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On Almost Lyapunov Functions for Non-vanishing Vector Fields

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***Abstract*— We study convergence properties of nonlinear systems in the presence of "almost Lyapunov" functions which decrease along solutions in a given region not everywhere but rather on the complement of a set of small volume. The structure is quite general except that the system dynamics never vanishes in a region that is away from the equilibrium. It is shown that solutions starting inside the region will approach a small set around the origin as long as the volume where the Lyapunov function does not decrease fast enough is sufficiently small. The main theorem of this paper is established by tracking the change of Lyapunov function value when the solution passes through the above mentioned volume and finding an upper bound of the volume swept out by a neighborhood along the solution before it can achieve an overall gain in its Lyapunov function value. The result shows that the convergence rate is traded off against the size of such small volume that the system can have. In the end a non-trivial example where our theorem is applicable is demonstrated.**

I. INTRODUCTION

Consider a general system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

Asymptotic stability of all solutions of (1) is typically shown through Lyapunov's direct method (see, e.g., [1]), which results in finding a Lyapunov function V whose derivative along solutions satisfies

$$\dot{V}(x) := \frac{\partial V}{\partial x} \cdot f(x) < 0 \quad \forall x \neq 0 \tag{2}$$

Notice that (2) requires the inequality to hold for all x except for the equilibrium and hence the challenge of this method arises in constructing a proper Lyapunov function that meets the condition. A less conservative approach will result in finding "an almost Lyapunov function" such that

$$\dot{V}(x) < 0 \quad \forall x \in D \backslash \Omega \tag{3}$$

where D is the region of interest and Ω is an unknown subset of D whose relative measure in D is bounded from above. In addition, a strictly decreasing Lyapunov function may also become an almost Lyapunov function defined in (3) by perturbing the system dynamics (1) (see, e.g., [2]). Hence it is reasonable to believe that convergence still holds for the system while such Ω regions are small.

On the other hand, while it is usually not difficult to compute the expression for $\dot{V}$ in practice, challenges remain for analytically checking that it is negative definite on a certain region. Instead of trying to establish the inequality in (2) by deterministic methods, techniques based on random sampling [3] can be considered. This method only requires to verify whether the inequality in (2) holds at a sequence of points x_i inside D randomly. The problem can be again converted into finding an "almost Lyapunov function" with the property (3). This paper serves as an extension to the earlier research of [2], in which a perturbation argument is used in deriving the results. It has been shown that if the measure of the set Ω where $\dot{V}$ is not decreasing fast enough is sufficiently small, then all the solutions starting inside D with some small distance away from the boundary will eventually converge to a sublevel set of V whose volume depends on the measure of Ω. Nevertheless, a non-trivial example with $\dot{V}$ being strictly positive inside Ω could not be found, which left an open question in [2]. An interesting observation shows that by perturbing the system dynamics without changing the existing bounds, an unstable equilibrium can be constructed away from the origin. Hence it is reasonable to consider a modified region D with the origin excluded and assume that the vector field of (1) is not vanishing in D. With this additional information on the system structure, this paper employs a different approach that is based on geometry of curves in Euclidean spaces and it yields a stronger result. By continuity, the solution has to pass through a transient region in Ω before $\dot{V}$ becomes strictly positive. Hence the gain of V inside Ω can be compensated with the decay in the transient region if the solution does not stay inside Ω for too long. It turns out that for non-vanishing vector fields, such constraint on the traveling time can be converted into a constraint on the volume of Ω because the volume swept out by a tubular neighborhood of certain radius along the trajectory needs to be contained inside Ω. As a result, V can only decrease after the solution passes through Ω, let alone when it is in $D \backslash \Omega$. Therefore a convergence result can be achieved. A non-trivial example is also constructed in this paper, which represents a considerable improvement compared with the result in [2].

Section II contains the necessary definitions. Our main result (Theorem 1) is stated in Section III. The sketch of proof for the main theorem is given in Section IV while proofs of some lemmas are omitted for space reasons. Section V contains the non-trivial numerical example. Section VI contains some discussion and suggested future work on this topic and Section VII concludes the paper.

