Design of Saturating State-Feedback with Sign-Indefinite Quadratic Forms

Isabelle Queinnec, Sophie Tarbouriech, Giorgio Valmorbida, Luca Zaccarian

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***Abstract*—We propose a novel class of piecewise smooth Lyapunov functions leading to LMI-based stability/performance analysis and control design for linear systems with saturating inputs. We provide conditions for global properties, and also conditions for local properties and guaranteed estimates of the basin of attraction. The backbone of our result consists in using quadratic forms with constant matrices that are not necessarily sign definite, thereby providing additional degrees of freedom. Using generalized sector conditions involving the dead-zone nonlinearity and its derivative, we formulate convex optimization conditions to verify their positivity in the region of interest. Several numerical examples with connections to existing results, illustrate the potential behind our novel construction.**

I. Introduction

The operating constraints of any practical control system often include magnitude limits on the actuators that must be taken into account in the control design to avoid potentially catastrophic effects (see, for example, [22] and [1]). The conservative (so-called low-gain) approach of restricting solutions to never reach the saturation bounds may limit excessively the system operation or induce poor performance. With linear plants, it is known that global asymptotic stability can be only achieved if the plant is not exponentially unstable (these plants are called Asymptotically Null Controllable with Bounded Controls - ANCBC) [21]. Moreover, some of these systems require nonlinear stabilizers [7]. When the plant is not ANCBC it is imperative to characterize the basin of attraction of the origin and to obtain tight estimates thus providing the user with a large set of operating conditions.

In the last two decades, constructive methods based on convex optimization led to direct design and anti-windup design strategies to mitigate performance degradation during saturation [11], [23], [31]. Most of these results use quadratic Lyapunov functions and the circle criterion to certify absolute stability. In this scenario, only the sector property of the saturation is considered and the resulting analysis is generally conservative. For the regional analysis and design which are necessary with non ANCBC plants, generalized sector conditions enjoyed by the saturation are used, typically leading to certified estimates of the basin of attraction based on sublevel sets of the Lyapunov functions. When using the Popov criterion [14], the nonlinearity is incorporated in the Lyapunov function and less conservative regional results can be obtained using information on the gradient of the nonlinearity [4], [17]. However it is only recently that the sector properties of the nonlinearity have been used to also relax the positivity of the Lyapunov functions for optimization-based analysis of Lure systems [6], [16], [26], [27].

In this paper, we introduce a new class of non-quadratic Lyapunov functions for stability analysis and control design providing global and non-global guarantees. The quadratic forms that we consider were first introduced in [19] and then revisited in [4] and [16]. However, in [4], [19] a constraint on the positivity of the matrix in the quadratic form was imposed. As in the analysis results of [16], we exploit here the flexibility of matrices that are not sign definite for the stability analysis and, for the first time, for the control synthesis of nonlinear state feedback laws for saturating systems. Both the global and regional cases are studied. In the framework of Linear Complementarity Systems, a similar Lyapunov function was presented in [2]. Moreover, in the discrete-time saturated systems context, [10] used a special case of the function studied here. Not many results focus on non-positive quadratic forms when addressing continuous-time saturated systems. In our recent work [27], we show that relaxing the positivity conditions allows improving the nonlinear $\mathcal{L}_{2m}$ gain estimates, when using piecewise polynomial storage functions. Also [16] suggested similar positivity relaxations when focusing on max of quadratics for analysis purposes. Additionally, for systems with slope bounds in a known set of the state space, regional gain and regions of attractions were studied in [26]. Finally, for systems with rational Jacobian one can circumvent the need for a pre-defined set in which the slope is bounded [6].

In this paper we propose synthesis conditions exploiting sign-indefinite quadratic forms. For the global case, our conditions are authentic linear matrix inequalities, and allow constructing certificates for examples where quadratic functions are proven to not exist. For the regional case we obtain suggestive functions whose positivity is only guaranteed in the estimated region of attraction, and which even become negative outside that certified domain, so that extra degrees of freedom are available for obtaining nonconservative (and nonconvex) contractive sublevel sets. An interesting parallel approach is provided in [15], where generalized quadratic forms are used to address the analysis problems considered in our paper. In addition to also providing design conditions (whereas only analysis is addressed in [15]), our conditions directly imply well posedness of the nonlinear algebraic loop, whereas this property is stated as an assumption in [15].

The paper is organized as follows. Section II describes

I. Queinnec, S. Tarbouriech and L. Zaccarian are with LAAS-CNRS, University of Toulouse, CNRS, Toulouse, France, Email: {tarbour,queinnec,zaccarian}@laas.fr. Luca Zaccarian is also with the Dept. of Industrial Engineering, University of Trento, Italy. G. Valmorbida is with Laboratoire des Signaux et Systèmes, CentraleSupélec, CNRS, Université Paris-Saclay and with Inria Saclay, 3 Rue Joliot-Curie, Gif–sur–Yvette 91192, France, Email: giorgio.valmorbida@l2s.centralesupelec.fr.

Research supported in part by ANR via grant HANDY, number ANR-18-CE40-0010.

the system under consideration and the analysis and design problems, together with some background material. Section III introduces our new sign-indefinite quadratic form and the arising Lyapunov functions, together with some new tools for establishing its positivity in the region of interest. The paper is completed by Section IV providing global analysis and design conditions, and Section V providing regional analysis and design conditions. Several examples taken from the literature are presented throughout the paper.

Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space, while $\mathbb{C}$ is the complex plane and $\Re(s)$ is the real part of number $s \in \mathbb{C}$. $\mathbb{S}^n_{\geq 0}$ (respectively $\mathbb{S}^n_{>0}$) is the sets of symmetric positive semi-definite (respectively, positive definite) matrices of dimension n. $\mathbb{D}^m_{\geq 0}$ (respectively $\mathbb{D}^m_{>0}$) is the set of diagonal positive semi-definite (respectively, positive definite) matrices of size m. Given any symmetric matrix A, $\lambda_m(A)$ and $\lambda_M(A)$ are, respectively, its minimum and maximum real eigenvalues. For any matrix A, $A^\top$ denotes its transpose and $\mathrm{He}\,(A) = A + A^\top$. For any square matrix A, $\mathrm{Tr}(A)$ denotes its trace. For q matrices A_i, $i = 1, ..., q$, $\mathrm{diag}(A_1, ..., A_q)$ denotes the block-diagonal matrix constructed from matrices A_i. I_n denotes the identity matrix of dimension n, whereas 0 stands for a matrix of zeros of appropriate dimensions. For any $y \in \mathbb{R}^m$, $|y|_\infty$ and $|y|$ stand respectively for its infinity and Euclidean norms.

II. Problem statement

A. Linear state feedback with deadzone loop

Consider the following plant

$$\dot{x} = Ax + B\mathrm{sat}(u) \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the input, respectively. A and B are constant matrices of appropriate dimensions and pair (A, B) is supposed to be stabilizable. Function $\mathrm{sat} : \mathbb{R}^m \to \mathbb{R}^m$ denotes the vector-valued symmetric decentralized unit saturation, whose components are defined for $i = 1, ..., m$ as $\mathrm{sat}_i(u_i) := \max\{-1, \min\{1, u_i\}\}$, with u_i being the i-th component of vector u. Model (1) well represents situations where the saturation limits are not unitary by simply rescaling the columns of B.

By defining the dead-zone function as $\mathrm{dz}(u) := u - \mathrm{sat}(u)$, system (1) reads:

$$\dot{x} = Ax + Bu - B\mathrm{dz}(u). \tag{2}$$

In this paper we are interested in state-feedback control laws defined by

$$u(x) = K_1 x + K_2 \mathrm{dz}(u(x)). \tag{3}$$

where $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$ are either known in the problem of stability analysis or to be designed in the synthesis problem.

The presence of the dead-zone at the right-hand side of (3) induces a nonlinear algebraic loop implicitly defining a piecewise affine feedback selection $x \mapsto u(x)$. By setting $K_2 = 0$ one retrieves the classical linear state-feedback case without algebraic loop [23]. If the algebraic loop is well-posed then one can guarantee the existence of a (piecewise affine) solution to (3). Necessary and sufficient well-posedness conditions can be for example derived from [12, Claim 2].

The following facts are straightforward consequences of [30, Prop. 1] and [12, Claim 1 and Remark 1], respectively.

Fact 1: If there exists $\Delta \in \mathbb{D}^m_{>0}$ such that $2\Delta - \Delta K_2 - K_2^\top \Delta > 0$, then the algebraic loop in (3) is well-posed.

Fact 2: If the algebraic loop in (3) is well-posed, then its unique solution is a globally Lipschitz piecewise affine function of x over a finite polytopic partition of $\mathbb{R}^n$.

B. Problem statement and sector conditions background

The closed-loop system resulting from the interconnection of (2) and (3) is given by

$$\begin{aligned} \dot{x} &= (A + BK_1)x + B(K_2 - I_m)\mathrm{dz}(u(x)) \\ u(x) &= K_1 x + K_2 \mathrm{dz}(u(x)). \end{aligned} \tag{4}$$

Due to the presence of the dead-zone, the closed-loop dynamics (4) is nonlinear and asymptotic stability of the origin can be ensured globally (that is for any initial condition $x(0) \in \mathbb{R}^n$) or only locally (that is, only for initial conditions in a neighborhood of the origin). Fundamental limitations of bounded stabilization of linear systems [20] imply that when A has exponentially unstable eigenvalues, the maximal set of initial conditions providing solutions that converge to zero (the so-called *basin of attraction*) is bounded. The exact characterization of the basin of attraction of the origin remains an open problem in general. Hence, a challenging problem consists in computing accurate estimates. Whenever we can provide a guarantee on the size of the basin of attraction (typically through an inner approximation), we talk about *regional* asymptotic stability (as opposed to merely *local* asymptotic stability, where the basin of attraction may be arbitrarily small). Feedback (4) provides an interesting context where (due to the properties of function dz) local asymptotic stability holds if and only if $A + BK_1$ is Hurwitz, but the characterization of a large estimate (namely an inner approximation) of the basin of attraction requires nontrivial derivations (see, e.g., [11], [23], [31]).

In this paper we use piecewise quadratic Lyapunov functions stemming from sign-indefinite quadratic forms with the goal in mind of addressing both the global and the regional exponential stability problems in the two cases where K_1 and K_2 are fixed (analysis) or to be designed (synthesis).

