2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

On norm-controllability of nonlinear systems

Matthias A. Müller, Daniel Liberzon, and Frank Allgöwer

Abstract— **In this paper, we introduce and study the notion of "norm-controllability" for nonlinear systems. This property captures the responsiveness of a system with respect to the applied inputs, which is quantified via the norm of an output map. As a main contribution, we obtain a Lyapunov-like sufficient condition for norm-controllability. Several examples illustrate the various aspects of the proposed concept, and we also further elaborate norm-controllability for the special case of linear systems.**

I. INTRODUCTION

Controllability is one of the fundamental properties in control theory. It is usually formulated as the ability to steer the state of the system from any point to any other point by choosing an appropriate control input. For linear control systems, controllability can be easily checked via necessary and sufficient matrix rank conditions, and controllable modes can be identified with the help of the Kalman controllability decomposition (see, e.g., [1]). Similar decompositions have been studied for nonlinear systems, and for some classes of nonlinear systems—most notably systems affine in controls—controllability can be characterized in terms of the rank of a certain Lie algebra of vector fields (see, e.g, [2], [3]). However, for general nonlinear control systems our understanding of point-to-point controllability is much less complete compared to the linear case, and even in those settings where tests for it are available they are more difficult to apply.

In this paper, we propose a new notion that characterizes the responsiveness of a nonlinear system to control inputs differently from point-to-point controllability. Namely, we look at the *norm* of the state and ask whether this norm can be made large by applying large enough inputs for sufficiently long time. A precise definition is given in Section III, where we actually take a more general approach and work with the norm of an output which identifies directions of interest in the state space (and may in particular be the entire state). This "norm-controllability" property can be viewed as a weaker/coarser version of the standard controllability. We believe that this concept is very natural and arises in many settings of practical interest. In process control, for example, one may want to know whether increasing the amount of reagent in a batch reaction yields an increase in the amount of product; in economics, it may be of interest to maximize the outcome or profit of a production unit, for which the effects of certain inputs such as the number of employees, production costs, etc. on the system have to be analyzed.

Our basic premise in this work is similar to that adopted in the paper [4] which introduced and studied the concept of norm-observability. Instead of observability, usually defined as the ability to reconstruct the state of the system from measurements of the output (and of the input if one is present), norm-observability is defined in terms of being able to obtain an upper bound on the norm of the state rather than the precise value of the state. For linear systems the two properties turn out to be equivalent, but for general nonlinear systems it is natural and useful to consider the latter, weaker notion, as demonstrated in [4]. In the present work we follow the corresponding path for the dual notion of controllability. (The conceptual similarity notwithstanding, the technical developments presented here and in [4] are completely different and there does not seem to be any direct duality relationship between norm-controllability and norm-observability.)

The proposed concept of norm-controllability can also be viewed as complementary to the well-known input-to-state-stability (ISS) property introduced in [5]. Loosely speaking, a system is ISS if small inputs lead to small states; norm-controllability, by contrast, asks that large inputs lead to large states. ISS has an equivalent characterization in terms of a Lyapunov function that decreases when the norm of the state is large compared to the norm of the input [6]; our main result (Theorem 1 in Section III) formulates a similar Lyapunov-like sufficient condition for norm-controllability, but a Lyapunov function should now *increase* when the norm of the state is *small* compared to the norm of the input. Our proof of this result (given in Section IV) proceeds by constructing a specific piecewise-constant control input which yields a trajectory with a suitably increasing norm. Parts of this proof were inspired by [7] where the authors construct a piecewise-constant control to asymptotically stabilize a system. (Again, while it is instructive to note the similarities, most of the technical ideas employed here are very different from those used in the above references.)

In Section V we study several examples to illustrate the new norm-controllability concept and the use of the Lyapunov-like sufficient condition for checking it. In Section VI we revisit the setting of linear systems and show, among other things, that every linear system is norm-controllable for initial conditions in its controllable subspace (and so, in particular, every controllable linear system is

Matthias A. Müller and Frank Allgöwer are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, 70550 Stuttgart, Germany. Their work was supported by the German Research Foundation (DFG) within the Priority Programme 1305 "Control Theory of Digitally Networked Dynamical Systems" and within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart. {mueller,allgower}@ist.uni-stuttgart.de

Daniel Liberzon is with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. His work was supported by the NSF ECCS-0821153 grant. liberzon@uiuc.edu

norm-controllable for all initial conditions). Finally, Section VII concludes the paper with a short summary and an outlook on future research.

II. Preliminaries and Setup

Let $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ denote the set of nonnegative real numbers. Let $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function, i.e., $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^n$. For a set $S \subseteq \mathbb{R}^n$, let $\mathrm{co}(S)$ denote its convex hull, $\overline{S}$ its closure, $\mathrm{int}(S)$ its interior, and ∂S its boundary. A function α: $\mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$. For a function $V : \mathbb{R}^n \to \mathbb{R}$, define as in [7] the lower directional derivative of V at a point $x \in \mathbb{R}^n$ in the direction of a vector $h \in \mathbb{R}^n$ as

$$V'(x; h) := \liminf_{t \searrow 0, \bar{h} \to h} \frac{V(x + t\bar{h}) - V(x)}{t}.$$

Note that at each point $x \in \mathbb{R}^n$ where V is continuously differentiable, it holds that $V'(x; h) = (\partial V/\partial x)h$. We say that a function $\omega : \mathbb{R}^n \to \mathbb{R}^l$ is *uniformly continuous on a set* $W \subseteq R^n$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in W$ and $y \in \mathbb{R}^n$ with $|x - y| < \delta$ it holds that $|\omega(x) - \omega(y)| < \varepsilon$. Note that this property is true if ω is continuous and the set W is compact. We will later use this property with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$; in this case, uniform continuity on W means that level sets of $|\omega|$ do not converge to the zero level set at ∞.