II. PRELIMINARIES

The system is given by (1), where the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be locally Lipschitz. Consider a candidate Lyapunov function $V : \mathbb{R}^n \to [0, \infty)$ which is positive definite and C^1 with locally Lipschitz gradient, which we

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denote by V_x. In addition, consider the region

$$D := V^{-1}([c_1, c_2]), \quad c_2 > c_1 > 0 \tag{4}$$

We need D to be compact and connected as in the comments of [2]. The word "non-vanishing" means

$$f(x) \neq 0 \quad \forall x \in D \tag{5}$$

It indeed requires the origin excluded from D. We now define some bounds on f:

$$\bar{L}_0 := \max_{x \in D} |f(x)|$$

L_1 is a Lipschitz constant of f over D:

$$|f(x_1) - f(x_2)| \leq L_1 |x_1 - x_2| \quad \forall x_1, x_2 \in D$$

In addition to these bounds from [2], define a lower bound for $|f(x)|$ over D:

$$\underline{L}_0 := \min_{x \in D} |f(x)|$$

Notice that the vector field f is non-vanishing in D if and only if $\underline{L}_0 > 0$.
We also define some bounds on V_x:

$$M_1 := \max_{x \in D} |V_x(x)|$$

M_2 is a Lipschitz constant of V_x over D:

$$|V_x(x_1) - V_x(x_2)| \leq M_2 |x_1 - x_2| \quad \forall x_1, x_2 \in D$$

Recall that $\dot{V}(x) = V_x(x) f(x)$ by (2). Fix $a > 0$ and for any $\eta \in [0, 1]$, define the set

$$\Omega_\eta := \{x \in D : \dot{V}(x) \geq -\eta a V(x)\}$$

which implies that

$$\dot{V}(x) \leq -aV(x) \quad \forall x \in D \backslash \Omega_1 \tag{6}$$

By this definition, Ω_η is the same as the notation $\overline{D \backslash G_{\eta a}}$ and Ω_1 is same as the notation of Ω in [2]. Thus a plays the role of a known general convergence rate parameter on D except for some small "bad region" Ω_1. Notice that the solution passing through Ω_1 does not necessarily imply divergence; it is only when $\dot{V} > 0$ that such divergence may occur. The time derivative of V is naturally bounded from above by $\bar{L}_0 M_1$, but we can also assume that there is a tighter bound on it in the region D:

$$b := \max_{x \in D} \{\dot{V}(x)\} \tag{7}$$

In order for the theorem to be non-trivial, b needs to be strictly positive.
At last, define the function vol to be the standard volume function induced by the Euclidean metric on $\mathbb{R}^n$. For convenience, vol(Ω_1) will be used to represent the volume of a single connected component of Ω_1 throughout the paper.

III. Main result

We are now ready to present our main result:

Theorem 1 *Consider a system (1) with a locally Lipschitz right-hand side f, and a candidate Lyapunov function V which is positive definite and C^1 with locally Lipschitz gradient. Let the region D be defined via (4) with some c_1, c_2 and assume it is compact and connected. Let $\Omega_1 \subset D$ be a measurable set such that (6) holds and f is non-vanishing in D as defined in (5). Then there exist $\bar{\epsilon} > 0, g > 0, h > 0$ and a decreasing function $\lambda : [0, \bar{\epsilon}] \to [0, a]$ such that for every $\epsilon \in (0, \bar{\epsilon}]$, if every connected component of Ω_1 has volume less than ϵ, then for every initial condition $x_0 \in D$ with $V(x_0) \leq c_2 - h - g\epsilon$, the corresponding solution $x(\cdot)$ of (1) has the following properties:*

1) *$V(x(T)) \leq c_1 + h$ for some $T \geq 0$,*
2) *$V(x(t)) \leq c_2 - h$ for all $0 \leq t < T$,*
3) *$V(x(t)) \leq c_1 + h + g\epsilon$ for all $t \geq T$*

It is obvious that for the result to be meaningful, we should have $\epsilon < \frac{c_2 - c_1 - 2h}{g}$. The first statement suggests that the solution will eventually enter the sublevel set of $V^{-1}([0, c_1 + h])$. The second statement tells that the solution is bounded before it converges to that smaller set. The last statement implies that once the solution arrives at this sublevel set, it will be trapped inside a slightly inflated one forever. Later in the proof of the theorem readers will see that the convergence before time T is in fact exponential.

IV. Proof of theorem

A. Limited time in Ω_η

A typical "bad region" Ω_1 with a trajectory passing through it will look like the one in Figure 1. If the solution trajectory never enters Ω_0, then we are safe since $\dot{V} < 0$ for all time. If the trajectory enters Ω_0, then it has to enter Ω_η for some $\eta \in (0, 1)$ first. In this case we denote by t_1, t_4 the times when the solution enters and leaves Ω_η, and by t_2, t_3 the times when the solution enters and leaves Ω_0. Let $B^n_\gamma(x)$ be the closed ball whose center is at x in $\mathbb{R}^n$ with radius γ. The first two lemmas on the next page show that there exists a ball of certain radius that is sweeping through Ω_1 along the solution when the solution is inside Ω_η:

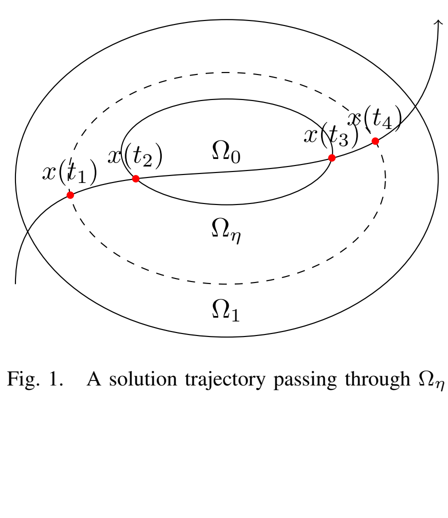

Fig. 1. A solution trajectory passing through Ω_η

Lemma 1 *For any* $x_1, x_2 \in D$,

$$|\dot{V}(x_1) - \dot{V}(x_2)| \leq \alpha |x_1 - x_2| \quad (8)$$

where $\alpha := M_1 L_1 + M_2 \bar{L}_0$

Lemma 2 *For any* $x \in \Omega_\eta$, $B^n_{\gamma_\eta}(x) \cap D \subseteq \Omega_1$, *where*

$$\gamma_\eta := \frac{(1-\eta) a c_1}{\alpha + \eta a M_1}$$

and α *defined in Lemma 1.*

The idea behind the proof of main theorem is that if there is a net increase of V due to the solution lying in Ω_η, it has to travel some amount of time inside. Hence the total swept volume of $B^n_{\gamma_\eta}(x)$ along the trajectory provides a lower bound for the volume of Ω_1. In other words, if the size of Ω_1 is below this bound, the solution will not have sufficient time staying inside Ω_η, which leads to the decrease of V.

Let $\mathcal{L}^t_s$ be the path length from time s to t. It is easy to see that

$$\underline{L}_0(t-s) \leq \mathcal{L}^t_s \leq \bar{L}_0(t-s) \quad (9)$$

Fix $\eta \in (0,1)$, denote $\bar{t} = t_4 - t_1$, which can be infinite if x never leaves Ω_η (then $t_4 = \infty$). Also define

$$S_{\eta,(s,t)} = \{x \in \mathbb{R}^n : x \in B^n_{\gamma_\eta}(x(\tau)), \\ (x - x(\tau)) \perp f(x(\tau)), s \leq \tau \leq t\} \quad (10)$$

which is the swept volume of radius γ_η due to the solution trajectory from time s to t. The second orthogonality condition in definition (10) specifies that the swept area is generated by the normal plane (the cross-section area of $B^{n-1}_{\gamma_\eta}(x(\tau))$) along the solution. This additional condition only excludes the starting and ending "dome" regions which are less significant, but it makes our later expression for volume neater. In addition we say the solution tube is *non-self-overlapping* from time s to t if for all $x \in S_{\eta,(s,t)}$,

$$x \in B^n_{\gamma_\eta}(x(t_i)) \text{ and } (x - x(t_i)) \perp f(x(t_i)) \\ \text{for both } i = 1, 2, t_1, t_2 \in [s,t) \Rightarrow t_1 = t_2. \quad (11)$$

It simply means that all the points in the swept volume are swept only once by such $B^{n-1}_{\gamma_\eta}(x(t))$ cross-section area at a unique time in $[s,t)$. Under the non-self-overlapping condition, the volume of $S_{\eta,(t_1,t_4)}$ can be precisely computed:

Lemma 3 *if the solution tube is non-self-overlapping for a time span of* $\bar{t}$, *then*

$$\mathrm{vol}(S_{\eta,(t_1,t_4)}) = \mathrm{vol}(B^{n-1}_{\gamma_\eta}) \mathcal{L}^{t_4}_{t_1} \quad (12)$$

Basically this lemma says that the swept volume is exactly the cross-section area times the trajectory length. A general expression for the volume of $(n-1)$-dimensional ball of radius γ_η is:

$$\mathrm{vol}(B^{n-1}_{\gamma_\eta}) = \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \gamma^{n-1}_\eta$$

where Γ is the standard gamma function [4]. The proof of this lemma is a direct conclusion from [4], [5]. It is worth noticing that the formula in [5] yields a signed volume with multiplicity; nevertheless, the non-self-overlapping condition we have ensures that there are no multiple counts of the integrated volume and the result is indeed the absolute volume that we want as a lower bound. More on non-self-overlapping condition will be discussed in the next subsection.

