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Lyapunov small-gain theorems for not necessarily ISS hybrid systems

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***Abstract*— We prove a novel Lyapunov-based small-gain theorem for interconnections of n hybrid systems, which are not necessarily input-to-state stable. This result unifies and extends several small-gain theorems for hybrid and impulsive systems, proposed in the last few years. Also we show how the average dwell-time (ADT) clocks and reverse ADT clocks can be used to modify the Lyapunov functions for subsystems and to enlarge the applicability of derived small-gain theorems.**

Keywords: hybrid systems, input-to-state stability, small-gain theorems.

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I. INTRODUCTION

The study of interconnected systems plays a significant role in the development of stability theory of dynamic systems, as it allows one to investigate the stability of complex systems by analyzing its less complicated components. In this context, the small-gain theorems have proved to be important tools in the analysis of feedback interconnections of multiple systems, which appear frequently in the control literature. A comprehensive overview of classical small-gain theorems involving input-output gains of linear systems can be found in [1]. This technique was then generalized to nonlinear feedback systems in [2], [3] within the input-output context. The next peak level in stability analysis of interconnected dynamic systems has been reached within the input-to-state stability (ISS) framework proposed by Sontag [4], which unified internal and external stability notions. In [5], [6] nonlinear small-gain theorems for interconnections of two ISS systems were established, which were then generalized to arbitrary interconnections of n dynamic systems in [7], [8]. A variety of nonlinear small-gain theorems were summarized in [9].

Theory described above has been developed for systems of ordinary differential equations. But often in the modeling of real phenomena one has to consider systems which exhibit both continuous and discrete behavior. A general framework for modeling of such phenomena is the hybrid systems theory [10], [11]. In our analysis we have adopted the modeling framework proposed by Goebel et al. [11], which is natural from the viewpoint of Lyapunov stability theory [12].

During last years a great effort has been devoted to development of small-gain theorems for interconnected hybrid systems. Trajectory-based small-gain theorems for interconnections of two hybrid systems were derived in [13], [14], [15], while Lyapunov-based formulations were proposed in [16], [17], [18]. Some of these results were extended to interconnections of n hybrid ISS systems in [15].

More challenging is a study of hybrid systems for which either continuous or discrete dynamics is not ISS. In this case input-to-state stability cannot be achieved unless the restrictions on the density of jumps of the state called dwell-time conditions are added. For interconnected hybrid systems, whose subsystems are of this kind, the small-gain theorems from [15], [18] cannot be used directly. In [18] it was shown, that it is possible to modify "bad" subsystems by adding of clock variables and to construct an ISS Lyapunov function for the modified subsystem so that it decreases after jumps and along the trajectory. The advantage of this method is that it can be applied even if instabilities of subsystems are of a different type (i.e. in some subsystems the continuous dynamics is ISS, and in some other ones the discrete dynamics is ISS). However, the Lyapunov gains of the modified systems increase exponentially with a chatter bound, which restricts the applicability of this method since enlarged gains may no longer satisfy the small-gain condition.

Another type of small-gain theorems has been proposed in [19], [20], [21], where ISS of interconnected impulsive systems with unstable discrete or continuous dynamics was investigated. The small-gain theorems developed therein provide a construction of an ISS Lyapunov function if the instabilities of subsystems are of the same type, that is if either continuous dynamics of all subsystems or discrete dynamics of all subsystems is ISS. The resulting ISS Lyapunov function can be then used to prove ISS of the overall system under suitable dwell-time conditions. In contrast to the previous method this method doesn't require the modification of subsystems, Lyapunov gains for subsystems are preserved and do not depend on a dwell-time condition (and on a chatter bound) which is used. However, this method has been developed only for impulsive systems and cannot be used for systems whose subsystems have instabilities of different type.

In this paper we unify the above two methods. In Section II we introduce the notation, main definitions and prove the Lyapunov sufficient condition for ISS of hybrid systems. Next we prove a general small-gain theorem for hybrid systems, leading to the construction of an ISS Lyapunov function for an interconnection of n hybrid systems, provided instabilities of its subsystems are of the same type, which generalizes the Lyapunov small-gain theorems from [17], [18], [15], [19], [21]. In the same section we derive several consequences of this general result, in particular, the small-gain theorem for interconnections of hybrid systems

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possessing exponential ISS Lyapunov functions with linear Lyapunov gains. In Section IV we propose a version of the method of modification of Lyapunov functions for subsystems from [18] which is less restrictive than the original method from [18] because a lesser number of systems has to be modified. It relies on the small-gain theorem from Section III. Next in Section V we combine the results of this work into the unified method for analysis of ISS of interconnected hybrid systems and conclude the paper.

II. Framework for hybrid systems

Let $\mathbb{R}_+ := [0, \infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$. For a vector $x \in \mathbb{R}^n$, $|x|$ is used to denote its Euclidean norm. For a set $\mathcal{A} \subset \mathbb{R}^n$, define its (Euclidean) distance to a vector x as $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x-y|$. For n vectors $x_1, x_2, \ldots, x_n$, their concatenation is denoted by $(x_1, x_2, \ldots, x_n) := (x_1^\top, x_2^\top, \ldots, x_n^\top)^\top$. We use id to denote the identity function.

Following [12], a hybrid system with input is modeled as

$$\begin{aligned} \dot{x} &\in f(x, u), \quad (x, u) \in C, \\ x^+ &\in g(x, u), \quad (x, u) \in D, \end{aligned} \tag{1}$$

where $x \in X \subset \mathbb{R}^N$, $u \in \mathbb{R}^M$, C and D are closed subsets of $\mathbb{R}^N \times \mathbb{R}^M$ and $f : C \rightrightarrows \mathbb{R}^N$ and $g : D \rightrightarrows X$ are set-valued maps. The hybrid system (1) is fully defined by its data $H := (f, g, C, D)$. The dynamics of (1) is continuous if $(x, u) \in C\backslash D$ and discrete if $(x, u) \in D\backslash C$. For $(x, u) \in D \cap C$, it can be either continuous or discrete.

A set $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \le t_1 \le \cdots \le t_{J+1}$. A set $E \subset \mathbb{R}_+ \times \mathbb{N}$ is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain.

A hybrid signal is a function defined on a hybrid time domain. A hybrid signal $u : \mathrm{dom}\, u \to \mathbb{R}^M$ is called a hybrid input if $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid signal $x : \mathrm{dom}\, x \to X$ is called a hybrid arc if $x(\cdot, j)$ is locally absolutely continuous for each j. A hybrid arc $x : \mathrm{dom}\, x \to X$ and a hybrid input $u : \mathrm{dom}\, u \to \mathbb{R}^m$ form a *solution pair* (x, u) to (1) if:

- $\mathrm{dom}\, x = \mathrm{dom}\, u$;
- for all $j \in \mathbb{N}$ and almost all $t \in \mathbb{R}_+$ such that $(t, j) \in \mathrm{dom}\, x$, $(x(t, j), u(t, j)) \in C$ and $\dot{x}(t, j) \in f(x(t, j), u(t, j))$;[1]
- for all $(t, j) \in \mathrm{dom}\, x$ such that $(t, j + 1) \in \mathrm{dom}\, x$, $(x(t, j), u(t, j)) \in D$ and $x(t, j + 1) \in g(x(t, j), u(t, j))$.