We consider nonlinear control systems of the type

$$\begin{aligned} \dot{x} &= f(x, u), \qquad x(0) = x_0, \\ y &= h(x) \end{aligned} \tag{1}$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^k$, and input $u \in U \subseteq \mathbb{R}^m$, where the set U of admissible input values can be any closed subset of $\mathbb{R}^m$ (or the whole $\mathbb{R}^m$). Suppose that f is locally Lipschitz in x and u. Input signals $u(\cdot)$ to the system (1) satisfy $u(\cdot) \in L^\infty_{loc}(\mathbb{R}_+, U)$, where $L^\infty_{loc}(\mathbb{R}_+, U)$ denotes the set of all measurable and locally bounded functions from $\mathbb{R}_+$ to U. We assume that the system (1) exhibits the *unboundedness observability* property (see [8] and the references therein), which means that for every trajectory of the system (1) with finite escape time t_{esc}, also the corresponding output becomes unbounded for $t \to t_{esc}$. This is a very reasonable assumption as one cannot expect to measure responsiveness of the system in terms of an output map h (as we will later do) if a finite escape time cannot be detected by this output map. All linear systems satisfy this assumption, as do all nonlinear systems with radially unbounded output maps.

Let $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_+$. For every $b > 0$, denote by

$$U_b := \{u \in U : b \leq |u| \leq \mu(b)\} \tag{2}$$

the set of all admissible input values with norm in the interval $[b, \mu(b)]$. Furthermore, for every $a, b > 0$, denote by

$$\mathcal{U}_{a,b} := \{u(\cdot) : u(t) \in U_b,\ \forall t \in [0, a]\} \subseteq L^\infty_{loc}(\mathbb{R}_+, U) \tag{3}$$

the set of all measurable and locally bounded input signals whose norm takes values in the interval $[b, \mu(b)]$ on the time interval $[0, a]$. Let $\mathcal{R}^\tau\{x_0, \mathcal{U}\} \subseteq \mathbb{R}^n \cup \{\infty\}$ be the reachable set of the system (1) at time $\tau \geq 0$, starting at the initial condition $x(0) = x_0$ and applying input signals $u(\cdot)$ in some set $\mathcal{U} \subseteq L^\infty_{loc}(\mathbb{R}_+, U)$. The reachable set $\mathcal{R}^\tau\{x_0, \mathcal{U}\}$ contains ∞ if for some $u(\cdot) \in \mathcal{U}$ a finite escape time $t_{esc} \leq \tau$ exists. Furthermore, let $\mathcal{R}^{\leq\tau}\{x_0, \mathcal{U}\} := \bigcup_{0 \leq t \leq \tau} \mathcal{R}^t\{x_0, \mathcal{U}\}$. Define $R^\tau_h(x_0, \mathcal{U})$ as the radius of the smallest ball in the output space centered at $y = 0$ which contains the image of the reachable set $\mathcal{R}^\tau\{x_0, \mathcal{U}\}$ under the output map $h(\cdot)$, or ∞ if this image is unbounded. In what follows, the reader may wish to keep in mind that, as pointed out in the introduction, $y = x$ (i.e., $h = \mathrm{id}$) is a meaningful special case for which the developments simplify a bit but still retain all their main features.

III. Norm-controllability: Definition and Sufficient Condition

In this section, we propose a definition of norm-controllability and establish sufficient conditions for a system to possess this property. We want to consider the "responsiveness" of the system (1) with respect to the input u, i.e., how far and in which direction can the system evolve if inputs of a certain magnitude are applied. We will quantify this responsiveness in terms of the output map h.

Definition 1: The system (1) is *norm-controllable* from x_0 with scaling function μ and gain function γ, if there exist a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_+$ and a function $\gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing in the first argument and a $\mathcal{K}_\infty$-function in the second argument, such that for all $a > 0$ and $b > 0$

$$R^a_h(x_0, \mathcal{U}_{a,b}) \geq \gamma(a, b), \tag{4}$$

where $\mathcal{U}_{a,b}$ is defined in (3). □

The above definition of norm-controllability can be interpreted in the following way. For each fixed time horizon a, if the magnitude of the applied inputs is increased, then also the smallest ball containing the image of the reachable set under the output map h has to increase, which means that the norm of the output y can be increased. On the other hand, for every fixed lower bound b on the input norm, increasing the time horizon a should not decrease the magnitude of the output. Thus norm-controllability captures both the "short-term" as well as the "long-term" responsiveness of the system (1) with respect to the input u in terms of the norm of the output map h, for which the gain γ is a quantitative measure.