In the global case, we address the following problems.

Problem 1 (Global Analysis): Given gains K_1, K_2 in (4), determine whether the origin is globally exponentially stable.

Problem 2 (Global Synthesis): Compute gains K_1, K_2 in (4) ensuring global exponential stability of the origin.

We address Problems 1 and 2 by exploiting the following well-known global sector condition (see, for example, [14]).

Fact 3 (Global Sector Condition): The inequality

$$\mathrm{dz}(u)^\top T(u - \mathrm{dz}(u)) \geq 0 \tag{5}$$

holds for any $u \in \mathbb{R}^m$ and for any $T \in \mathbb{D}^m_{\geq 0}$.

In the regional case, we address the following problems.

Problem 3 (Regional Analysis): Given gains K_1, K_2 in (4), determine whether the origin is locally exponentially stable and provide an estimate of its basin of attraction.

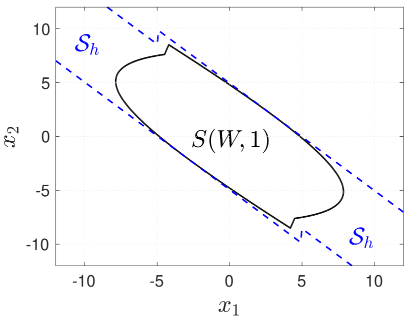

Fig. 1. Regional stability via inclusion of a sub-level set $S(V,1)$ of function V within set $\mathcal{S}_h$.

Problem 4 (Regional Synthesis): Compute gains K_1, K_2 in (4) ensuring local exponential stability of the origin, together with an estimate of the basin of attraction.

To study regional properties it is useful to generalize the global sector condition of Fact 3 in ways originally characterized in [9], [11]. As shown in Figure 1, regional asymptotic stability is assessed by focusing on an open sublevel set $S(W,1) := \{x \in \mathbb{R}^n : \ W(x) < 1\}$ of a Lyapunov function W restricted to the set

$$\mathcal{S}_h := \{x \in \mathbb{R}^n : |h(x)|_\infty \leq 1 \}, \tag{6}$$

where function $h : \mathbb{R}^n \to \mathbb{R}^m$ is typically vanishing at the origin. Most existing works [9], [11], [12], [23] use linear selections for h. In this paper we use the following implicit form, where H_1 and H_2 are arbitrary design parameters and function $x \mapsto u(x)$ is implicitly defined in (3) and characterized in Facts 1 and 2,

$$h(x) = H_1 x + H_2 \mathrm{dz}(u(x)). \tag{7}$$

With the above selections, the following result is a direct adaptation of [24, Lemma 1] (see also [13]).

Fact 4 (Regional Sector Condition): The inequality

$$\mathrm{dz}(u)^\top T(u - \mathrm{dz}(u) - h(x)) \geq 0 \tag{8}$$

holds for all $x \in \mathcal{S}_h$, all $u \in \mathbb{R}^m$ and all $T \in \mathbb{D}^m_{\geq 0}$.

We complete our background material by introducing a few equalities inspired by the bounds initially proposed in [4], exploiting the properties of the directional derivative $x \mapsto \dot{u}(x)$ of function $x \mapsto u(x)$ in (3) and the directional derivative $x \mapsto \dot{\mathrm{dz}}(u(x))$ of function $x \mapsto \mathrm{dz}(u(x))$ along the solutions of (4).

Fact 5 (Derivative of the dead-zone): The identities

$$\dot{\mathrm{dz}}(u(x))^\top T(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) \equiv 0, \tag{9}$$

$$\mathrm{dz}(u(x))^\top T(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) \equiv 0, \tag{10}$$

hold for almost all $x \in \mathbb{R}^n$, and for all $T \in \mathbb{D}^m$.

Proof. We prove the relations for all x such that $|u_i(x)| \neq 1$, for all $i = 1, \ldots, m$. For those values of x the proof follows by noting that each component of u satisfies

$$\dot{\mathrm{dz}}_i(u_i(x)) = \begin{cases} \dot{u}_i(x) & \text{if} \quad |\, u_i(x)\, | > 1 \\ 0 & \text{if} \quad |\, u_i(x)\, | < 1, \end{cases}$$

and

$$\dot{u}_i(x) - \dot{\mathrm{dz}}_i(u_i(x)) = \begin{cases} 0 & \text{if} \quad |\, u_i(x)\, | > 1 \\ \dot{u}_i(x) & \text{if} \quad |\, u_i(x)\, | < 1. \end{cases}$$

■

Remark 1: The control solution (4), requires the knowledge of the physical saturation output $\mathrm{sat_{ph}}$ present in the actuators, which might not be accessible in industrial applications. In this setting, a workaround is the introduction of an artificial (conservative) saturation function sat at the controller output, so that the actuation effort reaching the plant is $\mathrm{sat_{ph}}(\mathrm{sat}(u)) = \mathrm{sat}(u)$, where the last equality holds as long as the artificial saturation levels of sat are smaller than or equal to the physical saturation levels of $\mathrm{sat_{ph}}$. In this case, the artificial saturation provides an effective way to obtain the closed loop (4) even when the output of $\mathrm{sat_{ph}}$ is not accessible.

Besides this fact, mismatches between the saturation model sat and the actual saturation nonlinearity $\mathrm{sat_{ph}}$ may also emerge. In this case we emphasize that the Lyapunov-based results presented in this paper enjoy intrinsic robustness of asymptotic stability (see, e.g., [8, Chapter 7]), so that a certain level of nonzero mismatch is guaranteed to preserve the asymptotic stability properties in a semiglobal practical sense (see [8, Def. 7.18]). Due to this *robustness-in-the-small* guarantee, we expect some kind of graceful performance degradation when the saturation modeling is imprecise (e.g., the slope of the saturation may be not unitary in certain ranges) and in cases where the physical saturation levels might be smaller than the artificial ones (indeed in those cases $\mathrm{sat_{ph}}(\mathrm{sat}(u)) \neq \mathrm{sat}(u)$). Similarly, due to the stated robustness-in-the-small, it is straightforward to extend our proofs to the case where the state measurement used in our feedback is replaced by an estimated state provided by a sufficiently fast Luenberger observer. Providing quantitative bounds on the tolerated mismatch that does not destroy stability, namely a *robustness-in-the-large* guarantee, is possible by embedding the nonlinearities in suitable inflated sector conditions, but is beyond the scope of this work and provides an interesting direction of future investigation. ⋆

III. Extended sign indefinite Lyapunov function

A. Proposed structure

The four main results of this paper, providing solutions to Problems 1-4, rely on a generalization of the piecewise smooth Lyapunov function originally proposed in [4], corresponding to

$$\begin{aligned} V(x) &= \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix} \\ &= \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}^\top P \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}, \end{aligned} \tag{11}$$

where $\mathrm{dz}(u(x))$ is the solution to (3). As in [4], we do not need to explicitly compute the mapping $x \mapsto u(x)$ because the Lyapunov construction follows from the implicit equation (3).

Due the fact that there are points where V in (11) is not differentiable, it is useful to emphasize that the Lipschitz nature of V allows to ignore those areas when characterizing local or global exponential stability of the origin. This fact is well formalized in the next lemma, which is a direct consequence of the results in [3] (see also [25]).

Lemma 1: Consider dynamics (4) and assume that there exists a locally Lipschitz Lyapunov function $x \mapsto W(x)$, positive scalars β_1, β_2 and β_3 and an open sub-level set $S(W,r) := \{x \in \mathbb{R}^n : W(x) < r\}$ of W, for some $r > 0$ satisfying

$$\beta_1 |x|^2 \leq W(x) \leq \beta_2 |x|^2, \ \forall x \in S(W,r) \tag{12}$$

$$\dot{W}(x) := \langle \nabla W(x), Ax + B\text{sat}(u(x)) \rangle \leq -\beta_3 |x|^2, \tag{13}$$

$$\text{for almost all } x \in S(W,r),$$

then the origin is locally exponentially stable for (4) with basin of attraction containing $S(W,r)$. Moreover, if (12), (13) hold for any $r > 0$ with the same scalars β_i, $i = 1,2,3$, then the origin is globally exponentially stable for (4).

The following lemma uses Fact 3 to establish sufficient conditions for positive definiteness and radial unboundedness of V in (11), while not imposing positive definiteness of matrix P.

Lemma 2: Given V in (11) with u defined in (3), if the algebraic loop in (3) is well-posed and there exist matrices $P_{11} \in \mathbb{S}^n_{>0}$, $P_{12} \in \mathbb{R}^{n\times m}$, $P_{22} \in \mathbb{S}^m$ and $T_0 \in \mathbb{D}^m_{\geq 0}$ such that

$$\Phi_0 = \begin{bmatrix} P_{11} & P_{12} - K_1^\top T_0 \\ P_{12}^\top - T_0 K_1 & P_{22} - T_0 K_2 - K_2^\top T_0 + 2T_0 \end{bmatrix} > 0, \tag{14}$$

then there exist positive scalars β_1 and β_2 satisfying (12) for any $r > 0$.

Proof. The global sector condition (5) with $T = T_0$ implies

$$\begin{aligned} V(x) &\geq V(x) - 2\text{dz}(u(x))^\top T_0 (u(x) - \text{dz}(u(x))) \\ &= \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}^\top \Phi_0 \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix} \\ &\geq \lambda_m(\Phi_0) \left| \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix} \right|^2 \geq \lambda_m(\Phi_0) |x|^2, \end{aligned}$$

which provides the lower bound in (12) with $\beta_1 = \lambda_m(\Phi_0)$.

