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Analysis of different Lyapunov function constructions for interconnected hybrid systems

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Abstract— **This paper studies stability of interconnections of hybrid dynamical systems, in the general scenario that the continuous or discrete dynamics of subsystems may have destabilizing effects. We analyze two existing methods of constructing Lyapunov functions for the interconnection based on candidate ISS Lyapunov functions for subsystems, small-gain conditions, and auxiliary timers modeling restrictions on jump frequency in terms of average dwell-time and reverse average dwell-time. We compare their feasibility and limitations for different types of subsystem dynamics, and examine a case that the combination of them is needed to establish global asymptotic stability.**

I. INTRODUCTION

In studying practical phenomena one usually finds it beneficial to transform a complex system into an interconnection of simpler ones and to establish stability based on properties of the constituents via small-gain theorems. Classical input-output small-gain theorems for linear systems were detailed in [1] and were generalized to nonlinear feedback systems in [2], [3]. The notion of input-to-state stability (ISS) [4] is widely used in more recent works on interconnections, as it naturally unifies the concepts of internal and external stability. Small-gain theorems for interconnected ISS systems were introduced in [5], and furthered in [6], [7] and the references therein. Small-gain theorems are particularly useful in constructing Lyapunov functions for interconnections based on ISS Lyapunov functions for subsystems. Lyapunov-based small-gain theorems for general feedback interconnections were first reported in [8] for continuous-time systems and then in [9] for discrete-time ones.

Hybrid systems are dynamical systems exhibiting both continuous and discrete behaviors. This paper adopts the modeling framework in [10], which proves to be general and natural from the viewpoint of Lyapunov stability theory [11]. Trajectory-based small-gain theorems for interconnections of hybrid systems were introduced in [12], [13], and Lyapunov-based formulations were established in [14], [15].

In this work we study interconnections of hybrid subsystems with either continuous or discrete non-ISS dynamics—a more challenging case that the results above cannot be directly applied. In the presence of destabilizing dynamics, stability is often achieved by restricting the jump frequency in terms of average dwell-time (ADT) [16] (for non-ISS jumps) and/or reverse ADT (RADT) [17] (for non-ISS flows). In [15] it was shown that one can augment the subsystems by introducing ADT/RADT auxiliary timers to construct ISS Lyapunov functions that decrease along solutions both during flows and at jumps. However, it was observed in [18] that such modifications inevitably increase the feedback gains, making the small-gain condition afterwards more restrictive.

A different type of Lyapunov-based small-gain theorems was proposed in [19] for interconnected impulsive systems in a similar setting. This result generated a Lyapunov function by first constructing a candidate Lyapunov function (i.e., one that may increase during flows or at jumps) for the interconnection based on candidate ISS Lyapunov functions for subsystems and a small-gain condition, and then applying the suitable ADT/RADT modification to the interconnection instead of to the subsystems. While this method works under the same small-gain condition for interconnections of only ISS subsystems, it does not apply to the case when the non-ISS dynamics in subsystems are of different types (i.e., non-ISS flows in one while non-ISS jumps in the other).

In [18] we unified the two methods above (for interconnections of $n \geq 2$ hybrid systems) and established less restrictive small-gain conditions by showing that it suffices to only modify the subsystems with one type (either flow or jump) of destabilizing dynamics. In this paper we provide a thorough analysis of the Lyapunov function constructions above for interconnections of two hybrid subsystems. Compared with the previous results, explicit Lyapunov-based small-gain conditions for establishing stability of the interconnection via ADT/RADT modifications are derived. Moreover, our small-gain conditions indicate that, unlike ADT modifications, the RADT ones induce no substantial increase in the feedback gains. Striving for as simple a setting as possible, we distill the main features of the approaches in [15], [19] and highlight their relative advantages.

This paper is structured as follows: In Section II we introduce the hybrid system framework. A small-gain theorem for the baseline case of only ISS subsystems is provided in Section III. In Section IV we study different cases of destabilizing dynamics in subsystems, and identify a scenario that the combination of the two methods above is required.

