gamma, \Delta, \Xi \Rightarrow R}\,(sel*)$$

or

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta \xrightarrow{[\,][\Xi, P]} R}{\Delta, \Xi, P \Rightarrow R}\,(sel*)}{\Gamma, \Delta, \Xi \Rightarrow R}\,(cut_4)$$

reduces to:

$$\dfrac{\Gamma \Rightarrow P \quad \Delta \xrightarrow{[\,][\Xi,P]} R}{\Gamma, \Delta, \Xi \Rightarrow R}\ (cut_6)$$

(c) $(\multimap_{\mathcal{R}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P, S \Rightarrow T}{\Delta, P \Rightarrow S \multimap T}\ (\multimap_{\mathcal{R}})}{\Gamma, \Delta \Rightarrow S \multimap T}\ (cut_4)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P, S \Rightarrow T}{\Gamma, \Delta, S \Rightarrow T}\ (cut_4)}{\Gamma, \Delta \Rightarrow S \multimap T}\ (\multimap_{\mathcal{R}})$$

(d) $(\otimes_{\mathcal{R}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta_1, P \Rightarrow S \quad \Delta_2 \Rightarrow T}{\Delta_1, \Delta_2, P \Rightarrow S \otimes T}\ (\otimes_{\mathcal{R}})}{\Gamma, \Delta_1, \Delta_2 \Rightarrow S \otimes T}\ (cut_4)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta_1, P \Rightarrow S}{\Gamma, \Delta_1 \Rightarrow S}\ (cut_4) \quad \Delta_2 \Rightarrow T}{\Gamma, \Delta_1, \Delta_2 \Rightarrow S \otimes T}\ (\otimes_{\mathcal{R}})$$

(e) $(I_{\mathcal{R}})$ Not possible.

(f) $(\top_{\mathcal{R}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{}{\Delta, P \Rightarrow \top}\ (\top_{\mathcal{R}})}{\Gamma, \Delta \Rightarrow \top}\ (cut_4)$$

reduces to:

$$\dfrac{}{\Gamma, \Delta \Rightarrow \top}\ (\top_{\mathcal{R}})$$

(g) $(\oplus_{\mathcal{R}_1})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \Rightarrow S}{\Delta, P \Rightarrow S \oplus T}\ (\oplus_{\mathcal{R}_1})}{\Gamma, \Delta \Rightarrow S \oplus T}\ (cut_4)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow S}{\Gamma, \Delta \Rightarrow S}\ (cut_4)}{\Gamma, \Delta \Rightarrow S \oplus T}\ (\oplus_{\mathcal{R}_1})$$

(h) $(\oplus_{\mathcal{R}_2})$ Similar to above.

(i) $(\&_{\mathcal{R}})$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \Rightarrow S \quad \Delta, P \Rightarrow T}{\Delta, P \Rightarrow S \& T}\ (\&_{\mathcal{R}})}{\Gamma, \Delta \Rightarrow S \& T}\ (cut_4)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow S}{\Gamma, \Delta \Rightarrow S}\,(cut_4) \qquad \dfrac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow T}{\Gamma, \Delta \Rightarrow T}\,(cut_4)}{\Gamma, \Delta \Rightarrow S \& T}\,(\&_{\mathcal{R}})$$

5. $(cut_5)$ Analysis by cases on the right premiss.

   (a) $(tog)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \overset{[!\Phi,!S][\Psi]}{\twoheadrightarrow} R}{\Delta, P \overset{[!\Phi][!S,\Psi]}{\twoheadrightarrow} R}\,(tog)}{\Gamma, \Delta \overset{[!\Phi][!S,\Psi]}{\twoheadrightarrow} R}\,(cut_5)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \overset{[!\Phi,!S][\Psi]}{\twoheadrightarrow} R}{\Gamma, \Delta \overset{[!\Phi,!S][\Psi]}{\twoheadrightarrow} R}\,(cut_5)}{\Gamma, \Delta \overset{[!\Phi][!S,\Psi]}{\twoheadrightarrow} R}\,(tog)$$

   (b) $(\multimap_{\mathcal{L}}*)$

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Delta_1, P \Rightarrow S \quad \Delta_2 \overset{[!\Phi][T,\Psi]}{\twoheadrightarrow} R}{\Delta_1, \Delta_2, P \overset{[!\Phi][S \multimap T,\Psi]}{\twoheadrightarrow} R}\,(\multimap_{\mathcal{L}}*)}{\Gamma, \Delta_1, \Delta_2 \overset{[!\Phi][S \multimap T,\Psi]}{\twoheadrightarrow} R}\,(cut_5)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta_1, P \Rightarrow S}{\Gamma, \Delta_1 \Rightarrow S}\,(cut_4) \qquad \Delta_2 \overset{[!\Phi][T,\Psi]}{\twoheadrightarrow} R}{\Gamma, \Delta_1, \Delta_2 \overset{[!\Phi][S \multimap T,\Psi]}{\twoheadrightarrow} R}\,(\multimap_{\mathcal{L}}*)$$

or

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Delta_1 \Rightarrow S \quad \Delta_2, P \overset{[!\Phi][T,\Psi]}{\twoheadrightarrow} R}{\Delta_1, \Delta_2, P \overset{[!\Phi][S \multimap T,\Psi]}{\twoheadrightarrow} R}\,(\multimap_{\mathcal{L}}*)}{\Gamma, \Delta_1, \Delta_2 \overset{[!\Phi][S \multimap T,\Psi]}{\twoheadrightarrow} R}\,(cut_5)$$

reduces to:

$$\dfrac{\Delta_1 \Rightarrow S \qquad \dfrac{\Gamma \Rightarrow P \quad \Delta_2, P \overset{[!\Phi][T,\Psi]}{\twoheadrightarrow} R}{\Gamma, \Delta_2 \overset{[!\Phi][T,\Psi]}{\twoheadrightarrow} R}\,(cut_5)}{\Gamma, \Delta_1, \Delta_2 \overset{[!\Phi][S \multimap T,\Psi]}{\twoheadrightarrow} R}\,(\multimap_{\mathcal{L}}*)$$

(c) $(\&_{\mathcal{L}_1}*)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\gg R}{\Delta, P \xrightarrow{[!\Phi][S\&T,\Psi]}\!\!\!\gg R}\ (\&_{\mathcal{L}_1}*)}{\Gamma, \Delta \xrightarrow{[!\Phi][S\&T,\Psi]}\!\!\!\gg R}\ (cut_5)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\gg R}{\Gamma, \Delta \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\gg R}\ (cut_5)}{\Gamma, \Delta \xrightarrow{[!\Phi][S\&T,\Psi]}\!\!\!\gg R}\ (\&_{\mathcal{L}_1}*)$$

(d) $(\&_{\mathcal{L}_2}*)$ Similar to above.

(e) $(D*)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Delta, P \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\gg R}{\Delta, P \xrightarrow{[!\Phi][!S,\Psi]}\!\!\!\gg R}\ (D*)}{\Gamma, \Delta \xrightarrow{[!\Phi][!S,\Psi]}\!\!\!\gg R}\ (cut_5)$$

reduces to:

$$\dfrac{\dfrac{\Gamma \Rightarrow P \quad \Delta, P \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\gg R}{\Gamma, \Delta \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\gg R}\ (cut_5)}{\Gamma, \Delta \xrightarrow{[!\Phi][!S,\Psi]}\!\!\!\gg R}\ (D*)$$

(f) $(P)$ Not possible.

6. $(cut_6)$ Analysis by cases on the left premiss.

(a) $(sel)$ Three possibilities:

$$\dfrac{\dfrac{\begin{matrix}\Gamma' \Rightarrow P \\ \vdots \\ \Gamma \xrightarrow{S} P\end{matrix}}{\Gamma, S \Rightarrow P}\ (sel) \quad \Delta \xrightarrow{[\ ][\Xi,P]}\!\!\!\gg R}{\Gamma, \Delta, \Xi, S \Rightarrow R}\ (cut_6)$$

reduces to:

$$\dfrac{\begin{matrix}\dfrac{\Gamma' \Rightarrow P \quad \Delta \xrightarrow{[\ ][\Xi,P]}\!\!\!\gg R}{\Gamma', \Delta, \Xi \Rightarrow R}\ (cut_6) \\ \vdots \\ \Gamma, \Delta, \Xi \xrightarrow{S} R\end{matrix}}{\Gamma, \Delta, \Xi, S \Rightarrow R}\ (sel)$$

or

$$
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{}{\phi \xrightarrow{P} P}\,(ax) \\
\vdots \\
\Gamma \xrightarrow{S} P
\end{array}
}{\Gamma, S \Rightarrow P}\,(sel)
\qquad
\begin{array}{c}
\dfrac{\Delta' \overset{[!\Phi][P,\Psi]}{\twoheadrightarrow} R}{}\,(tog) \\
\vdots \\
\Delta \overset{[\ ][\Xi,P]}{\twoheadrightarrow} R
\end{array}
}{\Gamma, \Delta, \Xi, S \Rightarrow R}\,(cut_6)
$$

reduces to:

$$
\dfrac{
\begin{array}{c}
\Delta' \overset{[!\Phi][P,\Psi]}{\twoheadrightarrow} R \\
\vdots \\
\dfrac{\Gamma, \Delta' \overset{[!\Phi][S,\Psi]}{\twoheadrightarrow} R}{}\,(tog) \\
\vdots \\
\Gamma, \Delta \overset{[\ ][\Xi,S]}{\twoheadrightarrow} R
\end{array}
}{\Gamma, \Delta, \Xi, S \Rightarrow R}\,(sel*)
$$

or

$$
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{}{\Gamma' \xrightarrow{0} P}\,(0_{\mathcal{L}}) \\
\vdots \\
\Gamma \xrightarrow{S} P
\end{array}
}{\Gamma, S \Rightarrow P}\,(sel)
\qquad
\Delta \overset{[\ ][\Xi,P]}{\twoheadrightarrow} R
}{\Gamma, \Delta, \Xi, S \Rightarrow R}\,(cut_6)
$$

reduces to:

$$
\dfrac{
\begin{array}{c}
\dfrac{}{\Gamma', \Delta, \Xi \xrightarrow{0} R}\,(0_{\mathcal{L}}) \\
\vdots \\
\Gamma, \Delta, \Xi \xrightarrow{S} R
\end{array}
}{\Gamma, \Delta, \Xi, S \Rightarrow R}\,(sel)
$$

(b) $(sel*)$ Two possibilities.

$$
\dfrac{
\dfrac{
\begin{array}{c}
\dfrac{!Q_1, ...., !Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\ ]}{\twoheadrightarrow} !P}\,(P) \\
\vdots \\
\Gamma_1, ..., \Gamma_n \overset{[\ ][S_1,...,S_n]}{\twoheadrightarrow} !P
\end{array}
}{\Gamma_1, ..., \Gamma_n, S_1, ..., S_n \Rightarrow !P}\,(sel*)
\qquad
\begin{array}{c}
\dfrac{\dfrac{\Delta' \overset{[!\Phi][P,\Psi]}{\twoheadrightarrow} R}{\Delta' \overset{[!\Phi][!P,\Psi]}{\twoheadrightarrow} R}\,(D*)}{}\,(tog) \\
\vdots \\
\Delta \overset{[\ ][\Xi,!P]}{\twoheadrightarrow} R
\end{array}
}{\Gamma_1, ..., \Gamma_n, \Delta, \Xi, S_1, ..., S_n \Rightarrow R}\,(cut_6)
$$

reduces to:

$$
\dfrac{\Gamma, S_n \Rightarrow !Q_n \qquad \dfrac{!Q_1,\ldots,!Q_n \Rightarrow P \qquad \begin{array}{c}\dfrac{\Delta' \xrightarrow{[!\Phi][P,\Psi]}\!\!\gg R}{}\,(tog)\\ \vdots \\ \Delta \xrightarrow{[\,][\Xi,P]}\!\!\gg R\end{array}}{\Delta,\Xi,!Q_1,\ldots,!Q_n \Rightarrow R}\,(cut_6)}{\Delta,\Xi,!Q_1,\ldots,!Q_{n\perp 1},\Gamma_n,S_n \Rightarrow R}\,(cut_4)
$$

$$
\begin{array}{c} \vdots\; cut_4 s \\ \Gamma_1,\ldots,\Gamma_n,\Delta,\Xi,S_1,\ldots,S_n \Rightarrow R \end{array}
$$

or

$$
\dfrac{\begin{array}{c}\dfrac{!Q_1,\ldots,!Q_n \Rightarrow P}{\xrightarrow{[!Q_1,\ldots,!Q_n][\,]}\!\!\gg !P}\,(P)\\ \vdots \\ \dfrac{\Gamma_1,\ldots,\Gamma_n \xrightarrow{[\,][S_1,\ldots,S_n]}\!\!\gg !P}{\Gamma_1,\ldots,\Gamma_n,S_1,\ldots,S_n \Rightarrow !P}\,(sel*)\end{array} \qquad \begin{array}{c}\dfrac{!\Xi',!P \Rightarrow R}{\xrightarrow{[!\Xi',!P][\,]}\!\!\gg !R}\,(P)\\ \vdots \\ \dfrac{\dfrac{\Delta' \xrightarrow{[!\Phi,!P][\Psi]}\!\!\gg !R}{\Delta' \xrightarrow{[!\Phi][!P,\Psi]}\!\!\gg !R}\,(tog)}{}\,(tog)\\ \vdots \\ \Delta \xrightarrow{[\,][\Xi,!P]}\!\!\gg !R\end{array}}{\Gamma_1,\ldots,\Gamma_n,\Delta,\Xi,S_1,\ldots,S_n \Rightarrow !R}\,(cut_6)
$$

reduces to:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{!Q_1,\ldots,!Q_n \Rightarrow P}{\xrightarrow{[!Q_1,\ldots,!Q_n][\,]}\!\!\gg !P}\,(P) \qquad !\Xi',!P \Rightarrow R}{!\Xi',!Q_1,\ldots,!Q_n \Rightarrow R}\,(cut_8)}{\xrightarrow{[!\Xi',(!Q_1,\ldots,!Q_n)][\,]}\!\!\gg !R}\,(P)\\
\vdots\\
\dfrac{\Gamma_1,\ldots,\Gamma_n,\Delta \xrightarrow{[\,][\Xi,S_1,\ldots,S_n]}\!\!\gg !R}{\Gamma_1,\ldots,\Gamma_n,\Delta,\Xi,S_1,\ldots,S_n \Rightarrow !R}\,(sel*)
\end{array}
$$