From Lemma 2, $S_{\eta,(t_1,t_4)} \subseteq \Omega_1$. Notice that $S_{\eta,(t_1,t_4)}$ depends on $\bar{t}$. Hence with inequalities (9), Lemma 3 suggests that

Corollary 1 *if the solution is non-self-overlapping for a time span of* $\bar{t}$, *then*

$$\mathrm{vol}(\Omega_1) \geq \mathrm{vol}(B^{n-1}_{\gamma_\eta}) \underline{L}_0 \bar{t} =: \rho(\bar{t}) \quad (13)$$

Notice that ρ here is a positive, linear and increasing function, which means we can invert it to get an upper bound of $\bar{t}$ by the volume of Ω_1.

B. On non-self-overlapping condition

Here is a sufficient condition for non-self-overlapping:

Lemma 4 *A solution* $x(t)$ *tube is non-self-overlapping from time* s *to* t *when swept with ball* $B^n_{\gamma_\eta}$ *if*

$$\gamma_\eta < \frac{\underline{L}_0}{L_1}, \quad (14)$$

and

$$\mathcal{L}^t_s < \frac{2\underline{L}_0}{L_1} \left(\pi - \sin^{-1}\left(\frac{L_1 \gamma_\eta}{\underline{L}_0}\right) \right) \quad (15)$$

Basically the first condition (14) means that the sweeping ball radius should be bounded from above (by the radius of curvature of solution trajectory). It ensures that the swept volume does not "squeeze" too much so that part of the volume is repeated. In addition, (15) suggests that the longest path length of an arbitrary shaped solution trajectory (with bounded curvature) to be non-self-overlapping is exactly that of an arc of radius $\underline{L}_0/L_1$ all the way up to having a distance of $2\gamma_\eta$ between $x(0)$ and $x(t)$.

Proof of Lemma 4: The proof involves two classical results from differential geometry [6]:

Lemma 5 (Fenchel's Theorem) *The total curvature of any closed space curve is at least* 2π, *and equality holds if and only if the curve is a (convex) planar curve.*

Lemma 6 (Schur's Comparison Theorem) *Suppose C(s) is a plane curve with curvature* $\kappa(s)$ *which makes a convex curve when closed by the chord connecting its endpoints, and* $C^*(s)$ *is a curve of the same length with curvature* $\kappa^*(s)$. *Let* d *be the distance between the endpoints of C and* d^* *be the distance between the endpoints of* C^*. *If* $\kappa^*(s) \leq \kappa(s)$ *then* $d^* \geq d$.

It is worth mentioning that the total curvature in Fenchel's Theorem is defined to be the integral of curvature along the curve where curvature is well defined, plus the sum of tangent angles on the curve where curvature is not defined. Recall that for a space curve, curvature is defined to be the norm of time derivative of its tangent vector when parametrized by arc length and hence is only well defined for points on the curve with second order derivative. Although our vector field $f(x)$ is only Lipschitz, it is differentiable almost everywhere according to Rademacher's theorem and hence we can still compute its total curvature. While $\frac{\partial f(x)}{\partial x}$ exists, it is bounded by the Lipschitz constant L_1 and using the curvature formula for parameterized curve $x(t)$ in $\mathbb{R}^n$ from [7], the curvature $\kappa(x)$ can be found to be bounded from above by $\frac{L_1}{\underline{L}_0}$. For the states where $f(x)$ is not smooth, as $f(x)$ is continuous, tangent angle at x is 0. Bounded curvature is an important feature for non-vanishing vector fields since it prevents the system from some non-converging behavior which will not generate new sweeping volume, such as looping around inside a small region.

Next consider a circular arc from $y_1 \in \mathbb{R}^n$ to $y_2 \in \mathbb{R}^n$ with curvature $\frac{L_1}{\underline{L}_0}$ and central angle 2θ, $\theta \in [0, \frac{\pi}{2}]$. This arc is convex and has length of $\frac{2\underline{L}_0}{L_1}\theta$ and $|y_1 - y_2| = \frac{2\underline{L}_0}{L_1}\sin\theta$. According to Schur's Comparison Theorem, since the curvature of the solution trajectory is never larger than $\frac{L_1}{\underline{L}_0}$, if $\mathcal{L}_0^t = \frac{2\underline{L}_0}{L_1}\theta$, then $|x(t) - x(0)| \geq \frac{2\underline{L}_0}{L_1}\sin\theta$. On the other hand, suppose there is a multiple counted point z in $S_{\eta,(0,t)}$ which violates the condition (11). Without loss of generality, we can always assume such self-overlapping initiates at time t and hence z has to be on the boundary of $S_{\eta,(0,t)}$:

$$\begin{aligned} &|z - x(0)| = |z - x(t)| = \gamma_\eta, \\ &\qquad (z - x(0)) \perp f(x(0)),\ (z - x(t)) \perp f(x(t)) \end{aligned}$$