A solution pair (x, u) is *complete* if $\mathrm{dom}\, x$ is unbounded.

The essential supremum norm of a hybrid signal u up to hybrid time $(t, j) \in \mathrm{dom}\, u$ is defined as

$$\|u\|_{(t,j)} := \max\left\{ \operatorname*{ess\,sup}_{\substack{(s,l)\in\mathrm{dom}\, u,\\ s\le t, l\le j}} |u(s, l)|, \sup_{\substack{(s,l)\in\Phi(u),\\ s\le t, l\le j}} |u(s, l)| \right\},$$

where $\Phi(u) := \{(s, l) \in \mathrm{dom}\, u : (s, l + 1) \in \mathrm{dom}\, u\}$.

[1] Here $x(t, j)$ represents the state of the hybrid system at time t and after j jumps.

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite if $\gamma(x) = 0 \Leftrightarrow x = 0$. We say that $\gamma \in \mathcal{P}$ if it is continuous and positive definite. γ is of class $\mathcal{K}$ (denoted by $\gamma \in \mathcal{K}$) if $\gamma \in \mathcal{P}$ and is strictly increasing. γ is of class $\mathcal{K}_\infty$ (denoted by $\gamma \in \mathcal{K}_\infty$) if $\gamma \in \mathcal{K}$ and $\lim_{x\to\infty} \gamma(x) = \infty$. $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ (denoted by $\gamma \in \mathcal{L}$) if it is continuous, strictly decreasing and $\lim_{t\to\infty} \gamma(t) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ (denoted by $\beta \in \mathcal{KL}$) if $\beta(\cdot, t) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$ and $\beta(r, \cdot) \in \mathcal{L}$ for all $r \in (0, \infty)$.

Definition 1: Following [18], a set of solution pairs $\mathcal{S}$ of (1) is *pre-input-to-state stable (pre-ISS) w.r.t.* $\mathcal{A} \subset X$ if there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that for all $(x, u) \in \mathcal{S}$,

$$|x(t, j)|_{\mathcal{A}} \le \max\left\{\beta(|x(0, 0)|_{\mathcal{A}}, t + j), \gamma(\|u\|_{(t,j)})\right\} \tag{2}$$

for all $(t, j) \in \mathrm{dom}\, x$. If $\mathcal{S}$ contains all solution pairs (x, u) of (1), then we call (1) pre-ISS w.r.t. $\mathcal{A}$. If all solution pairs are in addition complete, then (1) is called *ISS w.r.t.* $\mathcal{A}$.

Remark 1: In [12], the input-to-state stability of hybrid systems is defined in terms of class $\mathcal{KLL}$ functions and without requiring all solution pairs to be complete, which is equivalent to our definition of the pre-ISS property of hybrid systems; cf. [22, Lemma 6.1].

In this work, our principal technique for investigation of the ISS of (1) is an ISS Lyapunov function.

Definition 2: A Lipschitz continuous function $V : X \to \mathbb{R}_+$ is called an *ISS Lyapunov function for* (1) *w.r.t.* $\mathcal{A} \subset X$ if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(|x|_{\mathcal{A}}) \le V(x) \le \psi_2(|x|_{\mathcal{A}}) \qquad \forall\ x \in X \tag{3}$$

holds and $\exists\ \chi \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that $V(x) \ge \chi(|u|)$ implies

$$\begin{cases} \dot{V}(x; y) \le -\varphi(V(x)) & \forall\ y \in f(x, u), (x, u) \in C, \\ V(y) \le \alpha(V(x)) & \forall\ y \in g(x, u), (x, u) \in D, \end{cases} \tag{4}$$

where $\dot{V}(x; y)$ is the Dini derivative of V at x in the direction y, namely

$$\dot{V}(x; y) = \overline{\lim_{h\to+0}} \frac{V(x + hy) - V(x)}{h}. \tag{5}$$

In addition, if for all $r \in \mathbb{R}_+$, $\varphi(r) = cr$ and $\alpha(r) = e^{-d}r$ for some $c, d \in \mathbb{R}$, then V is called an *exponential ISS Lyapunov function for* (1) *w.r.t.* $\mathcal{A}$ *with rate coefficients* c, d.

The following lemma gives an alternative description of the ISS Lyapunov function for a hybrid system, which is useful for the formulation of small-gain theorems in Section III.

Lemma 2: A Lipschitz continuous function $V : X \to \mathbb{R}_+$ is an *ISS Lyapunov function for* (1) *w.r.t.* $\mathcal{A}$ if and only if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$ such that (3) holds and $\exists\ \bar{\chi} \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that for all $(x, u) \in C$ and all $y \in f(x, u)$,

$$V(x) \ge \bar{\chi}(|u|) \ \Rightarrow\ \dot{V}(x; y) \le -\varphi(V(x)), \tag{6}$$

and for all $(x, u) \in D$ and all $y \in g(x, u)$,

$$V(y) \le \max\{\alpha(V(x)), \bar{\chi}(|u|)\}. \tag{7}$$

Proof: The proof goes along the lines of the proof [19, Proposition 1] and is omitted. ■

Similar restatement can also be provided for the exponential ISS Lyapunov function. Note that the gain functions χ in Definition 2 and $\bar{\chi}$ in Lemma 2 are different in general.

In the definition of an ISS Lyapunov function, we do not assume that $\varphi \in \mathcal{P}$ and $\alpha < \mathrm{id}$. If both of these conditions are satisfied, then the existence of an ISS Lyapunov function implies that (1) is pre-ISS, see [12, Proposition 2.7] (note that ISS in [12] means pre-ISS in this work; cf. Remark 1). If neither of them is valid, then we are not able to conclude anything about the ISS property of (1). However, if one of these conditions is satisfied[2], then we can still establish the ISS property for some subset of the solution pairs of (1) under additional restrictions on the density of jumps.

Proposition 1: Let V be an exponential ISS Lyapunov function for (1) w.r.t. $\mathcal{A} \subset X$ with rate coefficients $c, d \in \mathbb{R}$ with $d \neq 0$. For arbitrary $\mu \geq 1$ and $\eta, \lambda > 0$, let $\mathcal{S}[\eta, \lambda, \mu]$ denote the set of solution pairs (x, u) satisfying

$$-(d-\eta)(j-i) - (c-\lambda)(t-s) \leq \mu \quad \forall\ (t,j),(s,i) \in \operatorname{dom} x. \tag{8}$$

Then $\mathcal{S}[\eta, \lambda, \mu]$ is pre-ISS w.r.t. $\mathcal{A}$.