The scaling function μ can be interpreted as follows. If a system is norm-controllable with $\mu = \mathrm{id}$, then for each $b > 0$ there exists an input with magnitude exactly b which is "good" in the sense that the norm of y can be increased when applying this input to the system (1). If a system is only norm-controllable for some $\mu \neq \mathrm{id}$, then only inputs with magnitude in the interval $[b, \mu(b)]$ are "good". Furthermore, the upper bound $\mu(b)$ on the input norm is essential, because working with input signals with norm in the interval $[b, \infty)$ instead of $[b, \mu(b)]$ would result in $\mathcal{U}_{a,b_2} \subseteq \mathcal{U}_{a,b_1}$ for $b_2 > b_1$, and then (4) cannot hold with $\gamma(a, \cdot)$ of class $\mathcal{K}_\infty$ unless the left-hand side equals ∞ for all b. Examples 5 and 6 illustrate the dependence of the norm-controllability property on the choice of the scaling function μ in more detail.

We are now in a position to state our main result.

Theorem 1: Suppose there exist a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^l$, $1 \leq l \leq n$ which is uniformly continuous on the set $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which is continuously differentiable with Lipschitz gradient on $\mathbb{R}^n \setminus W$, functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$, a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_+$, and for each $x \in \mathbb{R}^n$ a set $U(x) \subseteq U$, such that the following holds:

- For all $x \in \mathbb{R}^n$,

$$\nu(|\omega(x)|) \leq |h(x)| \tag{5}$$

$$\alpha_1(|\omega(x)|) \leq V(x) \leq \alpha_2(|\omega(x)|) \tag{6}$$

- For each $b > 0$ and $x \in \mathbb{R}^n$,

$$U(x) \cap U_b \neq \emptyset. \tag{7}$$

- For all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $u \in U(x)$ and $|\omega(x)| \leq \rho(|u|)$, the lower directional derivative $V'(x; f(x, u))$ satisfies

$$V'(x; f(x, u)) \geq \chi(|u|). \tag{8}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with scaling function μ and gain function

$$\gamma(r, s) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{r\chi(s) + V(x_0), \alpha_1(\rho(s))\Big\}\Big)\Big). \tag{9}$$

Remark 1: We need to allow V to be not continuously differentiable for all x where $\omega(x) = 0$ because $V \in C^1$ together with (6) would imply that the gradient of V vanishes for all x where $\omega(x) = 0$, and thus it would be impossible to satisfy (8) there. In the examples given in Section V, a typical choice will be $V(x) = |\omega(x)|$. □

Remark 2: If (8) holds not just for $|\omega(x)| \leq \rho(|u|)$ but rather for all x, then we can let $\rho \to \infty$ and γ in (9) simplifies to $\gamma(r, s) = \nu\big(\alpha_2^{-1}(r\chi(s) + V(x_0))\big)$. Note that in this case, $\gamma(r, \cdot)$ might not be of class $\mathcal{K}_\infty$, as $\gamma(r, 0) \neq 0$ if $V(x_0) \neq 0$. Nevertheless, $\gamma(r, \cdot)$ still satisfies all other properties of a class $\mathcal{K}_\infty$ function, i.e., is continuous, strictly increasing and unbounded. □

When the output map h is such that $h(x_0) = 0$, norm-controllability captures the system's ability to "move away" from the initial state x_0. This could e.g. be of interest if one wants to know how far one can move away from an initial equilibrium state (x_0, u_0). In other settings, it makes sense to consider $h(x_0) \neq 0$, e.g. in a chemical process where initially already some product is available. This allows us, for fixed h, ω, and V, to vary the initial condition x_0, and the effect of this is given by the term $V(x_0)$ in (9).

Also, there might be several possible choices for the functions ω and V satisfying the conditions of Theorem 1. In this case the degrees of freedom in the choice of ω and V can be used to maximize the gain γ in (9). Example 4 will illustrate this in more detail.

Furthermore, if systems without outputs are considered, i.e., an output map h is not given a priori, we might first search for functions ω and V satisfying the relevant conditions of Theorem 1. Then, we can quantify the responsiveness of the system with respect to every a posteriori defined output map h satisfying (5). It is also useful to note that increasing the output dimension by appending extra variables to the output cannot destroy norm-controllability (it can only help attain it).

IV. Proof of Theorem 1

In the following, we will develop two technical lemmas (the proofs of which are omitted in this conference paper due to space restrictions) and then obtain the proof of Theorem 1 by combining them. Let $a, b > 0$ be arbitrary but fixed, and assume in the sequel that the hypotheses of Theorem 1 are satisfied. Furthermore, we assume that no $u(\cdot) \in \mathcal{U}_{a,b}$ leads to a finite escape time $t_{esc} \leq a$, for otherwise, by the unboundedness observability property, also $R_h^a(x_0, \mathcal{U}_{a,b}) = \infty$, and thus (4) is satisfied with γ as in (9) and we are done.

The idea of the proof is to show that we can construct an input signal $u \in \mathcal{U}_{a,b}$ such that when applying this input signal, $\dot{V}(x(t)) \geq \chi(|u(t)|)$ holds along the resulting state trajectory $x(\cdot)$ as long as $|\omega(x(t))| \leq \rho(|u(t)|)$, from which we can obtain the desired gain function γ. We will show that using continuity arguments, we can construct a piecewise constant input signal such that this is satisfied.