To study the upper bound in (12) from well-posedness of the algebraic loop (3) and Fact 2, the globally Lipschitz property of u, together with $u(0) = 0$, ensures the existence of $L_u > 0$ such that $|u(x)| \leq L_u |x|$, therefore using $|\text{dz}(u(x))| \leq |u(x)| \leq L_u |x|$,

$$\begin{aligned} V(x) &\leq \lambda_M(P_{11})|x|^2 + 2|P_{12}||x||\text{dz}(u(x))| + |\text{dz}(u(x))|^2 |P_{22}| \\ &\leq (\lambda_M(P_{11}) + 2|P_{12}|L_u + |P_{22}|L_u^2)|x|^2 =: \beta_2 |x|^2, \end{aligned}$$

which completes the proof. ∎

B. Advantages from using a sign indefinite P

A novelty of this paper is that we do not insist on the fact that matrix P in (11) be positive definite, which allows representing a broader class of Lyapunov functions as compared to previous works such as [4]. As an example, consider the following two Popov-like functions, the first one focusing on the deadzone and the second one focusing (more classically, as in [14, Ex. 7.4] and [29, §2.5 and 2.1.6]) on the saturation:

$$V_{\text{Dai}}(x) := x^\top P_0 x + \sum_{i=1}^m 2\lambda_i \int_0^{u_i(x)} \text{dz}_i(s)\text{d}s \tag{15}$$

$$V_{\text{Khalil}}(x) := x^\top P_0 x + \sum_{i=1}^m 2\lambda_i \int_0^{u_i(x)} \text{sat}_i(s)\text{d}s. \tag{16}$$

As observed in [4], using the identity $2\int_0^{u_i(x)} \text{dz}_i(s)ds = \text{dz}_i^2(u_i(x))$, the first function V_{Dai} in (15) can be obtained as a particular case of V in (11), by setting $P_{11} = P_0$, $P_{12} = 0$, $P_{22} = \text{diag}(\lambda_1, \ldots, \lambda_m) = \Lambda$ and $\lambda_i > 0$, which provides a positive definite P.

However, a positive definite P does not provide enough degrees of freedom for representing the second function V_{Khalil} in (16), which can be instead represented by a sign-indefinite P, as follows, with the last line inspired by (11),

$$\begin{aligned} V_{\text{Khalil}}(x) &= x^\top P_0 x + \sum_{i=1}^m 2\lambda_i \int_0^{u_i(x)} s - \text{dz}_i(s)\text{d}s \\ &= x^\top P_0 x + \sum_{i=1}^m \lambda_i \left(u_i(x)^2 - \text{dz}_i^2(u_i(x)) \right), \\ &= \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \begin{bmatrix} x \\ \text{dz}(u(x)) \end{bmatrix}. \end{aligned} \tag{17}$$

Using the expression of $u(x)$ in (3), we obtain the following selection of P in (17),

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} = \begin{bmatrix} P_0 & 0 \\ 0 & -\Lambda \end{bmatrix} + \begin{bmatrix} K_1^\top \\ K_2^\top \end{bmatrix} \Lambda \begin{bmatrix} K_1 & K_2 \end{bmatrix}, \tag{18}$$

where P_{22} is often nonpositive (e.g., when $K_2 = 0$) but yet satisfies the positivity conditions of Lemma 2 under mild conditions on matrix P_0 (these conditions are satisfied whenever $P_0 > 0$ but also allow for more general cases).

Proposition 1: If matrix $P_0 + K_1^\top \Lambda K_1$ is positive definite, then condition (14) is satisfied for any $T_0 > \frac{\Lambda}{2}$, namely under a well-posedness assumption, V_{Khalil} in (16) satisfies (12) for all $r > 0$.

Proof. With the structure of V in (11), using the derivations in (17), function V_{Khalil} in (16) corresponds to the selection of P in (18). With this selection of P, matrix Φ_0 in (14) reads

$$\begin{aligned} \Phi_0 = &\begin{bmatrix} P_0 & 0 \\ 0 & 2T_0 - \Lambda \end{bmatrix} + \begin{bmatrix} K_1^\top \\ K_2^\top \end{bmatrix} \Lambda \begin{bmatrix} K_1 & K_2 \end{bmatrix} \\ &+ \text{He}\left(\begin{bmatrix} 0 \\ -I_m \end{bmatrix} T_0 \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right). \end{aligned} \tag{19}$$

Using the elimination lemma (see, e.g., [18]), defining bases N_1 and N_2 of the null-spaces of $\begin{bmatrix} 0 & -I_m \end{bmatrix}$ and $\begin{bmatrix} K_1 & K_2 \end{bmatrix}$, respectively, it follows that $\Phi_0 > 0$ is equivalent to:

$$\begin{aligned} &N_1^\top \left(\begin{bmatrix} P_0 & 0 \\ 0 & 2T_0 - \Lambda \end{bmatrix} + \begin{bmatrix} K_1^\top \\ K_2^\top \end{bmatrix} \Lambda \begin{bmatrix} K_1 & K_2 \end{bmatrix} \right) N_1 \\ &\quad = P_0 + K_1^\top \Lambda K_1 > 0, \\ &N_2^\top \begin{bmatrix} P_0 & 0 \\ 0 & 2T_0 - \Lambda \end{bmatrix} N_2 > 0, \end{aligned}$$

which are both satisfied under the stated assumptions. ∎

C. Set inclusion for regional results

As depicted in Figure 1, regional asymptotic stability should rely on a set inclusion property ensuring that the sublevel set

$$S(W,1) := \{x \in \mathbb{R}^n : W(x) < 1 \} \tag{20}$$

of some suitable function W, based on function V in (11), is contained in the set $\mathcal{S}_h$ of (6) (associated with function h in (7)) so that the inequality of Fact 4 holds in $S(W,1)$. Not insisting on positivity of function V outside $\mathcal{S}_h$ can lead to extra degrees of freedom in the construction proposed in Section V (see for example the surface represented in Figure 4 and the corresponding discussion in Example 3). Therefore we propose selecting W in (20) as

$$W(x) := \begin{cases} \min\{V(x), 1\}, & \text{if } x \in \mathcal{S}_h \\ 1, & \text{otherwise.} \end{cases} \tag{21}$$

Note that (21) does not lead, in general, to a continuous W, but it guarantees the identity

$$S(W,1) = S(V,1) \cap \mathcal{S}_h, \tag{22}$$

which is useful for the Lyapunov analysis of Section V. [1] The following lemma establishes sufficient conditions for (Lipschitz) continuity of W and provides a tool, based on Fact 4, to guarantee bounds (12) with $r = 1$.

Lemma 3: If the algebraic loop in (3) is well-posed and there exist matrices $P_{11} \in \mathbb{S}^n_{>0}$, $P_{12} \in \mathbb{R}^{n\times m}$, $P_{22} \in \mathbb{S}^m$, $H_1 \in \mathbb{R}^{m\times n}$, $H_2 \in \mathbb{R}^{m\times m}$ and $T \in \mathbb{D}^m_{>0}$ such that for all $i = 1, \dots, m$

$$\Phi_1 = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12}^\top & P_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} 0 & (H_1^\top - K_1^\top)T & H_{1i}^\top \\ 0 & TH_2 - T(K_2 - I_m) & H_{2i}^\top \\ 0 & 0 & 0 \end{bmatrix}\right) > 0 \tag{23}$$

then defining V and W as in (11), (21), the following properties hold:

(i) function W in (21) is Lipschitz continuous;
(ii) the set inclusion $S(W,1) \subseteq \mathcal{S}_h$ holds, with $S(W,1)$ and $\mathcal{S}_h$ defined in (20) and (6), respectively;
(iii) function W satisfies (12) for $r = 1$ and for suitable scalars β_1 and β_2.

Proof. As a first step of the proof, we apply a Schur complement to (23), ensuring that, for all $i = 1, \dots, m$,

$$\bar{\Phi}_1 := P + \mathrm{He}\left(\begin{bmatrix} 0 & (H_1^\top - K_1^\top)T \\ 0 & T(H_2 - K_2) + T \end{bmatrix}\right) - \begin{bmatrix} H_{1i}^\top \\ H_{2i}^\top \end{bmatrix}\begin{bmatrix} H_{1i} & H_{2i} \end{bmatrix} > 0. \tag{24}$$

Then, based on (24), we can prove the following fact:

$$h_i(x)^2 \le V(x), \quad \forall x \in \mathcal{S}_h, \tag{25}$$

indeed, for all $x \in \mathcal{S}_h$ we can use the sector condition of Fact 4 ensuring that

$$\begin{aligned} V(x) - h_i(x)^2 &\ge V(x) - h_i(x)^2 - \mathrm{dz}(u)^\top T(u - \mathrm{dz}(u) - h(x)) \\ &= \left[\begin{smallmatrix} x \\ \mathrm{dz}(u) \end{smallmatrix}\right]^\top \bar{\Phi}_1 \left[\begin{smallmatrix} x \\ \mathrm{dz}(u) \end{smallmatrix}\right] \ge 0. \end{aligned} \tag{26}$$

It then follows from (25) that at the boundary of $\mathcal{S}_h$, where $h_i(x) = 1$ for some $i \in \{1, \dots, m\}$, it must hold that $V(x) \ge 1$, which implies that $W(x)$ is continuous at the patching surfaces in (21), which proves item (i).

Item (ii) follows immediately from (22), ensuring that

$$x \in S(W,1) \Rightarrow x \in \mathcal{S}_h. \tag{27}$$

As for item (iii), to show (12) for all $x \in S(W,1)$, first notice that from definition (21) we have $W(x) = V(x)$ for all $x \in S(W,1)$. Then, we may focus on function V, for establishing (12). In particular, for any $x \in S(W,1)$ we may use Fact 4 to conclude once again from (26) that

$$W(x) = V(x) \ge \lambda_m(\bar{\Phi}_1)|x|^2, \quad \forall x \in S(W,1).$$

As for the upper bound in (12), we may proceed as in Lemma 2 using well-posedness of the algebraic loop (3). ■

IV. Global results

A. Global analysis

Using function V in (11), this section proposes conditions to solve Problem 1, as stated in the next theorem.

Theorem 1: Given K_1 and K_2, if there exist matrices $P_{11} \in \mathbb{S}^n_{>0}$, $P_{12} \in \mathbb{R}^{n\times m}$, $P_{22} \in \mathbb{S}^m$, $T_i \in \mathbb{D}^m_{\ge 0}$ $i \in \{0, 1\}$, $T_2 \in \mathbb{D}^m_{>0}$, $T_3 \in \mathbb{D}^m$ such that (14) and (28) (given at the top of the next page) hold, then the origin of (4) is globally exponentially stable.

Proof. The proof uses Lemma 1 by showing both (12) and (13) for any $x \in \mathbb{R}^n$.