Notice that the total curvature along the solution from $x(0)$ to $x(t)$ is no larger than 2θ. By Fenchel's Theorem applied along the closed curve consisting of this solution and the two vectors $\overrightarrow{zx(0)}$ and $\overrightarrow{x(t)z}$, the interior angle φ between $\overrightarrow{zx(0)}$ and $\overrightarrow{zx(t)}$ has to be no larger than 2θ. Therefore using the condition (14), it can be computed that $|x(t) - x(0)| < \frac{2\underline{L}_0}{L_1}\sin\theta$, which leads to a contradiction on the chord length and hence non-self-overlapping for path length up to $\frac{\underline{L}_0\pi}{L_1}$. For $\mathcal{L}_0^t \in \left[\frac{\underline{L}_0\pi}{L_1}, \frac{2\underline{L}_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{\underline{L}_0})\right)\right)$, still consider the circular arc from y_1 to y_2 of same length and curvature of $\frac{L_1}{\underline{L}_0}$. Observe that in this case $|y_1 - y_2| > 2\gamma_\eta$. Once again by Schur's Comparison Theorem, solution of this length $\mathcal{L}_0^t$ has to have $|x(t) - x(0)| \geq |y_1 - y_2| > 2\gamma_\eta$ and hence self-overlapping will not occur. ∎

So far we have achieved the self-overlapping condition as a constraint on the path length. According to Lemma 3, it is concluded that the largest size of arbitrary $S_{\eta,(t_1,t_4)}$ which is non-self-overlapping is bounded:

$$\begin{aligned} \mathrm{vol}(S_{\eta,(t_1,t_4)}) &= \mathrm{vol}(B_{\gamma_\eta}^{n-1})\mathcal{L}_{t_1}^{t_4} \\ &< \mathrm{vol}(B_{\gamma_\eta}^{n-1})\frac{2\underline{L}_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{\underline{L}_0})\right) \end{aligned}$$

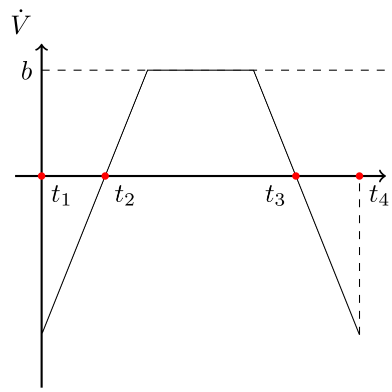

Fig. 2. $\dot V$ vs. t on the trajectory passing through Ω_η

Hence if we have

$$\mathrm{vol}(\Omega_1) < \mathrm{vol}(B_{\gamma_\eta}^{n-1})\frac{2\underline{L}_0}{L_1}\left(\pi - \sin^{-1}(\frac{L_1\gamma_\eta}{\underline{L}_0})\right) \quad (16)$$

then the condition (15) is automatically satisfied.

C. Change of V when passing through Ω_η

As discussed in subsection A, under non-self-overlapping conditions an upper bound for $\bar t$ can be derived. Hence corresponding to the trajectory in Figure 1, $\dot V$ as a function of time is shown in Figure 2. Recall that t_1, t_4 are the times when the solution enters and leaves Ω_η, and t_2, t_3 are the time when the solution enters and leaves Ω_0. It is easy to see that the sum of algebraic area under the curve of $\dot V$ gives the net change of V on the path through Ω_η and it is defined as follows:

$$\Delta V := V(t_4) - V(t_1) = \int_{t_1}^{t_4} \dot V dt$$

It follows from (8) that $\alpha\bar L_0$ is a Lipschitz bound of $\dot V$ as a function of time. Therefore finding largest value of ΔV becomes

$$\begin{aligned} &\text{maximize} \int_{t_1}^{t_4} \dot V dt \\ &\quad \text{subject to } \dot V \leq b, \dot V(t_1) \leq -\eta a c_1, \dot V(t_4) \leq -\eta a c_1, \\ &\qquad |\dot V(t_i) - \dot V(t_j)| \leq \alpha \bar L_0 |t_i - t_j| \ \forall t_i, t_j \in [t_1, t_4] \end{aligned}$$

In fact the curve plotted in Figure 2 maximizes ΔV for $t_4 - t_1 = \bar t$ and this fact is summarized as follows:

Lemma 7

$$\begin{aligned} &\Delta V \leq \phi(\bar t) \\ &:= \begin{cases} \frac{1}{4}\bar t^2 \alpha \bar L_0 - \bar t \eta a c_1 & \text{if } \ \bar t \alpha \bar L_0 < 2(b + \eta a c_1) \\ b\bar t - \frac{(b+\eta a c_1)^2}{\alpha \bar L_0} & \text{if } \ \bar t \alpha \bar L_0 \geq 2(b + \eta a c_1) \end{cases} \end{aligned} \quad (17)$$