The first lemma considers the initial phase and proves that we can move away from x_0 with the desired speed, and is in particular needed for the case where $\omega(x_0) = 0$, i.e., $x_0 \in W$. To this end, define the set

$$X_{b,\kappa} := \{x \in \mathbb{R}^n : \kappa \leq |\omega(x)| \leq \rho(b)\} \tag{10}$$

with κ satisfying $0 \leq \kappa \leq \rho(b)$.

Lemma 1: Let $u_0 \in U(x_0) \cap U_b$ and assume that $x_0 \in X_{b,0}$. For all $\varepsilon > 0$, there exists some $t_1 > 0$ such that for all $t \in [0, t_1]$, it holds that

$$V(x(t)) \geq (1 - \varepsilon)t\chi(b) + V(x_0), \tag{11}$$

where $x(\cdot)$ is the trajectory of the system (1) that results from applying the constant input u_0 during this time interval. □

Next we consider the situation where the state x is already away from the set W. Then, according to our assumptions, V is continuously differentiable with Lipschitz gradient, which we can use to show that if some input u_i is "good" at some point x_i, it is also "good" for nearby x_i.

Lemma 2: Consider some time instant $0 \leq t_i < a$ with $x(t_i) =: x_i \in \mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}$, and assume that $x_i \in X_{b,\delta}$ for some $\delta > 0$; furthermore, pick an arbitrary $u_i \in U(x_i) \cap U_b$. Then, for each $0 < \varepsilon \leq 1$, there exists a number $\Delta(\varepsilon, \delta) > 0$ such that

$$V(x(t)) - V(x_i) \geq (1 - \varepsilon)(t - t_i)\chi(b) \tag{12}$$

for all $t \in [t_i, t_i + \Delta(\varepsilon, \delta)) \cap [0, a]$, where $x(\cdot)$ is the trajectory that results from applying the constant input u_i during this time interval. □

Compared to Lemma 1, the statement of Lemma 2 is stronger in the sense that the time interval Δ, over which (12) holds, is valid for all $x_i \in X_{b,\delta}$, whereas t_1 in Lemma 1 in general depends on x_0.

Combining Lemmas 1 and 2, we are now able to prove Theorem 1. Fix an arbitrary $0 < \bar{\varepsilon} < 1$. Denote by Λ_b the sublevel set $\Lambda_b := \{x \in \mathbb{R}^n : V(x) \le \alpha_1(\rho(b))\}$. We construct a desired input signal in a recursive fashion using the following algorithm.

Step 0: If $x_0 \in \text{int}(\Lambda_b)$, then by (6) we have $|\omega(x_0)| < \rho(b)$ and so $x_0 \in X_{b,0}$ according to (10). We can then pick some $u_0 \in U(x_0) \cap U_b$, which exists by (7), and apply Lemma 1 with $\varepsilon = \bar{\varepsilon}$ to find a time $t_1 > 0$ such that the trajectory corresponding to the constant control $u \equiv u_0$ satisfies (11) with $\varepsilon = \bar{\varepsilon}$ for all $0 \le t \le t_1$. If already $x_0 \notin \Lambda_b$, then pick an arbitrary $u_0 \in U_b$ and let $t_1 := a$. In either case, apply the constant input $u \equiv u_0$ on the interval $[0, \min\{t_1, a\})$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in [0, \min\{t_1, a\})$, then denote this time t by $\check{t}_1$ and skip to Step 2. If this does not happen but $t_1 \ge a$, then skip to Step 3.

Step 1: If $x(t_1) \in \partial\Lambda_b$, then let $\check{t}_1 := t_1$ and skip to Step 2. Otherwise, $x(t_1) \in \text{int}(\Lambda_b)$. Let $\bar{\delta} := \alpha_2^{-1}\big(V(x(t_1))\big)$. From (6) and the definition of Λ_b we have $\bar{\delta} \le |\omega(x(t_1))| < \rho(b)$, hence $x(t_1) \in X_{b,\bar{\delta}}$ by (10). We can thus pick some $u_1 \in U(x(t_1)) \cap U_b$ and apply Lemma 2 with $\varepsilon = \bar{\varepsilon}$ and $\delta = \bar{\delta}$ to find a $\Delta(\bar{\varepsilon}, \bar{\delta})$ such that the trajectory corresponding to the constant control $u \equiv u_1$ on the interval $[t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$ satisfies (12) with $\varepsilon = \bar{\varepsilon}$ on this interval. Apply the constant input $u \equiv u_1$ on this interval for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$, then denote this time t by $\check{t}_1$ and skip to Step 2. If this does not happen but $t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}) \ge a$, then skip to Step 3. Otherwise, let $t_2 := t_1 + \Delta(\bar{\varepsilon}, \bar{\delta})$. In this case, $x(t_2) \in \Lambda_b$ and $V(x(t_2)) > V(x(t_1))$, and so we can check that $x(t_2) \in X_{b,\bar{\delta}}$ in the same way as we did earlier for $x(t_1)$. Therefore, we can repeat Step 1 for the times t_2, t_3, ... (but without changing the value of $\bar{\delta}$).