II. HYBRID SYSTEM

Following [11], a *hybrid system* with state $x \in \mathcal{X} \subset \mathbb{R}^n$ and input $u \in \mathcal{U} \subset \mathbb{R}^m$ is modeled by

$$\begin{aligned} \dot{x} &\in F(x,u), && (x,u) \in \mathcal{C}, \\ x^+ &\in G(x,u), && (x,u) \in \mathcal{D}, \end{aligned} \tag{1}$$

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where $F : \mathcal{X} \times \mathcal{U} \rightrightarrows \mathbb{R}^n, G : \mathcal{X} \times \mathcal{U} \rightrightarrows \mathcal{X}$ are set-valued mappings. Solutions of (1) are defined on a so-called *hybrid time domain* $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, which is a union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last one (if existent) possibly of the form $[t_j, T) \times \{j\}$ with $T \in \mathbb{R}$ or $T = \infty$. A *hybrid input* is a function $u : \operatorname{dom} u \to \mathcal{U}$ defined on a hybrid time domain such that $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t, j) \in \operatorname{dom} u\}$ for each fixed j. A *solution* $x : \operatorname{dom} x \to \mathcal{X}$ of (1) with a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ satisfies $x(\cdot, j)$ is locally absolutely continuous on $\{t : (t, j) \in \operatorname{dom} x\}$ for each fixed j, $\operatorname{dom} x = \operatorname{dom} u$, $(x(0, 0), u(0, 0)) \in \mathcal{C} \cup \mathcal{D}$, and[1]

1. $(x(t, j), u(t, j)) \in \mathcal{C}$ and $\dot{x}(t, j) \in F(x(t, j), u(t, j))$ for all j and almost all t such that $(t, j) \in \operatorname{dom} x$;
2. $(x(t, j), u(t, j)) \in \mathcal{D}$ and $x(t, j+1) \in G(x(t, j), u(t, j))$ for all $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$.

With proper assumptions on the data $\mathcal{H} = (\mathcal{C}, F, \mathcal{D}, G)$, one can establish the local existence of solutions of (1), which are not necessarily unique (see, e.g., [10, Prop. 2.10]). A solution is *maximal* if it cannot be extended, and is *complete* if its domain is unbounded.

The hybrid system (1) is *input-to-state stable (ISS)* w.r.t. a set $\mathcal{A} \subset \mathcal{X}$ if there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that every solution x with a hybrid input u satisfies[2]

$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j) + \gamma(\|u\|_{(t,j)})$$

for all $(t, j) \in \operatorname{dom} x$, where $|\cdot|_{\mathcal{A}}$ denotes the Euclidean distance to the set $\mathcal{A}$, and $\|\cdot\|_{(t,j)}$ the essential supremum Euclidean norm up to hybrid time (t, j), that is,

$$\|u\|_{(t,j)} := \max\left\{ \operatorname*{ess\,sup}_{\substack{(s,l)\in \operatorname{dom} u,\\ s\leq t,\ l\leq j}} |u(s, l)|, \sup_{\substack{(s,l)\in J(u),\\ s\leq t,\ l\leq j}} |u(s, l)| \right\},$$

where $J(u) := \{(s, l) \in \operatorname{dom} u : (s, l + 1) \in \operatorname{dom} u\}$.[3]

The *Clarke derivative* [20] of a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ at x in the direction $v \in \mathbb{R}^n$ is defined by

$$V^\circ(x; v) := \limsup_{s\to 0^+, y\to x} \frac{V(y + sv) - V(y)}{s}.$$

Definition 1. A locally Lipschitz function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a *candidate ISS Lyapunov function w.r.t.* $\mathcal{A}$ for the hybrid system (1) with *bounds* ψ_1, ψ_2, a *gain* χ, and *rates* ϕ, α if

1. there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that
$$\psi_1(|x|_{\mathcal{A}}) \leq V(x) \leq \psi_2(|x|_{\mathcal{A}}) \qquad \forall\, x \in \mathcal{X};$$
2. there exist a $\chi \in \mathcal{K}_\infty$ and a continuous $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $\phi(0) = 0$ such that
$$V(x) \geq \chi(|u|) \Rightarrow V^\circ(x; v) \leq -\phi(V(x)) \tag{2}$$
for all $(x, u) \in \mathcal{C}$ and $v \in F(x, u)$; and
3. there exists a continuous, positive definite $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that
$$V(y) \leq \max\{\alpha(V(x)), \chi(|u|)\} \tag{3}$$
for all $(x, u) \in \mathcal{D}$ and $y \in G(x, u)$ (with the same χ).