(c) $(\multimap_{\mathcal{R}})$

$$
\dfrac{\dfrac{\Gamma,P \Rightarrow Q}{\Gamma \Rightarrow P\multimap Q}\,(\multimap_{\mathcal{R}}) \qquad \begin{array}{c}\dfrac{\dfrac{\Delta_1 \Rightarrow P \quad \Delta_2' \xrightarrow{[!\Phi][Q,\Psi]}\!\!\gg R}{\Delta_1,\Delta_2' \xrightarrow{[!\Phi][P\perp\!\circ Q,\Psi]}\!\!\gg R}\,(\multimap_{\mathcal{L}}*)}{}\,(tog)\\ \vdots \\ \Delta_1,\Delta_2 \xrightarrow{[\,][\Xi,P\perp\!\circ Q]}\!\!\gg R\end{array}}{\Gamma,\Delta_1,\Delta_2,\Xi \Rightarrow R}\,(cut_6)
$$

reduces to:

$$\dfrac{\Delta_1 \Rightarrow P \qquad \dfrac{\Gamma, P \Rightarrow Q \qquad \begin{array}{c} \dfrac{\Delta_2' \xrightarrow{[!\Phi][Q,\Psi]} R}{}\,(tog) \\ \vdots \\ \Delta_2 \xrightarrow{[\ ][\Xi,Q]} R \end{array}}{\Gamma, \Xi, \Delta_2, P \Rightarrow R}\,(cut_6)}{\Gamma, \Delta_1, \Delta_2, \Xi \Rightarrow R}\,(cut_4)$$

(d) $(\otimes_{\mathcal{R}})$ Not possible.

(e) $(I_{\mathcal{R}})$ Not possible.

(f) $(\top_{\mathcal{R}})$ Not possible.

(g) $(\oplus_{\mathcal{R}_1})$ Not possible.

(h) $(\oplus_{\mathcal{R}_2})$ Not possible.

(i) $(\&_{\mathcal{R}})$

$$\dfrac{\dfrac{\Gamma \Rightarrow P \qquad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P\&Q}\,(\&_{\mathcal{R}}) \qquad \begin{array}{c} \dfrac{\dfrac{\Delta' \xrightarrow{[!\Phi][P,\Psi]} R}{\Delta' \xrightarrow{[!\Phi][P\&Q,\Psi]} R}\,(\&_{\mathcal{L}_1}*)}{}\,(tog) \\ \vdots \\ \Delta \xrightarrow{[\ ][\Xi,P\&Q]} R \end{array}}{\Gamma, \Delta, \Xi \Rightarrow R}\,(cut_6)$$

reduces to:

$$\dfrac{\Gamma \Rightarrow P \qquad \begin{array}{c} \dfrac{\Delta' \xrightarrow{[!\Phi][P,\Psi]} R}{}\,(tog) \\ \vdots \\ \Delta \xrightarrow{[\ ][\Xi,P]} R \end{array}}{\Gamma, \Delta, \Xi \Rightarrow R}\,(cut_6)$$

7. $(cut_7)$ Analysis by cases on the right premiss.

(a) $(ax)$

$$\dfrac{\xrightarrow{[!Q_1,\ldots,!Q_n][\ ]} !P \qquad \dfrac{}{\phi \xrightarrow{!P} !P}\,(ax)}{!Q_1, \ldots, !Q_n \Rightarrow !P}\,(cut_7)$$

reduces to:

$$\dfrac{\begin{array}{c} \xrightarrow{[!Q_1,\ldots,!Q_n][\ ]} !P \\ \vdots \\ \xrightarrow{[\ ][!Q_1,\ldots,!Q_n]} !P \end{array}}{!Q_1, \ldots, !Q_n \Rightarrow !P}\,(sel*)$$

(b) $(\multimap_{\mathcal{L}})$ Not possible.

(c) $(\otimes_{\mathcal{L}})$ Not possible.

(d) $(I_{\mathcal{L}})$ Not possible.

(e) $(0_{\mathcal{L}})$ Not possible.

(f) $(\oplus_{\mathcal{L}})$ Not possible.

(g) $(\&_{\mathcal{L}_1})$ Not possible.

(h) $(\&_{\mathcal{L}_2})$ Not possible.

(i) $(W)$

$$\dfrac{\dfrac{!Q_1,...,!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\ggg} !P}\,(P) \qquad \dfrac{\Delta, (!P)^i \Rightarrow R}{\Delta, (!P)^i \overset{!P}{\longrightarrow} R}\,(W)}{\Delta, (!Q_1,...,!Q_n)^{i+1} \Rightarrow R}\,(cut_7)$$

if $i = 0$ then this reduces to:

$$\dfrac{\dfrac{\Delta \Rightarrow R}{\Delta \overset{!Q_n}{\longrightarrow} R}\,(W)}{\Delta, !Q_n \Rightarrow R}\,(sel)$$
$$\vdots\ weakenings$$
$$\Delta, !Q_1, ...., !Q_n \Rightarrow R$$

if $i \neq 0$ then the reduction is to:

$$\dfrac{\dfrac{\dfrac{\dfrac{!Q_1,...!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\ggg} !P}\,(P) \qquad \Delta, (!P)^i \Rightarrow R}{\Delta, (!Q_1,...,!Q_n)^i \Rightarrow R}\,(cut_8)}{\Delta, (!Q_1,...,!Q_n)^i \overset{!Q_n}{\longrightarrow} R}\,(W)}{\Delta, (!Q_1,...,!Q_n)^i, !Q_n \Rightarrow R}\,(sel)$$
$$\vdots\ weakenings$$
$$\Delta, (!Q_1,...,!Q_n)^{i+1} \Rightarrow R$$

(j) $(C)$

$$\dfrac{\dfrac{!Q_1,...,!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\ggg} !P}\,(P) \qquad \dfrac{\Delta, (!P)^{i+2} \Rightarrow R}{\Delta, (!P)^i \overset{!P}{\longrightarrow} R}\,(C)}{\Delta, (!Q_1,...,!Q_n)^{i+1} \Rightarrow R}\,(cut_7)$$

reduces to:

$$\dfrac{\dfrac{\dfrac{\dfrac{!Q_1,...,!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\ggg} !P}\,(P) \qquad \Delta, (!P)^{i+2} \Rightarrow R}{\Delta, (!Q_1,...,!Q_n)^{i+1}, !Q_1,...,!Q_n \Rightarrow R}\,(cut_8)}{\Delta, (!Q_1,...,!Q_n)^i, !Q_1,...,!Q_{n\perp 1}, !Q_1,...,!Q_{n\perp 1} \overset{!Q_n}{\longrightarrow} R}\,(C)}{\Delta, (!Q_1,...,!Q_n)^i, !Q_1,...,!Q_{n\perp 1}, !Q_1,...,!Q_{n\perp 1}, !Q_n \Rightarrow R}\,(sel)$$
$$\vdots\ contractions$$
$$\Delta, (!Q_1,...,!Q_n)^{i+1} \Rightarrow R$$

(k) $(D)$

$$\dfrac{\dfrac{!Q_1,...,!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\twoheadrightarrow} !P}\,(P) \qquad \dfrac{\Delta,(!P)^i \xrightarrow{P} R}{\Delta,(!P)^i \xrightarrow{!P} R}\,(D)}{\Delta,(!Q_1,...,!Q_n)^{i+1} \Rightarrow R}\,(cut_7)$$

if $i = 0$ the reduction is to:

$$\dfrac{!Q_1,...,!Q_n \Rightarrow P \quad \Delta \xrightarrow{P} R}{\Delta, !Q_1,...,!Q_n \Rightarrow R}\,(cut_3)$$

if $i \neq 0$ the reduction is to:

$$\dfrac{!Q_1,...,!Q_n \Rightarrow P \qquad \dfrac{\dfrac{!Q_!,...,!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\twoheadrightarrow} !P}\,(P) \qquad \Delta,(!P)^i \xrightarrow{P} R}{\Delta,(!Q_1,...,!Q_n)^i \xrightarrow{P} R}\,(cut_9)}{\Delta,(!Q_1,...,!Q_n)^{i+1} \Rightarrow R}\,(cut_3)$$

8. $(cut_8)$ Analysis by cases on the right premiss.

(a) $(sel)$ Two possibilities

$$\dfrac{\overset{[!\Theta][\,]}{\twoheadrightarrow} !P \qquad \dfrac{\Delta,(!P)^i \xrightarrow{S} R}{\Delta,(!P)^i, S \Rightarrow R}\,(sel)}{\Delta,(!\Theta)^i, S \Rightarrow R}\,(cut_8)$$

reduces to:

$$\dfrac{\dfrac{\overset{[!\Theta][\,]}{\twoheadrightarrow} !P \quad \Delta,(!P)^i \xrightarrow{S} R}{\Delta,(!\Theta)^i \xrightarrow{S} R}\,(cut_9)}{\Delta,(!\Theta)^i, S \Rightarrow R}\,(sel)$$

or

$$\dfrac{\overset{[!\Theta][\,]}{\twoheadrightarrow} !P \qquad \dfrac{\Delta,(!P)^i \xrightarrow{!P} R}{\Delta,(!P)^{i+1} \Rightarrow R}\,(sel)}{\Delta,(!\Theta)^{i+1} \Rightarrow R}\,(cut_8)$$

reduces to:

$$\dfrac{\overset{[!\Theta][\,]}{\twoheadrightarrow} !P \quad \Delta,(!P)^i \xrightarrow{!P} R}{\Delta,(!\Theta)^{i+1} \Rightarrow R}\,(cut_7)$$

(b) $(sel*)$ We give two reductions which, when combined, give the desired reduction.

$$\dfrac{\dfrac{!Q_1,...,!Q_n \Rightarrow P}{\overset{[!Q_1,...,!Q_n][\,]}{\twoheadrightarrow} !P}\,(P) \qquad \dfrac{\begin{array}{c} !\Xi', (!P)^k \Rightarrow R \\ \vdots \\ \Delta,(!P)^i \overset{[\,][\Xi,(!P)^{j+k}]}{\twoheadrightarrow} !R \end{array}}{\Delta,(!P)^{i+j+k}, \Xi \Rightarrow R}\,(sel*)}{\Delta,(!Q_1,...,!Q_n)^{i+j+k}, \Xi \Rightarrow !R}\,(cut_8)$$

if $i \neq 0$ then the reduction is to:

$$
\dfrac{
\dfrac{!Q_1,\ldots,!Q_n \Rightarrow P}{\xrightarrow{[!Q_1,\ldots,!Q_n][\ ]}\!\!\gg\ !P}\ (P)
\qquad
\dfrac{
\dfrac{
\xrightarrow{[!Q_1,\ldots,!Q_n][\ ]}\!\!\gg\ !P
\quad
\begin{array}{c} !\Xi', (!P)^k \Rightarrow R \\ \vdots \\ \Delta, (!P)^i \xrightarrow{[\ ][\Xi,(!P)^{j+k}]}\!\!\gg\ !R \end{array}
}{\Delta, (!Q_1,\ldots,!Q_n)^i \xrightarrow{[\ ][\Xi,(!P)^{j+k}]}\!\!\gg\ !R}\ (cut_{10})
}{\Delta, (!Q_1,\ldots,!Q_n)^i, \Xi, (!P)^{j+k} \Rightarrow !R}\ (sel*)
}{\Delta, (!Q_1,\ldots,!Q_n)^{i+j+k}, \Xi \Rightarrow !R}\ (cut_8)
$$

if $i = 0$ the then reduction is to the following:

$$
\dfrac{
!Q_1,\ldots,!Q_n \Rightarrow P
\qquad
\dfrac{
\begin{array}{c}
\dfrac{
\dfrac{\xrightarrow{[!Q_1,\ldots,!Q_n][\ ]}\!\!\gg\ !P \quad !\Xi, (!P)^k \Rightarrow R}{!\Xi', (!Q_1,\ldots,!Q_n)^k \Rightarrow R}\ (cut_8)
}{\xrightarrow{[!\Xi,(!Q_1,\ldots,!Q_n)^k][\ ]}\!\!\gg\ !R}\ (P) \\
\vdots \\
\Delta, (!Q_1,\ldots,!Q_n)^i \xrightarrow{[\ ][\Xi,(!Q_1,\ldots,!Q_n)^k,(P)^j]}\!\!\gg\ !R
\end{array}
}{\Delta, \Xi, (!Q_1,\ldots,!Q_n)^{i+k}, (P)^j \Rightarrow !R}\ (sel*)
}{
\begin{array}{c}
\Delta, \Xi, (!Q_!,\ldots,!Q_n)^{i+k+1}, (P)^{j\perp 1} \Rightarrow !R \\
\vdots\ cut_4 s \\
\Delta, \Xi, (!Q_1,\ldots,!Q_n)^{i+j+k} \Rightarrow !R
\end{array}
}\ (cut_4)
$$

By doing the first reduction, eliminating the $(cut_{10})$s (they move into the minor premisses of the promotion) and then performing the second reduction, we get one large reduction, which is the one we really want to consider.