Step 2: We have $x(\check{t}_1) \in \partial\Lambda_b$, i.e., $V(x(\check{t}_1)) = \alpha_1(\rho(b))$. If $\check{t}_1 = a$ then skip to Step 3. Otherwise, pick some $\check{u}_1 \in U(x(\check{t}_1)) \cap U_b$, which exists by (7). Apply the constant input $u \equiv \check{u}_1$ on the interval $[\check{t}_1, \check{t}_2)$ where $\check{t}_2 := \min\{\inf\{t : t > \check{t}_1,\, x(t) \in \partial\Lambda_b\}, a\}$. This interval is non-empty; in fact, $\check{t}_2 \ge \min\{\check{t}_1 + \Delta(1, \check{\delta}), a\}$ where $\check{\delta} := \alpha_2^{-1}(\alpha_1(\rho(b)))$ and $\Delta(\cdot,\cdot)$ comes from Lemma 2. To see why this is true, note that $\check{\delta} \le |\omega(x(\check{t}_1))| \le \rho(b)$ according to (6) and the definition of $\check{t}_1$. Hence we can apply Lemma 2 with $\varepsilon = 1$ and $\delta = \check{\delta}$ in order to conclude that $V(x(t)) - V(x(\check{t}_1)) \ge 0$ for all $t \in \big[\check{t}_1, \min\{\check{t}_1 + \Delta(1, \check{\delta}), a\}\big)$. And we know (setting ε slightly below 1 in Lemma 2) that $V(x(t))$ starts out strictly increasing for $t \ge \check{t}_1$. This implies that indeed $\check{t}_2 \ge \min\{\check{t}_1 + \Delta(1, \check{\delta}), a\}$. Moreover, if $\check{t}_2 < a$ then $x(\check{t}_2) \in \partial\Lambda_b$ and we can repeat Step 2 for the times $\check{t}_2$, $\check{t}_3$, and so on.

Step 3: We have now reached the time $t = a$ and we have constructed the following control input defined on the interval $[0, a)$, with the control values $u_i, \check{u}_j$ and the times $t_i, \check{t}_j$ as specified above (those times that are never defined are treated as ∞):

$$u(t) = \begin{cases} u_0 & 0 \le t < \min\{t_1, \check{t}_1, a\} \\ u_i & t_i \le t < \min\{t_{i+1}, \check{t}_1, a\}, \quad i = 1, 2, \dots \\ \check{u}_j & \check{t}_j \le t < \min\{\check{t}_{j+1}, a\}, \quad j = 1, 2, \dots \end{cases}$$

This input, extended with the last value (u_0, u_i or $\check{u}_j$) at $t = a$ (and arbitrarily for $t > a$), satisfies $u \in \mathcal{U}_{a,b}$, as by construction, $u(t) \in U_b$ for all $t \in [0, a]$. For each $0 < \bar{\varepsilon} < 1$, this input signal is piecewise constant in the interval $[0, a]$ with only finitely many different values u_i and $\check{u}_j$; this follows from the construction in Step 1 and the argument given in Step 2. The state trajectory $x(\cdot)$ resulting from the application of the control input $u(\cdot)$ to the system (1) has the following properties. First, for $0 \le t \le \min\{\check{t}_1, a\}$, using (11) and then recursively applying (12) we have $V(x(t)) \ge (1 - \bar{\varepsilon})t\chi(b) + V(x_0)$. Second, if $\check{t}_1 < a$, then for $\check{t}_1 \le t \le a$ the construction guarantees that $V(x(t)) \ge V(x(\check{t}_1)) = \alpha_1(\rho(b))$. This yields

$$V(x(a)) \ge \min\big\{(1 - \bar{\varepsilon})a\chi(b) + V(x_0), \alpha_1(\rho(b))\big\}.$$

Hence, using (6), we have

$$|\omega(x(a))| \ge \alpha_2^{-1}\Big(\min\big\{(1 - \bar{\varepsilon})a\chi(b) + V(x_0), \alpha_1(\rho(b))\big\}\Big).$$

Finally, using (5), we obtain

$$|h(x(a))| \ge \nu\Big(\alpha_2^{-1}\Big(\min\big\{(1-\bar{\varepsilon})a\chi(b) + V(x_0), \alpha_1(\rho(b))\big\}\Big)\Big).$$

As $u(\cdot)$ is contained in $\mathcal{U}_{a,b}$ and as the above calculations hold for arbitrary x_0, it follows that

$$R_h^a(x_0, \mathcal{U}_{a,b}) \ge \nu\Big(\alpha_2^{-1}\Big(\min\big\{(1 - \bar{\varepsilon})a\chi(b) + V(x_0), \alpha_1(\rho(b))\big\}\Big)\Big) \quad (13)$$

for all $x_0 \in \mathbb{R}^n$. But as (13) holds for every $0 < \bar{\varepsilon} \le 1$ and its left-hand side is independent of $\bar{\varepsilon}$, we can let $\bar{\varepsilon} \to 0$ and arrive at the desired bound (4) with γ as defined in (9). The function γ satisfies the required properties of Definition 1, i.e., $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot) \in \mathcal{K}_\infty$ for each fixed $a > 0$. This concludes the proof of Theorem 1. □