Proof: Let $\chi \in \mathcal{K}_\infty$ be as in Definition 2. Consider an arbitrary solution pair $(x, u) \in \mathcal{S}[\eta, \lambda, \mu]$. For all $(t_1, j_1), (t_0, j_0) \in \operatorname{dom} x$ such that $t_1 \geq t_0, j_1 \geq j_0$, if

$$V(x(s,i)) \geq \chi(\|u\|_{(s,i)}) \quad \forall\ (s,i) \in \operatorname{dom} x \cap ([t_0, t_1] \times \{j_0, \dots, j_1\}), \tag{9}$$

by (8) and (4) we have

$$\begin{aligned} V(x(t_1, j_1)) &\leq e^{-d(j_1 - j_0) - c(t_1 - t_0)} V(x(t_0, j_0)) \\ &\leq e^{-\eta(j_1 - j_0) - \lambda(t_1 - t_0) + \mu} V(x(t_0, j_0)). \end{aligned} \tag{10}$$

For an arbitrary $(t, j) \in \operatorname{dom} x$, if (9) holds with $(t_1, j_1) = (t, j)$ and $(t_0, j_0) = (0, 0)$, then (10) implies

$$|x(t,j)|_{\mathcal{A}} \leq \beta(|x(0,0)|_{\mathcal{A}}, t+j), \tag{11}$$

where $\beta(r, l) := \psi_1^{-1}(e^{-l \min\{\eta, \lambda\} + \mu} \psi_2(r))$ is of class $\mathcal{KL}$. On the other hand, for any $(t, j) \in \operatorname{dom} x$ define $(t', j') := \arg\max_{\operatorname{dom} x \cap ([0,t] \times \{0,\dots,j\})} \{s + i : V(x(s,i)) \leq \chi(\|u\|_{(s,i)})\}$, then (10) implies $V(x(t,j)) \leq e^{-\eta(j-j') - \lambda(t-t') + \mu} e^{|d|} V(x(t', j')) \leq e^{\mu + |d|} \chi(\|u\|_{(t',j')})$, which further implies

$$|x(t,j)|_{\mathcal{A}} \leq \gamma(\|u\|_{(t,j)}), \tag{12}$$

where $\gamma(r) := \psi_1^{-1}(e^{\mu + |d|} \psi_2(r))$ is of class $\mathcal{K}_\infty$.

Combining (11) and (12) shows that (2) is satisfied for all $(x, u) \in \mathcal{S}[\eta, \lambda, \mu]$ and all $(t, j) \in \operatorname{dom} x$. ■

Remark 3: Notice that (8) cannot be satisfied if c and d are both negative. It is easily to verify that for $c > 0$, the claim of Proposition 1 holds with $\eta = 0$; analogously, for $d > 0$, it holds with $\lambda = 0$.

Remark 4: For $d < 0$, condition (8) can be transformed to the average dwell-time condition [24] via division by $-(d - \eta)$; analogously, for $c < 0$, it can be transformed to the reverse average dwell-time condition [25] via division by $-(c - \lambda)$.

[2] That is, either the continuous or the discrete dynamics taken alone is ISS, but not both; see [4] and [23] for the definition of ISS for continuous and discrete dynamics, respectively.

With an exponential ISS Lyapunov function, we can find the pre-ISS set of solution pairs of (1) via Proposition 1. In the following section we investigate the construction of an exponential ISS Lyapunov function for an interconnection of hybrid systems.

III. Interconnections and small-gain theorems

Consider an interconnection of n hybrid subsystems with states $x_i \in X_i \subset \mathbb{R}^{N_i}$, $i = 1, \dots, n$, and a common external input $u \in U \subset \mathbb{R}^M$. Let $x := (x_1, \dots, x_n)$, the interconnection can be modeled as

$$\Sigma : \begin{array}{ll} \dot{x}_i \in f_i(x,u), & (x,u) \in C, \\ x_i^+ \in g_i(x,u), & (x,u) \in D, \end{array} \quad i = 1, \dots, n \tag{13}$$

with $f_i : C \rightrightarrows \mathbb{R}^{N_i}$, $g_i : D \rightrightarrows X_i$, $C := C_1 \times \cdots \times C_n \times C_u$ and $D := D_1 \times \cdots \times D_n \times D_u$, where $C_i, D_i \subset X_i$, $i = 1, \dots, n$ and $C_u, D_u \subset U$.

For the i-th subsystem of (13) (denoted by Σ_i), the states x_j of Σ_j, $j \neq i$ are treated as (internal) inputs. Note that the sets C and D coincide for all subsystems as well as for the interconnection. This justifies the view of the system (13) as an interconnection of n hybrid subsystems.

We define $N := \sum_{i=1}^n N_i$ and $X := X_1 \times \cdots \times X_n$. The interconnection (13) can be viewed as a single hybrid system (1) if we define functions $f : C \rightrightarrows \mathbb{R}^N$ by $f := f_1 \times \ldots \times f_n$ and $g : D \rightrightarrows X$ by $g := g_1 \times \ldots \times g_n$.

In the following, we specialize the definition of an ISS Lyapunov function for Σ_i according to Lemma 2.

Definition 3: A Lipschitz continuous function $V_i : X_i \to \mathbb{R}_+$ is an *ISS Lyapunov function for Σ_i w.r.t. $\mathcal{A}_i \subset X_i$*, if the following three properties hold:

1) There exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that

$$\psi_{i1}(|x_i|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \psi_{i2}(|x_i|_{\mathcal{A}_i}) \quad \forall\ x_i \in X_i. \tag{14}$$

2) There exist $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1, \dots, n$ and a continuous function $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi_i(0) = 0$ such that $\chi_{ii} \equiv 0$, and for all $(x, u) \in C$ and all $y_i \in f_i(x, u)$,

$$V_i(x_i) \geq \max\left\{ \max_{j=1}^n \chi_{ij}(V_j(x_j)), \chi_i(|u|) \right\} \tag{15}$$

implies

$$\dot{V}_i(x_i; y_i) \leq -\varphi_i\left(V_i(x_i)\right). \tag{16}$$

3) There exists $\alpha_i \in \mathcal{P}$ such that for the gains χ_{ij}, χ_i defined above, for all $(x, u) \in D$ and all $y_i \in g_i(x, u)$, we have

$$V_i(y_i) \leq \max\left\{ \alpha_i(V_i(x_i)), \max_{j=1}^n \chi_{ij}(V_j(x_j)), \chi_i(|u|) \right\}. \tag{17}$$

In addition, if $\varphi_i(r) = c_i r$ and $\alpha_i(r) = e^{-d_i} r$ for all $r \in \mathbb{R}_+$ for some $c_i, d_i \in \mathbb{R}$, then V_i is an *exponential ISS Lyapunov function for Σ_i w.r.t. $\mathcal{A}_i$ with rate coefficients c_i, d_i*.

As we will see, the question, whether the interconnection (13) is ISS, depends on the properties of the *gain operator* $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined for $s = (s_1, \dots, s_n) \in \mathbb{R}_+^n$ by

$$\Gamma(s) := \left(\max_{j=1}^n \chi_{1j}(s_j), \dots, \max_{j=1}^n \chi_{nj}(s_j) \right). \tag{18}$$

To construct an ISS Lyapunov function for the interconnection (13), we adopt the notion of Ω-path [8].