It is an *ISS Lyapunov function w.r.t.* $\mathcal{A}$ if ϕ is positive definite and α is a contraction on $\mathbb{R}_{>0}$, that is, if

$$\phi(r) > 0,\ \alpha(r) < r \qquad \forall\, r > 0. \tag{4}$$

For brevity, we omit the term "w.r.t. $\mathcal{A}$" when $\mathcal{A} = \{0\}$. The candidate ISS Lyapunov function is defined for studying the effects of destabilizing flows or jumps. When it is actually decreasing along solutions of (1), the term "candidate" is dropped (cf. [11, Def. 2.5 and Prop. 2.6]) and ISS is achieved.

Proposition 1 ([11, Prop. 2.7]). *The hybrid system* (1) *is ISS w.r.t.* $\mathcal{A}$ *if it admits an ISS Lyapunov function w.r.t.* $\mathcal{A}$.

Remark 1. In the absence of inputs, ISS is equivalent to the notion of *global pre-asymptotic stability (pre-GAS)*, that is, there exist $\beta \in \mathcal{KL}$ such that all solutions satisfy

$$|x(t, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, t + j) \qquad \forall\, (t, j) \in \operatorname{dom} x. \tag{5}$$

Pre-GAS corresponds to the standard notion of global asymptotic stability without requiring completeness of all maximal solutions, that is, all solutions are stable, bounded, and all complete solutions converge to the origin; cf. [10, Def. 3.6] and [15, p. 1397]. In this case, candidate ISS Lyapunov functions become candidate Lyapunov functions, and ISS Lyapunov functions become Lyapunov functions, which ensure pre-GAS of the hybrid system (1).

III. Interconnection

Consider a hybrid system with state $x = (x_1, x_2) \in \mathcal{X}$ transformed into an interconnection of two subsystems with states $x_1 \in \mathcal{X}_1 \subset \mathbb{R}^{n_1}$ and $x_2 \in \mathcal{X}_2 \subset \mathbb{R}^{n_2}$ modeled by

$$\begin{aligned} \dot{x}_i &= f_i(x_1, x_2), & (x_1, x_2) &\in \mathcal{C}, \\ x_i^+ &= g_i(x_1, x_2), & (x_1, x_2) &\in \mathcal{D}, \end{aligned} \qquad i = 1, 2. \tag{6}$$

For $i = 1, 2$, we let $j := \{1, 2\}\backslash\{i\}$ throughout this paper,. Each x_i-subsystem of (6) treats the state x_j as an input.

Remark 2. The analysis and results in this and the following sections also apply to the case of interconnections with external inputs and/or set-valued maps, that is,

$$\begin{aligned} \dot{x} &\in F_i(x, u), & (x, u) &\in \mathcal{C} \times \mathcal{U}, \\ x_i^+ &\in G_i(x, u), & (x, u) &\in \mathcal{D} \times \mathcal{U}, \end{aligned} \qquad i = 1, 2.$$

We omit the external inputs and consider only the special case of single-valued maps here to better focus on investigating effects and implications of the interconnection.

A. Small-gain condition

Assuming that each subsystem admits a candidate ISS Lyapunov function, we investigate stability of the interconnection (6) based on small-gain conditions.

[1] Here $x(t, j)$ represents the state of (1) at time t and after j jumps.

[2] A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing and positive definite. It is of *class* $\mathcal{K}_\infty$ if it is unbounded in addition. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t, and $\beta(r, \cdot)$ is strictly decreasing to 0 for each fixed r.

[3] Note that the set of hybrid jump times $J(u)$ with measure 0 cannot be ignored when computing the essential supremum norm.