Remark 3: Under the conditions of Theorem 1, the system (1) is norm-controllable in $\mathbb{R}^n$, i.e., from all initial conditions $x_0 \in \mathbb{R}^n$. However, it might be the case that the conditions in Theorem 1 do not hold for all $x \in \mathbb{R}^n$, but only for all x in some (control-) invariant subset $\mathcal{B} \subseteq \mathbb{R}^n$. In this case, the system (1) is norm-controllable from all initial conditions $x_0 \in \mathcal{B}$. This can be proved in the exact same way as in Theorem 1, because the set $\mathcal{B}$ is assumed to be (control-) invariant. Example 4 and Proposition 3 in Section VI will illustrate norm-controllability on control-invariant sets in more detail. □

V. EXAMPLES

In this section, we will illustrate the concept of norm-controllability and the verification of this property via Theorem 1 with several examples.

Example 1: Consider the linear one-dimensional system $\dot{x} = u$ with output $h(x) = x$, and take $\omega(x) = x$ and $V(x) =$

$|x|$, which means that $\nu = \alpha_1 = \alpha_2 = \text{id}$. For all $x \neq 0$ we obtain $\dot{V} = \text{sign}(x)u = |u| =: \chi(|u|)$ if we choose u such that $xu \geq 0$. For $x = 0$ we obtain $V'(0, f(0,u)) = |u| = \chi(|u|)$. Hence we can take the set $U(x)$ as

$$U(x) := \begin{cases} \mathbb{R} & \text{if } x = 0 \\ \{u \in \mathbb{R} : xu \geq 0\} & \text{if } x \neq 0 \end{cases} \qquad (14)$$

and thus (7) holds with $\mu := \text{id}$. Furthermore, as $V'(x, f(x,u)) \geq \chi(|u|)$ for all x and $u \in U(x)$, we can choose $\rho \to \infty$ according to Remark 2. Thus we can invoke Theorem 1 to conclude that the considered system is norm-controllable from all $x_0 \in \mathbb{R}$ with scaling function $\mu = \text{id}$ and gain $\gamma(r,s) = rs + |x_0|$.

Example 2: Consider the system $\dot{x} = -x^3 + u$ with output $h(x) = x$, and take $\omega(x) = x$ and $V(x) = |x|$, which means that $\nu = \alpha_1 = \alpha_2 = \text{id}$. For all $x \neq 0$ we obtain

$$\begin{aligned} \dot{V} &= -|x|^3 + \text{sign}(x)u \\ &= -|x|^3 + \theta\text{sign}(x)u + (1-\theta)\text{sign}(x)u \\ &\geq (1-\theta)|u| =: \chi(|u|), \qquad 0 < \theta < 1 \end{aligned}$$

for all $|x| \leq \sqrt[3]{\theta|u|} =: \rho(|u|)$ if we choose u such that $xu \geq 0$. For $x = 0$ we obtain $V'(0, f(0,u)) = |u| \geq \chi(|u|)$. Hence the set $U(x)$ is given as in (14), and we can again take $\mu := \text{id}$. Thus by Theorem 1 we obtain that the considered system is norm-controllable from all $x_0 \in \mathbb{R}$ with scaling function $\mu = \text{id}$ and gain $\gamma(r,s) = \min\{(1-\theta)rs + |x_0|, \sqrt[3]{\theta s}\}$.

Example 3: Consider the system $\dot{x} = \frac{u}{1+|u|}$ with $h(x) = x$. For this system, no scaling function μ exists such that it is norm-controllable. Namely, it is easy to see that $|\dot{x}| \leq 1$ for all x and u. But this means that we cannot find a function $\gamma(\cdot,\cdot)$ which is a $\mathcal{K}_\infty$ function in the second argument such that (4) holds, as for a given a time horizon a, the norm of the output cannot go to infinity as $b \to \infty$. Hence this system lacks the "short-term" responsiveness captured by the norm-controllability property. Nevertheless, one can see that for $a \to \infty$, also $|h(x)| \to \infty$ for every constant control $u > 0$.

Example 4: Consider the system

$$\dot{x} = f(x,u) = \begin{bmatrix} -x_1^3 + x_2 + u \\ -x_2 + x_1 + u \end{bmatrix}, \qquad h(x) = x. \quad (15)$$