Definition 4: Given a gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, a function $\sigma = (\sigma_1, \dots, \sigma_n) : \mathbb{R}_+ \to \mathbb{R}^n_+$, where $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$ is called an *Ω-path* (w.r.t. Γ), if the following properties hold:

1) For all $i \in \{1, \dots, n\}$, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;
2) For every compact set $P \subset (0, \infty)$, there are finite constants $0 < K_1 < K_2$ such that for all $i \in \{1, \dots, n\}$ and all points of differentiability of σ_i^{-1}, we have
$$0 < K_1 \le (\sigma_i^{-1})'(r) \le K_2 \qquad \forall\ r \in P;$$
3) It holds that
$$\Gamma(\sigma(r)) < \sigma(r) \qquad \forall\ r > 0. \tag{19}$$

We say that Γ satisfies the *small-gain condition* if
$$\Gamma(s) \not\ge s \qquad \forall\ s \in \mathbb{R}^n_+ \backslash \{0\}. \tag{20}$$

As reported in [26, Proposition 2.7 and Remark 2.8], we are able to construct an Ω-path σ w.r.t. Γ provided (20) is satisfied. Furthermore, σ can be made smooth via standard mollification arguments; cf. [27, Appendix B.2]. The following theorem shows that, if Γ satisfies the small-gain condition (20), an ISS Lyapunov function for the interconnection (13) can be constructed from the ISS Lyapunov functions for the subsystems and the corresponding Ω-path.

Theorem 2: Consider the interconnection (13). Let V_i be an ISS Lyapunov function for the subsystem Σ_i w.r.t. $\mathcal{A}_i \subset X_i$ with corresponding gains $\chi_{ij}, \chi_i \in \mathcal{K}$. If the gain operator Γ defined by (18) satisfies the small-gain condition (20), the function $V : X \to \mathbb{R}_+$ defined as
$$V(x) := \max_{i=1}^{n} \sigma_i^{-1}(V_i(x_i)), \tag{21}$$
where $\sigma = (\sigma_1, \dots, \sigma_n)$ is a smooth Ω-path w.r.t. Γ, is an ISS Lyapunov function for (13) w.r.t. $\mathcal{A} := \mathcal{A}_1 \times \dots \times \mathcal{A}_n$.

Proof: For all $i \in \{1, \dots, n\}$, let $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ and $\alpha_i \in \mathcal{P}$ be given in (14), (16) and (17). We will show that V defined by (21) satisfies the conditions of Lemma 2. First we will show that (3) holds for ψ_1, ψ_2 on $\mathbb{R}_+$ defined as
$$\psi_1(r) := \min_{i=1}^{n} \sigma_i^{-1}(\psi_{i1}(r/\sqrt{n})),$$
$$\psi_2(r) := \max_{i=1}^{n} \sigma_i^{-1}(\psi_{i2}(r)).$$
Since $\sigma_i, \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ for all i, we know that $\psi_1, \psi_2 \in \mathcal{K}_\infty$. Then (3) is satisfied according to (14). In particular,
$$\begin{aligned}\psi_1(|x|_\mathcal{A}) &= \min_{i=1}^{n} \sigma_i^{-1}(\psi_{i1}(|x|_\mathcal{A}/\sqrt{n})) \\ &\le \min_{i=1}^{n} \sigma_i^{-1}\left(\psi_{i1}\left(\max_{j=1}^{n} |x_j|_{\mathcal{A}_j}\right)\right) \\ &\le \max_{j=1}^{n} \sigma_j^{-1}(\psi_{j1}(|x_j|_{\mathcal{A}_j})) \\ &\le \max_{j=1}^{n} \sigma_j^{-1}(V_j(x_j)) = V(x).\end{aligned}$$

Now we will prove that (6) holds for χ, φ on $\mathbb{R}_+$ defined as
$$\chi(r) := \max_{i=1}^{n} \sigma_i^{-1}(\chi_i(r)). \tag{22}$$
$$\varphi(r) := \min_{i=1}^{n} (\sigma_i^{-1})'(\sigma_i(r))\varphi_i(\sigma_i(r)). \tag{23}$$
Since for all i, $\sigma_i \in \mathcal{K}_\infty$ is smooth, $\chi_i \in \mathcal{K}$, φ_i is continuous and $\varphi_i(0) = 0$, we know that $\chi \in \mathcal{K}_\infty$, φ is continuous and $\varphi(0) = 0$. For each i, define a set M_i as
$$M_i := \left\{x \in X : \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)) \quad \forall\ j : j \ne i\right\}$$
From the continuity of all V_i and σ_i^{-1}, it follows that all M_i are open, $X = \cup_{i=1}^n \overline{M}_i$ and $M_i \cap M_j = \varnothing$ for all i, j such that $i \ne j$. Thus for all $(x, u) \in C$, there are two possibilities:

1) There exists a unique $i \in \{1, \dots, n\}$ such that $x \in M_i$. Then $V(x) = \sigma_i^{-1}(V_i(x_i))$ and
$$V(x) > \sigma_j^{-1}(V_j(x_j)) \qquad \forall\ j : j \ne i.$$
Assume $V(x) \ge \chi(|u|)$, the definition of χ implies that
$$\sigma_i^{-1}(V_i(x_i)) \ge \sigma_i^{-1}(\chi_i(|u|)),$$
and thus $V_i(x_i) \ge \chi_i(|u|)$. Furthermore, (19) implies that
$$\begin{aligned}V_i(x_i) &= \sigma_i(V(x)) \\ &\ge \max_{j=1}^{n} \chi_{ij}(\sigma_j(V(x))) \\ &> \max_{j=1}^{n} \chi_{ij}(\sigma_j(\sigma_j^{-1}(V_j(x_j)))) \\ &= \max_{j=1}^{n} \chi_{ij}(V_j(x_j)).\end{aligned}$$
Thus (15), and therefore (16), is satisfied. Since $x \in M_i$, by the definition of Dini derivative, (16) implies that,
$$\begin{aligned}\dot{V}(x; y) &= \frac{\mathrm{d}\sigma_i^{-1}(V_i(x_i))}{\mathrm{d}t} \\ &= (\sigma_i^{-1})'(V_i(x_i))\frac{\mathrm{d}V_i(x_i)}{\mathrm{d}t} \\ &\le -(\sigma_i^{-1})'(V_i(x_i))\varphi_i(V_i(x_i)) \\ &= -(\sigma_i^{-1})'(\sigma_i(V(x)))\varphi_i(\sigma_i(V(x))) \\ &\le -\varphi(V(x))\end{aligned}$$
for all $y \in f(x, u)$.
2) Let $x \in \cap_{i \in I(x)} \partial M_i$ for an index set $I(x) \subset \{1, \dots, n\}$ with $|I(x)| \ge 2$. Then we know that, according to the standard arguments (see, e.g., [28, proof of Theorem 4]),
$$\begin{aligned}\dot{V}(x; y) &= \max_{i \in I(x)} \frac{\mathrm{d}\sigma_i^{-1}(V_i(x_i))}{\mathrm{d}t} \\ &\le \max_{i \in I(x)} -(\sigma_i^{-1})'(\sigma_i(V(x)))\varphi_i(\sigma_i(V(x))) \\ &\le -\varphi(V(x))\end{aligned}$$
for all $y \in f(x, u)$.

