2021 61st IEEE Conference on Decision and Control (CDC)
December 6-9, 2022. Cancún, Mexico

Stability of linear systems with slow and fast time variation and switching

Daniel Liberzon and Hyungbo Shim

***Abstract*— We establish exponential stability for a class of linear systems with slow and fast time variation and switching. We use the averaging method to approximate the original system by the average system which only exhibits slow time variation and switching. We then apply a stability criterion recently developed for such systems to prove stability of the average system and, consequently, of the original system.**

***Index Terms*— Switched systems, time-varying systems, stability of linear systems.**

I. Introduction

For systems with time-varying parameters, there are well-known sufficient conditions for stability that ask the system to be stable for each frozen value of the parameters and the variation to be sufficiently slow (typically by placing some type of upper bound on the time derivative of the parameters). Such results are by now standard, especially for linear time-varying systems, and appear in textbooks; see, e.g., [4, Section 3.4], [5, Section 9.6] and the references therein.

For switched systems, which are characterized by instantaneous switching instead of continuous variation, there exist stability criteria which parallel the ones mentioned above for time-varying systems and which are also well known. They are formulated in terms of stability of each individual mode of the switched system and a slow-switching condition, typically in terms of sufficiently large (average) dwell time; see, e.g., [6] for an introduction to this class of systems and representative basic results.

D. Liberzon is with the Coordinated Science Laboratory, University of Illinois Urbana-Champaign, Urbana, IL 61801 USA (email: liberzon@uiuc.edu). His work was supported in part by the NSF grant CMMI-2106043.

H. Shim is with ASRI and the Department of Electrical and Computer Engineering, Seoul National University, Korea (email: hshim@snu.ac.kr). His work was supported in part by the National Research Foundation of Korea grant funded by the Korea government (Ministry of Science and ICT), (NRF-2017R1E1A1A03070342).

The recent work [2], [3] made apparently the first attempt to unify these two sets of results. For systems combining continuous variation and switching, these papers utilized the concept of *total variation*. This is the quantity obtained, loosely speaking, by integrating the norm of the derivative of the time-varying parameter vector (or matrix) and adding, at each switching instant, the norm of the jump. For linear systems, it was shown in [3] that exponential stability is preserved if the total variation is suitably small. It was also demonstrated that this approach allows one to recover known results for systems with only continuous variation and only switching, with the results in the latter category actually going beyond the basic ones appearing in [6, Section 3.2]. An extension to nonlinear systems was presented in [2]. See also [7] for an interesting recent extension to switched linear systems with unstable modes (and disturbances).

For systems with *fast* time variation, a well-known analysis method is based on *averaging*. In its classical formulation (see, e.g., [5, Section 10.4], [8]), it deals with periodic or nearly periodic fast-varying signals by defining the average system which is by construction time-invariant, and proving via perturbation arguments that the behavior of the original system is close to that of the average system. The averaging method has also been applied to other system classes, including switched systems [11].

While in the above references the average system is time-invariant[1], the paper [1] (see also [9]) studied the case when both slow and fast variation are present in the system, leading to a slowly time-varying average system (with the fast variation "averaged out"). The idea behind the main result of [1] is that perturbation analysis can still be used to show that the behaviors of the original system and the average system are close, and if the remaining variation in the average system is sufficiently slow then known results (mentioned earlier)

[1]The paper [10] did consider averaging constructions for hybrid systems in which the average system is also hybrid (with the same jumps as the original system).

can be used to establish stability of the average system and, consequently, of the original system.

In the setting of [1], the slow variation present both in the original and in the average system is continuous (and almost everywhere differentiable). In view of the recent results developed in [3] and [2], it seems natural to ask whether we may now be able to relax this assumption. In other words, can we now develop stability criteria for systems in which both continuous time variation and switching are present, and these can be both slow and fast?

The purpose of this paper is to give a positive answer to the above question for a special class of linear systems with slow and fast time variation and switching. The average system exhibits slow time variation and switching, and we tap into the results from [3] to show that it is exponentially stable under a suitable upper bound on its total variation. We then carry out perturbation analysis, in the spirit of [5] and [1], to establish that the original system is exponentially stable as well, provided that its fast-variation component is sufficiently fast.