Assumption 1. Each x_i-subsystem (with input x_j) admits a candidate ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ with bounds ψ_{1i}, ψ_{2i}, a gain χ_i, and rates ϕ_i, α_i.

We say that two gains $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ satisfy the *small-gain condition* if

$$\gamma_1(\gamma_2(r)) < r \qquad \forall\, r > 0. \tag{7}$$

Lemma 1 ([8, Lemma A.1])**.** *Provided that the small-gain condition* (7) *holds, there exists a $\rho \in \mathcal{K}_\infty$ such that $\rho \in \mathcal{C}^1$ with $\rho' > 0$ on $\mathbb{R}_{>0}$, and*

$$\gamma_1^{-1}(r) > \rho(r) > \gamma_2(r) \qquad \forall\, r > 0. \tag{8}$$

Based on Lemma 1, we are able to combine the candidate ISS Lyapunov functions for subsystems to construct a candidate Lyapunov function for the interconnection (6).

Proposition 2 ([15, Th. III.1])**.** *Consider the interconnection* (6)*. Suppose that Assumption 1 and* (7) *hold for*

$$\gamma_i := \chi_i \circ \psi_{1j}^{-1}, \qquad (i,j) = (1,2), (2,1). \tag{9}$$

With ρ in (8)*, the function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ defined by*

$$V(x) := \max\{\rho(V_1(x_1)), V_2(x_2)\} \tag{10}$$

is a candidate Lyapunov function for (6) *with bounds ψ_1, ψ_2 defined by $\psi_1(r) := \min\{\rho(\psi_{11}(r/\sqrt{2})), \psi_{12}(r/\sqrt{2})\}$ and $\psi_2(r) := \max\{\rho(\psi_{21}(r)), \psi_{22}(r)\}$, and rates ϕ, α defined by $\phi(r) := \min\{\rho'(\rho^{-1}(r))\phi_1(\rho^{-1}(r)), \phi_2(r)\}$ and $\alpha(r) := \max\{\rho(\alpha_1(\rho^{-1}(r))), \alpha_2(r), \rho(\chi_{12}(r)), \chi_{21}(\rho^{-1}(r))\}$.*

If V is a Lyapunov function then we can conclude pre-GAS of (6) via Proposition 1 and Remark 1. In the following we establish pre-GAS for various cases of subsystems.

B. ISS subsystems

Consider the case that each subsystem admits an ISS Lyapunov function.

Proposition 3. *Consider the interconnection* (6)*. Suppose that Assumption 1 holds with V_1, V_2 being ISS Lyapunov functions, and* (7) *with γ_1, γ_2 defined by* (9)*. Then the function V defined by* (10) *with ρ in* (8) *is a Lyapunov function for* (6)*, and hence* (6) *is pre-GAS.*

If (4) doesn't hold for one of $\phi_1, \phi_2, \alpha_1, \alpha_2$ then it doesn't for the corresponding ϕ, α in Proposition 2, either. Thus the function V defined by (10) is not a Lyapunov function, and we cannot conclude pre-GAS. In the following section, we consider such cases and establish pre-GAS for particular sets of solutions.

IV. Modifying ISS Lyapunov functions

In this section, we investigate stability of the interconnection (6) for the cases that the flows or the jumps in its subsystems have destabilizing effects, via suitable ADT/RADT modifications. Candidate ISS Lyapunov functions with linear rates are required for such modifications.

A. Candidate exponential ISS Lyapunov functions

Definition 2. A candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for the hybrid system (1) with the rates ϕ, α in (2), (3) satisfying $\phi(r) = cr$ and $\alpha(r) = e^{-d}r$ for all $r \geq 0$ with some constants $c, d \in \mathbb{R}$ is called a *candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}$* with *rate coefficients* c, d. It is an *exponential ISS Lyapunov function w.r.t. $\mathcal{A}$* if $c, d > 0$.