Consequently, equation (6) in Lemma 2 is satisfied. Now we proceed to prove that (7) holds for α on $\mathbb{R}_+$ defined as
$$\alpha(r) := \max_{i=1}^{n} \left\{\sigma_i^{-1}(\alpha_i(\sigma_i(r))), \max_{j=1}^{n} \sigma_i^{-1}(\chi_{ij}(\sigma_j(r)))\right\}. \tag{24}$$

For all $(x,u) \in D$ and all $y \in g(x,u)$, there exists $i \in \{1,\dots,n\}$ such that

$$V(y) = \sigma_i^{-1}(V_i(y_i)).$$

Since $\sigma_i \in \mathcal{K}_\infty$, (17) implies that

$$V(y) \le \max\Big\{\sigma_i^{-1}(\alpha_i(V_i(x_i))), \max_{j=1}^{n} \sigma_i^{-1}(\chi_{ij}(V_j(x_j))), \sigma_i^{-1}(\chi_i(|u|))\Big\}.$$

Moreover, the definition of V implies that

$$\sigma_j(V(x)) \ge V_j(x_j) \qquad \forall\, j \in \{1,\dots,n\}.$$

Thus, from the definition of α, we know that

$$\alpha(V(x)) \ge \max\left\{\sigma_i^{-1}(\alpha_i(V_i(x_i))), \max_{j=1}^{n} \sigma_i^{-1}(\chi_{ij}(V_j(x_j)))\right\}.$$

Furthermore, the definition of χ implies that

$$\sigma_i^{-1}(\chi_i(|u|)) \le \chi(|u|).$$

Hence

$$V(y) \le \max\{\alpha(V(x)), \chi(|u|)\},$$

for all $(x,u) \in D$ and all $y \in g(x,u)$, that is, equation (7) in Lemma 2 is satisfied. Therefore, by Lemma 2 we know that V is an ISS Lyapunov function for the interconnection (13) w.r.t. $\mathcal{A}$. ■

Theorem 2 is a powerful tool in the study of the ISS property of interconnections of hybrid systems. In the remaining part of this section we will inspect some of its implications.

If all the subsystems of (13) are pre-ISS, Theorem 2 implies the following result, which generalizes [18, Theorem III.1] and [15, Theorem 3.6].

Corollary 3: Consider the interconnection (13). Let V_i be an ISS Lyapunov function for the subsystem Σ_i w.r.t. $\mathcal{A}_i \subset X_i$ with corresponding gains $\chi_{ij}, \chi_i \in \mathcal{K}$. Assume that, for all $i \in \{1,\dots,n\}$, $\varphi_i \in \mathcal{P}$ and $\alpha_i < \mathrm{id}$ in (16) and (17). If the gain operator Γ satisfies the small-gain condition (20), then the interconnection (13) is pre-ISS w.r.t. $\mathcal{A}$.

Proof: By Theorem 2, we know that V defined by (21) is an ISS Lyapunov function for (13) w.r.t. $\mathcal{A}$. Also, φ defined by (23) satisfies $\varphi \in \mathcal{P}$ since $\sigma_i \in \mathcal{K}_\infty$ and $\varphi_i \in \mathcal{P}$ for all $i \in \{1,\dots,n\}$. Meanwhile, for all i, the facts $\alpha_i < \mathrm{id}$ and $\sigma_i \in \mathcal{K}_\infty$ imply that $\sigma_i^{-1}(\alpha_i(\sigma_i(r))) < r$ for all $r \in \mathbb{R}_+$. Moreover, since σ is an Ω-path, (19) implies that for all $i,j \in \{1,\dots,n\}$, $\sigma_i^{-1}(\chi_{ij}(\sigma_j(r))) < r$ for all $r \in \mathbb{R}_+$. Hence α defined by (23) satisfies $\alpha < \mathrm{id}$. Consequently, V is decreasing during the flows as well as at the jumps, and thus (13) is pre-ISS w.r.t. $\mathcal{A}$, according to [12, Proposition 2.7] and Remark 1. ■

Since the assumptions in the Corollary 3 are quite restrictive, we now investigate the case in which there may exist some i so that either $\varphi_i \notin \mathcal{P}$ or $\alpha_i(r) > r$ for some $r > 0$ (cf. footnote 2). In this case, we are not able to show that (13) is pre-ISS, but we can establish the pre-ISS property of some set of solution pairs which neither jump too fast nor too slowly using Proposition 1. However, in general Theorem 2 is not sufficient to provide the exponential ISS Lyapunov function for the interconnection (13) needed in Proposition 1. Next we will show that such an exponential ISS Lyapunov function can be constructed if all V_i are exponential ISS Lyapunov functions with linear internal gains χ_{ij}. Denote the spectral radius of a matrix A by $\rho(A)$ and define the *gain matrix* as $\Gamma_M := (\chi_{ij})_{n\times n}$.

Theorem 4: Consider the interconnection (13). Let V_i be an exponential ISS Lyapunov function for the subsystem Σ_i w.r.t. $\mathcal{A}_i \subset X_i$ with rate coefficients $c_i, d_i \in \mathbb{R}$ and corresponding gains $\chi_{ij}, \chi_i \in \mathcal{K}$ such that $d_i \neq 0$ and χ_{ij} is linear. If $\rho(\Gamma_M) < 1$, the function $V : X \to \mathbb{R}_+$ defined as

$$V(x) := \max_{i=1}^{n} \tfrac{1}{s_i} V_i(x_i), \tag{25}$$

where $s = (s_1,\dots,s_n) \in \mathbb{R}_+^n$ and $\sigma : r \in \mathbb{R}_+ \mapsto sr$ is an Ω-path w.r.t. Γ, is an exponential ISS Lyapunov function for (13) w.r.t. $\mathcal{A}$ with rate coefficients $c, d \in \mathbb{R}$ defined as

$$c := \min_{i=1}^{n} c_i, \quad d := \min_{i,j:j\neq i}\left\{d_i, -\ln\left(\frac{s_j}{s_i}\chi_{ij}\right)\right\}. \tag{26}$$

Proof: Following [7, p. 110], the condition $\rho(\Gamma_M) < 1$ implies that the gain matrix Γ_M satisfies the small-gain condition (20), which further implies the existence of a linear Ω-path; cf. [29, p. 78]. From the definition of an exponential ISS Lyapunov function, we know that $\varphi_i(r) = c_i r$ and $\alpha_i(r) = e^{-d_i} r$ for all $r \in \mathbb{R}_+$. Moreover, by (23) and (24) we can compute that for all $r \in \mathbb{R}_+$,

$$\varphi(r) = \min_{i=1}^{n} c_i r, \quad \alpha(r) = \max\left\{\max_{i=1}^{n} e^{-d_i}, \max_{i,j=1}^{n} \tfrac{s_j}{s_i}\chi_{ij}\right\} r.$$

With $c, d \in \mathbb{R}$ defined by (26), Theorem 2 implies that V is an exponential ISS Lyapunov function for (13) w.r.t. $\mathcal{A}$ with rate coefficients c, d. ■

The following remark shows that the formula of d can be simplified in some important particular cases.

Remark 5: If the gain matrix Γ_M is irreducible, then $1 > \rho(\Gamma_M) > 0$ is an eigenvalue of Γ_M, and the corresponding eigenvector s satisfies $s \in \mathbb{R}_+^n$; cf. [30, Theorem 2.1.3, p. 27]. Since $(\Gamma(s))_i \le (\Gamma_M s)_i < s_i$ for all i, we know that s is an Ω-path w.r.t. Γ. Hence $\max_j \frac{s_j}{s_i}\chi_{ij} \le \sum_j \frac{s_j}{s_i}\chi_{ij} = \rho(\Gamma_M)$ for all i, and thus $d \ge \min_i\{d_i, -\ln(\rho(\Gamma_M))\}$.