II. Set-up and statement of the main result

A. The system and its average

We consider the system

$$\dot{x} = \big(A(t) + B(t/\varepsilon)\big)x. \tag{1}$$

Here $x \in \mathbb{R}^n$, $A(\cdot)$ and $B(\cdot)$ are piecewise continuous functions from $[0,\infty)$ to $\mathbb{R}^{n\times n}$, $B(\cdot)$ is periodic with a known period $T > 0$, and $\varepsilon > 0$. For small ε, we think of $A(t)$ as describing "slow" time-variation and switching in the system (this will be made precise by the assumptions below), and of $B(t/\varepsilon)$ as describing "fast" variation and switching.

Without loss of generality, we assume that $B(\cdot)$ has zero average, i.e.,

$$\frac{1}{T}\int_0^T B(s)ds = 0.$$

(We can always achieve this by redefining $A(\cdot)$ and $B(\cdot)$ to absorb the average of $B(\cdot)$ into $A(\cdot)$, if needed.) Then the *average system* corresponding to (1) is given by

$$\dot{x} = A(t)x. \tag{2}$$

We note that, by introducing a correspondence between vectors and matrices as done in [2, Section V], we can show that this definition of the average system is a special case of the more general definition of the average for nonlinear systems considered in [1], which in turn is consistent with the standard definition in [5, Section 10.4] after a time rescaling. (In the approach of [1] both the state and the slowly time-varying input are "frozen" when defining the average.)

B. Assumptions on average system

We impose assumptions on the slow variation $A(\cdot)$ to ensure that Theorem 3 from [3] applies to the average system (2). These are as follows.

Assumption 1: $A(\cdot)$ is uniformly bounded: for some $L > 0$, we have

$$\|A(t)\| \le L \quad \forall\, t \ge 0 \tag{3}$$

(here $\|\cdot\|$ is the induced matrix norm corresponding to the Euclidean norm in $\mathbb{R}^n$).

Assumption 2: The matrices $A(t)$ are uniformly Hurwitz, in the sense that for some $\kappa > 0$ the real parts of their eigenvalues satisfy

$$\operatorname{Re}\lambda_i(A(t)) \le -\kappa \quad \forall\, t \ge 0,\ i = 1, 2, \dots, n.$$

Assumptions 1 and 2 together imply that for each $\lambda \in (0,\kappa)$ there exists a $c > 0$ such that

$$\|e^{A(t)s}\| \le ce^{-\lambda s} \quad \forall\, t \ge 0,\ s \ge 0 \tag{4}$$

(see [5, Section 9.6, proof of Lemma 9.9]).

We adopt the same regularity assumptions as in [3, Assumption 2] to ensure that the results from [3] apply:

Assumption 3: $A(\cdot)$ has finitely many discontinuities on any bounded interval, is càdlàg[2], is C^1 between discontinuities, and $\dot{A}(\cdot)$ and $\|\dot{A}(\cdot)\|$ are Riemann integrable between discontinuities.

Next, we consider the *total variation* of $A(\cdot)$ as defined in [3]. On an arbitrary time interval $[a,b]$, this is given by

$$\int_a^b \|dA\| := \sum_{i=0}^{m}\int_{d_i}^{d_{i+1}} \|\dot{A}(t)\|dt + \sum_{i=1}^{m}\|A(d_i) - A(d_i^-)\| \tag{5}$$

where $d_1,\dots,d_m$ are discontinuities of $A(\cdot)$ with $a =: d_0 < d_1 < \cdots < d_m < d_{m+1} := b$, and $A(d_i^-)$ denotes the left limit of $A(\cdot)$ at d_i (the right limit equals $A(d_i)$ by Assumption 3). We refer the reader to [3] for a more intrinsic but equivalent definition of the total variation and for further discussion. Our fourth

[2]Continuous from the right, has limits from the left; this assumption is made for notational convenience.

and final assumption here places an upper bound on the total variation.