Remark 3. Using arguments similar to the proof of [17, Th. 2, (b) ⇒ (c)], one can show that the existence of an ISS Lyapunov function is equivalent to that of an exponential one. In general, even for continuous-time systems, the existence of a candidate ISS Lyapunov function is not equivalent to that of an exponential one. This can be seen readily from the following simple example. The scalar system $\dot{x} = x^3$ admits a candidate ISS Lyapunov function $V(x) := x^2$ with $\nabla V(x)x^3 = 2x^4 = 2V(x)^2$. However, it is not forward complete, and hence does not admit a candidate exponential ISS Lyapunov function [21, Th. 2].

Assumption 2. Each x_i-subsystem (with input x_j) admits a candidate exponential ISS Lyapunov function $V_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ with bounds ψ_{1i}, ψ_{2i}, a gain χ_i, and rate coefficients c_i, d_i.

The next corollary follows directly from Propositions 2, 3.

Corollary 4. *Consider the interconnection* (6)*. Suppose that Assumption 2 and* (7) *hold for γ_1, γ_2 defined by* (9)*. With ρ in* (8)*, the function V defined by* (10) *is a candidate Lyapunov function for* (6) *with rates ϕ, α defined by $\phi(r) := \min\{c_2 r, c_1\rho'(\rho^{-1}(r))\rho^{-1}(r)\}$, $\alpha(r) := \max\{\rho(e^{-d_1}\rho^{-1}(r)), e^{-d_2}r, \rho(\chi_{12}(r)), \chi_{21}(\rho^{-1}(r))\}$. If $c_1, c_2, d_1, d_2 > 0$ then it is a Lyapunov function, and hence* (6) *is pre-GAS.*

Suppose that each subsystem of (6) admits a candidate exponential ISS Lyapunov function. If in one subsystem both the flows and the jumps are destabilizing (i.e., $c_i, d_i \leq 0$ for $i = 1$ or 2), we cannot establish stability by restricting the jump frequency. In the following, we consider the other cases when at least one of c_1, c_2, d_1, d_2 is non-positive.

B. Destabilizing flows: RADT modification

Consider the case that Assumption 2 holds with $c_1, c_2 \leq 0 < d_1, d_2$, that is, the flows in both subsystems have destabilizing effects.[4] Pre-GAS can be established for solutions that jump fast enough, in the sense of reverse average dwell-time [17]. We say a solution x of (1) admits a *reverse average dwell-time (RADT)* $\tau_a^* > 0$ if

$$j - k \geq (t - s)/\tau_a^* - N_0^* \tag{11}$$

for all $(s,k), (t,j) \in \operatorname{dom} x$ such that $t + j \geq s + k$ with an integer $N_0^* \geq 1$. (If $N_0^* = 1$ then τ_a^* becomes a *reverse dwell-time*: any two consecutive jumps are separated by at most τ_a^*.) Following [22], a solution x satisfies (11) if and

[4]The cases that only one of c_1, c_2 is non-positive can be handled by a similar approach, see the discussion after Theorem 5.

only if $\mathrm{dom}\, x = \mathrm{dom}\, \tau$ for an RADT timer τ modeled by[5]

$$\begin{aligned} \dot{\tau} &= 1/\tau_a^*, & \tau &\in [0, N_0^*], \\ \tau^+ &= \max\{0, \tau - 1\}, & \tau &\in [0, N_0^*]. \end{aligned}$$

Consider the augmented interconnection[6]

$$\begin{aligned} \dot{x}_i &= f_i(x),\ i = 1,2,\ \dot{\tau} = 1/\tau_a^*, & (x,\tau) &\in \bar{\mathcal{C}}^*, \\ x_i^+ &= g_i(x),\ i = 1,2,\ \tau^+ = \max\{0, \tau - 1\}, & (x,\tau) &\in \bar{\mathcal{D}}^* \end{aligned} \tag{12}$$

with $\bar{\mathcal{C}}^* = \mathcal{C} \times [0, N_0^*]$ and $\bar{\mathcal{D}}^* = \mathcal{D} \times [0, N_0^*]$. Following [18, Prop. 6] (which extends [15, Prop. IV.4]), for each (x_i, τ)-subsystem of (12) (with input x_j), the function $W_i : \mathcal{X}_i \times [0, N_0^*] \to \mathbb{R}_{\geq 0}$ defined by