(c) $(\multimap_{\mathcal{R}})$

$$
\dfrac{
\xrightarrow{[!\Theta][\ ]}\!\!\gg !P
\quad
\dfrac{\Delta, (!P)^i, S \Rightarrow T}{\Delta, (!P)^i \Rightarrow S \multimap T}\ (\multimap_{\mathcal{R}})
}{\Delta, (!\Theta)^i \Rightarrow S \multimap T}\ (cut_8)
$$

reduces to:

$$
\dfrac{
\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \Delta, (!P)^i, S \Rightarrow T}{\Delta, (!\Theta)^i, S \Rightarrow T}\ (cut_8)
}{\Delta, (!\Theta)^i \Rightarrow S \multimap T}\ (\multimap_{\mathcal{R}})
$$

(d) $(\otimes_{\mathcal{R}})$

$$
\dfrac{
\xrightarrow{[!\Theta][\ ]}\!\!\gg !P
\quad
\dfrac{\Delta_1, (!P)^i \Rightarrow S \quad \Delta_2, (!P)^j \Rightarrow T}{\Delta_1, \Delta_2, (!P)^{i+j} \Rightarrow S \otimes T}\ (\otimes_{\mathcal{R}})
}{\Delta_1, \Delta_2, (!\Theta)^{i+j} \Rightarrow S \otimes T}\ (cut_8)
$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \Delta_1, (!P)^i \Rightarrow S}{\Delta_1, (!\Theta)^i \Rightarrow S}\ (cut_8) \quad \dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \Delta_2, (!P)^j \Rightarrow T}{\Delta_2, (!\Theta)^j \Rightarrow T}\ (cut_8)}{\Delta_1, \Delta_2, (!\Theta)^{i+j} \Rightarrow S \otimes T}\ (\otimes_{\mathcal{R}})$$

(e) $(I_{\mathcal{R}})$ Not possible.

(f) $(\top_{\mathcal{R}})$

$$\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \dfrac{}{\Delta, (!P)^i \Rightarrow \top}\ (\top_{\mathcal{R}})}{\Delta, (!\Theta)^i \Rightarrow \top}\ (cut_8)$$

reduces to:

$$\dfrac{}{\Delta, (!\Theta)^i \Rightarrow \top}\ (\top_{\mathcal{R}})$$

(g) $(\oplus_{\mathcal{R}_1})$

$$\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \dfrac{\Delta, (!P)^i \Rightarrow S}{\Delta, (!P)^i \Rightarrow S \oplus T}\ (\oplus_{\mathcal{R}_1})}{\Delta, (!\Theta)^i \Rightarrow S \oplus T}\ (cut_8)$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \Delta, (!P)^i \Rightarrow S}{\Delta, (!\Theta)^i \Rightarrow S}\ (cut_8)}{\Delta, (!\Theta)^i \Rightarrow S \oplus T}\ (\oplus_{\mathcal{R}_1})$$

(h) $(\oplus_{\mathcal{R}_2})$ Similar to above.

(i) $(\&_{\mathcal{R}})$

$$\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \dfrac{\Delta, (!P)^i \Rightarrow S \quad \Delta, (!P)^i \Rightarrow T}{\Delta, (!P)^i \Rightarrow S \& T}\ (\&_{\mathcal{R}})}{\Delta, (!\Theta)^i \Rightarrow S \& T}\ (cut_8)$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \Delta, (!P)^i \Rightarrow S}{\Delta, (!\Theta)^i \Rightarrow S}\ (cut_8) \quad \dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \Delta, (!P)^i \Rightarrow T}{\Delta, (!\Theta)^i \Rightarrow T}\ (cut_8)}{\Delta, (!\Theta)^i \Rightarrow S \& T}\ (\&_{\mathcal{R}})$$

9. $(cut_9)$ Analysis by cases on the right premiss.

(a) $(ax)$ Not possible.

(b) $(\multimap_{\mathcal{L}})$

$$\dfrac{\xrightarrow{[!\Theta][\,]}\!\!\gg !P \quad \dfrac{\Delta_1, (!P)^i \Rightarrow S \quad \Delta_2, (!P)^j \xrightarrow{T} R}{\Delta_1, \Delta_2, (!P)^{i+j} \xrightarrow{S \multimap T} R}\ (\multimap_{\mathcal{L}})}{\Delta_1, \Delta_2, (!\Theta)^{i+j} \xrightarrow{S \multimap T} R}\ (cut_9)$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \Delta_1, (!P)^i \Rightarrow S}{\Delta_1, (!\Theta)^i \Rightarrow S}\,(cut_8) \qquad \dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \Delta_2, (!P)^j \xrightarrow{T} R}{\Delta_2, (!\Theta)^j \xrightarrow{T} R}\,(cut_9)}{\Delta_1, \Delta_2, (!\Theta)^{i+j} \xrightarrow{S \multimap T} R}\,(\multimap_{\mathcal{L}})$$

(c) $(\otimes_{\mathcal{L}})$

$$\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \qquad \dfrac{\Delta, (!P)^i, S, T \Rightarrow R}{\Delta, (!P)^i \xrightarrow{S \otimes T} R}\,(\otimes_{\mathcal{L}})}{\Delta, (!\Theta)^i \xrightarrow{S \otimes T} R}\,(cut_9)$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \Delta, (!P)^i, S, T \Rightarrow R}{\Delta, (!\Theta)^i, S, T \Rightarrow R}\,(cut_8)}{\Delta, (!\Theta)^i \xrightarrow{S \otimes T} R}\,(\otimes_{\mathcal{L}})$$

(d) $(I_{\mathcal{L}})$

$$\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \dfrac{\Delta, (!P)^i \Rightarrow R}{\Delta, (!P)^i \xrightarrow{I} R}\,(I_{\mathcal{L}})}{\Delta, (!\Theta)^i \xrightarrow{I} R}\,(cut_9)$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \Delta, (!P)^i \Rightarrow R}{\Delta, (!\Theta)^i \Rightarrow R}\,(cut_8)}{\Delta, (!\Theta)^i \xrightarrow{I} R}\,(I_{\mathcal{L}})$$

(e) $(0_{\mathcal{L}})$

$$\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \quad \dfrac{}{\Delta, (!P)^i \xrightarrow{0} R}\,(0_{\mathcal{L}})}{\Delta, (!\Theta)^i \xrightarrow{0} R}\,(cut_9)$$

reduces to:

$$\dfrac{}{\Delta, (!\Theta)^i \xrightarrow{0} R}\,(0_{\mathcal{L}})$$

(f) $(\oplus_{\mathcal{L}})$

$$\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\gg !P \qquad \dfrac{\Delta, (!P)^i, S \Rightarrow R \quad \Delta, (!P)^i, T \Rightarrow R}{\Delta, (!P)^i \xrightarrow{S \oplus T} R}\,(\oplus_{\mathcal{L}})}{\Delta, (!\Theta)^i \xrightarrow{S \oplus T} R}\,(cut_9)$$

reduces to:

$$\cfrac{\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \Delta, (!P)^i, S \Rightarrow R}{\Delta, (!\Theta)^i, S \Rightarrow R}\ (cut_8) \quad \cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \Delta, (!P)^i, T \Rightarrow R}{\Delta, (!\Theta)^i, T \Rightarrow R}\ (cut_8)}{\Delta, (!\Theta)^i \xrightarrow{S \oplus T} R}\ (\oplus_{\mathcal{L}})$$

(g) $(\&_{\mathcal{L}_1})$

$$\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \cfrac{\Delta, (!P)^i \xrightarrow{S} R}{\Delta, (!P)^i \xrightarrow{S \& T} R}\ (\&_{\mathcal{L}_1})}{\Delta, (!\Theta)^i \xrightarrow{S \& T} R}\ (cut_9)$$

reduces to:

$$\cfrac{\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \Delta, (!P)^i \xrightarrow{S} R}{\Delta, (!\Theta)^i \xrightarrow{S} R}\ (cut_9)}{\Delta, (!\Theta)^i \xrightarrow{S \& T} R}\ (\&_{\mathcal{L}_1})$$

(h) $(\&_{\mathcal{L}_2})$ Similar to above.

(i) $(W)$

$$\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \cfrac{\Delta, (!P)^i \Rightarrow R}{\Delta, (!P)^i \xrightarrow{!S} R}\ (W)}{\Delta, (!\Theta)^i \xrightarrow{!S} R}\ (cut_9)$$

reduces to:

$$\cfrac{\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \Delta, (!P)^i \Rightarrow R}{\Delta, (!\Theta)^i \Rightarrow R}\ (cut_8)}{\Delta, (!\Theta)^i \xrightarrow{!S} R}\ (W)$$

(j) $(C)$

$$\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \cfrac{\Delta, (!P)^i, !S, !S \Rightarrow R}{\Delta, (!P)^i \xrightarrow{!S} R}\ (C)}{\Delta, (!\Theta)^i \xrightarrow{!S} R}\ (cut_9)$$

reduces to:

$$\cfrac{\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \Delta, (!P)^i, !S, !S \Rightarrow R}{\Delta, (!\Theta)^i, !S, !S \Rightarrow R}\ (cut_8)}{\Delta, (!\Theta)^i \xrightarrow{!S} R}\ (C)$$

(k) $(D)$

$$\cfrac{\overset{[!\Theta][\ ]}{\longrightarrow\!\!\gg} !P \quad \cfrac{\Delta, (!P)^i \xrightarrow{S} R}{\Delta, (!P)^i \xrightarrow{!S} R}\ (D)}{\Delta, (!\Theta)^i \xrightarrow{!S} R}\ (cut_9)$$

reduces to:

$$\dfrac{\dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \Delta, (!P)^i \xrightarrow{S} R}{\Delta, (!\Theta)^i \xrightarrow{S} R}\,(cut_9)}{\Delta, (!\Theta)^i \xrightarrow{!S} R}\,(D)$$

10. $(cut_{10})$ Analysis by cases on the right premiss.

(a) $(tog)$

$$\dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \dfrac{\Delta, (!P)^i \overset{[!\Phi, !S][\Psi]}{\gg} R}{\Delta, (!P)^i \overset{[!\Phi][!S, \Psi]}{\gg} R}\,(tog)}{\Delta, (!\Theta)^i \overset{[!\Phi][!S, \Psi]}{\gg} R}\,(cut_{10})$$

reduces to:

$$\dfrac{\dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \Delta, (!P)^i \overset{[!\Phi, !S][\Psi]}{\gg} R}{\Delta, (!\Theta)^i \overset{[!\Phi, !S][\Psi]}{\gg} R}\,(cut_{10})}{\Delta, (!\Theta)^i \overset{[!\Phi][!S, \Psi]}{\gg} R}\,(tog)$$

(b) $(\multimap_{\mathcal{L}}*)$

$$\dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \dfrac{\Delta_1, (!P)^i \Rightarrow S \quad \Delta_2, (!P)^j \overset{[!\Phi][T, \Psi]}{\gg} R}{\Delta_1, \Delta_2, (!P)^{i+j} \overset{[!\Phi][S \perp\!\!\circ T, \Psi]}{\gg} R}\,(\multimap_{\mathcal{L}}*)}{\Delta_1, \Delta_2, (!\Theta)^{i+j} \overset{[!\Phi][S \perp\!\!\circ T, \Psi]}{\gg} R}\,(cut_{10})$$

reduces to:

$$\dfrac{\dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \Delta, (!P)^i \Rightarrow S}{\Delta_1, (!\Theta)^i \Rightarrow S}\,(cut_8) \quad \dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \Delta_2, (!P)^j \overset{[!\Phi][T, \Psi]}{\gg} R}{\Delta_2, (!\Theta)^j \overset{[!\Phi][T, \Psi]}{\gg} R}\,(cut_{10})}{\Delta_1, \Delta_2, (!\Theta)^{i+j} \overset{[!\Phi][S \perp\!\!\circ T, \Psi]}{\gg} R}\,(\multimap_{\mathcal{L}}*)$$

(c) $(\&_{\mathcal{L}_1}*)$

$$\dfrac{\overset{[!\Theta][\ ]}{\gg} !P \quad \dfrac{\Delta, (!P)^i \overset{[!\Phi][S, \Psi]}{\gg} R}{\Delta, (!P)^i \overset{[!\Phi][S \& T, \Psi]}{\gg} R}\,(\&_{\mathcal{L}_1}*)}{\Delta, (!\Theta)^i \overset{[!\Phi][S \& T, \Psi]}{\gg} R}\,(cut_{10})$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\!\!\gg !P \quad \Delta, (!P)^i \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\!\gg R}{\Delta, (!\Theta)^i \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\!\gg R}\,(cut_{10})}{\Delta, (!\Theta)^i \xrightarrow{[!\Phi][S\&T,\Psi]}\!\!\!\!\gg R}\,(\&_{\mathcal{L}_1}*)$$

(d) $(\&_{\mathcal{L}_2}*)$ Similar to above.

(e) $(D*)$

$$\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\!\!\gg !P \quad \dfrac{\Delta, (!P)^i \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\!\gg R}{\Delta, (!P)^i \xrightarrow{[!\Phi][!S,\Psi]}\!\!\!\!\gg R}\,(D*)}{\Delta, (!\Theta)^i \xrightarrow{[!\Phi][!S,\Psi]}\!\!\!\!\gg R}\,(cut_{10})$$

reduces to:

$$\dfrac{\dfrac{\xrightarrow{[!\Theta][\ ]}\!\!\!\!\gg !P \quad \Delta, (!P)^i \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\!\gg R}{\Delta, (!\Theta)^i \xrightarrow{[!\Phi][S,\Psi]}\!\!\!\!\gg R}\,(cut_{10})}{\Delta, (!\Theta)^i \xrightarrow{[!\Phi][!S,\Psi]}\!\!\!\!\gg R}\,(D*)$$

(f) $(P)$ Not possible.