Notice the function ϕ is continuous. Lemma 7 also leads us to the following conclusion:

Corollary 2 $\Delta V < 0$ *if*

$$\eta a c_1 \geq b \quad (18)$$

and

$$\bar{t} < \frac{(b+\eta a c_1)^2}{\alpha \bar{L}_0 b} \tag{19}$$

Now suppose vol(Ω_1) is known or bounded from above. Pick some $\eta \in (0,1)$ such that the non-self-overlapping conditions are met: that is, both equations (14) and (16) are satisfied. Then we can invert the ρ function defined in (13) to achieve an upper bound for $\bar{t}$. Further check if the two conditions (18), (19) in Corollary 2 are satisfied. If all of them are true, ΔV will be negative.

On the other hand, combining (13) with (19) gives a direct bound on vol(Ω_1) which yields negative ΔV:

$$\text{vol}(\Omega_1) \le \text{vol}(B^{n-1}_{\gamma_\eta}) \frac{\underline{L}_0 (b+\eta a c_1)^2}{\alpha \bar{L}_0 b}$$

This is another bound on vol(Ω_1) in addition to (16). Hence, by picking proper $\eta \in (0,1)$ and vol(Ω_1) being sufficiently small, the following conclusion can be achieved:

Lemma 8 $\Delta V < 0$ *if the following conditions are satisfied for some* $\eta \in (0,1)$*:*

1) $\gamma_\eta < \underline{L}_0 / L_1$,
2) $\eta a c_1 \ge b$,
3) vol$(\Omega_1) < \bar{\epsilon}$

where

$$\bar{\epsilon} := \text{vol}(B^{n-1}_{\gamma_\eta}) \min\left\{ \frac{\underline{L}_0 (b+\eta a c_1)^2}{\alpha \bar{L}_0 b}, \frac{2\underline{L}_0 \left(\pi - \sin^{-1}(\frac{L_1 \gamma_\eta}{\underline{L}_0})\right)}{L_1} \right\}.$$

Now assume that vol$(\Omega_1) < \epsilon$ for some $\epsilon \in (0, \bar{\epsilon}]$. In this case V will decrease each time it passes through Ω_η, let alone when it is outside of Ω_η. It turns out that the overall solution is also bounded and converges to a smaller set. To show this, consider a solution $x(t)$ from time 0 up to t. Define

$$0 = \check{t}_0 \le \hat{t}_1 < \check{t}_1 \le \cdots < \check{t}_{n-1} \le \hat{t}_n = t$$

so that $x(t) \in \Omega_\eta$ for all $t \in [\check{t}_{i-1}, \hat{t}_i],\ i = 1, 2, \cdots, n$ and $x(t) \in D \backslash \Omega_\eta$ for all $t \in (\hat{t}_i, \check{t}_i),\ i = 1, 2, \cdots, n-1$. If $x(0) \notin D$, set $\hat{t}_1 = 0$ and if $x(t) \notin D$, set $\check{t}_{n-1} = t$. Now if $n = 1$, that is, solution from time 0 to t completely stays inside a connected component of Ω_η, we must have $t \le \bar{t}$ and

$$V(t) - V(0) \le b\bar{t} \le b\rho^{-1}(\epsilon) = \frac{b\epsilon}{\underline{L}_0 \text{vol}(B^{n-1}_{\gamma_\eta})} =: g\epsilon$$

where g is the positive constant coefficient and this bound is proportional to ϵ.

Otherwise, we consider the change of V along the solution by parts. Notice that we have $\dot{V}(\hat{t}_1) \le -\eta a c_1$, hence to find the maximum gain of V up to time $\hat{t}_1$ is the same as the following maximization problem:

$$\begin{aligned} &\text{maximize} \int_0^{\hat{t}_1} \dot{V} dt \\ &\quad \text{subject to } \hat{t}_1 \le \bar{t}, \dot{V} \le b, \dot{V}(\hat{t}_1) \le -\eta a c_1, \\ &\quad |\dot{V}(t_i) - \dot{V}(t_j)| \le \alpha L_0 |t_i - t_j| \ \forall t_i, t_j \in [0, \hat{t}_1] \end{aligned}$$

Notice that this problem is exactly the same as finding maximum ΔV across Ω_1 as we did for Lemma 7, but without the constraint on initial $\dot{V}(0)$. The result is found to be bounded from above by $\frac{1}{2} g\epsilon$. Similarly for finding $V(t) - V(\check{t}_{n-1})$, same problem formulation but with opposite boundary condition and the bound is also found to be bounded from above by $\frac{1}{2} g\epsilon$.