Assumption 4: The total variation satisfies the bound

$$\int_{t_1}^{t_2} \|dA\| \le \mu(t_2 - t_1) + \alpha \quad \forall t_2 \ge t_1 \ge 0 \tag{6}$$

with some $\alpha > 0$ and

$$0 < \mu < \frac{\beta_1}{2\beta_2^3} \tag{7}$$

where

$$\beta_1 := \frac{1}{2L}, \quad \beta_2 := \frac{c^2}{2\lambda}$$

and L, c, λ come from (3) and (4).

C. Main result

The main result of this paper states that under the above assumptions and for sufficiently small ε, the system (1) is globally exponentially stable (in the classical sense, with respect to the equilibrium at the origin).

Theorem 1 *Let Assumptions 1–4 hold. Then there exists an $\varepsilon^* > 0$ such that the system* (1) *is globally exponentially stable for all $\varepsilon \in (0, \varepsilon^*)$.*

The proof of this result is developed in the next section.

III. Proof of Theorem 1

The proof proceeds by, first, invoking the results from [3] to establish exponential stability of the average system (2); next, expressing the original system (1) as a perturbation of the average system; and, finally, using perturbation analysis to verify exponential stability of the original system.

A. Stability of average system

Theorem 3 from [3] establishes that, under Assumptions 1–4, the system (2) is exponentially stable. We reproduce the main steps of the proof of this result here, as they will be needed to extend the stability analysis to (1). For each $t \ge 0$ we let $P(t)$ be the unique symmetric positive definite solution to the Lyapunov equation

$$P(t)A(t) + A^T(t)P(t) = -I \tag{8}$$

and consider the candidate Lyapunov function

$$V(t,x) := x^T P(t)x \tag{9}$$

whose derivative along solutions of (2) is given, in view of (8), by

$$\dot V = -|x|^2 + x^T \dot P(t)x. \tag{10}$$

Here and below, by $\dot V$ we mean the quantity $\frac{d}{dt}V(t,x(t))$ which is only defined away from discontinuities of $A(\cdot)$, where $\dot A(t)$ exists (by Assumption 3) and consequently $\dot P(t)$ exists as well. (The last claim easily follows from the well-known formula $P(t) = \int_0^\infty e^{A^T(t)s}e^{A(t)s}ds$.) By Lemma 9.9 from [5] we have

$$\beta_1 \le \|P(t)\| \le \beta_2 \quad \forall t \ge 0 \tag{11}$$

hence

$$\beta_1|x|^2 \le V(t,x) \le \beta_2|x|^2 \quad \forall x \in \mathbb{R}^n,\ \forall t \ge 0. \tag{12}$$

Next, following the proof of Lemma 9.9 in [5] or Theorem 3.4.11 in [4], we can show that

$$\|\dot P(t)\| \le 2\beta_2^2\|\dot A(t)\|. \tag{13}$$

Plugging this bound into (10), we have

$$\dot V \le -|x|^2 + 2\beta_2^2\|\dot A(t)\||x|^2.$$

Using the bounds (12), we obtain

$$\begin{aligned}\dot V &\le -\beta_2^{-1}V + 2\beta_2^2\beta_1^{-1}\|\dot A(t)\|V\\ &= -\left(\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\|\dot A(t)\|\right)V.\end{aligned}$$

Applying the standard comparison principle, we conclude that

$$V(t_2^-) \le e^{-\int_{t_1}^{t_2^-}(\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\|\dot A(s)\|)ds}V(t_1) \tag{14}$$

for every interval $[t_1, t_2)$ containing no discontinuities of $A(\cdot)$; here $V(t)$ is a shorthand for $V(t, x(t))$.