### 6.6.2 Weighting Cuts in SILL

In this section we give a weight to simple cut instances (defined in Definition 3.2) in SILL. This measure is then used with a cut reduction strategy to prove the (weak) cut-elimination theorem.

**Definition 6.2** *Associated with every formula occurrence in a SILL proof is an* **elimination number**. *The elimination number of a formula is zero if it has form* $!P$ *and was not introduced by a promotion. Otherwise it has elimination number one.*

**Definition 6.3** *The* **weight** *of a simple cut instance in a SILL*$^{cut}$ *derivation is the quadruple:*

$$(e, |P|, h_2, h_1)$$

*where*

- $e$ *is the elimination number of the left cut formula.*
- $|P|$ *is the size of the cut formula.*
- $h_2$ *is the height of the right premiss.*

– $h_1$ *is the height of the left premiss.*

*The quadruple is lexicographically ordered from the left.*

**Lemma 6.3** *The weights defined in Definition 6.3 are well-ordered.*

**Theorem 6.4** *The rules* $(cut_1), ..., (cut_{10})$ *are admissible in SILL.*

PROOF: We give a reduction strategy:

– pick any simple cut instance and reduce

– recursively reduce any simple cut instances in the result

By induction on the weight of the simple cut instances, and induction on the number of simple cut instances, this strategy terminates.

This can easily be seen by inspection. ■

The reason for introducing the elimination number is now obvious. Reductions 3(b) and 6(b) introduce cuts whose cut formulae can be of greater size than the cut being reduced. Inspection of these new cuts reveals that they are easily eliminable (consider, for example, the second of the possibilities for reduction 6(a)). All that is needed is a measure which captures this. This the elimination number achieves: a simple cut whose left premiss has elimination number zero has a form such that the cut can easily be eliminated (independently of the elimination of the cuts above it).

### 6.6.3 More on Cut Elimination

The '⇒' sequents are the basic judgement form for SILL. Therefore, elimination of $(cut_4)$ is of primary interest. Indeed, the other nine cuts result from the attempt to algorithmically eliminate the first – they naturally arise in the reduction of $(cut_4)$. However, the simple admissibility of $(cut_4)$ can be proved without recourse to the other cuts and all the complicated work above. We prove the admissibility again:

**Theorem 6.5** *The following rule is admissible in SILL:*

$$\frac{\Gamma \Rightarrow P \quad \Delta, P \Rightarrow R}{\Gamma, \Delta \Rightarrow R} \ (cut)$$

PROOF: Given that the premisses are provable in SILL, they are provable in ILL (from Theorem 6.1). We know ([Bie94]) that cut is admissible in ILL, hence the conclusion is provable in ILL. Again from Theorem 6.1, the conclusion is provable in SILL. ■

We could use similar arguments to prove the admissibility of the other cut rules described above.

Cut elimination is of interest for two reasons. Firstly, from a logic programming point of view, we are interested in backwards proof search, and a complete cut-free system is desirable for this. We have already described a cut-free system and proved its completeness, hence from the logic programming perspective the cut-elimination theorem is of lesser importance.

Another reason to be interested in a cut-elimination theorem for a system such as SILL is that it can be seen as a computation process – calculating a normal-form (cut-free proof) from a program (proof with cuts). This is the motivation for the cut reductions given in section 6.6.1 and the cut-elimination theorem as proved in Theorem 6.4. The proof given there is that the reduction strategy terminates – we have a syntactic algorithm that will produce a normal form. This is akin to normalisation of lambda terms as computation as seen in functional programming. We would ideally like to prove that the set of reductions given (and the associated proof terms not given) strongly normalise, but such a proof is beyond the scope of this thesis. Indeed, given that normalisation for natural deduction is not confluent, we are unsure whether or not cut is strongly admissible. We do not, however, have a counter-example.

It was said above that the ten cut rules arise from the process of algorithmically eliminating the 'basic' $(cut_4)$. That this is so is easily seen from the reduction rules. Of course, picking the right form for the cut rules is tricky. The rules have to be sound with respect to provability in cut-free SILL and for all cases to reduce to valid SILL sequents. This necessitates the 'big step reductions' which can be seen in, for example, 6(a). The decomposition of promotions leads to several other complicated reductions, such as 3(b). Finally notice that $(cut_7)$,...,$(cut_{10})$ are multicut (or mix) rules – one formula on the left is cut with many formulae on the right. This has led to very little complication in the cut-elimination process, but the use of multicuts for any purpose at all is unattractive, and using them with a calculus with several judgement forms and focused formulae seemed best avoided. However, without the use of multicuts, we were unable to find a measure on the size of a cut which would always decrease. The situation is similar to that for multiplicative formulations of intuitionistic logic (such as G6 in the appendix). Indeed, multicuts were first introduced by Gentzen ([Gen69]) when trying to prove cut-elimination for this calculus by similar methods to those we are using here. We know of no treatment of cut-elimination for calculi such as G6 which do not use a multicut. However, we know of no work showing that the use of the multicut is necessary.

## 6.7 SILL and Logic Programming

One of the motivations for the development of SILL is the link between 'permutation-free' calculi and logic programming. Linear logic has been extensively studied in relation to logic programming, in particular by Hodas & Miller and Harland & Pym. Hodas & Miller have developed two systems for linear logic programming.

The first of these, Lolli ([HM94]), is based on a fragment of Intuitionistic Linear Logic. The second, Forum ([Mil96]), is based on full classical linear logic. Harland & Pym's system, Lygon ([HP94]), is based on a fragment of classical linear logic. This section will briefly describe Lolli (the language most closely related to SILL) and compare it with SILL.

### 6.7.1 SILL and Lolli

In [MNPS91], the idea of a uniform proof was introduced. A uniform proof is one where the goal formula can be broken up until atomic before the context is considered (see Definition 1.1). Hereditary Harrop logic with uniform proofs and a backchaining calculus allows goal directed proof search. This can be seen as a logical foundation of logic programming. Lolli is the linear logic programming language most closely corresponding to SILL. Lolli is a calculus (introduced in [HM94]) for the fragment of ILL similar to hereditary Harrop logic as a fragment of intuitionistic logic. This fragment is the largest fragment of ILL for which uniform proofs are complete with respect to provability. Lolli is a backchaining calculus suitable for goal-directed proof search.

In order to avoid problems with the structural rules, the Lolli calculus is formulated with contexts split into linear and non-linear parts, like the calculus ILL$^{\Sigma}$ seen in Figure 2.2. The calculus is presented with two implications ($\multimap$ and $\rightarrow$). $\multimap$ can be thought of as the usual linear implication for formula $P \multimap Q$ where $P$ is not banged. $\rightarrow$ can be thought of as linear implication for formulae of the form $!P \multimap Q$. Therefore, unlike ILL$^{\Sigma}$, there is no rule for moving banged formulae on the left into the non-linear context. The usual left rules are replaced by two backchaining rules.

Lolli is a calculus for the following fragment of ILL. We call this fragment UILL. Formulae are generated according to the following grammar.

**R**::=

$$A \mid \top \mid G \multimap R \mid G \rightarrow R \mid R \& R \mid \forall V.R$$

**G**::=

$$A \mid \top \mid I \mid R \multimap G \mid R \rightarrow G \mid G \& G \mid G \otimes G \mid G \oplus G \mid !G \mid \forall V.G \mid \exists V.G$$

UILL has **G** formulae as goals. On the left, **R** formulae and banged **R** formulae are allowed.

Lolli is displayed in Figure 6.8 (with a minor change from [HM94] – we have given two backchaining rules for the cases where the resource formula is and is not in the linear context, whereas Hodas & Miller give one backchaining rule and a dereliction rule). All banged formulae on the left are in $\Sigma$ and all formulae which are not banged are in $\Gamma$.

We need the following definition from [HM94].

$$\frac{}{\Sigma;\Gamma \Rightarrow \top}\ (\top_{\mathcal{R}}) \qquad \frac{}{\Sigma;\Rightarrow I}\ (I_{\mathcal{R}})$$

$$\frac{\Sigma;\Gamma,P \Rightarrow Q}{\Sigma;\Gamma \Rightarrow P{-\!\circ}Q}\ ({-\!\circ}_{\mathcal{R}}) \qquad \frac{\Sigma,P;\Gamma \Rightarrow Q}{\Sigma;\Gamma \Rightarrow P \rightarrow Q}\ (\rightarrow_{\mathcal{R}})$$

$$\frac{\Sigma;\Gamma_1 \Rightarrow P \quad \Sigma;\Gamma_2 \Rightarrow Q}{\Sigma;\Gamma_1,\Gamma_2 \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}}) \qquad \frac{\Sigma;\Gamma \Rightarrow P \quad \Sigma;\Gamma \Rightarrow Q}{\Sigma;\Gamma \Rightarrow P\&Q}\ (\&_{\mathcal{R}})$$

$$\frac{\Sigma;\Gamma \Rightarrow P}{\Sigma;\Gamma \Rightarrow P \oplus Q}\ (\oplus_{\mathcal{R}_1}) \qquad \frac{\Sigma;\Gamma \Rightarrow Q}{\Sigma;\Gamma \Rightarrow P \oplus Q}\ (\oplus_{\mathcal{R}_2})$$

$$\frac{\Sigma;\Rightarrow P}{\Sigma;\Rightarrow !P}\ (P)$$

$$\frac{\Sigma;\Gamma \Rightarrow P[y/x]}{\Sigma;\Gamma \Rightarrow \forall x P}\ (\forall_{\mathcal{R}})\dagger \qquad \frac{\Sigma;\Gamma \Rightarrow P[t/x]}{\Sigma;\Gamma \Rightarrow \exists x.P}\ (\exists_{\mathcal{R}})$$

$$\frac{\Sigma;\Rightarrow P_1 \quad ... \quad \Sigma;\Rightarrow P_n \quad \Sigma;\Gamma_1 \Rightarrow Q_1 \quad ... \quad \Sigma;\Gamma_m \Rightarrow Q_m}{\Sigma;\Gamma_1,...,\Gamma_m,P \Rightarrow A}\ (BC_1)\ddagger$$

$$\frac{\Sigma,P;\Rightarrow P_1 \quad ... \quad \Sigma,P;\Rightarrow P_n \quad \Sigma,P;\Gamma_1 \Rightarrow Q_1 \quad ... \quad \Sigma,P;\Gamma_m \Rightarrow Q_m}{\Sigma,P;\Gamma_1,...,\Gamma_m \Rightarrow A}\ (BC_2)\ddagger$$

† $y$ not free in $\Sigma;\Gamma$
‡ $n,m \geq 0$ and $< \{P_1,...,P_n\},\{Q_1,...,Q_m\},A > \in \parallel P \parallel$
Notice that $(BC_1),(BC_2)$ can be nullary, providing the leaf nodes for the calculus.

**Figure 6.8:** Lolli

**Definition 6.4** *Let $P$ range over logical formulae built using the connectives $\top$, $\&$, ${-\!\circ}$, $\rightarrow$ and $\forall$. Then $\parallel P \parallel$ is the smallest set of triples of the form $< \Sigma,\Gamma,Q >$ where $\Sigma$ is a set of formulae and $\Gamma$ is a multiset of formulae, such that*

1. $< \phi,\phi,P > \in \parallel P \parallel$
2. *if* $< \Sigma,\Gamma,S\&T > \in \parallel P \parallel$ *then both* $< \Sigma,\Gamma,S > \in \parallel P \parallel$ *and* $< \Sigma,\Gamma,T > \in \parallel P \parallel$
3. *if* $< \Sigma,\Gamma,\forall x.S > \in \parallel P \parallel$ *then for all closed terms* $t$, $< \Sigma,\Gamma,S[t/x] > \in \parallel P \parallel$
4. *if* $< \Sigma,\Gamma,S \rightarrow T > \in \parallel P \parallel$ *then* $< \Sigma \cup \{S\},\Gamma,T > \in \parallel P \parallel$
5. *if* $< \Sigma,\Gamma,S{-\!\circ}T > \in \parallel P \parallel$ *then* $< \Sigma,\Gamma \cup \{S\},T > \in \parallel P \parallel$

Proofs in Lolli proceed by applying right rules in order to break up the goal formula until it is atomic, then backchaining and repeating the process.

How does Lolli compare with derivations in SILL? Lolli has contexts split into linear and non-linear parts, and hence no structural rules. SILL does not have this feature. Therefore a direct comparison of the two systems is not possible – treatment of the structural rules cannot be compared. Instead we show that (over the UILL fragment of ILL) every Lolli derivation can be interpreted as a SILL derivation and that every SILL derivation can be interpreted as a Lolli derivation. These interpretations rest on the fact that the premisses of the backchaining rule with backchaining formula $P$ are exactly the minor premisses of the chain of stoup inference ending in an axiom that arise from selecting $P$ as the stoup formula. The axiom itself is unnecessary in the backchaining rule. Note we make a slight change to SILL – we restrict axioms to atomic formulae.

**Proposition 6.2** *Every Lolli derivation can be interpreted as a* SILL *derivation.*

PROOF: We take each Lolli inference in turn and interpret it as a series of one or more SILL inferences.

Where $\Pi$ is a Lolli proof, we call this interpretation $\mathcal{J}(\Pi)$.