With an exponential ISS Lyapunov function V with either $c > 0$ or $d > 0$, Proposition 1 can be used to prove ISS of the interconnection (13) for the set of solution pairs satisfying certain average dwell-time condition. However, if both $c < 0$ and $d < 0$, Proposition 1 is not applicable. In the following section we will provide a method of modifying ISS Lyapunov functions for the subsystems to handle this situation.

IV. Modifications of ISS Lyapunov functions for subsystems

Consider the interconnection (13). For all $i \in \{1,\dots,n\}$, let V_i be an exponential ISS Lyapunov function for the subsystem Σ_i w.r.t. $\mathcal{A}_i \subset X_i$ with rate coefficients c_i, d_i and linear internal gains. As mentioned in the previous section, if there exist $j, k \in \{1,\dots,n\}$ such that $c_j < 0$ and $d_k < 0$, then (26) implies that $c < 0$ and $d < 0$ in Theorem 4 and Proposition 1 is not applicable. In the following we will construct new exponential ISS Lyapunov functions for the subsystems with rate coefficients $\tilde{c}_i$, $\tilde{d}_i$ such that either $\tilde{c}_i > 0$ for all i (continuous dynamics are ISS) or $\tilde{d}_i > 0$ for all i (discrete dynamics are ISS). To achieve this we first restrict the frequency of jumps and then modify the ISS Lyapunov functions for the corresponding subsystems.

A. Making discrete dynamics ISS

In this subsection, we will construct exponential ISS Lyapunov functions with rate coefficients $\tilde{d}_i > 0$ for all i.

Define $I_d := \{i \in \{1, \ldots, n\} : d_i < 0\}$. Pick any solution pair (x, u) of (13) and let $(t, j), (s, k) \in \operatorname{dom} x$. For any $i \in I_d$, we restrict the frequency of jumps of the subsystem Σ_i by the *average dwell-time (ADT) condition*:

$$j - k \le \delta_i(t - s) + N_0^i, \tag{27}$$

where $\delta_i, N_0^i > 0$.

It is shown (cf. [31, Appendix]) that a hybrid time domain satisfies (27) if and only if it is the domain of some solution pair to the following hybrid system of the *clock* τ_i:

$$\begin{aligned} \dot{\tau}_i &\in [0, \delta_i], \quad \tau_i \in [0, N_0^i], \\ \tau_i^+ &= \tau_i - 1, \quad \tau_i \in [1, N_0^i]. \end{aligned} \tag{28}$$

Define $z_i := x_i$, $Z_i := X_i$ for $i \notin I_d$ and $z_i := (x_i, \tau_i)$, $Z_i := X_i \times [0, N_0^i]$ for $i \in I_d$. Let $z := (z_1, \ldots, z_n)$, a modified interconnection $\tilde{\Sigma}$ can be modeled as

$$\tilde{\Sigma} : \begin{array}{ll} \dot{z}_i \in \tilde{f}_i(z, u), & (z, u) \in \tilde{C}, \\ z_i^+ \in \tilde{g}_i(z, u), & (z, u) \in \tilde{D}, \end{array} \quad i = 1, \ldots, n, \tag{29}$$

where $\tilde{f}_i(z, u) := f_i(x, u)$, $\tilde{g}_i(z, u) := g_i(x, u)$ for $i \notin I_d$;

$$\tilde{f}_i(z, u) := \begin{bmatrix} f_i(x, u) \\ [0, \delta_i] \end{bmatrix}, \quad \tilde{g}_i(z, u) := \begin{bmatrix} g_i(x, u) \\ \{\tau_i - 1\} \end{bmatrix}$$

for $i \in I_d$, $\tilde{C} := \tilde{C}_1 \times \cdots \times \tilde{C}_n \times C_u$ and $\tilde{D} := \tilde{D}_1 \times \cdots \times \tilde{D}_n \times D_u$, where $\tilde{C}_i = C_i$, $\tilde{D}_i = D_i$ for $i \notin I_d$ and $\tilde{C}_i = C_i \times [0, N_0^i]$, $\tilde{D}_i = D_i \times [1, N_0^i]$ for $i \in I_d$. Let $\tilde{f} := \tilde{f}_1 \times \cdots \times \tilde{f}_n$, $\tilde{g} := \tilde{g}_1 \times \cdots \times \tilde{g}_n$, the interconnection with clock (29) is fully defined by its data $\tilde{H} := (\tilde{f}, \tilde{g}, \tilde{C}, \tilde{D})$.

To study the ISS property of $\tilde{\Sigma}$, consider the following ISS Lyapunov function candidate for the subsystem $\tilde{\Sigma}_i$:

$$W_i(z_i) := \begin{cases} V_i(x_i), & i \notin I_d, \\ e^{L_i \tau_i} V_i(x_i), & i \in I_d \end{cases} \tag{30}$$

for some constant $L_i > 0$.

Let $\tilde{\mathcal{A}}_i := \mathcal{A}_i$ for $i \notin I_d$ and $\tilde{\mathcal{A}}_i := \mathcal{A}_i \times [0, N_0^i]$ for $i \in I_d$, we will prove the following proposition:

Proposition 5: Function W_i is an exponential ISS Lyapunov function for the subsystem $\tilde{\Sigma}_i$ w.r.t. $\tilde{\mathcal{A}}_i$. In particular:

1) There exist $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that

$$\tilde{\psi}_{i1}(|z_i|_{\tilde{\mathcal{A}}_i}) \le W_i(z_i) \le \tilde{\psi}_{i2}(|z_i|_{\tilde{\mathcal{A}}_i}) \quad \forall\, z_i \in Z_i. \tag{31}$$

2) For all $(z, u) \in \tilde{C}$ and all $y_i \in \tilde{f}_i(z, u)$,

$$W_i(z_i) \ge \max \left\{ \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u| \right\} \tag{32}$$

implies that

$$\dot{W}_i(z_i; y_i) \le -\tilde{c}_i W_i(z_i), \tag{33}$$

where $\tilde{c}_i = c_i$ for $i \notin I_d$; $\tilde{c}_i = c_i - L_i \delta_i$ for $i \in I_d$ and the new gains $\tilde{\chi}_i$ and $\tilde{\chi}_{ij}$, $j = 1, \ldots, n$ are defined as

$$\begin{aligned} \tilde{\chi}_i &:= \chi_i, & \tilde{\chi}_{ij} &:= \chi_{ij}, & i &\notin I_d, \\ \tilde{\chi}_i &:= e^{L_i N_0^i} \chi_i, & \tilde{\chi}_{ij} &:= e^{L_i N_0^i} \chi_{ij}, & i &\in I_d. \end{aligned} \tag{34}$$

3) For all $(z, u) \in \tilde{D}$ and all $y_i \in \tilde{g}_i(z, u)$,

$$W_i(y_i) \le \max \left\{ e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(z_j), \tilde{\chi}_i |u| \right\}, \tag{35}$$

where $\tilde{d}_i = d_i$ for $i \notin I_d$ and $\tilde{d}_i = d_i + L_i$ for $i \in I_d$.