$$W_i(x_i, \tau) := e^{-L_i \tau} V_i(x_i) \tag{13}$$

with $L_i > 0$ is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}_i := \{0\} \times [0, N_0^*]$ with bounds $e^{-L_i N_0^*}\psi_{1i}, \psi_{2i}$, gain χ_i, and rate coefficients $\bar{c}_i^*, \bar{d}_i^*$ defined by

$$\bar{c}_i^* := c_i + L_i/\tau_a^*, \qquad \bar{d}_i^* := d_i - L_i. \tag{14}$$

Therefore, if the RADT τ_a^* satisfies

$$-c_i \tau_a^* < d_i \tag{15}$$

then there exists an $L_i \in (-c_i \tau_a^*, d_i)$ such that $\bar{c}_i^*, \bar{d}_i^* > 0$, and hence that W_i is an exponential ISS Lyapunov function.

To establish pre-GAS of (12), we construct a Lyapunov function for (12) via Corollary 4, which requires (7) with

$$\gamma_i(r) := \chi_i(\psi_{1j}^{-1}(e^{L_j N_0^*} r)), \quad (i,j) = (1,2), (2,1). \tag{16}$$

Lemma 2. *Suppose that there is an $\varepsilon > 0$ such that*

$$(1+\varepsilon)\chi_1\big(\psi_{12}^{-1}\big((1+\varepsilon)\chi_2(\psi_{11}^{-1}(r))\big)\big) < r \qquad \forall\, r > 0, \tag{17}$$

and that τ_a^* and N_0^* satisfy (15) *and*

$$-c_i N_0^* \tau_a^* < \ln(1+\varepsilon) \tag{18}$$

for all $i \in \{1, 2\}$. Then there exist $L_1 \in (-c_1\tau_a^*, d_1)$, $L_2 \in (-c_2\tau_a^*, d_2)$ such that (7) *holds for γ_1, γ_2 defined by* (16).

From Corollary 4 and the analysis above it follows that:

Theorem 5. *Consider the interconnection* (6). *Suppose that Assumption 2 holds with $c_1, c_2 \leq 0 < d_1, d_2$, and that there is an $\varepsilon > 0$ such that* (17) *holds. For each RADT τ_a^* and integer $N_0^* \geq 1$ satisfying* (15) *and* (18) *for all $i \in \{1, 2\}$, there exist $L_1 \in (-c_1\tau_a^*, d_1)$ and $L_2 \in (-c_2\tau_a^*, d_2)$ such that* (7) *holds for γ_1, γ_2 defined by* (16). *Then the function $W : \mathcal{X} \times [0, N_0^*] \to \mathbb{R}_{\geq 0}$ defined by $W(x, \tau) := \max\{\rho(e^{-L_1\tau}V_1(x_1)), e^{-L_2\tau}V_2(x_2)\}$ with ρ in* (8) *is a Lyapunov function w.r.t. $\mathcal{A} := \{0\} \times [0, N_0^*]$ for the augmented interconnection* (12), *and hence the pre-GAS estimate* (5) *holds for all solutions satisfying* (11).

For each $c_1, c_2 \leq 0 < d_1, d_2$ and each $\varepsilon > 0$ satisfying (17), there exists an RADT τ_a^* small enough that (15) and (18) hold for all $i \in \{1, 2\}$. Meanwhile, if $c_i > 0$ then (15) and (18) always hold; and if $c_1, c_2 \geq 0$ then the conclusions of Theorem 5 hold for arbitrary $\tau_a^* > 0$ and $N_0^* \geq 1$.

The coefficient $1 + \varepsilon$ in (17) can be made arbitrarily close to 1 with a sufficiently small ε. Consider the following small-gain conditions:

(SG1) The inequality (7) holds for γ_1, γ_2 defined by (9).
(SG2) There is an $\varepsilon > 0$ such that (17) holds.

We say that (SG2) is generic in (SG1), in the sense that every pair of gains χ_1, χ_2 satisfying (SG1) can be approximated by a pair satisfying (SG2). Therefore, the RADT modification induces no substantial increase in the feedback gains.