To show (12), let us first observe that the (3,3) term of (28) reads $2T_2 - T_2K_2 - K_2^\top T_2 > 0$, where $T_2 > 0$ by assumption. Therefore Fact 1 implies well-posedness of the algebraic loop (3) and, from Lemma 2, hypothesis (14) implies (12) globally (for any $r > 0$).

To show (13), first note that

$$\dot{V}(x) = 2\begin{bmatrix} x \\ \mathrm{dz}(u(x)) \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\mathrm{dz}}(u(x)) \end{bmatrix}. \tag{29}$$

Using Facts 3 and 5, we may bound $\dot{V}(x)$ for almost all $x \in \mathbb{R}^n$ as follows:

$$\begin{aligned} \dot{V}(x) \le \dot{V}(x) &+ 2\mathrm{dz}(u(x))^\top T_1(u - \mathrm{dz}(u(x))) \\ &+ 2\dot{\mathrm{dz}}(u(x))^\top T_2(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) \\ &+ 2\mathrm{dz}(u(x))^\top T_3(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) = -\eta^\top \mathcal{G}\eta, \end{aligned} \tag{30}$$

[1] Condition (22) is especially useful in cases where $V(x)$ becomes smaller than 1 outside $\mathcal{S}_h$ (so that $S(V,1)$ may possibly be an unbounded disconnected set), as in the example shown in Figure 4.

$$\Psi_1 := \mathrm{He}\left(-\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A+BK_1 & B(K_2-I_m) & 0 \\ 0 & 0 & I_m \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ T_1K_1+T_3K_1(A+BK_1) & T_1(K_2-I_m)+T_3K_1B(K_2-I_m) & T_3(K_2-I_m) \\ T_2K_1(A+BK_1) & T_2K_1B(K_2-I_m) & T_2(K_2-I_m) \end{bmatrix} \right) > 0 \tag{28}$$

where $\eta := \begin{bmatrix} x \\ \mathrm{dz}(u(x)) \\ \dot{\mathrm{dz}}(u(x)) \end{bmatrix}$ is an extended state, and matrix $\mathcal{G}$ can be easily evaluated as follows:

$$\begin{aligned} \mathcal{G} = \mathrm{He}&\left(-\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A+BK_1 & B(K_2-I_m) & 0 \\ 0 & 0 & I_m \end{bmatrix} \right) \\ -\mathrm{He}&\left(\begin{bmatrix} 0 \\ T_1 \\ 0 \end{bmatrix} \begin{bmatrix} K_1 & K_2-I_m & 0 \end{bmatrix} \right) \\ -\mathrm{He}&\left(\begin{bmatrix} 0 \\ 0 \\ T_2 \end{bmatrix} \begin{bmatrix} K_1(A+BK_1) & K_1B(K_2-I_m) & K_2-I_m \end{bmatrix} \right) \\ -\mathrm{He}&\left(\begin{bmatrix} 0 \\ T_3 \\ 0 \end{bmatrix} \begin{bmatrix} K_1(A+BK_1) & K_1B(K_2-I_m) & K_2-I_m \end{bmatrix} \right). \end{aligned}$$

It is straightforward to check that the above expression corresponds to the matrix in (28), which then implies $\mathcal{G} > 0$, and by (30) also implies (13) for almost all $x \in \mathbb{R}^n$ and completes the proof. ■

For the analysis result of Theorem 1, since K_1 and K_2 are a priori given, the inequalities (14) and (28) are linear in P_{11}, P_{12}, P_{22} and T_i, $i = 0, \dots, 3$, therefore they can be efficiently solved by a linear matrix inequality (LMI) solver (as a feasibility Semi-Definite Program).

Example 1: We illustrate here the reduced conservativeness of the conditions of Theorem 1, as compared to the use of quadratic Lyapunov functions. For global exponential stabilisation of the dynamics presented in [31, Example 4.3.1], one may attempt using static anti-windup gains and quadratic Lyapunov certificates. Unfortunately, as discussed in [31], these certificates fail to satisfy the necessary conditions for quadratic stability analysis and synthesis. It turns out that even with the anti-windup gains set to zero (mere stability analysis), the LMI conditions of our Theorem 1 ensure GES of the origin.

More specifically, the dynamics in [31, Example 4.3.1] before anti-windup corresponds to closed loop (4) with selections

$$A = \begin{bmatrix} -0.05 & 1 \\ -10 & -0.5 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad K_1 = \begin{bmatrix} 9.9 & 0.495 \end{bmatrix}; \quad K_2 = 0.$$

By solving LMIs (14) and (28) as per Theorem 1 one obtains matrices P and T_i, $i = 0,1,2,3$ certifying global exponential stability. In particular, for this example, Lyapunov function V is characterized by matrix $P = \begin{bmatrix} 1.0052 & -0.1815 & 0.0411 \\ -0.1815 & 8.4599 & -0.1985 \\ 0.0411 & -0.1985 & 0.7077 \end{bmatrix}$, with the eigenvalues $\{ 0.6982 \quad 1.0051 \quad 8.4694 \}$, which reveals that the degree of freedom of not constraining $P_{22} > 0$ is not necessary for establishing GAS of the origin, in this example.

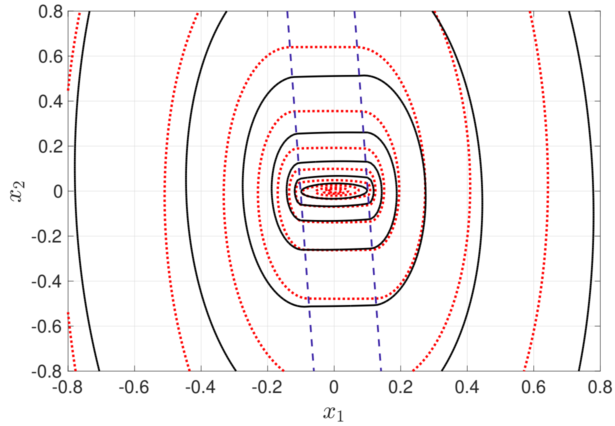

Fig. 2. Level sets of the piecewise quadratic Lyapunov function (11) (black solid). Trajectories exponentially converging to the origin (red dotted). The dashed blue lines correspond to the boundary of the set containing the origin where $\mathrm{dz}(u(x)) = 0$.

Figure 2 depicts in red some trajectories of the closed-loop system (4). Note that the level sets of the Lyapunov function (in black) are stretched horizontally in the proximity of the origin and vertically away from the origin. This peculiar shape of the trajectories causes the infeasibility of quadratic Lyapunov conditions, well overcome by Theorem 1. ◦

Remark 2: The conditions of Theorem 1 readily extend to the case where dynamics (4) is generalized to

$$\begin{aligned} \dot{x} &= A_{\mathrm{cl}}x + B_{\mathrm{cl}}\mathrm{dz}(u(x)) \\ u(x) &= K_1x + K_2\mathrm{dz}(u(x)). \end{aligned} \tag{31}$$

Indeed, it suffices to replace all the occurrences of $A + BK_1$ by A_{cl}, and those of $B(K_2 - I_m)$ by B_{cl} in conditions (28) and the proof of the theorem remains unchanged.

Expression (31) arises when analyzing the output feedback interconnection of a linear plant with an n_c-order dynamic controller possibly involving an anti-windup gain [23, ch. 8]:

$$\begin{aligned} \dot{x}_p &= A_px_p + B_p\mathrm{sat}(y_c), \quad y_p = C_px_p \\ \dot{x}_c &= A_cx_c + B_cy_p + E_c\mathrm{dz}(y_c) \\ y_c &= C_cx_c + D_cy_p + F_c\mathrm{dz}(y_c), \end{aligned}$$

which is represented by (31) with

$$\begin{aligned} \left[A_{\mathrm{cl}} \mid B_{\mathrm{cl}} \right] &= \left[\begin{array}{cc|c} A_p + B_pD_cC_p & B_pC_p & -B_p(F_c - I_m) \\ B_cC_p & A_c & -E_c \end{array} \right] \\ \left[K_1 \mid K_2 \right] &= \left[\begin{array}{cc|c} D_cC_p & C_c & F_c \end{array} \right]. \end{aligned}$$

⋆

B. Global synthesis

Using function V in (11), we propose here conditions solving Problem 2, that is designing gains K_1 and K_2 ensuring global exponential stability of the origin.

Theorem 2: If there exist matrices $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $S \in \mathbb{D}^m_{>0}$, $M \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{m\times n}$ and $Y_2 \in \mathbb{R}^{m\times m}$ such that

$$\Phi_2 := \begin{bmatrix} Q_{11} & Q_{12} - Y_1^\top \\ Q_{12}^\top - Y_1 & Q_{22} - Y_2 - Y_2^\top + 2S \end{bmatrix} > 0 \tag{32}$$

$$\Psi_2 := \mathrm{He}\left(-\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \right. \tag{33}$$
$$\left. - \begin{bmatrix} AM + BY_1 & BY_2 - BS & -M & 0 \\ Y_1 & Y_2 - S & Y_1 & Y_2 - S \\ AM + BY_1 & BY_2 - BS & -M & 0 \\ 0 & 0 & Y_1 & Y_2 - S \end{bmatrix} \right) > 0,$$

then M is nonsingular and the origin of system (4) with

$$K_1 = Y_1 M^{-1}, \qquad K_2 = Y_2 S^{-1}, \tag{34}$$

is globally exponentially stable.

Proof. First note that M is nonsingular, indeed $M + M^\top > 0$ is implied by the (3,3) entry of (33).

The rest of the proof mimics the one of Theorem 1 by using Lemma 1. In particular, to show (12), consider the change of variables $K_1 M = Y_1$ and $K_2 S = Y_2$ and note that inequality (32) coincides with (14) pre- and post-multiplied by $\mathrm{diag}(M^\top, S)$ and $\mathrm{diag}(M, S)$, respectively. Moreover, the (4,4) term in (33) with $Y_2 = K_2 S$ reads $2S - K_2 S - S K_2^\top > 0$, which implies well-posedness of (3) from Fact 1. Then, (12) is guaranteed by Lemma 2.

The rest of the proof focuses on showing (13) by exploiting Fact 5 in addition to considering an enlarged space (comprising x and $\dot{x}$ as in [5], [18]) so that the Lyapunov certificate is suitably separated from the dynamics matrices.