Proof: For $i \notin I_d$ the claim is obvious. Thus, let $i \in I_d$. For all $z_i = (x_i, \tau_i) \in Z_i$, (14) and (30) implies that

$$\begin{aligned} W_i(z_i) &\ge V_i(x_i) \ge \psi_{i1}(|x_i|_{\mathcal{A}_i}), \\ W_i(z_i) &\le e^{L_i N_0^i} V_i(x_i) \le e^{L_i N_0^i} \psi_{i2}(|x_i|_{\mathcal{A}_i}). \end{aligned} \tag{36}$$

Thus (31) holds for $\tilde{\psi}_{i1} := \psi_{i1}$ and $\tilde{\psi}_{i2} := e^{L_i N_0^i} \psi_{i2}$.

Because of (36), we know that (32) implies (15), which in turn implies (16). From (16) and the dynamics of the clock (28), we know that, for all $(z, u) \in \tilde{C}$ and all $y_i = (y_i^1, y_i^2) \in \tilde{f}_i(z, u)$,

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{L_i \tau_i} \dot{V}_i(x_i; y_i^1) + L_i e^{L_i \tau_i} V_i(x_i) y_i^2 \\ &\le -(c_i - L_i \delta_i) W_i(z_i). \end{aligned}$$

For all $(z, u) \in \tilde{D}$ and all $y_i = (y_i^1, y_i^2) \in \tilde{g}_i(z, u)$, (17) and (30) implies that

$$\begin{aligned} & W_i(y_i) = e^{L_i(\tau_i - 1)} V_i(y_i^1) \\ &\le e^{L_i(\tau_i - 1)} \max \left\{ e^{-d_i} V_i(x_i), \max_{j=1}^{n} \chi_{ij} V_j(x_j), \chi_i |u| \right\} \\ &\le e^{L_i(\tau_i - 1)} \max \left\{ e^{-d_i - L_i \tau_i} W_i(z_i), \max_{j=1}^{n} \chi_{ij} W_j(z_j), \chi_i |u| \right\} \\ &\le \max \left\{ e^{-d_i - L_i} W_i(z_i), \max_{j=1}^{n} e^{L_i N_0^i} \chi_{ij} W_j(x_j), e^{L_i N_0^i} \chi_i |u| \right\} \\ &\le \max \left\{ e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(x_j), \tilde{\chi}_i |u| \right\}. \end{aligned}$$

Thus W_i is an exponential ISS Lyapunov function for the subsystem $\tilde{\Sigma}_i$ w.r.t. $\tilde{\mathcal{A}}_i$ with rate coefficients $\tilde{c}_i, \tilde{d}_i$. ■

Remark 6: Proposition 5 shows that we can make $\tilde{d}_i > 0$ by choosing large enough L_i, at the cost of decreasing $\tilde{c}_i$ and increasing the internal gains $\tilde{\chi}_{ij}$ (which is worse). According to (34), an increase of the chatter bound N_0^i leads to the increase of the gains $\tilde{\chi}_{ij}$, and thus for large enough N_0^i, gain operator $\tilde{\Gamma}$ will fail to satisfy the small-gain condition (20) (unless the interconnection is not a cascade).

To see the consequences of this fact clearer, let us consider for simplicity an interconnection of 2 systems, possessing exponential ISS Lyapunov functions V_1, V_2 with linear internal gains. Let also $c_1 > 0$, $d_1 < 0$, $c_2 < 0$ and $d_2 > 0$. Then Theorem 2 is not applicable, and we perform the scheme developed in this section. After adding clocks to the first subsystem, the modified gain operator $\tilde{\Gamma}$ takes the form

$$\tilde{\Gamma} = \begin{bmatrix} 0 & \tilde{\chi}_{12} \\ \tilde{\chi}_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{L_1 N_0^1}\chi_{12} \\ \chi_{21} & 0 \end{bmatrix}.$$

For $n = 2$ the small-gain condition (20) for Γ is equivalent to $\chi_{12}(\chi_{21}(r)) < r$ for all $r > 0$ [7, p.108]. Applying this fact to $\tilde{\Gamma}$ gives that $\tilde{\Gamma}$ satisfies the small-gain condition iff

$$\chi_{12}\chi_{21} < e^{-L_1 N_0^1}. \tag{37}$$

For any given N_0^1 and δ_1, we have the freedom to select any $L_1 > -d_1$. Thus we choose $L_1 = -d_1 + \varepsilon$ for $\varepsilon > 0$ arbitrarily small, which implies that

$$\chi_{12}\chi_{21} \le e^{d_1 N_0^1} < 1.$$

Moreover, for $N_0^1 = 1$ the average dwell-time condition is resolved to the fixed dwell-time condition [19, p. 1976], and if $N_0^1 < 1$, then the jumps are not allowed at all (this can be seen directly from (27) by taking $t - s$ small enough). Thus if the original gains do not satisfy the condition

$$\chi_{12}\chi_{21} \le e^{d_1},$$

the construction developed in this section cannot be applied.

Above observations hint us that one should add clocks to as small number of subsystems as possible, and it may be better to make all $\tilde{c}_i > 0$ (instead of all $\tilde{d}_i > 0$ as we have in this subsection).

B. Making continuous dynamics ISS

In this subsection, we will construct exponential ISS Lyapunov functions with rate coefficients $\tilde{c}_i > 0$ for all i.

Define $I_c := \{i \in \{1, \ldots, n\} : c_i < 0\}$. Pick any solution pair (x, u) of (13) and let $(t, j), (s, k) \in \operatorname{dom} x$. For any $i \in I_c$, we restrict the frequency of jumps of the subsystem Σ_i by the *reverse average dwell-time (RADT) condition*:

$$t - s \le \delta_i(j - k) + N_0^i \delta_i, \tag{38}$$

where $\delta_i, N_0^i > 0$.

It is shown (cf. [31, Appendix]) that a hybrid time domain satisfies (38) if and only if it is the domain of some solution pair to the following hybrid system of the *clock* τ_i:

$$\begin{aligned} \dot{\tau}_i &= 1, & \tau_i &\in [0, N_0^i \delta_i], \\ \tau_i^+ &= \max\{0, \tau_i - \delta_i\}, & \tau_i &\in [0, N_0^i \delta_i]. \end{aligned} \tag{39}$$

Define $z_i := x_i$, $Z_i := X_i$ for $i \notin I_c$ and $z_i := (x_i, \tau_i)$, $Z_i := X_i \times [0, N_0^i \delta_i]$ for $i \in I_c$. Let $z := (z_1, \ldots, z_n)$, a modified interconnection $\tilde{\Sigma}$ can be modeled as (29) with $\tilde{f}_i(z, u) := f_i(x, u)$, $\tilde{g}_i(z, u) := g_i(x, u)$ for $i \notin I_c$;

$$\tilde{f}_i(z, u) := \begin{bmatrix} f_i(x, u) \\ \{1\} \end{bmatrix}, \quad \tilde{g}_i(z, u) := \begin{bmatrix} g_i(x, u) \\ \max\{0, \tau_i - \delta_i\} \end{bmatrix}$$

for $i \in I_c$, $\tilde{C} := \tilde{C}_1 \times \cdots \times \tilde{C}_n \times C_u$ and $\tilde{D} := \tilde{D}_1 \times \cdots \times \tilde{D}_n \times D_u$, where $\tilde{C}_i = C_i$, $\tilde{D}_i = D_i$ for $i \notin I_c$ and $\tilde{C}_i = C_i \times [0, N_0^i \delta_i]$, $\tilde{D}_i = D_i \times [0, N_0^i \delta_i]$ for $i \in I_c$.