We now want to find an exponential type bound on the ratio from $V(\check{t}_{i-1})$ to $V(\hat{t}_i),\ i = 2, 3, \cdots, n-1$. Define

$$k(t) := -\frac{\ln(1 + \frac{\phi(t)}{c_2})}{t}$$

where ϕ is defined in (17). It can be shown that

$$V(\hat{t}_i) \le V(\check{t}_{i-1}) e^{-k \circ \rho^{-1}(\epsilon) \bar{t}}$$

Denote $\lambda(\epsilon) := k \circ \rho^{-1}(\epsilon)$ which is non-negative, continuous and decreasing on $[0, \bar{\epsilon}]$. Notice that $\lambda(0) = k \circ \rho^{-1}(0)$ is indeed defined via L'Hôpital's rule:

$$k \circ \rho^{-1}(0) := \lim_{\delta \to 0^+} k \circ \rho^{-1}(\delta) = \frac{c_1}{c_2} \eta a < \eta a$$

This result indeed suggests that the solution converges slower inside Ω_η than outside. Hence

$$\begin{aligned} V(\check{t}_{n-1}) \le& V(\hat{t}_{n-1}) e^{-\eta a (\check{t}_{n-1} - \hat{t}_{n-1})} \\ \le& V(\check{t}_{n-2}) e^{-\lambda(\epsilon)(\hat{t}_{n-1} - \check{t}_{n-2}) - \eta a (\check{t}_{n-1} - \hat{t}_{n-1})} \\ &\vdots \\ \le& V(\hat{t}_1) e^{-\lambda(\epsilon)(\check{t}_{n-1} - \hat{t}_1)} \end{aligned}$$

Therefore combining the 3 parts together,

$$\begin{aligned} V(t) &\le V(\check{t}_{n-1}) + g\epsilon \\ &\le V(\hat{t}_1) e^{-\lambda(\epsilon)(\check{t}_{n-1} - \hat{t}_1)} + \frac{1}{2} g\epsilon \\ &\le (V(0) + \frac{1}{2} g\epsilon) e^{-\lambda(\epsilon)(\check{t}_{n-1} - \hat{t}_1)} + \frac{1}{2} g\epsilon \\ &\le V(0) e^{-\lambda(\epsilon)(\check{t}_{n-1} - \hat{t}_1)} + g\epsilon \end{aligned} \tag{20}$$

Define $h := M_1 \gamma_\eta$. When $V(x(0)) \le c_2 - h - 2g\epsilon$, we have $V(x(t)) \le c_2 - h$. Further assume at this time $V(x(t)) \ge c_1 + h$. Let $y \in B_{\gamma_\eta}(x(t))$. By Lemma 1 in [2],

$$\begin{aligned} V(y) \le V(x(t)) + M_1 |y - x(t)| \le c_2, \\ V(y) \ge V(x(t)) - M_1 |y - x(t)| \ge c_1 \end{aligned}$$

which means $y \in D$. Hence the entire $B_{\gamma_\eta}(x(t))$ lies inside D and our volume sweeping formula (12) is indeed valid. Hence (20) is true for all $x(0) \in D,\ V(0) \le c_2 - h - g\epsilon$ up to time t such that $V(t) \ge c_1 + h$. As this is an exponential type of bound on V, we have proven the first and second statement in the main theorem.

When $t = T$ such that eventually $V(t) \le c_1 + h$, some bounds are no longer valid on the subsequent solution and the solution may have a chance to return to the boundary $V^{-1}(c_1 + h)$. However, the solution afterwards can be again treated as a new solution starting from $x(0) \in D$ with $V(0) \le c_2 - h - g\epsilon$ and by the same analysis above we know that it can have an overshoot of $g\epsilon$ at most. This proves the last statement in the main theorem.

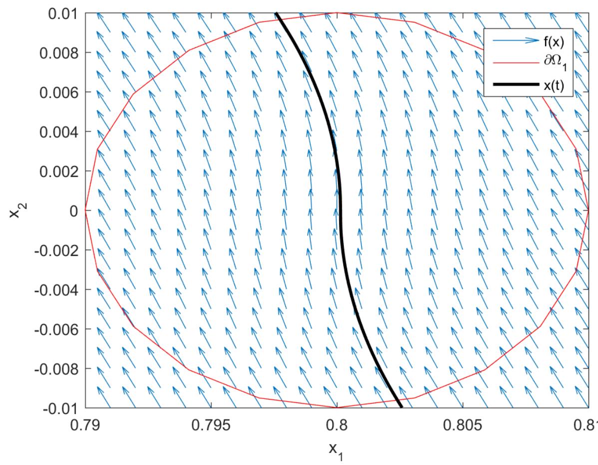

Fig. 3. Local behavior of the example system

V. A NON-TRIVIAL EXAMPLE

The system (1) is explicitly defined as follows:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -\lambda(x) & \mu \\ -\mu & -\lambda(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{21}$$

with

$$\lambda(x) = 1.01 \min\left\{\frac{|x-x_c|}{\rho}, 1\right\} - 0.01, \qquad x_c = (0.8, 0)^T, \mu = 2, \rho = 0.01$$