More specifically, consider the extended vector $\eta := \begin{bmatrix} x^\top & \mathrm{dz}(u(x))^\top & \dot{x}^\top & \dot{\mathrm{dz}}(u(x))^\top \end{bmatrix}^\top$. From the dynamic equation (4) we get, for any matrix N,

$$\eta^\top N(-\dot{x} + (A + BK_1)x + B(K_2 - I_m)\mathrm{dz}(u(x))) = 0, \tag{35}$$

which may be evaluated with $N = \begin{bmatrix} M^{-\top} & 0 & M^{-\top} & 0 \end{bmatrix}^\top$, where $M \in \mathbb{R}^{n\times n}$. To establish bound (13) we may write the following bounds originating from Facts 3 and 5 evaluated with $T = S^{-1}$, and from (35):

$$\begin{aligned} \dot{V}(x) \le\, & \dot{V}(x) + 2\mathrm{dz}(u(x))^\top S^{-1}(u - \mathrm{dz}(u(x))) \\ & + 2\dot{\mathrm{dz}}(u(x))^\top S^{-1}(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) \\ & + 2\mathrm{dz}(u(x))^\top S^{-1}(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) \\ & + 2\eta^\top N(-\dot{x} + (A + BK_1)x + B(K_2 - I_m)\mathrm{dz}(u(x))) \\ =\, & -\eta^T \mathcal{G}_d \eta, \qquad \text{for almost all } x \in \mathbb{R}^n. \end{aligned} \tag{36}$$

Thus exponential stability can be established by proving $\mathcal{G}_d > 0$, which would prove a negative upper bound on $\dot{V}(x)$, quadratic in x (and other variables too). To show that $\mathcal{G}_d > 0$, let us equivalently study the sign of matrix

$$\mathcal{G}_1 := \mathrm{diag}(M^\top, S, M^\top, S)\mathcal{G}_d \mathrm{diag}(M, S, M, S),$$

and consider the change of variables $Q_{11} = M^\top P_{11} M$, $Q_{12} = M^\top P_{12} S$ and $Q_{22} = S P_{22} S$, $K_1 M = Y_1$ and $K_2 S = Y_2$, so that the expression $\mathcal{G}_1$ can be written as:

$$\begin{aligned} \mathcal{G}_1 =\, & \mathrm{He}\left(-\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \right) \\ & - \mathrm{He}\left(\begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} Y_1 & Y_2 - S & 0 & 0 \end{bmatrix} \right) \\ & - \mathrm{He}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ I_m \end{bmatrix} \begin{bmatrix} 0 & 0 & Y_1 & Y_2 - S \end{bmatrix} \right) \\ & - \mathrm{He}\left(\begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & Y_1 & Y_2 - S \end{bmatrix} \right) \\ & - \mathrm{He}\left(\begin{bmatrix} I_n \\ 0 \\ I_n \\ 0 \end{bmatrix} \begin{bmatrix} AM + BY_1 & BY_2 - BS & -M & 0 \end{bmatrix} \right), \end{aligned}$$

whose five addends clearly sum up to (33). Then, from (33) we have $\mathcal{G}_1 > 0$ and therefore $\mathcal{G}_d > 0$, which implies, via (36), that bound (13) holds for almost all $x \in \mathbb{R}^n$, thus completing the proof. ■

Remark 3: While the analysis and synthesis results of Theorems 1 and 2 both rely on the technical tools introduced in Sections II-B and III, the extra unknowns appearing in the synthesis conditions require additional transformations and an extended space corresponding to $\eta = (x, \mathrm{dz}(u(x)), \dot{x}, \dot{\mathrm{dz}}(u(x)))$, while $\dot{x}$ does not appear in the representation of the analysis conditions. In addition to this fact, we expect the synthesis conditions to show increased conservativeness, because the coordinate transformation in the proof of Theorem 2 requires imposing $T_1 = T_2 = T_3 = S^{-1}$. Due to this reason, after the gains have been selected through the synthesis conditions of Theorem 2, we expect less conservative estimates to emerge from the solution of the analysis conditions of Theorem 1. ⋆

While the conditions of Theorem 2 ensure global exponential stability, the convergence rate of the resulting feedback law could be arbitrarily small. The next proposition provides a means to ensure that the eigenvalues of matrix $A + BK_1$ (namely the linear dynamics governing the tail of the converging responses) are placed in the half plane $\{s \in \mathbb{C} : \Re(s) \le -\alpha\}$ for some desired convergence rate $\alpha > 0$. We emphasize that the convergence rate $\alpha > 0$, as stated in the next proposition, is a local property (related to $A + BK_1$) and therefore is ensured only in a neighborhood of the origin where the saturation is not active.

Proposition 2: Given a desired convergence rate $\alpha > 0$, if there exist matrices $R \in \mathbb{S}^n_{>0}$, $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $S \in \mathbb{D}^m_{>0}$, $M \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{m\times n}$ and $Y_2 \in \mathbb{R}^{m\times m}$ satisfying (32), (33) and

$$R = R^\top > 0, \ \mathrm{He}\begin{bmatrix} \alpha M + AM + BY_1 & -M \\ \alpha M + AM + BY_1 + R & -M \end{bmatrix} < 0, \tag{37}$$

then matrix M is nonsingular, the origin of system (4) with selections (34) is globally exponentially stable, and the eigenvalues of $A + BK_1$ have real part smaller than $-\alpha$.

Proof. Global exponential stability has been already proven in Theorem 2. To show that the eigenvalues of $A + BK_1$ have real part smaller than $-\alpha$, it is enough to show that matrix $\bar{A} := A + BK_1 + \alpha I$ is Hurwitz. To this end, condition (37) pre-multiplied by $\mathrm{diag}(M^{-\top}, M^{-\top})$ and post-multiplied by $\mathrm{diag}(M^{-1}, M^{-1})$ reads

$$\begin{bmatrix} 0 & \bar{R} \\ \bar{R} & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} M^{-\top} \\ M^{-\top} \end{bmatrix} \begin{bmatrix} \bar{A} & -I_n \end{bmatrix} \right) < 0,$$

with the Lyapunov certificate $\bar{R} = M^{-\top} R M^{-1} > 0$. Pre- and post-multiplying the above inequality by $\left[\begin{smallmatrix} z \\ \bar{A}z \end{smallmatrix} \right]^\top$ and its transpose, we obtain

$$2z^\top \bar{R}\bar{A}z = 2z^\top \bar{R}(A + BK_1 + \alpha I)z < 0, \ \forall z \neq 0,$$

which completes the proof. ∎

V. Regional stability

In this section, we address Problems 3 and 4 following the idea of the set inclusion in Figure 1 and exploiting the result in Lemma 3. We first present analysis results and then synthesis results, paralleling the previous section.

A. Regional analysis

Using Lemma 3, the next theorem proposes conditions ensuring local exponential stability of the origin for system (4), thus providing a solution to Problem 3.

Theorem 3: Given any scalar $\tau > 0$, if there exist matrices $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $T_1 = \mathrm{diag}\{t_1, \ldots, t_m\}, T_2 \in \mathbb{D}^m_{>0}$, $T_3 \in \mathbb{D}^m$, $Z_1 \in \mathbb{R}^{m\times n}$, $Z_2 \in \mathbb{R}^{m\times m}$ such that

$$T_1 > \tau I_m, \tag{38}$$

$$\Phi_3 := \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12}^\top & P_{22} & 0 \\ 0 & 0 & \tau t_i \end{bmatrix} + \mathrm{He} \begin{bmatrix} 0 & Z_1^\top - K_1^\top T_1 & Z_{1i}^\top \\ 0 & Z_2 - T_1(K_2 - I_m) & Z_{2i}^\top \\ 0 & 0 & 0 \end{bmatrix} > 0, \tag{39}$$

for all $i = 1, \ldots, m$, and condition (40) (given at the top of the next page) holds, then the origin is locally exponentially stable for closed loop (4) and its basin of attraction contains set $S(W, 1)$ defined in (20), with functions V and W in (11), (21) characterized by $P = \left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{smallmatrix} \right]$.

Proof. The proof relies on Lemma 1 by showing the two bounds (12) and (13) with $r = 1$.

To show (12), let us first observe that the (3,3) term of (40) reads $2T_2 - T_2K_2 - K_2^\top T_2 > 0$, where $T_2 > 0$ by assumption, so that Fact 1 implies well-posedness of the algebraic loop (3). Moreover, defining $H_1 = T_1^{-1}Z_1$ and $H_2 = T_1^{-1}Z_2$ and combining (38) with (39), we get

$$\bar{\Phi}_3 := \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12}^\top & P_{22} & 0 \\ 0 & 0 & t_i^2 \end{bmatrix} + \mathrm{He} \begin{bmatrix} 0 & Z_1^\top - K_1^\top T_1 & Z_{1i}^\top \\ 0 & Z_2 - T_1K_2 + T_1 & Z_{2i}^\top \\ 0 & 0 & 0 \end{bmatrix} > \Phi_3 > 0.$$

Left and right multiplying $\bar{\Phi}_3 > 0$ by $\mathrm{diag}(I_n, I_m, t_i^{-1})$, we obtain (23) so that all the assumptions of Lemma 3 hold. Therefore, from item (i), function W in (21) is Lipschitz continuous and coincides with V (of (11) in the open set $S(W, 1)$). Moreover, from item (iii), bound (12) holds.