To study the ISS property of $\tilde{\Sigma}$, consider the following ISS Lyapunov function candidate for the subsystem $\tilde{\Sigma}_i$:

$$W_i(z_i) := \begin{cases} V_i(x_i), & i \notin I_d, \\ e^{-L_i \tau_i} V_i(x_i), & i \in I_d \end{cases} \tag{40}$$

for some constant $L_i > 0$.

Let $\tilde{\mathcal{A}}_i := \mathcal{A}_i$ for $i \notin I_c$ and $\tilde{\mathcal{A}}_i := \mathcal{A}_i \times [0, N_0^i \delta_i]$ for $i \in I_c$, we will prove the following proposition:

Proposition 6: Function W_i is an exponential ISS Lyapunov function for the subsystem $\tilde{\Sigma}_i$ w.r.t. $\tilde{\mathcal{A}}_i$. In particular:

1) There exist $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that (31) holds.
2) For all $(z, u) \in \tilde{C}$ and all $y_i \in \tilde{f}_i(z, u)$, (32) implies (33), where $\tilde{c}_i = c_i$ for $i \notin I_c$; $\tilde{c}_i = c_i - L_i$ for $i \in I_d$ and the new gains $\tilde{\chi}_i$ and $\tilde{\chi}_{ij}$, $j = 1, \ldots, n$ are defined as

$$\begin{aligned} &\tilde{\chi}_i := \chi_i, \quad \tilde{\chi}_{ij} := \chi_{ij}, & i \notin I_c, \\ &\tilde{\chi}_i := \chi_i, \quad \tilde{\chi}_{ij} := e^{L_j N_0^j \delta_j} \chi_{ij}, & i \in I_c. \end{aligned} \tag{41}$$

3) For all $(z, u) \in \tilde{D}$ and all $y_i \in \tilde{g}_i(z, u)$, (35) holds with $\tilde{d}_i = d_i$ for $i \notin I_c$ and $\tilde{d}_i = d_i - L_i \delta_i$ for $i \in I_d$.

Proof: For $i \notin I_c$ the claim is obvious. Thus, let $i \in I_c$. For all $z_i = (x_i, \tau_i) \in Z_i$, (14) and (40) implies that

$$\begin{aligned} &W_i(z_i) \ge e^{-L_i N_0^i \delta_i} V_i(x_i) \ge e^{-L_i N_0^i \delta_i} \psi_{i1}(|x_i|_{\mathcal{A}_i}), \\ &W_i(z_i) \le V_i(x_i) \le \psi_{i2}(|x_i|_{\mathcal{A}_i}). \end{aligned} \tag{42}$$

Thus (31) holds for $\tilde{\psi}_{i1} := e^{-L_i N_0^i \delta_i} \psi_{i1}$ and $\tilde{\psi}_{i2} := \psi_{i2}$.

Because of (42), we know that (32) implies (15), which in turn implies (16). From (16) and the dynamics of the clock (39), we know that, for all $(z, u) \in \tilde{C}$ and all $y_i = (y_i^1, y_i^2) \in \tilde{f}_i(z, u)$,

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{-L_i \tau_i} \dot{V}_i(x_i; y_i^1) - L_i e^{-L_i \tau_i} V_i(x_i) y_i^2 \\ &\le -(c_i - L_i) W_i(z_i). \end{aligned}$$

For all $(z, u) \in \tilde{D}$ and all $y_i = (y_i^1, y_i^2) \in \tilde{g}_i(z, u)$, (17) and (40) implies that

$$\begin{aligned} &W_i(y_i) = e^{-L_i \max\{0, \tau_i - \delta_i\}} V_i(y_i^1) \\ &\le e^{-L_i \max\{0, \tau_i - \delta_i\}} \max\Big\{ e^{-d_i} V_i(x_i), \max_{j=1}^{n} \chi_{ij} V_j(x_j), \chi_i |u| \Big\} \\ &\le \max\Big\{ e^{-d_i - L_i(\tau_i - \delta_i)} V_i(x_i), \max_{j=1}^{n} \chi_{ij} V_j(x_j), \chi_i |u| \Big\} \\ &\le \max\Big\{ e^{-d_i + L_i \delta_i} W_i(z_i), \max_{j=1}^{n} e^{L_j N_0^j \delta_j} \chi_{ij} W_j(z_j), \chi_i |u| \Big\} \\ &\le \max\Big\{ e^{-(d_i - L_i \delta_i)} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij} W_j(x_j), \tilde{\chi}_i |u| \Big\}. \end{aligned}$$

Thus W_i is an exponential ISS Lyapunov function for the subsystem $\tilde{\Sigma}_i$ w.r.t. $\tilde{\mathcal{A}}_i$ with rate coefficients $\tilde{c}_i, \tilde{d}_i$. ■

V. DISCUSSION, CONCLUSION AND DIRECTIONS FOR A FUTURE RESEARCH

In this paper we have proved several small-gain theorems for interconnected hybrid systems, resulting in the construction of an ISS Lyapunov function for the interconnection. These results unify various Lyapunov-based small-gain theorems for hybrid [18], [15], [17] and impulsive systems [19],

[20], [21] and pave the way to the following general scheme of analysis of ISS for interconnected hybrid systems:

1) For all $i \in \{1, \ldots, n\}$, construct an exponential ISS Lyapunov function V_i for Σ_i with linear internal gains and rate coefficients c_i, d_i.
2) Compute I_d, I_c.
3) Modify Σ_i *either for all* $i \in I_d$ *or for all* $i \in I_c$.
4) Use Theorem 4 to construct an exponential ISS Lyapunov function W for $\tilde{\Sigma}$ with rate coefficients c, d.
5) Obtain the conditions on ISS of $\tilde{\Sigma}$ via Proposition 1.
6) Obtain the conditions on ISS of the original system Σ.

As we know from Section IV, the modification of Lyapunov functions (step 3) leads to the substantial increase of internal gains. Therefore a considerable improvement of the above scheme in compare to the method described in [18] lies in the fact that only subsystems with the indices in I_d or I_c should be modified instead of all subsystems from $I_d \cup I_c$, as it is in [18]. If either $I_d = \varnothing$ or $I_c = \varnothing$, then subsystems do not have to be modified at all. Moreover, the above method is valid for arbitrary interconnections of $n \geq 2$ systems.

In the above scheme it is assumed that all V_i are exponential ISS Lyapunov functions with linear gains. However, the modification method works without any changes for exponential Lyapunov functions with *nonlinear* gains, and Theorem 2 has been proved for arbitrary ISS Lyapunov functions with nonlinear internal gains. Having generalized Proposition 1 to the case of non-exponential Lyapunov functions one can use the above scheme also for the Lyapunov functions with nonlinear gains. Such a theorem has been proved in [28] for impulsive systems and we believe, that it can be generalized to the hybrid systems. This is one of the possible directions for future research.

More challenging is the question whether one can get rid of the modifications of Lyapunov functions at all. At the time this question remains completely open.

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