Consider the standard Lyapunov function

$$V = |x|^2 = x_1^2 + x_2^2$$

Notice that everywhere else the system is linear except for a small ball (which is our Ω_1) of radius ρ at x_c. When x is very close to x_c, $\lambda(x)$ becomes negative and hence system (21) will have eigenvalues with positive real parts hence $\dot{V}$ becomes positive. Hence for this system $\Omega_0 \neq \emptyset$ and the theorem result is nontrivial.

Choose $d_1 = 0.7, d_2 = 1, c_1 = d_1^2, c_2 = d_2^2$. It then can be calculated on the set $D = \{x : d_1 \leq |x| \leq d_2\}$ that $\bar{L}_0 = d_2 \times \sqrt{\max\{\lambda(x)^2\} + \mu^2} = \sqrt{5}, \underline{L_0} = d_1\mu = 1.4, M_1 = 2d_2 = 2, M_2 = 2, b = -2\min\{\lambda(x)\} \times 0.8^2 = 0.0128$. L_1 is derived numerically to be 90.78. Pick $a = 2 \Rightarrow \Omega_1 = \{x : |x - (0.8, 0)^T| \leq \rho\} \Rightarrow \text{vol}(\Omega_1) = \pi\rho^2 \approx 3.14 \times 10^{-4}$. Also note that this Ω_1 is completely inside D.

Pick $\eta = 0.6$. Checking the conditions in Lemma 8, it can be calculated that $\gamma_\eta \approx 0.0021 \leq 0.0154 = \frac{\underline{L_0}}{L_1}$, $\eta a c_1 = 0.588 > b$, $\bar{\epsilon} \approx 3.86 \times 10^{-4} \geq \text{vol}(\Omega_1)$, and $h \approx 0.0042$, $g\epsilon \approx 6.9 \times 10^{-4}$. Hence all the conditions are verified and the system will converge to the set of $\{x : V(x) \leq c_1 + h + g\epsilon\} \approx B_{0.7044}(0)$ if starts at $x(0)$ with $V(0) \leq c_2 - h - g\epsilon \approx 0.9951$. Since the Lyapunov function is chosen to be quadratic, this convergence is in fact exponentially fast by our theorem.

As seen in Figure 3, the spiral vector fields are distorted at the region of $B_\rho(x_c)$. A solution $x(t)$ passing through this region will be deviated from getting closer to the origin. Nevertheless, convergence is preserved as the effect of such "bad region" is not strong.

VI. DISCUSSION

The explicit example in the previous section represents a nontrivial system with a subdomain where $\dot{V} > 0$ and yet where convergence of solutions can be established by application of the main theorem. Thus, the example gives an affirmative answer to the question asked in [2]. By contrast, the main theorem in [2], when applied to the above example, does not give a conclusive result. This indicates that the additional assumption of non-vanishing required in this paper (which results in the positive bound $\underline{L_0}$) is indeed crucial to establish the convergence result.

Recall that the significance of our main theorem appears when there are multiple disconnected "bad regions" with the volume of each of them bounded above. For example, by modifying the vector field of above example such that there are other symmetric Ω_1 regions distributed along radius of 0.8 away from the origin, our main theorem is still applicable and will lead to the same conclusion.

In addition, once η is chosen, a sweeping ball of constant radius is employed for the analysis. We can make γ_η time-varying based on the level set of Ω_η that x is in. Since it is known that the radius of the sweeping ball becomes larger when $\dot{V}$ becomes positive, the bound for $\text{vol}(\Omega_1)$ will be larger and this modification should yield a better result.

VII. CONCLUSIONS

We presented a result (Theorem 1) which establishes convergence of system trajectories from a given set to a smaller set, based on an "almost Lyapunov" function which is known to decrease along solutions on the complement of a set of small enough volume. The result is established by tracking the change of Lyapunov function value when the solution passes through the above-mentioned small volume and finding an upper bound on the volume swept out by a neighborhood along the solution before it can achieve an overall gain in its Lyapunov function value. With some knowledge of the structure of the system dynamics, it is shown that convergence will still hold even if there is some temporary gain in Lyapunov function value. Future work can be done on finding a tighter bound on the allowed "bad region" volume, such as exploiting a time-varying sweeping radius or imposing more structure on the system.

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