As pointed out in Section III, with this example we illustrate how different functions ω and V can be used to establish the norm-controllability property of the system (15). To this end, consider the two functions $\omega_1(x) = x_1$ and $\omega_2(x) = x_2$, as well as $V_1(x) = |x_1|$ and $V_2(x) = |x_2|$. It holds that $|h(x)| \geq |\omega_i(x)|$ for $i = 1,2$; thus in both cases we can choose $\nu_i = \alpha_{1,i} = \alpha_{2,i} = \text{id}$. Furthermore, the positive orthant $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}_+$ is a control-invariant set for the system (15) if inputs u with $u \geq 0$ are applied, which can be easily seen by noting that the vector field f points inside the positive orthant for all x on its boundary and all $u \geq 0$. Considering ω_1 and V_1, by similar calculations as in the previous examples one can show via Theorem 1 that the system (15) is norm-controllable from all $x_0 = [x_{10}\ x_{20}]^T \in \mathbb{R}^2_+$ with gain $\gamma_1(r,s) = \min\{(1-\theta)rs + |x_{10}|, \sqrt[3]{\theta s}\}$. Similar calculations using ω_2 and V_2 yield that the system (15) is norm-controllable from all $x_0 \in \mathbb{R}^2_+$ with gain $\gamma_2(r,s) = \min\{(1-\theta)rs + |x_{20}|, \theta s\}$. Hence we can conclude that the system (15) is norm-controllable from all $x_0 \in \mathbb{R}^2_+$ with gain $\gamma = \max\{\gamma_1, \gamma_2\}$, which shows how the possible degrees of freedom in the choice of the functions ω and V can be used to maximize the gain γ. Furthermore, by the choice of the functions ω_1, ω_2 and V_1, V_2, we also have proved norm-controllability of the system (15) for the a posteriori defined output maps $h_1(x) = x_1$ and $h_2(x) = x_2$.

The following two examples illustrate the effect of an appropriate choice of the scaling function μ on norm-controllability.

Example 5: Consider the system

$$\dot{x} = f(x,u) = \begin{cases} |u|, & |u| \leq 1/2 \\ 1 - |u|, & 1/2 < |u| \leq 1 \\ 0, & 1 < |u| \leq 2 \\ |u| - 2, & 2 < |u| \end{cases} \qquad (16)$$

with output $h(x) = x$, for which $\mathbb{R}_+$ is an invariant set. This system is not norm-controllable in $\mathbb{R}_+$ with $\mu = \text{id}$, as $f(x,u) = 0$ for all $1 < |u| \leq 2$ and thus no function γ can be found such that (4) holds. Nevertheless, the system is norm-controllable in $\mathbb{R}_+$ with scaling function $\mu(s) = cs$ for every $c > 2$. To see why this is true, consider e.g. the functions $\omega(x) = x$ and $V(x) = |x|$, and for all $x \in \mathbb{R}_+$ define the set

$$U(x) := \Big\{u \in \mathbb{R} : |u| \in \big[0, \frac{c+2}{2c}\big] \cup \big[\frac{c+2}{2}, \infty\big)\Big\}$$

with $c > 2$. For each $b > 0$ and $x \in \mathbb{R}_+$, $U(x) \cap \{u \in \mathbb{R} : b \leq |u| \leq cb\} \neq \emptyset$. Furthermore, for all $x \in \mathbb{R}_+$ and $u \in U(x)$, we obtain $V'(x, f(x,u)) = f(x,u) \geq \frac{c-2}{c+2}|u| =: \chi(|u|)$. Thus we can infer from Theorem 1 that the system (16) is norm-controllable in $\mathbb{R}_+$ with scaling function $\mu(s) = cs$ for every $c > 2$. Note that $c > 2$ is also necessary for the system (16) to be norm-controllable in $\mathbb{R}_+$ with scaling function $\mu(s) = cs$. Namely, if $c \leq 2$, then for $b = 1$ we have $f(x,u) = 0$ for all $u \in [1, \mu(1)]$, and thus no function γ can be found such that (4) holds. One can see that the choice of the scaling function μ affects χ and thus, according to (9), also the gain γ. Namely, for $c \searrow 2$, $\chi \to 0$, whereas for $c \to \infty$, $\chi \to \text{id}$. Furthermore, by similar calculations as above, one can also show that the system (16) is norm-controllable in $\mathbb{R}_+$ with scaling function $\mu(s) = s + d$ if and only if $d > 1$.

Example 6: Let $0 < \varepsilon < 1$. Consider the system

$$\dot{x} = f(x,u) = \begin{cases} f_1(u), & |u| < 1 \\ f_2(u), & k \leq |u| < k + \varepsilon, \\ f_3(u), & k + \varepsilon \leq |u| < k + \dfrac{1+\varepsilon}{2}, \\ f_4(u), & k + \dfrac{1+\varepsilon}{2} \leq |u| < k+1 \end{cases} \qquad (17)$$

for $k = 1, 2, \ldots$, with $f_1(u) = |u|$, $f_2(u) = 1$, $f_3(u) = k|u| + 1 - k(k+\varepsilon)$, $f_4(u) = -k|u| + 1 + k(k+1)$, and output $h(x) = x$. By similar calculations as in Example 5 one can show that the system (17) is norm-controllable in $\mathbb{R}_+$ with scaling function $\mu(s) = cs$ if and only if $c > 1$, and also with scaling function $\mu(s) = s + d$ if and only if $d > 1 + \varepsilon$. On the other hand, if the vector field f is modified so that the segments where $f(x, u) = 1$, i.e. the segments in between two "peaks", have length $k\varepsilon$ instead of ε, then the system is norm-controllable in $\mathbb{R}_+$ with scaling function $\mu(s) = cs$ if and only if $c > 1 + \varepsilon$, whereas it is not norm-controllable with scaling function $\mu(s) = s + d$ for any $d \geq 0$.