Now, let us consider a time instant t at which $A(\cdot)$ is discontinuous: $A(t) \ne A(t^-)$. In [3] a relationship was derived between the jump in $P(\cdot)$ and the jump in $A(\cdot)$ at t, which is essentially a discrete counterpart of (13). Namely, Proposition 1 and Lemma 3 in [3] establish that

$$\|P(t) - P(t^-)\| \le 2\beta_2^2\|A(t) - A(t^-)\|$$

and that, consequently,

$$\begin{aligned}V(t) - V(t^-) &= x^T(t^-)(P(t) - P(t^-))x(t^-)\\ &\le \|P(t) - P(t^-)\||x(t^-)|^2\\ &\le 2\beta_2^2\beta_1^{-1}\|A(t) - A(t^-)\|V(t^-)\end{aligned}$$

which, applying the fact that $z + 1 \le e^z$ with $z = 2\beta_2^2\beta_1^{-1}\|A(t) - A(t^-)\|$, results in

$$V(t) \le e^{2\beta_2^2\beta_1^{-1}\|A(t)-A(t^-)\|}V(t^-). \tag{15}$$

We can now iteratively combine the two bounds (14) and (15), as done in the proof of Theorem 3 in [3], to show that for every interval $[t_1, t_2]$, possibly containing discontinuities of $A(\cdot)$, we have

$$V(t_2) \leq e^{-\beta_2^{-1}(t_2-t_1)+2\beta_2^2\beta_1^{-1}\int_{t_1}^{t_2}\|dA\|} V(t_1).$$

Finally, using the total variation bound (6), we arrive at

$$V(t_2) \leq e^{2\beta_2^2\beta_1^{-1}\alpha} e^{(2\beta_2^2\beta_1^{-1}\mu-\beta_2^{-1})(t_2-t_1)} V(t_1) \tag{16}$$

which establishes exponential stability of (2) because $2\beta_2^2\beta_1^{-1}\mu - \beta_2^{-1} < 0$ thanks to (7). For convenience, we define

$$\gamma := e^{2\beta_2^2\beta_1^{-1}\alpha}, \qquad \delta := \beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\mu > 0 \tag{17}$$

in terms of which the estimate (16) takes the form

$$V(t_2) \leq \gamma e^{-\delta(t_2-t_1)} V(t_1).$$

B. Approximation by average system

To approximate the original system (1) by the average system (2), we consider for (1) the change of variables

$$x = y + \varepsilon \int_0^{t/\varepsilon} B(s)ds \cdot y. \tag{18}$$

We can check that this change of variables is consistent with the one considered for the more general nonlinear case in [1] and, modulo time rescaling, in [5, Section 10.4], when specialized to the linear system (1). The time derivative of the left-hand side of (18) is $\dot{x}$ which, in view of (1) and (18), equals

$$A(t)y + A(t)\,\varepsilon \int_0^{t/\varepsilon} B(s)ds \cdot y + B(t/\varepsilon)y + B(t/\varepsilon)\,\varepsilon \int_0^{t/\varepsilon} B(s)ds \cdot y.$$

On the other hand, differentiating the right-hand side of (18) with respect to time gives

$$\dot{y} + \varepsilon \int_0^{t/\varepsilon} B(s)ds \cdot \dot{y} + B(t/\varepsilon)y.$$

Equating the two expressions and canceling and collecting terms, we obtain

$$\Big(I + \varepsilon \int_0^{t/\varepsilon} B(s)ds\Big)\dot{y} = A(t)y + \varepsilon\big(A(t) + B(t/\varepsilon)\big) \int_0^{t/\varepsilon} B(s)ds \cdot y.$$

Since $B(\cdot)$ is periodic and has zero average, $\int_0^\tau B(s)ds$ is bounded uniformly over τ. Thus, for ε sufficiently small, the matrix $I + \varepsilon \int_0^{t/\varepsilon} B(s)ds$ is invertible and we can write

$$\dot{y} = \Big(I + \varepsilon \int_0^{t/\varepsilon} B(s)ds\Big)^{-1} \times \Big(A(t)y + \varepsilon\big(A(t) + B(t/\varepsilon)\big) \int_0^{t/\varepsilon} B(s)ds \cdot y\Big).$$