To show (13), first consider item (ii) of Lemma 3 ensuring that for all $x \in S(W, 1)$ we have $x \in \mathcal{S}_h$ and we may use the local condition (8) of Fact 4. Then proceeding as in the proof of Theorem 1 but using Fact 4 in place of Fact 3 (because we focus on local results), using expressions (29), we get for almost all $x \in \mathbb{R}^n$

$$\begin{aligned} \dot{W}(x) \leq \dot{W}(x) &+ 2\mathrm{dz}(u(x))^\top T_1(u - \mathrm{dz}(u(x)) - h(x)) \\ &+ 2\dot{\mathrm{dz}}(u(x))^\top T_2(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) \\ &+ 2\mathrm{dz}(u(x))^\top T_3(\dot{u}(x) - \dot{\mathrm{dz}}(u(x))) = -\eta^\top \mathcal{G}_l \eta, \end{aligned} \tag{41}$$

where $\eta := \left[\begin{smallmatrix} x \\ \mathrm{dz}(u(x)) \\ \dot{\mathrm{dz}}(u(x)) \end{smallmatrix} \right]$ is an extended state, and matrix $\mathcal{G}_l$ can be easily evaluated, using $Z_1 = T_1H_1$ and $Z_2 = T_1H_2$, as follows:

$$\begin{aligned} \mathcal{G}_l = \mathrm{He}&\left(-\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A + BK_1 & B(K_2 - I_m) & 0 \\ 0 & 0 & I_m \end{bmatrix} \right) \\ -\mathrm{He}&\left(\begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} \begin{bmatrix} T_1K_1 - Z_1 & T_1(K_2 - I_m) - Z_2 & 0 \end{bmatrix} \right) \\ -\mathrm{He}&\left(\begin{bmatrix} 0 \\ 0 \\ T_2 \end{bmatrix} \begin{bmatrix} K_1(A + BK_1) & K_1B(K_2 - I_m) & K_2 - I_m \end{bmatrix} \right) \\ -\mathrm{He}&\left(\begin{bmatrix} 0 \\ T_3 \\ 0 \end{bmatrix} \begin{bmatrix} K_1(A + BK_1) & K_1B(K_2 - I_m) & K_2 - I_m \end{bmatrix} \right). \end{aligned}$$

It is straightforward to check that the above expression corresponds to the matrix in (40), which then implies $\mathcal{G}_l > 0$, and by (41) implies (13) for almost all $x \in S(W, 1)$ thus completing the proof. ∎

Since the dz function is zero around the origin, a necessary and sufficient condition for local asymptotic (and exponential) stability of the origin for closed loop (4) is that matrix $A + BK_1$ is Hurwitz. The advantage of using more advanced conditions, such as those of Theorem 3, is that of obtaining nontrivial estimates $S(W, 1)$ of the basin of attraction. The next proposition provides a useful means for maximizing the size of this estimate by guaranteeing the inclusion

$$\mathcal{E}(\hat{P}, 1) := \{x \in \mathbb{R}^n : \ x^\top \hat{P} x < 1\} \subset S(W, 1), \tag{42}$$

so that minimizing $\hat{P}$ in some suitable way ensures enlarged estimates $S(W, 1)$.

Proposition 3: Given any scalar $\tau > 0$, if there exist matrices $\hat{P} \in \mathbb{S}^n_{>0}$, $\hat{T} \in \mathbb{D}^m_{>0}$, $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $T_1 = \mathrm{diag}\{t_1, \ldots, t_m\}, T_2 \in \mathbb{D}^m_{>0}$, $T_3 \in \mathbb{D}^m$, $Z_1 \in \mathbb{R}^{m\times n}$, $Z_2 \in \mathbb{R}^{m\times m}$ satisfying (38), (39), (40) and

$$\frac{1}{2}\mathrm{He} \begin{bmatrix} \hat{P} - P_{11} & -2(P_{12} + K_1^\top \hat{T}) \\ 0 & -P_{22} - (K_2 - I_m)^\top \hat{T} - \hat{T}(K_2 - I_m) \end{bmatrix} > 0, \tag{43}$$

then the origin is locally exponentially stable for closed loop (4) and its basin of attraction contains set $S(W, 1)$ defined in (20), which satisfies inclusion (42).

$$\Psi_3 := \mathrm{He}\left(- \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A+BK_1 & B(K_2-I_m) & 0 \\ 0 & 0 & I_m \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} 0 & 0 & 0 \\ T_1K_1 - Z_1 + T_3K_1(A+BK_1) & T_1(K_2-I_m) - Z_2 + T_3K_1B(K_2-I_m) & T_3(K_2-I_m) \\ T_2K_1(A+BK_1) & T_2K_1B(K_2-I_m) & T_2(K_2-I_m) \end{bmatrix} \right) > 0 \tag{40}$$

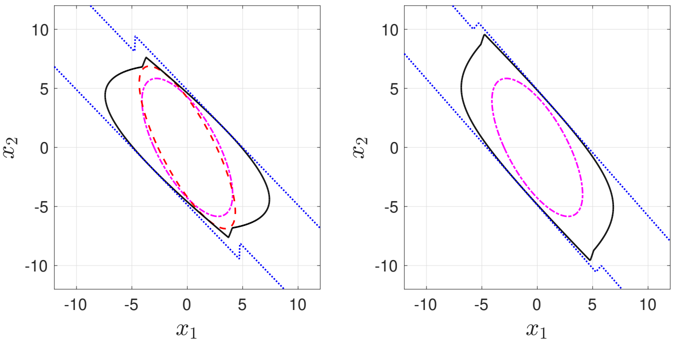

Fig. 3. Example 2: the left figure shows the set $S(W,1)$ (in black) obtained when solving (44) and the maximized ellipsoid $\mathcal{E}(\hat{P},1) \subset S(W,1)$ (dashed red ellipse). On the right we see the set $S(W,1)$ obtained when minimizing the trace of P_{11}.

Proof. The proof of local exponential stability with $S(W,1)$ contained in the basin of attraction has been already given in Theorem 3. We prove below that the additional LMI (43) ensures (42). To this end, it is enough to show that $(x^\top \hat{P} x < 1) \Rightarrow (S(W,1) < 1)$, which holds if we show that

$$x^\top \hat{P} x > V(x), \quad \forall x \in \mathcal{S}_h.$$

By using the global sector condition in (5) with $T = \hat{T}$, the inequality above is ensured if

$$x^\top \hat{P} x - V(x) - \mathrm{dz}(u)^\top \hat{T}(u - \mathrm{dz}(u)) > 0,$$

which can be checked to be a quadratic form in $\left[\begin{smallmatrix} x \\ \mathrm{dz}(u(x)) \end{smallmatrix}\right]$ involving the positive definite matrix in (44), thus completing the proof. ∎

Remark 4: As expressed in Theorem 3 and its proof, the trick used to handle t_i^2 which appears in $\bar{\Phi}_3$ consists in replacing it by τt_i, where τ is upper bounded by T_1. Moreover, τ may be either a scalar (as suggested in Theorem 3) or, when the dimension of the input is small, may be set as a vector, thus allowing us to replace (39) by $t_i > \tau_i$, for all $i = 1, \dots, m$. The selection of a suitable τ may be performed either with a simple grid search or cast in a nonlinear optimization procedure (such as the `fminsearch` Matlab function) including a convex optimization problem based on Proposition 3, corresponding to

$$\min_{\substack{\hat{P}, Q_{11}, Q_{12}, Q_{22}, \\ \hat{T}, T_1, T_2, T_3, Z_1, Z_2}} \mathrm{Tr}(\hat{P}), \text{ subject to: } (38), (39), (40), (43). \tag{44}$$

The objective function given by the trace of $\hat{P}$ is used as an indication of the size of the estimate of the basin of attraction. Example 2 reported below illustrates the use of this optimization criterion. ⋆

Example 2: Consider the load balancing example reported in [23, Ex. 1.1] corresponding to the closed loop (4) with:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ -5 \end{bmatrix}, \; K_1 = \begin{bmatrix} 2.6 & 1.4 \end{bmatrix}, \; K_2 = 0.$$

For this example, we first solve the conditions of Theorem 3 and Proposition 3 adopting the optimization strategy proposed in Remark 4 and minimizing the trace of matrix $\hat{P}$ as in (44). The resulting level set $S(W,1)$ is shown in black at the left of Figure 3 (already reported in Figure 1). The boundary of the set $\mathcal{E}(\hat{P},1)$ is shown in dashed red and is contained in $S(W,1)$, as proven in Proposition 3. The dashed-dotted magenta ellipse in the same figure corresponds to a quadratic estimate of the basin of attraction obtained with the analysis results in [23, Prop. 3.1], illustrating the increased ability of providing improved estimates of the basin of attraction with sign-indefinite quadratic forms and Theorem 3. The dotted blue lines denote the boundaries of set $\mathcal{S}_h$ in (6), where the local sector condition (8) is valid (and where function W coincides with V). The peculiar shape of these boundaries is obtained thanks to the term $H_2\mathrm{dz}(u(x))$ in (7).

The black level set $S(W,1)$ shown at the right of Figure 3 corresponds to replacing problem (44) by a heuristic approach minimizing the trace of P_{11} (the upper left subentry of P associated to vector x, as per (11)) under constraints (38)–(40). While this criterion only indirectly addresses the maximization of the set $S(W,1)$, the right plot indicates that it leads to a slightly larger set $S(W,1)$, as compared to the solution proposed in Proposition 3. In both cases the obtained estimates are significantly larger than the quadratic estimates obtained with [23, Prop. 3.1] (dashed-dotted magenta).

For this example, the optimization of Remark 4 returns a sign-indefinite solution P corresponding to

$$P = \begin{bmatrix} 0.0428 & 0.0030 & 0.0833 \\ 0.0030 & 0.0100 & 0.0570 \\ 0.0833 & 0.0570 & -0.0697 \end{bmatrix},$$

where we can see that $P_{22} = -0.0697$ is negative. ∘

Example 3: We present an example showing that the Lyapunov-like function V may become negative outside $\mathcal{S}_h$, which motivates introducing function W in (21), because $S(V,1)$ is an unbounded set, while $S(W,1)$ is bounded and contained in $\mathcal{S}_h$. Figure 4 reports the surface of V and shows in black the level set $V(x) = 1$. Function V is obtained for the exponentially unstable plant (4) with:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \; K_1 = \begin{bmatrix} -0.1 & 2 \end{bmatrix}, \; K_2 = 0.9,$$

by solving (38)–(40) while minimizing the trace of matrix P_{11}, as in the right picture in Figure 3. Note that $S(V,1)$

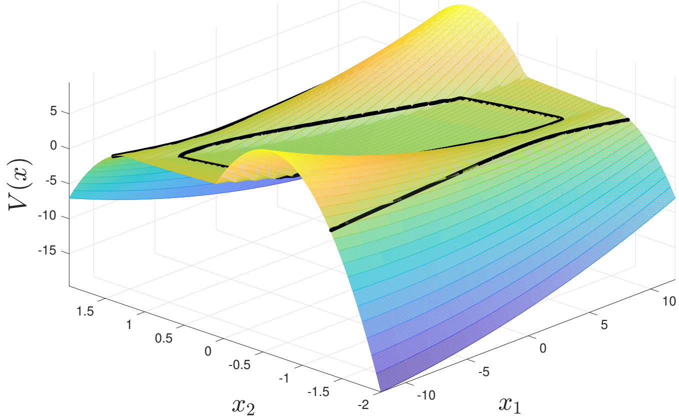

Fig. 4. Example 3: surface of the Lyapunov-like function $V(x)$ and the sub-level corresponding to $V(x) = 1$ (in black).