VI. Norm-controllability of linear systems

In this section, we further elaborate the property of norm-controllability for linear systems. We show how norm-controllability for special linear output maps corresponding to left eigenvectors of the system matrix can be established, and how this can be related to the standard controllability notion. Furthermore, we show how invariant sets can be used in order to establish norm-controllability for a large class of output maps h.

Consider the linear system

$$\dot{x} = Ax + Bu \tag{18}$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Denote the controllable subspace of the system (18) by[1] $\mathcal{S} := \text{span}[B, AB, \ldots, A^{n-1}B]$. Furthermore, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A (with algebraic and geometric multiplicity possibly greater than 1) and $\ell_1^T, \ldots$ the corresponding left eigenvectors of A.

Proposition 1: A linear system (18) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with output map $h(x) = \ell_i^T x$, for each left eigenvector ℓ_i of A which is not orthogonal to $\mathcal{S}$. □

Proposition 1 can be proven via Theorem 1 by considering the functions $\omega(x) = h(x)$ and $V(x) = |\omega(x)|$. For every linear system (18) which is controllable, the controllable subspace is $\mathcal{S} = \mathbb{R}^n$, and hence none of the left eigenvectors of A is orthogonal to $\mathcal{S}$, which according to Propositon 1 means that the system is norm-controllable for each output map $h(x) = \ell_i^T x$ with ℓ_i being a left eigenvector of A. In fact, it turns out that also the converse is true, which leads to the following Proposition:

Proposition 2: A linear system (18) is controllable if and only if it is norm-controllable from all $x_0 \in \mathbb{R}^n$ with output map $h(x) = \ell_i^T x$, for all left eigenvectors ℓ_i^T of A. □

Necessity follows from Proposition 1. For proving sufficiency, one can first show by a contradiction argument that if the system (18) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with output map $h(x) = \ell_i^T x$, then $\ell_i^T B \neq 0$, for all left eigenvectors ℓ_i^T of A. But then it follows via the Hautus test that the system (18) is controllable.

Furthermore, for each scalar linear function $\omega(x) = c^T x$ there exists a $\mathcal{K}_\infty$-function ν such that (5) is satisfied for the output map $h = \text{id}$ (e.g., we can take $\nu(r) = r/|c|$). Hence we can state the following corollary:

Corollary 1: Every controllable linear system with output $y = x$ is norm-controllable from every $x_0 \in \mathbb{R}^n$.

The above propositions, which consider norm-controllability of a linear system from all initial conditions $x_0 \in \mathbb{R}^n$, allow only such linear output maps h which correspond to one of the left eigenvectors of the matrix A which is not orthogonal to $\mathcal{S}$. This might be quite restrictive, as one might either not be able to find such eigenvectors or want to prove norm-controllability for a different output map. Thus in the following, we will consider situations where the conditions of Theorem 1 are not fulfilled globally, but only on a control-invariant set $\mathcal{B}$, as explained in Remark 3. In this conference paper we will only consider the controllable subspace for $\mathcal{B}$; results for more general control-invariant sets for the system (18) are not included for space reasons.

Without loss of generality, consider the linear system (18) in the Kalman controllability decomposition (see, e.g., [1])

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \tilde{B}u$$
$$\dot{x}_2 = A_{22}x_2.$$

Herein, $\dim(x_1) = \text{rank}[B, AB, \ldots, A^{n-1}B] =: n_1$, and $\dim(x_2) = n - n_1$.

Proposition 3: A linear system (18) with $B \neq 0$ is norm-controllable from all $x_0 \in \mathcal{S}$ with output map $h(x) = [\tilde{\ell}_i^T \; 0]x$, for each left eigenvector $\tilde{\ell}_i^T$ of A_{11}. □

VII. Conclusions and Future Research

In this paper, we introduced the notion of norm-controllability as a framework to quantify the responsiveness of a dynamical system to applied inputs. A Lyapunov-like sufficient condition was given under which norm-controllability can be established. Furthermore, the various aspects of the proposed concept were illustrated with several examples and further elaborated for linear systems.

There are many possibilities for future research. For example, it would be interesting to obtain converse Lyapunov results for norm-controllability. Also, further research is needed to specify the class of nonlinear systems which are norm-controllable with scaling function $\mu = \text{id}$. Moreover, relaxed sufficient conditions for norm-controllability involving higher-order derivatives, as well as weaker notions of norm-controllability, are currently under investigation.

[1] For a matrix $X \in \mathbb{R}^{n \times m}$ with $n \geq m$, denote by span(X) the subspace of $\mathbb{R}^n$ spanned by the columns of X.

References

[1] J. P. Hespanha, *Linear Systems Theory*. Princeton University Press, 2009.

[2] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer, 1990.

[3] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed. New York: Springer, 1998.

[4] J. P. Hespanha, D. Liberzon, D. Angeli, and E. D. Sontag, "Nonlinear norm-observability notions and stability of switched systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 154–168, 2005.

[5] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, pp. 435–443, 1989.

[6] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, pp. 351–359, 1995.

[7] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, "Asymptotic controllability implies feedback stabilization," *IEEE Transactions on Automatic Control*, vol. 42, no. 10, pp. 1394–1407, 1997.

[8] D. Angeli and E. D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *System & Control Letters*, vol. 38, pp. 209–217, 1999.