Moreover, from the expansion $(I + \varepsilon\Lambda)^{-1} = I - \varepsilon\Lambda + \varepsilon^2\Lambda^2 - \cdots = I + O(\varepsilon)$ we see that $\big(I + \varepsilon \int_0^{t/\varepsilon} B(s)ds\big)^{-1} = I + O(\varepsilon)$. This yields the equation of the form

$$\dot{y} = A(t)y + \varepsilon C(t, \varepsilon)y \tag{19}$$

where $C(\cdot,\cdot)$ is piecewise continuous in t for fixed ε and bounded in t uniformly over ε, provided ε is small enough. We see that the y-dynamics take the form of the average system dynamics with an additional perturbation term represented by a "vanishing perturbation" in the sense of [5, Section 9.1].

C. Stability of original system by perturbation analysis

We are now ready to prove Theorem 1. We know from our earlier analysis in Section III-A that away from discontinuities of $A(\cdot)$, the derivative of the Lyapunov function $V(t, y) = y^T P(t) y$ along solutions of the system $\dot{y} = A(t)y$ satisfies

$$\begin{aligned}\dot{V} &= -|y|^2 + y^T \dot{P}(t) y \\ &\leq -|y|^2 + 2\beta_2^2 \|\dot{A}(t)\| |y|^2 \\ &\leq -\big(\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\|\dot{A}(t)\|\big)V.\end{aligned}$$

In (19) we also have the second term on the right-hand side, whose effect on $\dot{V}$ can be upper-bounded as follows:

$$\begin{aligned}\left|\frac{\partial V}{\partial y}\varepsilon C(t,\varepsilon)y\right| &\leq 2\|P(t)\|\varepsilon\|C(t,\varepsilon)\||y|^2 \\ &\leq 2\beta_2\varepsilon\|C(t,\varepsilon)\||y|^2 \\ &\leq 2\beta_2\beta_1^{-1}\varepsilon\|C(t,\varepsilon)\|V\end{aligned}$$

where we used (11) and (12). Let $\varepsilon^* > 0$ and $\nu > 0$ be such that the derivation of (19) is valid and $\|C(t,\varepsilon)\| \leq \nu$ for all $t \geq 0$ and all $\varepsilon \in (0, \varepsilon^*)$. Then for such values of ε the derivative of V along solutions of (19) satisfies

$$\dot{V} \leq -(\beta_2^{-1} - 2\beta_2\beta_1^{-1}\varepsilon\nu - 2\beta_2^2\beta_1^{-1}\|\dot{A}(t)\|)V. \tag{20}$$

On the other hand, if t is a discontinuity of $A(\cdot)$, it is clear from (18) that y is continuous at t (because so is x). Thus the formula (15) is still valid for $V(t) =$

$V(t, y(t))$. Combining this with (20) and applying the comparison principle and iterating as we did earlier for the average system, we obtain

$$V(t_2) \leq e^{-(\beta_2^{-1} - 2\beta_2\beta_1^{-1}\varepsilon\nu)(t_2 - t_1) + 2\beta_2^2\beta_1^{-1}\int_{t_1}^{t_2}\|dA\|} V(t_1).$$

From this, the total variation bound (6) and the notation introduced in (17) bring us to

$$V(t_2) \leq \gamma e^{-(\delta - 2\beta_2\beta_1^{-1}\varepsilon\nu)(t_2 - t_1)} V(t_1).$$

We now see that exponential stability is indeed preserved for $\varepsilon \in (0, \varepsilon^*)$ after we reduce ε^* if necessary so that $\delta - 2\beta_2\beta_1^{-1}\varepsilon^*\nu > 0$. In the x-coordinates, the same conclusion then holds for the original system (1) by (18), and the proof is complete.

IV. Conclusions

We studied stability of a class of linear systems with slow and fast time variation and switching. This was accomplished by combining the averaging method as used in [1] with the recent result from [3] on stability of linear systems with slow time variation and switching.

Although the class of systems (1) considered in this paper is rather special, our methodology can be extended to more general linear systems as well as to some nonlinear systems. These results will be reported in forthcoming publications.

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