1. The last inference in the Lolli derivation is one of: $(\top_{\mathcal{R}})$, $(I_{\mathcal{R}})$, $(\multimap_{\mathcal{R}})$, $(\rightarrow_{\mathcal{R}})$, $(\&_{\mathcal{R}})$, $(\oplus_{\mathcal{R}_1})$, $(\oplus_{\mathcal{R}_2})$, $(\forall_{\mathcal{R}})$, $(\exists_{\mathcal{R}})$. Then the last inference in SILL is the corresponding inference. For example:

$$\frac{\Sigma, P; \Gamma \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \rightarrow Q}\ (\rightarrow_{\mathcal{R}}) \quad \text{is interpreted as} \quad \frac{!\Sigma, !P, \Gamma \Rightarrow Q}{!\Sigma, \Gamma \Rightarrow !P \multimap Q}\ (\multimap_{\mathcal{R}})$$

2. The last inference in Lolli is $(\otimes_{\mathcal{R}})$. Then

$$\frac{\Sigma; \Gamma_1 \Rightarrow P \quad \Sigma; \Gamma_2 \Rightarrow Q}{\Sigma; \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}})$$

is interpreted as

$$\cfrac{!\Sigma, \Gamma_1 \Rightarrow P \quad !\Sigma, \Gamma_2 \Rightarrow Q}{\cfrac{!\Sigma, !\Sigma, \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q}{\vdots\ contractions \atop !\Sigma, \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q}}\ (\otimes_{\mathcal{R}})$$

3. The last inference in Lolli is $(P)$. Then

$$\frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P}\ (P) \quad \text{is interpreted as} \quad \cfrac{\cfrac{!\Sigma \Rightarrow P}{\xrightarrow{[!\Sigma][\,]} \gg !P}\ (P) \atop {\vdots \atop \xrightarrow{[\,][!\Sigma]} \gg !P}}{!\Sigma \Rightarrow !P}\ (sel*)$$

4. The last inference in Lolli is $(BC_1)$. Then

$$\frac{\Sigma; \Rightarrow P_1 \quad \ldots \quad \Sigma; \Rightarrow P_n \quad \Sigma; \Gamma_1 \Rightarrow Q_1 \quad \ldots \quad \Sigma; \Gamma_m \Rightarrow Q_m}{\Sigma; \Gamma_1, \ldots, \Gamma_m, P \Rightarrow A}\ (BC_1)$$

(where $< \{P_1, \ldots, P_n\}, \{Q_1, \ldots, Q_m\}, A > \in \parallel P \parallel$)

is interpreted as (noting that an **R** formula once in the stoup has stoup premisses ending in an axiom or failure)

$$\dfrac{\dfrac{\begin{matrix}\text{minor premisses} & \dfrac{}{\phi \xrightarrow{A} A}\,(ax) \\ & \vdots \end{matrix}}{\begin{matrix} !\Sigma, ..., !\Sigma, \Gamma_1, ..., \Gamma_m \xrightarrow{P} A \end{matrix}}}{!\Sigma, ..., !\Sigma, \Gamma_1, ..., \Gamma_m, P \Rightarrow A}\,(sel)$$

$$\vdots\ contractions$$

$$!\Sigma, \Gamma_1, ..., \Gamma_m, P \Rightarrow A$$

where $n+m$ copies of $!\Sigma$ are made and the minor premisses are the interpretations of the premisses of $(BC_1)$ (via promotion for the $\Sigma; \Rightarrow P_i$).

5. The last inference in Lolli is $(BC_2)$. Then

$$\frac{\Sigma, P; \Rightarrow P_1 \quad ... \quad \Sigma, P; \Rightarrow P_n \quad \Sigma, P; \Gamma_1 \Rightarrow Q_1 \quad ... \quad \Sigma, P; \Gamma_m \Rightarrow Q_m}{\Sigma, P; \Gamma_1, ..., \Gamma_m \Rightarrow A}\,(BC_2)$$

(where $< \{P_1, ..., P_n\}, \{Q_1, ..., Q_m\}, A > \in \| P \|$)

is interpreted as

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\text{minor premisses} & \dfrac{}{\phi \xrightarrow{A} A}\,(ax) \\ & \vdots \end{matrix}}{!\Sigma, !P, ..., !\Sigma, !P, \Gamma_1, ..., \Gamma_m \xrightarrow{P} A}}{!\Sigma, !P, ..., !\Sigma, !P, \Gamma_1, ..., \Gamma_m \xrightarrow{!P} A}\,(D)}{!\Sigma, !P, ..., !\Sigma, !P, !P, \Gamma_1, ..., \Gamma_m \Rightarrow A}\,(sel)$$

$$\vdots\ contractions$$

$$!\Sigma, !P\Gamma_1, ..., \Gamma_m \Rightarrow A$$

where $n+m$ copies of $!\Sigma, !P$ are made and the minor premisses are the interpretations of the premisses of $(BC_2)$ (via promotion for the $\Sigma; \Rightarrow P_i$).

■

**Lemma 6.4** *The following rules are admissible in Lolli:*

$$\frac{\Sigma; \Gamma \Rightarrow G}{\Sigma, \Delta; \Gamma \Rightarrow G}\,(W*) \qquad \frac{\Sigma, \Delta, \Delta; \Gamma \Rightarrow G}{\Sigma, \Delta; \Gamma \Rightarrow G}\,(C*)$$

**Lemma 6.5** UILL *sequents of the form* $!\Sigma, S \Rightarrow !P$*, where* $S$ *is not banged, are unprovable.*

**Proposition 6.3** *Every* SILL *derivation over the* UILL *fragment of* ILL *can be interpreted as a Lolli derivation.*

PROOF: We analyse the cases for the '$\Rightarrow$' form of sequent. We interpret SILL sequent $!\Sigma, \Gamma \Rightarrow G$ as Lolli sequent $\Sigma; \Gamma \Rightarrow G$.

Where $\Pi$ is a SILL proof, we call this interpretation $\mathcal{I}(\Pi)$.

1. The last inference in SILL is one of $(\top_{\mathcal{R}})$, $(I_{\mathcal{R}})$, $(\&_{\mathcal{R}})$, $(\oplus_{\mathcal{R}_1})$, $(\oplus_{\mathcal{R}_2})$, $(\forall_{\mathcal{R}})$, $(\exists_{\mathcal{R}})$. The Lolli inference is the corresponding inference.

2. The last inference in SILL is $(-\!\circ_{\mathcal{R}})$. Two cases:

$$\frac{!\Sigma, \Gamma, P \Rightarrow Q}{!\Sigma, \Gamma \Rightarrow P -\!\circ Q}\ (-\!\circ_{\mathcal{R}}) \quad \text{is interpreted as} \quad \frac{\Sigma; \Gamma, P \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P -\!\circ Q}\ (-\!\circ_{\mathcal{R}})$$

or

$$\frac{!\Sigma, \Gamma, !P \Rightarrow Q}{!\Sigma, \Gamma \Rightarrow !P -\!\circ Q}\ (-\!\circ_{\mathcal{R}}) \quad \text{is interpreted as} \quad \frac{\Sigma, P; \Gamma \Rightarrow Q}{\Sigma; \Gamma \Rightarrow P \to Q}\ (\to_{\mathcal{R}})$$

3. The inference in SILL is $(\otimes_{\mathcal{R}})$. Then

$$\frac{!\Sigma_1, \Gamma_1 \Rightarrow P \quad !\Sigma_2, \Gamma_2 \Rightarrow Q}{!\Sigma_1, !\Sigma_2, \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}})$$

is interpreted as

$$\frac{\dfrac{\Sigma_1; \Gamma_1 \Rightarrow P}{\Sigma_1, \Sigma_2; \Gamma_1 \Rightarrow P}\ (W*) \quad \dfrac{\Sigma_2; \Gamma_2 \Rightarrow Q}{\Sigma_1, \Sigma_2; \Gamma_2 \Rightarrow Q}\ (W*)}{\Sigma_1, \Sigma_2; \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q}\ (\otimes_{\mathcal{R}})$$

4. The last inference in SILL is $(sel*)$. Then we have (because of Lemma 6.5)

$$\frac{\dfrac{!\Sigma \Rightarrow P}{\xrightarrow{[!\Sigma][\ ]} \gg !P}\ (P) \\ \vdots \\ \xrightarrow{[\ ][!\Sigma]} \gg !P}{!\Sigma \Rightarrow !P}\ (sel*) \quad \text{is interpreted as} \quad \frac{\Sigma; \Rightarrow P}{\Sigma; \Rightarrow !P}\ (P)$$

5. The last inference in SILL is $(sel)$. Then

$$\frac{\text{minor premisses} \qquad \dfrac{}{\phi \xrightarrow{A} A}\ (ax) \\ \vdots \\ !\Sigma_1, ..., !\Sigma_{n+m}, \Gamma_1, ..., \Gamma_m \xrightarrow{P} A}{!\Sigma_1, ..., !\Sigma_{n+m}, \Gamma_1, ..., \Gamma_m, P \Rightarrow A}\ (sel)$$

(where $< \{P_1, ..., P_n\}, \{Q_1, ..., Q_m\}, A > \in \parallel P \parallel$)

is interpreted as (omitting the weakenings that are needed)

$$\frac{\Sigma_1; \Rightarrow P_1 \quad ... \quad \Sigma_n; \Rightarrow P_n \quad \Sigma_{n+1}; \Gamma_1 \Rightarrow Q_1 \quad ... \quad \Sigma_{n+m}; \Gamma_m \Rightarrow Q_m}{\Sigma_1, ..., \Sigma_{n+m}; \Gamma_1, ..., \Gamma_m, P \Rightarrow A} \ (BC_1)$$

If the selected formula has a bang as its top formula, then either the next step is dereliction, and the case is similar to before with $(BC_2)$, or the next step is weakening or contraction and we get the result by the admissibility of these rules in Lolli.

■

**Proposition 6.4** *For any Lolli derivation* $\Pi$, $\mathcal{I}(\mathcal{J}(\Pi)) = \Pi$, *modulo the elimination of weakenings and contractions.*

PROOF: Follows from the interpretations given in the preceding propositions. We illustrate with the following example:

$$\frac{\Sigma; \Gamma_1 \Rightarrow P \quad \Sigma; \Gamma_2 \Rightarrow Q}{\Sigma; \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q} \ (\otimes_{\mathcal{R}})$$

is interpreted as

$$\cfrac{\cfrac{!\Sigma, \Gamma_1 \Rightarrow P \quad !\Sigma, \Gamma_2 \Rightarrow Q}{!\Sigma, !\Sigma, \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q} \ (\otimes_{\mathcal{R}})}{!\Sigma, \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q} \ \vdots \ contractions$$

which is then interpreted as

$$\cfrac{\cfrac{\cfrac{\Sigma; \Gamma_1 \Rightarrow P}{\Sigma, \Sigma; \Gamma_1 \Rightarrow P} \ (W) \quad \cfrac{\Sigma; \Gamma_2 \Rightarrow Q}{\Sigma, \Sigma; \Gamma_2 \Rightarrow Q} \ (W)}{\Sigma, \Sigma; \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q} \ (\otimes_{\mathcal{R}})}{\Sigma; \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q} \ (C*)$$

and eliminating the structural rules:

$$\frac{\Sigma; \Gamma_1 \Rightarrow P \quad \Sigma; \Gamma_2 \Rightarrow Q}{\Sigma; \Gamma_1, \Gamma_2 \Rightarrow P \otimes Q} \ (\otimes_{\mathcal{R}})$$

■

Propositions 6.2 and 6.3 show that up to the treatment of structural rules SILL and Lolli coincide for UILL. Many SILL proofs are interpreted as the same Lolli proof, but this is simply because of the greater flexibility in positioning weakening and contractions. $\mathcal{J}(\mathcal{I}(\Pi))$ brings all weakenings to axioms and contractions to immediately below context splitting rules. In fact, several SILL proofs are interpreted as one Lolli proof due to $(sel*)$ in SILL, as each ordering of the formulae selected gives a different proof.

As has been noted several times in this thesis, we might want to consider reformulations of natural deduction, in particular one which has linear and non-linear assumptions – such a natural deduction system might give a good correspondence with a split context version of SILL. We would expect such a calculus to match Lolli even more closely than SILL matches Lolli. Syntactically, we could easily give a split context version of SILL, but this would lack the correspondence with semantics that motivates the calculus.

One of the reasons for developing SILL is as a logic programming language based on ILL. MJ can be seen as extending the view of logic programming as backwards proof search in a backchaining calculus for hereditary Harrop logic to proof search over the whole of intuitionistic logic. In doing so, it gives a semantic rationale to the calculus used. Lolli is a logic programming language with a backchaining calculus for a fragment of ILL. As MJ extends logic programming founded on intuitionistic logic, so SILL extends Lolli. SILL contains all the Lolli proofs, and extends the calculus to cover the whole of ILL, producing a calculus with a semantic rationale. However, proof search in the resulting calculus is no longer goal directed. Whilst for MJ this isn't too problematic, SILL is a very complicated calculus, especially because of the unrestricted occurrences of bang. SILL appears to be too complicated to be practically used as a logic programming language, and its interest is restricted to its theoretical properties of naturally corresponding in a 1–1 with normal natural deductions, and hence giving a semantic rationale to Lolli.

### 6.7.2 SILL and Forum

Forum ([Mil96]) is another linear logic programming language. It is based on full classical linear logic and exploits the symmetries of linear logic to give a calculus for the whole of linear logic whilst avoiding the use of connectives that have rules which do not fit well with goal-directedness. The calculus is not given with a backchaining rule as the presence of query formulae on the left prevents a calculus with this as the only left rule from being complete. The rules are in fact presented with single stoup rules, much like those of SILL. If we restrict Forum to its single succedent subsystem, with sequents in the fragment of UILL built from the connectives allowed in Forum, we find a subsystem of Lolli inside Forum. This subsystem of Forum then matches SILL in the same way that Lolli matches SILL.

It would be interesting to see what a sequent system matching natural deduction for classical linear logic ([Bie96]) would look like and how it would compare with Forum. Of course one could argue that we should be interested in proof nets rather than natural deduction for classical linear logic, and that we should direct our efforts towards finding a sequent system reflecting these. This system might be similar to Andreoli's ([And92]) focusing calculus.