is a disconnected unbounded set (unbounded in directions where x_2 grows unbounded) while $S(W, 1)$ only corresponds to the inner square-shaped connected set. Also in this case, the optimal value of P is not sign definite and corresponds to

$$P = \begin{bmatrix} 0.0152 & 0.0264 & -0.4719 \\ 0.0264 & 0.1066 & -5.8609 \\ -0.4719 & -5.8609 & -9.6106 \end{bmatrix}.$$

Based on Remark 1, we may numerically investigate the robustness of stability when the artificial saturation level $u_{\max} = 1$ embedded in the controller is different from the actual physical saturation level $u_{\text{max,ph}}$ trimming the plant input. Clearly, only the cases where $u_{\text{max,ph}} \leq u_{\max} = 1$ are interesting because the case $u_{\text{max,ph}} \geq u_{\max}$ does not pose any problem. We characterize three cases below, by picking an initial condition close to the boundary of $S(W, 1)$ and perturbing the initial condition and the physical saturation level:

- Case 1. Consider the initial condition $x_0 = \begin{bmatrix} 4 & -0.82 \end{bmatrix}^\top$ belonging to the guaranteed stability set $S(W, 1)$ (indeed $V(x_0) = 0.9781 < 1$). When shrinking the saturation level up to $u_{\text{max,ph}} = 0.8205$ the closed-loop trajectories preserve convergence the origin. With $u_{\text{max,ph}} = 0.8200$, the closed-loop trajectory diverges.
- Case 2. Consider the initial condition $x_0 = \begin{bmatrix} 4 & -0.95 \end{bmatrix}^\top$, which does not belong to $S(W, 1)$ (indeed $V(x_0) = 1.431 > 1$). The closed-loop trajectory converges to zero with $u_{\text{max,ph}} = 1$ but diverges with $u_{\text{max,ph}} = 0.95$.
- Case 3. Consider the initial condition $x_0 = \begin{bmatrix} 4 & -1 \end{bmatrix}^\top$, which does not belong to $S(W, 1)$ (indeed $V(x_0) = 1.435 > 1$). The closed-loop solution already diverges with $u_{\text{max,ph}} = 1$.

Both Cases 1 and 2 show some level of robustness with uncertain saturation levels, whereas Case 3 show that the estimate of the basin of attraction is not too conservative. ◦

Remark 5: Similar to the observations in Remark 2 for the global case, the analysis conditions in Theorem 3 and the optimization problem in (44) readily extend to the case where dynamics (4) is replaced by the more general form (31) and may be used to address output feedback dynamic controllers with anti-windup action. ⋆

B. Regional synthesis

The following theorem provides conditions to solve Problem 4 by suitably selecting feedback gains K_1, K_2.

Theorem 4: If there exist matrices $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $M \in \mathbb{R}^{n\times n}$, $S \in \mathbb{D}^m_{>0}$, $Z_1 \in \mathbb{R}^{m\times n}$, $Z_2 \in \mathbb{R}^{m\times m}$, $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{m\times m}$ such that

$$\Phi_4 := \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12}^\top & Q_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathrm{He}\begin{bmatrix} 0 & Z_1^\top - Y_1^\top & Z_{1i}^\top \\ 0 & Z_2 - Y_2 + S & Z_{2i}^\top \\ 0 & 0 & 0 \end{bmatrix} > 0 \tag{45}$$

and condition (46) (given at the top of next page) hold then matrix M is nonsingular. Moreover, the origin is locally exponentially stable for the closed loop (4) with K_1 and K_2 chosen as in (34), and its basin of attraction contains the set $S(W, 1)$ defined in (20), with functions V and W in (11), (21) characterized by

$$P = \begin{bmatrix} M^{-\top} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} M^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}. \tag{47}$$

Proof. We first show that M is nonsingular, which is a direct consequence of $M + M^\top > 0$ implied by the (3,3) entry of Ψ_4 in (46). The rest of the proof relies on Lemma 1, by showing the two bounds (12) and (13) with $r = 1$.

To show (12), let us first observe that with $Y_2 = K_2 S$. the (4,4) term of (46) reads $2S - SK_2 - K_2^\top S > 0$, where $S > 0$ by assumption, so that Fact 1 implies well-posedness of the algebraic loop (3). Moreover, left and right multiplying $\Phi_4 > 0$ by $\mathrm{diag}(M^\top, S, 1)$ and $\mathrm{diag}(M, S, 1)$, respectively, and also using $Y_1 = K_1 M$ and $Y_2 = K_2 S$ and P as in (47), we obtain $\Phi_1 > 0$, with Φ_1 as in (23). As a consequence, Lemma 3 holds and function W in (21) is Lipschitz continuous (item (i)) and satisfies (12) with $r = 1$ (item (iii)).

To show (13), item (ii) of Lemma 3 ensures $S(W, 1) \subset \mathcal{S}_h$, which allows applying the local conditions (8) of Fact 4 to all $x \in S(W, 1)$. In particular, we exploit Facts 4 and 5 evaluated with $T = S^{-1}$ and the change of variables

$$\begin{aligned} &Z_1 = H_1 M, \qquad Z_2 = H_2 S, \\ &Q_{11} = M^\top P_{11} M, \quad Q_{12} = M^\top P_{12} S, \quad Q_{22} = S P_{22} S, \end{aligned} \tag{48}$$

following similar steps to the proof of Theorem 2 on an extended space parametrized by $\eta := \begin{bmatrix} x^\top & \mathrm{dz}(u(x))^\top & \dot x^\top & \dot{\mathrm{dz}}(u(x))^\top \end{bmatrix}^\top$, which satisfies bound (35). Recalling that $W(x) = V(x)$ for all x in the open set $S(W, 1)$, we have

$$\begin{aligned} \dot W(x) &\leq \dot W(x) + 2\mathrm{dz}(u)^\top S^{-1}(u - \mathrm{dz}(u) - h(x)) \\ &\quad + 2\dot{\mathrm{dz}}(u(x))^\top S^{-1}(\dot u(x) - \dot{\mathrm{dz}}(u(x))) \\ &\quad + 2\dot{\mathrm{dz}}(u(x))^\top S^{-1}(\dot u(x) - \dot{\mathrm{dz}}(u(x))) \\ &\quad - 2\eta^\top N(-\dot x + (A + BK_1)x + B(K_2 - I_m)\mathrm{dz}(u(x))) \\ &= -\eta^\top \bar\Psi_4 \eta, \qquad \text{for almost all } x \in S(W, 1). \end{aligned} \tag{49}$$

$$\Psi_4 := \mathrm{He}\left(-\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} - \begin{bmatrix} AM+BY_1 & BY_2-BS & -M & 0 \\ Y_1-Z_1 & Y_2-Z_2-S & Y_1 & Y_2-S \\ AM+BY_1 & BY_2-BS & -M & 0 \\ 0 & 0 & Y_1 & Y_2-S \end{bmatrix} \right) > 0 \tag{46}$$

By selecting $N = \begin{bmatrix} M^{-\top} & 0 & M^{-\top} & 0 \end{bmatrix}^\top$ in (49), and using the change of variables (48), together with $Y_1 = K_1 M$ and $Y_2 = K_2 S$, we get

$$\begin{aligned}
\hat{\Psi}_4 &= \mathrm{diag}(M^\top, S, M^\top, S)\bar{\Psi}_4 \mathrm{diag}(M, S, M, S) \\
&= \mathrm{He}\left(-\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \right) \\
&- \mathrm{He}\left(\begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} K_1M - Z_1 & K_2S - Z_2 - S & 0 & 0 \end{bmatrix} \right) \\
&- \mathrm{He}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ I_m \end{bmatrix} \begin{bmatrix} 0 & 0 & K_1M & K_2S - S \end{bmatrix} \right) \\
&- \mathrm{He}\left(\begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & K_1M & K_2S - S \end{bmatrix} \right) \\
&- \mathrm{He}\left(\begin{bmatrix} I_n \\ 0 \\ I_n \\ 0 \end{bmatrix} \begin{bmatrix} AM + BK_1M & BK_2S - BS & -M & 0 \end{bmatrix} \right),
\end{aligned}$$

which coincides with Ψ_4 appearing in (46), and is positive definite by hypothesis. As a consequence, $\bar{\Psi}_4 > 0$ which, by (49), implies (13) with $r = 1$, thus concluding the proof. ■

Remark 6: From Theorems 3 and 4, one can recover the global conditions of Theorems 1 and 2 by setting $Z_1 = 0$ and $Z_2 = 0$. In particular, the observations in Remark 3 apply also to the regional case, and we expect the main source of conservativeness to come from the fact that the multipliers T_i of the sector-like conditions are set to be the same (and equal to S^{-1}) in the synthesis LMIs, to preserve convexity. ⋆

Similar to the discussion after Theorems 2 and 3, is it useful to provide guaranteed performance criteria that may be enforced together with the LMI conditions (45), (46) of Theorem 4. The next two propositions provide parallel results to those of Propositions 2 and 3 for the regional synthesis results of Theorem 4.

Proposition 4: Given a desired convergence rate $\alpha > 0$, if there exist matrices $R \in \mathbb{S}^n_{>0}$, $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $M \in \mathbb{R}^{n\times n}$, $S \in \mathbb{D}^m_{>0}$, $Z_1 \in \mathbb{R}^{m\times n}$, $Z_2 \in \mathbb{R}^{m\times m}$, $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{m\times m}$ satisfying (45), (46), and (37), then matrix M is nonsingular. Moreover, the origin is locally exponentially stable for the closed loop (4) with K_1 and K_2 chosen as in (34), and its basin of attraction contains set $S(W, 1)$ defined in (20), with functions V and W in (11), (21) characterized by (47). Moreover, the eigenvalues of $A + BK_1$ have real part smaller than $-\alpha$.