## 6.8 SILL and Permutations of Inference Rules

In Chapter 2, the calculus ILLF was presented for ILL. This calculus gives proofs for sequents giving only one proof for each equivalence class of proofs equivalent up to permutation of inferences (P-equivalent proofs). Calculi such as SILL have been described as permutation-free elsewhere in this thesis, and so we compare SILL with ILLF. It has already been mentioned that we consider 'permutation-free' to be a poor description of a calculus such as SILL since many permutations are still possible in SILL. These permutations may not be semantically sound with respect to normal natural deductions, but this suggests rather that natural deduction is a poor proof-theoretic semantics for ILL than that the permutations are not important.

If we restrict formulae to those in the UILL fragment of ILL, we find a calculus very similar to Lolli. Apart from issues to do with context management, there is only one difference. When focusing on a formula, ILLF as formulated in Figure 2.1 allows atoms to be returned to the context (by the $(\Downarrow_{\mathcal{L}_2})$ rule), whereas in Lolli the backchaining that this is the end of would not be allowed. Otherwise the calculi are the same. We could further restrict the $(\Downarrow_{\mathcal{L}_2})$ rule so that atoms would not be returned the context. This calculus would match Lolli over the UILL fragment of ILL.

We could give interpretation of the various systems into each other, much as we did with Lolli and SILL. We omit the details of these interpretations, but name them:

- Lolli into ILLF, $\mathcal{K}(\Pi)$
- ILLF into Lolli, $\mathcal{L}(\Pi)$
- ILLF into SILL, $\mathcal{M}(\Pi)$
- SILL into ILLF, $\mathcal{N}(\Pi)$

**Proposition 6.5** *For any Lolli proof* $\Pi$, $\mathcal{L}(\mathcal{K}(\Pi)) = \Pi$.

PROOF: By putting together the interpretations as in Proposition 6.4. ■

**Proposition 6.6** *For any* ILLF *proof* $\Pi$, $\mathcal{K}(\mathcal{L}(\Pi)) = \Pi$.

PROOF: Similar to Proposition 6.4. ■

**Proposition 6.7** *For any* ILLF *proof* $\Pi$, $\mathcal{N}(\mathcal{M}(\Pi)) = \Pi$.

PROOF: Similar to Proposition 6.4. ■

Interpreting SILL proofs as ILL proofs and then interpreting back again will move occurrences of weakening to the axioms and occurrences of contractions to immediately below context splitting rules (as with interpretation in Lolli and back).

## 6.9 Conclusion

In this chapter we have presented a calculus, SILL, for Intuitionistic Linear Logic, the proofs in which correspond in a 1–1 to the normal natural deductions for ILL. We have proved that SILL has this property. We have also given a weak cut-elimination theorem for SILL$^{cut}$.

We have compared SILL with the linear logic programming language Lolli. We have shown that the Lolli can be interpreted in SILL, but that over the fragment of ILL for which Lolli is defined, the calculi do not coincide owing to the treatment of structural rules.

We have also discussed the formulation of natural deduction for ILL, and have suggested that a more refined notion of normal natural deduction might have more attractive properties. A sequent calculus matching this as SILL matches the formulation used in this chapter might correspond more closely to Lolli, and have better proof-theoretic properties.

# Chapter 7

# Conclusion

This thesis has been a study of proof search systems for a variety of non-classical logics. Proof search has two meanings. Firstly it can mean the search for a single proof, a simple yes/no answer to a query. Secondly it can mean the search for all answers of a query, the enumeration of all proofs of a sequent. This thesis has investigated proof search in both these senses.

– Chapter 2 was a study of the permutations of inference rules in Intuitionistic Linear Logic. A Gentzen calculus for the logic, called ILLF, was presented and shown to be sound and complete with respect to provability in ILL. ILLF gives only one proof in each P-equivalence class (proofs equivalent up to permutations). This calculus can be seen as an efficient calculus for searching for a yes/no answer to a query (although as ILL is only semi-decidable, it is not guaranteed that a negative answer will be produced). ILLF can also be as a calculus for enumerating proofs, and thus as a basis of a logic programming language (but without the semantic properties later argued for).

– Chapter 3 was a study in the application of 'permutation-free' techniques to an intuitionistic modal logic, Lax Logic. A 'permutation-free' calculus is one with proofs naturally corresponding in a 1–1 way to the normal natural deductions for that logic. For well behaved fragments of logics, these proofs are also the normal forms for sequent proofs up to permutation of inferences. The calculus for Lax Logic, called PFLAX, is proved to have the correspondence, hence is sound and complete. PFLAX is a suitable calculus for enumerating all proofs. Links with constraint logic programming are discussed. Cut-elimination is also studied, and both weak and strong cut-elimination are proved for the calculus.

– In Chapter 4 a method for turning suitable propositional calculi into decision procedures using a history mechanism is given. A history mechanism keeps track of which sequents have appeared so far on a branch, and prevents looping. The mechanism is applied to the G3 and MJ calculi for intuitionistic propositional logic, as well as PFLAX and intuitionistic S4. These calculi are intended for delivering yes/no answers to queries.

– Chapter 5 is a short investigation into the embedding of intuitionistic logic in linear logic.

– Chapter 6 applies ‘permutation-free’ techniques to Intuitionistic Linear Logic. The resulting calculus, SILL, is proved to be in 1–1 correspondence with normal natural deductions. A weak cut-elimination theorem is proved. Connections with linear logic programming languages are discussed. SILL is seen to contain the language Lolli. There is also discussion of the formulation of natural deduction for ILL and its suitability as a proof-theoretic semantics for ILL.

The work in this thesis achieves several things. Firstly, the work in Chapter 2 clarifies material already to be found in the literature for single succedent classical linear logic by applying it to two-sided Intuitionistic Linear Logic. How to turn the CLL studies into ILL studies is not obvious and it is worth spelling out in detail the calculus ILLF. Chapter 3 gives a Gentzen sequent calculus for Lax Logic corresponding to normal natural deductions. Not only is this calculus attractive because of the focusing involved, but it gives a suitable proof search calculus for constraint logic programming (if it is accepted that constraint logic programming can be based on Lax Logic). Chapter 4 gives a new decision procedure for propositional Lax Logic. It also gives a general method for turning calculi into decision procedures, which can often be useful. Chapter 5 raises some interesting questions as to what calculi are induced by embedding one logic into another and partially answers these questions. Chapter 6 again gives a Gentzen sequent calculus (this time for ILL) corresponding to normal natural deductions which can be related to logic programming. This calculus can be seen as giving a proof-theoretic semantics to the linear logic programming language Lolli.

There are, however, points where although the work is technically correct and has achieved its aims, we are a little disappointed with the outcome. The application of the history mechanism to intuitionistic logic in Chapter 4 did not give as efficient a theorem prover as had been hoped. It had also been hoped that we could make improvements to decision procedures to classical modal logics, but unfortunately this did not prove possible. In Chapter 5 we were unable to find an embedding of intuitionistic logic into linear logic that induced the whole of MJ (or any other attractive sequent calculus). We were disappointed, if not completely surprised, by this. Finally, the system SILL for Intuitionistic Linear Logic given in Chapter 6 seems unattractive. In order to make the system correspond to the normal natural deductions, the multistoup is needed (this corresponds to the $n$ minor premisses of promotion). There is a large amount of non-determinism in the selection of the multistoup – not only do you have to decide which formulae go into the stoup, but in what order they appear in the multistoup. Each ordering of the formulae in the multistoup corresponds to a different proof, despite the fact that they appear to be the same in many senses. This suggests that it might be good to rework the theory with the promotion rule in natural deduction having unordered premisses, and therefore SILL with an unordered multistoup. But not only does this involve a lot of work to make sense of proof terms, it also makes it hard to see how to make the normalisation process in ILL confluent.... A more radical reworking of natural deduction might be more successful, starting with Mints's suggestion for an $n + 1$ premiss tensor elimination rule. SILL is undoubtably too unwieldy for practical use, but it does still provide a semantics for, and extension of, Lolli.

## 7.1 Permutation-free Calculi as a Foundation for Logic Programming

Permutation-free calculi are of interest for several reasons. Theoretically it is interesting that the structure of normal natural deductions can be captured in an elegant sequent calculus system for a wide range of constructive logics. The cut-elimination process for these calculi can also be viewed as a computation procedure. In this thesis we have been concentrating on the connections between cut-free permutation-free sequent calculi and logic programming viewed as backwards proof search in constructive logics. This section gives concluding remarks on this relationship.

It has already been seen that: the backchaining calculus for hereditary Harrop formulae is contained in MJ; that Lolli is contained in SILL; that over the intersection of their languages, Forum and SILL coincide.

Logic programming is about the search for proofs. A query is given and the interpreter for the logic programming language gives an answer. It can then be asked for another answer, and so on until all answers have been given. These answers are the different proofs possible in the logic. The question is, which proofs are wanted? Proofs which are the same will give the same solution to a query. This is wasteful. The uniform proof calculi are syntactically developed devices for giving proofs in a reduced search space. It might be better to have a justification for the canonical proofs from the proof-theoretic semantics of the logic. Normal natural deductions provide a good proof-theoretic semantics to many constructive logics. This suggest that the proofs that are interesting for logic programming are normal natural deductions. It is normal natural deductions that are found by 'permutation-free' calculi, hence it seems that these are the natural calculi to base logic programming languages on.

As stated above, over suitable fragments of the logic, the permutation-free calculi coincide with the corresponding backchaining calculi. Backchaining calculi are defined over fragments where the permutations in the sequent calculus match those in natural deductions and which are suitable for goal-directed proof search. That is, the 'nice' fragments of the logics.

For these fragments the calculi coincide, and although it is useful to have a semantic underpinning of the calculi used, no change is suggested for the actual bases of the programming languages. They now have a semantic justification. As the fragments do not cover the whole of the logic, their power of expression could be extended, by extending them to the whole calculus. At this point, goal-directed proof search is lost. What the calculi should be is not entirely obvious from a syntactic point of view. Some sort of Andreoli-style focusing calculus, such as ILLF, appears to be the answer. However, these do not have the proof-theoretic semantics argued for above, and so we think that extensions to logic programming should be the extensions of the corresponding permutation-free calculi. Of course, losing the goal directedness of proof search is a disadvantage from an implementational point of view, whereas an Andreoli style calculus would keep this to a greater extent.

Having argued for permutation-free calculi in general as good extensions to logic programming languages, we now consider SILL in particular as an extension of Lolli. Whereas, for example, normal natural deductions are generally considered to be the correct proof-theoretic semantics for intuitionistic logic, it is not so clear that they are for ILL. Many natural deductions (hence SILL proofs) that one would intuitively want to identify are not identified for ILL. SILL seems to lack determinism compared to a calculus such as ILLF. It is possible that a refinement of the notion of normal natural deduction for ILL would give a much better proof-theoretic semantics, and the resulting SILL-like calculus would seem a much more suitable extension of Lolli.

In conclusion, it is the proof-theoretic semantics of logics that are of primary importance in logic programming. Proof search is search for exactly the proofs given by the proof-theoretic semantics. Permutation-free calculi are of interest as they seem particularly well suited for the enumeration of normal natural deductions, the semantics for many non-classical logics. However, for logics whose proof-theoretic semantics is not normal natural deductions, these calculi are less well suited and less interesting as the basis of a logic programming language based on the logics. For these logics we are interested in good ways of enumerating all proofs, whether in the logical system itself, or in another more suitable. Hence MJ is a good calculus to base a logic programming language on, and SILL less good. A better understanding of the proof-theoretic semantics of ILL might suggest a more suitable calculus for extending Lolli.

## 7.2 Semantics of ILL

The preceding discussion leads one to question whether or not natural deduction is a suitable proof-theoretic semantics for ILL. At first glance it seems to be the obvious semantics for the logic. ILL can, after all, be seen as a refinement of the usual intuitionistic logic. That its semantics should be a refinement of usual semantics for intuitionistic logic seems to follow. Indeed, as can be seen, as a calculus, natural deduction for ILL seems attractive. It is only when one considers which proofs intuitively seem to be the same that natural deduction seems less well suited. Different normal natural deductions appear as though they should be identified. This could be fixed by adding more normalisation steps as suggested by Mints. There is also the problem of non-confluence of normalisation and its relation to the ordering of the premisses of promotion in natural deduction. Some sort of solution might be given, but the suggestion is that the proof-theoretic semantics is not normal natural deduction.

If normal natural deductions do not provide a good proof-theoretic semantics for ILL, then what does? The first thought is that a version of proof nets might provide the solution, but we know of no satisfactory treatment of proof nets for ILL. (Treatments include [Lam94], [BCST96], [BCS96], [CS97]). We do think that a study of the syntactic system ILLF, along with the categorical semantics for ILL might suggest some suitable system, but this is pure speculation.

Finally we should ask whether the problem is that ILL is not a sensible logic? Perhaps we should consider a larger fragment, or full CLL. However, as syntactically ILL is perfectly well defined, we think that ILL is interesting and worthy of study. ILL is a proper logic in its own right.

## 7.3 Future Work

This thesis leaves some immediate questions to be answered, as well as posing some more open ended problems. Most of these problems are subject to ongoing investigation by the author.

The most obvious unsolved problems left by this thesis are the questions of strong cut-elimination and strong normalisation for SILL. This is complicated and hard to formulate properly, and we are unsure whether or not the results hold. Strong cut-elimination can hopefully be proved by using a suitable modification of the definition of elimination number, and choosing suitable measures to build a weight for the cut instance. We would also like to extend the term calculus to cut terms. For presentation purposes, this would involve giving different proof terms for the cut-free calculus too (showing the internal structure of promotion). Once we have this with its extension to the cuts, we would hope that by using the recursive path order, strong normalisation could be proved without too much difficulty. We would then have a second proof of strong cut-elimination. Of course, if the results do not hold, we would like simple counter-examples!

It has been noted in [BdP96] that the introduction and elimination rules for the modality in intuitionistic S4 can be formulated in the same way as the introduction and elimination rules for promotion used here. With this knowledge, it should be a simple task to give a ‘permutation-free’ calculus for IS4.

A study of the relationship between cut-elimination and normalisation for natural deduction in ILL (as carried out in [Zuc74], [Pot77] for intuitionistic logic) would also be an interesting and a useful study.

In several places in this thesis, the work of Mints on ILL has been mentioned and outlined. A more extensive investigation of this, and its development, together with a permutation-free calculus for the resulting natural deduction system would be another interesting study, one which might be more fruitful and lead to attractive results. Trying to built in split linear and non-linear assumptions would be a part of this investigation. A study of ILLF and its relationship to the semantics of ILL would form another part of this study.

In Chapter 5 we tried to induce the calculus MJ for intuitionistic logic from an embedding of intuitionistic logic into ILL. This was only successful for hereditary Harrop logic. We would like the result to hold for the whole of intuitionistic logic. Although we are pessimistic of success, we feel the question is worth investigating and an understanding of the failure to get the desired result would be useful.

In Chapter 4 we gave a decision procedure for propositional Lax Logic. It was said there that a contraction-free (or terminating) calculus for the logic would be interesting and useful. Although we are again pessimistic of finding such a calculus the investigation would be useful. Again, analysis and proof of the failure might also be interesting. The decision procedure, the calculus PFLAX$^{Hist}$, has yet to be properly implemented, tested and developed. This would be a useful task to complete.

Finally, the Scottish history calculi in Chapter 4 could be used to enumerate all loop-free proofs for a sequent calculus. Syntactically, this seems like a well-defined (and finite) subset of proofs. It would be interesting to see whether or not this corresponds to a well defined subset of proofs in the semantics. The author has no intuition as to the answer to this question, but if it is a yes, then there is an interesting field of development in logic and possibly logic programming to consider.

# Appendix A

# Logical Calculi

This appendix contains the logical calculi mentioned, but not presented, in the body of this thesis.

## A.1 G3

This is not quite the same calculus as in [TS96], but it is exactly G3 as presented by Kleene in [Kle52a].

$$\frac{}{\Gamma, P \Rightarrow P}\ (ax) \qquad \frac{}{\Gamma, - \Rightarrow P}\ (-)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma, P \supset Q \Rightarrow P \quad \Gamma, P \supset Q, Q \Rightarrow R}{\Gamma, P \supset Q \Rightarrow R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma, P \Rightarrow -}{\Gamma \Rightarrow \neg P}\ (\neg_{\mathcal{R}}) \qquad \frac{\Gamma, \neg P \Rightarrow P}{\Gamma, \neg P \Rightarrow R}\ (\neg_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma, P \wedge Q, P \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma, P \wedge Q, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \vee Q, P \Rightarrow R \quad \Gamma, P \vee Q, Q \Rightarrow R}{\Gamma, P \vee Q \Rightarrow R}\ (\vee_{\mathcal{L}})$$

## A.2 G4

This is the contraction-free calculus of [Vor58], [Dyc92], [Hud93].

$$\frac{}{\Gamma, A \Rightarrow A}\,(ax) \qquad \frac{}{\Gamma, - \Rightarrow R}\,(-)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\,(\supset_{\mathcal{R}})$$

$$\frac{\Gamma, A, Q \Rightarrow R}{\Gamma, A \supset Q, A \Rightarrow R}\,(\supset_{\mathcal{L}1}) \qquad \frac{\Gamma, S \supset (T \supset Q) \Rightarrow R}{\Gamma, (S \wedge T) \supset Q \Rightarrow R}\,(\supset_{\mathcal{L}2})$$

$$\frac{\Gamma, S \supset Q, T \supset Q \Rightarrow R}{\Gamma, (S \vee T) \supset Q \Rightarrow R}\,(\supset_{\mathcal{L}3}) \qquad \frac{\Gamma, T \supset Q \Rightarrow S \supset T \quad \Gamma, Q \Rightarrow R}{\Gamma, (S \supset T) \supset Q \Rightarrow R}\,(\supset_{\mathcal{L}4})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\,(\wedge_{\mathcal{R}}) \qquad \frac{\Gamma, P, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\,(\wedge_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma, P \vee Q \Rightarrow R}\,(\vee_{\mathcal{L}})$$

Here, A is atomic. Note that $-$ is not atomic.

## A.3 G6

The multiplicative formulation of intuitionistic logic.

$$\frac{}{P \Rightarrow P}\,(ax) \qquad \frac{}{\Gamma, - \Rightarrow P}\,(-)$$

$$\frac{\Gamma \Rightarrow R}{\Gamma, P \Rightarrow R}\,(W) \qquad \frac{\Gamma, P, P \Rightarrow R}{\Gamma, P \Rightarrow R}\,(C)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\,(\supset_{\mathcal{R}}) \qquad \frac{\Gamma_1 \Rightarrow P \quad \Gamma_2, Q \Rightarrow R}{\Gamma_1, \Gamma_2, P \supset Q \Rightarrow R}\,(\supset_{\mathcal{L}})$$

$$\frac{\Gamma_1 \Rightarrow P \quad \Gamma_2 \Rightarrow Q}{\Gamma_1, \Gamma_2 \Rightarrow P \wedge Q}\,(\wedge_{\mathcal{R}}) \qquad \frac{\Gamma, P, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\,(\wedge_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}_2}) \qquad \frac{\Gamma_1, P \Rightarrow R \quad \Gamma_2, Q \Rightarrow R}{\Gamma_1, \Gamma_2, P \vee Q \Rightarrow R}\,(\vee_{\mathcal{L}})$$

## A.4 NJ

We present the natural deduction calculus for propositional intuitionistic logic, first in ‘tree-style’, and then in sequent style.

### A.4.1 Trees

$$\frac{}{P}\,(-)$$

$$\frac{\begin{array}{c}[P]\\ \vdots\\ Q\end{array}}{P \supset Q}\,(\supset_{\mathcal{I}}) \qquad \frac{P \supset Q \quad P}{Q}\,(\supset_{\varepsilon})$$

$$\frac{P \quad Q}{P \wedge Q}\,(\wedge_{\mathcal{I}}) \qquad \frac{P \wedge Q}{P}\,(\wedge_{\varepsilon_1}) \qquad \frac{P \wedge Q}{Q}\,(\wedge_{\varepsilon_2})$$

$$\frac{P}{P \vee Q}\,(\vee_{\mathcal{I}_1}) \qquad \frac{Q}{P \vee Q}\,(\vee_{\mathcal{I}_2}) \qquad \frac{P \vee Q \quad \begin{array}{c}[P]\\ \vdots\\ R\end{array} \quad \begin{array}{c}[Q]\\ \vdots\\ R\end{array}}{R}\,(\vee_{\varepsilon})$$

### A.4.2 Sequents

$$\frac{}{\Gamma, P \vdash P}\,(ax) \qquad \frac{}{\Gamma, - \vdash P}\,(-)$$

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \supset Q}\,(\supset_{\mathcal{I}}) \qquad \frac{\Gamma \vdash P \supset Q \quad \Gamma \vdash P}{\Gamma \vdash Q}\,(\supset_{\varepsilon})$$

$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\,(\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P}\,(\wedge_{\varepsilon_1}) \qquad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}\,(\wedge_{\varepsilon_2})$$

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\,(\vee_{\mathcal{I}_1}) \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\,(\vee_{\mathcal{I}_2})$$

$$\frac{\Gamma \vdash P \vee Q \quad \Gamma, P \vdash R \quad \Gamma, Q \vdash R}{\Gamma \vdash R}\,(\vee_{\varepsilon})$$

## A.5 CLL

### A.5.1 Single-sided

CLL: single sided, multiple succedent calculus for classical linear logic.

$$\frac{}{\Rightarrow P, P^{\perp}}\,(ax) \qquad \frac{\Rightarrow \Gamma, P \quad \Rightarrow \Delta, P^{\perp}}{\Rightarrow \Gamma, \Delta}\,(cut)$$

$$\frac{}{\Rightarrow I}\,(I) \qquad \frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, -}\,(-) \qquad \frac{}{\Rightarrow \Gamma, \top}\,(\top)$$

$$\frac{\Rightarrow \Gamma, P \quad \Rightarrow \Delta, Q}{\Rightarrow \Gamma, \Delta, P \otimes Q}\,(\otimes) \qquad \frac{\Rightarrow \Gamma, P \quad \Rightarrow \Gamma, Q}{\Rightarrow \Gamma, P \& Q}\,(\&)$$

$$\frac{\Rightarrow \Gamma, P, Q}{\Rightarrow \Gamma, P \parr Q}\ (\parr) \qquad \frac{\Rightarrow \Gamma, P}{\Rightarrow \Gamma, P \oplus Q}\ (\oplus_1) \qquad \frac{\Rightarrow \Gamma, Q}{\Rightarrow \Gamma, P \oplus Q}\ (\oplus_2)$$

$$\frac{\Rightarrow \Gamma\Gamma, P}{\Rightarrow \Gamma\Gamma, !P}\ (P) \qquad \frac{\Rightarrow \Gamma, P}{\Rightarrow \Gamma, \Gamma P}\ (D)$$

$$\frac{\Rightarrow \Gamma}{\Rightarrow \Gamma, \Gamma P}\ (W) \qquad \frac{\Rightarrow \Gamma, \Gamma P, \Gamma P}{\Rightarrow \Gamma, \Gamma P}\ (C)$$

### A.5.2 Two-sided

CLL$^2$: two-sided, multiple-succedent sequent calculus - including implication.

$$\frac{}{P \Rightarrow P}\ (ax) \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, P \quad \Gamma_2, P \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (cut)$$

$$\frac{}{\Rightarrow I}\ (I_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, I \Rightarrow \Delta}\ (I_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow -, \Delta}\ (-_{\mathcal{R}}) \qquad \frac{}{- \Rightarrow}\ (-_{\mathcal{L}})$$

$$\frac{}{\Gamma \Rightarrow \top, \Delta}\ (\top_{\mathcal{R}}) \qquad \frac{}{\Gamma, 0 \Rightarrow \Delta}\ (0_{\mathcal{L}})$$

$$\frac{\Gamma, P \Rightarrow Q, \Delta}{\Gamma \Rightarrow P \multimap Q}\ (\multimap_{\mathcal{R}}) \qquad \frac{\Gamma_1 \Rightarrow P, \Delta_1 \quad \Gamma_2 \Rightarrow Q, \Delta_2}{\Gamma_1, \Gamma_2, P \multimap Q \Rightarrow \Delta_1, \Delta_2}\ (\multimap_{\mathcal{L}})$$

$$\frac{\Gamma_1 \Rightarrow P, \Delta_1 \quad \Gamma_2 \Rightarrow Q, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow P \otimes Q, \Delta_1, \Delta_2}\ (\otimes_{\mathcal{R}}) \qquad \frac{\Gamma, P, Q \Rightarrow \Delta}{\Gamma, P \otimes Q \Rightarrow \Delta}\ (\otimes_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P, \Delta \quad \Gamma \Rightarrow Q, \Delta}{\Gamma \Rightarrow P \& Q, \Delta}\ (\&_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \Delta}{\Gamma, P \& Q \Rightarrow \Delta}\ (\&_{\mathcal{L}_1})$$

$$\frac{\Gamma, Q \Rightarrow \Delta}{\Gamma, P \& Q \Rightarrow \Delta}\ (\&_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P, Q, \Delta}{\Gamma \Rightarrow P \parr Q, \Delta}\ (\parr_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \Delta \quad \Gamma', Q \Rightarrow \Delta'}{\Gamma_1, \Gamma_2, P \parr Q \Rightarrow \Delta_1, \Delta_2}\ (\parr_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P, \Delta}{\Gamma \Rightarrow P \oplus Q, \Delta}\ (\oplus_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q, \Delta}{\Gamma \Rightarrow P \oplus Q, \Delta}\ (\oplus_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \Rightarrow \Delta \quad \Gamma, Q \Rightarrow \Delta}{\Gamma, P \oplus Q \Rightarrow \Delta}\ (\oplus_{\mathcal{L}})$$

$$\frac{!\Gamma \Rightarrow P, \Gamma\Delta}{!\Gamma \Rightarrow !P, \Gamma\Delta}\ (P_{\mathcal{R}}) \qquad \frac{!\Gamma, P \Rightarrow \Gamma\Delta}{!\Gamma, \Gamma P \Rightarrow \Gamma\Delta}\ (P_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P, \Delta}{\Gamma \Rightarrow \Gamma P, \Delta}\ (D_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \Delta}{\Gamma, !P \Rightarrow \Delta}\ (D_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Gamma P, \Delta}\ (W_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, !P \Rightarrow \Delta}\ (W_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow \Gamma P, \Gamma P, \Delta}{\Gamma \Rightarrow \Gamma P, \Delta}\ (C_{\mathcal{R}}) \qquad \frac{\Gamma, !P, !P \Rightarrow \Delta}{\Gamma, !P \Rightarrow \Delta}\ (C_{\mathcal{L}})$$

## A.6 IS4

Two sided single succedent calculus for intuitionistic S4.