Proof. The proof is a straightforward combination of the proof of Theorem 4 and Proposition 2. ■

Proposition 5: If there exist matrices $\hat{P} \in \mathbb{S}^n_{>0}$, $Q_{11} \in \mathbb{S}^n_{>0}$, $Q_{12} \in \mathbb{R}^{n\times m}$, $Q_{22} \in \mathbb{S}^m$, $M \in \mathbb{R}^{n\times n}$, $S \in \mathbb{D}^m_{>0}$, $Z_1 \in \mathbb{R}^{m\times n}$, $Z_2 \in \mathbb{R}^{m\times m}$, $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{m\times m}$ satisfying (45), (46), and

$$\begin{bmatrix} M + M^\top - Q_{11} & I_n & -Q_{12} - Y_1^\top \\ I_n & \hat{P} & 0 \\ -Q_{12}^\top - Y_1 & 0 & -Q_{22} + 2S - Y_2 - Y_2^\top \end{bmatrix} > 0, \tag{50}$$

then matrix M is nonsingular. Moreover, the origin is locally exponentially stable for the closed loop (4) with K_1 and K_2 chosen as in (34). Finally, its basin of attraction contains set $S(W, 1)$ defined in (20) (with V in (11) characterized by (47)), and the set $S(W, 1)$ satisfies inclusion (42).

Proof. The proof of local exponential stability of the origin with basin of attraction containing the set $S(W, 1)$ has been given in Theorem 4. We prove below that inclusion (42) holds. To this end, by Proposition 3, it is enough to show that (50) implies (43), which we do in the rest of the proof.

To prove (43) apply a Schur complement to (50) to obtain

$$\begin{bmatrix} M + M^\top - \hat{P}^{-1} - Q_{11} & -Q_{12} - Y_1^\top \\ -Q_{12}^\top - Y_1 & -Q_{22} + 2S - Y_2 - Y_2^\top \end{bmatrix} > 0,$$

and then observe that $(\hat{P}^{-1} - M^\top)\hat{P}(\hat{P}^{-1} - M) \geq 0$ implies $M + M^\top - \hat{P}^{-1} \leq M^\top \hat{P} M$, so that

$$\begin{bmatrix} M^\top \hat{P} M - Q_{11} & -Q_{12} - Y_1^\top \\ -Q_{12}^\top - Y_1 & -Q_{22} + 2S - Y_2 - Y_2^\top \end{bmatrix} > 0.$$

Finally, pre- and post-multiplying the previous inequality by $\mathrm{diag}(M^\top, S)$ and $\mathrm{diag}(M, S)$, respectively, and recalling that $Y_1 = K_1 M$ and $Y_2 = K_2 S$, we obtain (43). ■

Remark 7: Combining the results of Propositions 4 and 5, we may optimally choose the gains K_1 and K_2 by first fixing a desired convergence rate α, and then solving the optimization

$$\min_{\substack{R,\hat{P},Q_{11},Q_{12}Q_{22},\\ M,S,Z_1,Z_2,Y_1,Y_2}} \mathrm{Tr}(\hat{P}), \text{ subject to: } (45), (46), (37), (50) \tag{51}$$

so that the size of the estimate of the basin of attraction $S(W, 1)$ is maximized by minimizing the trace of $\hat{P}$, while guaranteeing the local convergence rate α. This type of trade-off is well explored in Example 4 discussed below. ⋆

Example 4: Consider the longitudinal dynamics of an F8 aircraft from [28], modeled by plant (1) with matrices

$$\begin{bmatrix} A \mid B \end{bmatrix} = \left[\begin{array}{cccc|cc} -0.8 & -0.006 & -12 & 0 & -19 & -3 \\ 0 & -0.014 & -16.64 & -32.2 & -0.66 & -0.5 \\ 1 & -0.0001 & -1.5 & 0 & -0.16 & -0.5 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

To design K_1 and K_2, we solve optimization (51) for a number of selections of target convergence rates $\alpha \in [1, 4]$.

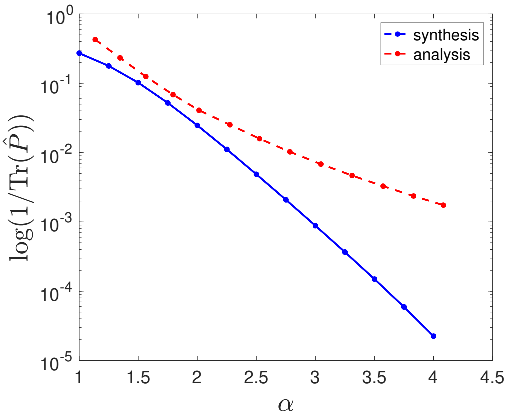

Fig. 5. Example 4: trade-off between the local convergence rate α and the optimization criterion $\mathrm{Tr}(\hat{P})$ in the synthesis conditions of Theorem 4 (blue curve) and the analysis conditions of Theorem 3.

The corresponding optimal costs $(\mathrm{Tr}(\hat{P}))^{-1}$ are represented in logarithmic scale by the blue dots in Figure 5. The trade-off blue curve interpolates the evaluated points. We recall that the minimized trace of $\hat{P}$ is only an indication of the size of the basin of attraction $\mathcal{B}$, due to the property $\mathcal{E}(\hat{P},1) \subset S(W,1) \subset \mathcal{B}$.

To illustrate the reduced conservativeness of the analysis conditions of Theorem 3 and Proposition 3, as discussed in Remarks 3 and 6, the red dots in the same figure report the actual local convergence rate α (easily evaluated as the maximum real part of the eigenvalues of $A + BK_1$) and the performance criterion $\mathrm{Tr}(\hat{P})$ evaluated by solving the optimization in (44) with the corresponding gains. As one may see, both the estimated α (by inequality (37) of Proposition 4) and the estimated trace of $\hat{P}$ (by inequality (50) of Proposition 5) in the synthesis conditions are significantly worse than those of the analysis case.

To illustrate the numerical conditioning of the obtained matrices, we report below the values of the gains obtained for $\alpha = 3.5$, corresponding to

$$K_1 = \begin{bmatrix} -0.0716 & 0.1813 & -2.2395 & -0.8175 \\ 3.0745 & -1.2914 & 10.8585 & 12.8640 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} 0.8391 & 0.0293 \\ 0.7123 & 0.8391 \end{bmatrix}$$

inducing the following eigenvalues of $A + BK_1 \in \mathbb{R}^{4\times 4}$: $-3.5733 \pm 0.5560i$; $-3.7880 \pm 1.6975i$, which confirm the desired convergence rate higher than 3.5. $\circ$

VI. Conclusions

We proposed a new class of sign-indefinite quadratic forms for the stability analysis and control design of nonlinear control laws for linear systems with input saturation. Both local and global stability analysis and synthesis results are presented, in terms of convex optimization problems, using the arising piecewise smooth Lyapunov functions.

Numerical examples testify the substantial improvement over the existing results, showing that non-quadratic Lyapunov function have the ability to show global stability of some systems for which a quadratic function fails to exist. Moreover the numerical results highlight that improvements also emerge from relaxing the positivity of the quadratic form. For regional stability, a non-linear generalized sector condition is introduced and numerical examples illustrate the improvement of the estimates of the basin of attraction. Indeed, more general shapes than mere ellipsoids are obtained thanks to the implicit definition of the Lyapunov function.

Future work includes the investigation of efficient schemes for the solution of implicit functions induced by the use of nonlinear control laws. We will also investigate the design of dynamic non-linear output feedback laws and their application to the design of anti-windup compensators. Furthermore, another interesting direction is to consider other nonlinearities, possibly combined with saturation, as for example quantizers, backlash or friction functions, with the aim of improving the characterization of an asymptotically stable attractor.

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Isabelle Queinnec is currently CNRS researcher at LAAS-CNRS, Toulouse University. She received her PhD degree and HDR degree in automatic control in 1990 and 2000, respectively, from University Paul Sabatier, Toulouse. Her current research interests include constrained control and robust control of processes with limited information, with particular interest in applications on aeronautical systems, robotic, electronic, biochemical and environmental processes. She has been serving as member of the IFAC technical committees on “Biosystems and Bioprocesses” and on “Modelling and Control of Environmental Systems”, respectively from 2002 and 2005 and of the IEEE CSS-CEB from 2013. She is currently AE for IEEE Transactions on Automatic Control. She is co-author of a book on saturated systems and of more than 50 journal papers, both in control theory and process engineering.

Sophie Tarbouriech received the Ph.D. and HDR degrees in automatic control from University Paul Sabatier, Toulouse, France, in 1991 and 1998, respectively. She is full-time researcher (Directeur de Recherche) in LAAS-CNRS, Toulouse. Her main research interests include analysis and control of linear and nonlinear systems with constraints (limited information), hybrid dynamical systems. She is currently Associate Editor for IEEE Transactions on Automatic Control, Automatica and European Journal of Control. She is also Senior Editor for IEEE Control Systems Letters and member of IFAC and IEEE Technical Committees.

Giorgio Valmorbida received the Ph.D. degree in control theory from the Institut National des Sciences Appliquées, Toulouse, France, in 2010 for the research developed at the LAAS/CNRS. He has held post-doctoral research positions at the Università di Roma “Tor Vergata”, Rome, Italy, from 2010 to 2012 and at the Department of Engineering Science, University of Oxford, Oxford, UK, from 2013 to 2015. He was a Junior Research Fellow at Somerville College, University of Oxford, Oxford, UK, from 2013 to 2015. He is currently an Associate Professor in the L2S, CentraleSupélec, Gif-sur-Yvette, France. His main research interests include analysis and control of nonlinear systems with input constraints, hybrid systems, infinite-dimensional systems, and polynomial optimization.

Luca Zaccarian (F’16) received the Ph.D. from the University of Roma Tor Vergata (Italy) in 2000, where he then became assistant and then associate professor. Since 2011 he is Directeur de Recherche at the LAAS-CNRS, Toulouse (France) and since 2013 he holds a part-time professor position at the University of Trento, Italy. Luca Zaccarian’s main research interests include analysis and design of nonlinear and hybrid control systems, modeling and control of mechatronic systems. He was a recipient of the 2001 O. Hugo Schuck Best Paper Award given by the American Automatic Control Council.