$$\frac{}{\Gamma, P \Rightarrow P}\ (ax) \qquad \frac{}{\Gamma, - \Rightarrow R}\ (-)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma, P \supset Q \Rightarrow P \quad \Gamma, P \supset Q, Q \Rightarrow R}{\Gamma, P \supset Q \Rightarrow R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma, P \wedge Q, P \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_1}) \qquad \frac{\Gamma, P \wedge Q, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}_2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}_2})$$

$$\frac{\Gamma, P \vee Q, P \Rightarrow R \quad \Gamma, P \vee Q, Q \Rightarrow R}{\Gamma, P \vee Q \Rightarrow R}\ (\vee_{\mathcal{L}})$$

$$\frac{\Box\Gamma \Rightarrow P}{\Box\Gamma, \Delta \Rightarrow \Box P}\ (\Box_{\mathcal{R}}) \qquad \frac{\Gamma, \Box P, P \Rightarrow R}{\Gamma, \Box P \Rightarrow R}\ (\Box_{\mathcal{L}})$$

## A.7 S4

This is a single sided (multiple succedent) sequent calculus.

$$\frac{}{\Pi, P, \neg P}\ (ax) \qquad \frac{\Pi, P}{\Pi, \neg\neg P}\ (\neg\neg)$$

$$\frac{\Pi, \neg P, Q}{\Pi, P \supset Q}\ (\supset) \qquad \frac{\Pi, P \quad \Pi, \neg Q}{\Pi, \neg(P \supset Q)}\ (\neg \supset)$$

$$\frac{\Pi, P \quad \Pi, Q}{\Pi, P \wedge Q}\ (\wedge) \qquad \frac{\Pi, \neg P, \neg Q}{\Pi, \neg(P \wedge Q)}\ (\neg\wedge)$$

$$\frac{\Pi, P, Q}{\Pi, P \vee Q}\ (\vee) \qquad \frac{\Pi, \neg P \quad \Pi, \neg Q}{\Pi, \neg(P \vee Q)}\ (\neg\vee)$$

$$\frac{\Pi, \Diamond P, P}{\Pi, \Diamond P}\ (\Diamond) \qquad \frac{\Box\Pi_2, P}{\Pi_1, \Box\Pi_2, \neg\Diamond P}\ (\neg\Diamond)$$

$$\frac{\Box\Pi_2, P}{\Pi_1, \Box\Pi_2, \Box P}\ (\Box) \qquad \frac{\Pi, \neg\Box P, P}{\Pi, \neg\Box P}\ (\neg\Box)$$

# Appendix B

# Benchmark Formulae

This Appendix gives the benchmark formulae used in Chapter 4. The benchmark formulae in Figure A.1 are from a comparison of propositional intuitionistic theorem provers at the TABLEAUX'98 conference. A description of them can be found in [Dyc97]. The formulae in Figures A.2 and A.3 are taken from [How97]. As can be seen from the table of results in Table 4.2, the formulae with quantifiers are instantiated over finite universes to give propositional formulae.

1. `de_bruijn_p(n)`:LHS$(2n+1)$ ⊃RHS$(2n+1)$
   `de_bruijn_n(n)`:LHS$(2n)$ ⊃ $(p(0)\vee$RHS$(2n) \vee \neg p(0))$

   RHS$(m) := \wedge_{i=1}^{m} p(i)$
   LHS$(M) := \wedge_{i=1}^{m}((p(i) \leftrightarrow p(i+1)) \supset$RHS$(m))$
   addition modulo $m$

2. `ph_p(n)`:= left_p($n$) ⊃ right($n$)
   `ph_n(n)`:= left_n($n$) ⊃ right($n$)

   left_p($n$):=$\wedge_{i=1}^{n+1}(\vee_{j=1}^{n}$ occ$(i, j))$
   left_n($n$):=$\wedge_{i=1}^{n+1}(\vee_{j\perp 1}^{n\perp 1}$ (occ$(i, j) \vee \neg\neg$ occ$(i, n)))$
   right($n$):=$\wedge_{i1=1}^{n} \vee_{i2=1}^{n+1} \vee_{j=i2+1}^{n+1}$ s$(i1, i2, j)$
   s$(l, m, n)$:=occ$(l, n) \wedge$ occ$(m, n)$

3. `con_p(n)`:=((conjs($n$)$\vee$ disjs_p($n$)) ⊃ $p(f)) \supset p(f)$
   `con_n(n)`:=((conjs($n$)$\vee$ disjs_n($n$)) ⊃ $p(f)) \supset p(f)$

   conjs($n$):=$\wedge_{i=1}^{n} p(i)$
   disjs_p($n$):=$\vee_{i=1}^{n}(p(i) \supset p(f))$
   disjs_n($n$):=$(\neg\neg p(1) \supset p(f)) \vee \vee_{i=2}^{n}(p(i) \supset p(f))$

4. `schwicht_p(n)`:=(ant_p($n$)⊃ $p(0))$
   `schwicht_n(n)`:=(ant_n($n$)⊃ $p(0))$

   ant_p($n$):=$p(n) \wedge \wedge_{i=1}^{n}(p(i) \supset p(i) \supset p(i-1))$
   ant_n($n$):=$\neg\neg p(n) \wedge \wedge_{i=1}^{n}(p(i) \supset p(i) \supset p(i-1))$

5. `kk_p(`$n$`)`:=(kk_pp($n$, $n$) ⊃ $p(f)) \wedge$ (kkr($n$, $n$)⊃ $p(f))$
   `kk_n(`$n$`)`:=kk_nn($n$, $n$)

   kk_pp($n$, 0):=(pr(a, 0)⊃ $p(f)) \wedge$ ((pr(b, $n$)⊃ pr(b, 0)) ⊃ pr(a, $n$))
   kk_pp($n$, $m$):=kk_pp($n$, $m-1$) ∧ ((pr(b, $m-1$) ⊃ pr(a, $m$)) ⊃ pr(a, $m-1$))
   kkr($n$, 0):=((pr(b, $n$)⊃ pr(b, 0)) ⊃ pr(a, $n$)) ∧ (pr(a, 0) ⊃ p(f))
   kkr($n$, $m$):=((pr(b, $m-1$)⊃ pr(a, $m$)) ⊃ pr(a, $m-1$)) ∧ kkr($n$, $m-1$)
   kk_nn($n$, 0):=(pr(a, 0)⊃ p(f)) ∧ ((¬¬ pr(b, $n$) ⊃ pr(b, 0)) ⊃ pr(a, $n$))
   kk_nn($n$, $m$):=kk_nn($n$, $m-1$) ∧ ((¬¬ pr(b, $m-1$) ⊃ pr(a, $m$)) ⊃ pr(a, $m$))

6. `equiv_p(n)`:=eq_pf($n$) ↔ eq_b($n$)
   `equiv_n(n)`:=eq_nf($n$) ↔ eq_b($n$)

   eq_pf(1):=$p(1)$
   eq_pf($n$):=eq_pf($n-1$) ↔ $p(n)$
   eq_nf(1):=$\neg\neg p(1)$
   eq_nf($n$):=eq_nf($n-1$) ↔ $p(n)$
   eq_b(1):=$p(1)$
   eq_b($n$):=$p(n)$ ↔ eq_pb($n-1$)

**Figure B.1:** Benchmark Formulae

**1.** $((A \vee B) \wedge (D \vee E) \wedge (G \vee H)) \supset ((A \wedge D) \vee (A \wedge G) \vee (D \wedge G) \vee (B \wedge E) \vee (B \wedge H) \vee (E \wedge H))$

**2.** $((A \vee B \vee C) \wedge (D \vee E \vee F) \wedge (G \vee H \vee J) \wedge (K \vee L \vee M)) \supset (A \wedge D) \vee (A \wedge G) \vee (A \wedge K) \vee (D \wedge G) \vee (D \wedge K) \vee (G \wedge K) \vee (B \wedge E) \vee (B \wedge H) \vee (B \wedge L) \vee (E \wedge H) \vee (E \wedge L) \vee (H \wedge L) \vee (C \wedge F) \vee (C \wedge J) \vee (C \wedge M) \vee (F \wedge J) \vee (F \wedge M) \vee (J \wedge M)$

**3.** $((A \vee B \vee C) \wedge (D \vee E \vee F)) \supset ((A \wedge B) \vee (B \wedge E) \vee (C \wedge F))$

**4.** $(A \supset B) \supset (A \supset C) \supset (A \supset (B \wedge C))$

**5.** $(A \wedge \neg A) \supset B$

**6.** $(A \vee C) \supset (A \supset B) \supset (B \vee C)$

**7.** $((((A \supset B) \wedge (B \supset A)) \supset (A \wedge B \wedge C)) \wedge (((B \supset C) \wedge (C \supset B)) \supset (A \wedge B \wedge C)) \wedge (((C \supset A) \wedge (A \supset C)) \supset (A \wedge B \wedge C))) \supset (A \wedge B \wedge C)$

**8.** $((\neg\neg P \supset P) \supset P) \vee (\neg P \supset \neg P) \vee (\neg\neg P \supset \neg\neg P) \vee (\neg\neg P \supset P)$

**9.** $(((G \supset A) \supset J) \supset D \supset E) \supset (((H \supset B) \supset I) \supset C \supset J) \supset (A \supset H) \supset F \supset G \supset (((C \supset B) \supset I) \supset D) \supset (A \supset C) \supset (((F \supset A) \supset B) \supset I) \supset E$

**10.** $A \supset B \supset ((A \supset B \supset C) \supset C) \supset (A \supset B \supset C)$

**11.** $((\neg\neg(\neg A \vee \neg B) \supset (\neg A \vee \neg B)) \supset (\neg\neg(\neg A \vee \neg B) \vee \neg(\neg A \vee \neg B))) \supset (\neg\neg(\neg A \vee \neg B) \vee \neg(\neg A \vee \neg B))$

**12.** $B \supset (A \supset (((A \wedge B) \supset C_1) \supset (((A \wedge B) \supset C_2) \supset (((A \wedge B) \supset C_3) \supset (((A \wedge B) \supset (B \supset C_1 \supset C_2 \supset C_3 \supset B)) \supset (A \wedge B))))))$

**13.** $((A \wedge B \vee C) \supset (C \vee (C \wedge D))) \supset (\neg A \vee ((A \vee B) \supset C))$

**14.** $\neg\neg((\neg A \supset B) \supset (\neg A \supset \neg B) \supset A)$

**15.** $\neg\neg(((A \leftrightarrow B) \leftrightarrow C) \leftrightarrow (A \leftrightarrow (B \leftrightarrow C)))$

**16.** $\forall x \exists y \forall z (p(x) \wedge q(y) \wedge r(z)) \leftrightarrow \forall z \exists y \forall x (p(x) \wedge q(y) \wedge r(z))$

**17.** $\exists x_1 \forall y_1 \exists x_2 \forall y_2 \exists x_3 \forall y_3 (p(x_1, y_1) \wedge q(x_2, y_2) \wedge r(x_3, y_3)) \supset \forall y_3 \exists x_3 \forall y_2 \exists x_2 \forall y_1 \exists x_1 (p(x_1, y_1) \wedge q(x_2, y_2) \wedge r(x_3, y_3))$

**18.** $\neg \exists x \forall y (mem(y, x) \leftrightarrow \neg mem(x, x))$

**19.** $\neg \exists x \forall y (q(y) \supset r(x, y)) \wedge \exists x \forall y (s(y) \supset r(x, y)) \supset \neg \forall x (q(x) \supset s(x))$

**Figure B.2:** Example formulae

**20.** $\forall z_1 \forall z_2 \forall z_3 (q(z_1, z_2, z_3, z_1, z_2, z_3)) \supset \exists x_1 \exists x_2 \exists x_3 \exists y_1 \exists y_2 \exists y_3 ((p(x_1) \wedge p(x_2) \wedge p(x_3) \leftrightarrow p(y_1) \wedge p(y_2) \wedge p(y_3)) \wedge q(x_1, x_2, x_3, y_1, y_2, y_3))$

**21.** $((\exists x(p \supset f(x))) \wedge (\exists x_1(f(x_1) \supset p))) \supset (\exists x_2((p \supset f(x_2)) \wedge (f(x_2) \supset p)))$

**22.** $(\exists x(p(x)) \wedge (\forall x_1(f(x_1) \supset (\neg g(x_1) \wedge r(x_1))) \wedge (\forall x_2(p(x_2) \supset (g(x_2) \wedge f(x_2))) \wedge (\forall x_3(p(x_3) \supset q(x_3)) \vee \exists x_4(p(x_4) \wedge r(x_4)))))) \supset \exists x_5(q(x_5) \wedge p(x_5))$

**23.** $((\exists x(p(x)) \leftrightarrow \exists x_1(q(x_1))) \wedge \forall x_2 \forall y((p(x_2) \wedge q(y)) \supset (r(x_2) \leftrightarrow s(y)))) \supset (\forall x_3(p(x_3) \supset r(x_3)) \leftrightarrow \forall x_4(q(x_4) \supset s(x_4)))$

**24.** $(\forall x((f(x) \vee g(x)) \supset \neg h(x)) \wedge \forall x_1((g(x_1) \supset \neg i(x_1)) \supset (f(x_1) \wedge h(x_1)))) \supset \forall x_2(i(x_2))$

**25.** $(\neg \exists x(f(x) \wedge (g(x) \vee h(x))) \wedge (\exists x_1(i(x_1) \wedge f(x_1)) \wedge \forall x_2(\neg h(x_2) \supset j(x_2)))) \supset \exists x_3(i(x_3) \wedge j(x_3))$

**26.** $(\forall x((f(x) \wedge (g(x) \vee h(x))) \supset i(x)) \wedge (\forall x_1((i(x_1) \wedge h(x_1)) \supset j(x_1)) \wedge \forall x_2(k(x_2) \supset h(x_2)))) \supset \forall x_3((f(x_3) \wedge k(x_3)) \supset j(x_3))$

**27.** $\neg \exists y \forall x(f(x, y) \leftrightarrow \neg \exists z(f(x, z) \wedge f(z, x)))$

**Figure B.3:** Example Formulae

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