C. Destabilizing jumps: ADT modification

Consider the case that Assumption 2 holds with $c_1, c_2 > 0 \geq d_1, d_2$, that is, the jumps in both subsystems have destabilizing effects.[7] Pre-GAS can be established for solutions that jump slow enough, in the sense of average dwell-time [16]. We say a solution x of (1) admits an *average dwell-time (ADT)* $\tau_a > 0$ if

$$j - k \leq (t - s)/\tau_a + N_0 \tag{19}$$

for all $(s, k), (t, j) \in \mathrm{dom}\, x$ such that $t + j \geq s + k$ with an integer $N_0 \geq 1$. (If $N_0 = 1$ then τ_a becomes a *dwell-time* [23]: any two jumps are separated by at least τ_a.) Following [22] (cf. [24]), a solution x satisfies (19) if and only if $\mathrm{dom}\, x = \mathrm{dom}\, \tau$ for an ADT timer τ modeled by

$$\begin{aligned} \dot{\tau} &\in [0, 1/\tau_a], & \tau &\in [0, N_0], \\ \tau^+ &= \tau - 1, & \tau &\in [1, N_0]. \end{aligned}$$

Consider the augmented interconnection

$$\begin{aligned} \dot{x}_i &= f_i(x),\ i = 1,2,\ \dot{\tau} \in [0, 1/\tau_a], & (x,\tau) &\in \bar{\mathcal{C}}, \\ x_i^+ &= g_i(x),\ i = 1,2,\ \tau^+ = \tau - 1, & (x,\tau) &\in \bar{\mathcal{D}}, \end{aligned} \tag{20}$$

where $\bar{\mathcal{C}} = \mathcal{C} \times [0, N_0]$ and $\bar{\mathcal{D}} = \mathcal{D} \times [1, N_0]$. Following [18, Prop. 5] (which extends [15, Prop. IV.1]), for each (x_i, τ)-subsystem of (20) (with input x_j), the function $W_i : \mathcal{X}_i \times [0, N_0] \to \mathbb{R}_{\geq 0}$ defined by

$$W_i(x_i, \tau) := e^{L_i \tau} V_i(x_i) \tag{21}$$

with $L_i > 0$ is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}_i := \{0\} \times [0, N_0]$ with bounds $\psi_{1i}, e^{L_i N_0}\psi_{2i}$, gain $e^{L_i N_0}\chi_i$, and rates coefficients $\bar{c}_i, \bar{d}_i$ defined by

$$\bar{c}_i := c_i - L_i/\tau_a, \qquad \bar{d}_i := d_i + L_i. \tag{22}$$

Therefore, if the ADT τ_a satisfies

$$c_i \tau_a > -d_i, \tag{23}$$

then there exists an $L_i \in (-d_i, c_i \tau_a)$ such that $\bar{c}_i, \bar{d}_i > 0$, and hence that W_i is an exponential ISS Lyapunov function.

To establish pre-GAS of (20), we construct a Lyapunov function for (20) via Corollary 4, which requires (7) with

$$\gamma_i(r) := e^{L_i N_0}\chi_i(\psi_{1j}^{-1}(r)), \qquad (i,j) = (1,2), (2,1) \tag{24}$$

[5] There is a scaling difference between the RADT timers here and in [22].

[6] Compared to [15], [18], we use one timer for all subsystems, due to the fact that the RADT is a property of the hybrid timer domain, which is the same for both subsystems.

[7] The cases that only one of d_1, d_2 is non-positive can be handled by a similar approach, see the discussion after Theorem 6.

Lemma 3. *Suppose that there is an $\varepsilon > 0$ such that*

$$(1+\varepsilon)e^{-d_1}\chi_1\big(\psi_{12}^{-1}\big((1+\varepsilon)e^{-d_1}\chi_2(\psi_{11}^{-1}(r))\big)\big) < r \quad \forall\, r > 0, \tag{25}$$

and that τ_a and N_0 satisfy (23) *and*

$$-d_i(N_0 - 1) < \ln(1+\varepsilon). \tag{26}$$

for all $i \in \{1,2\}$. Then there exist $L_1 \in (-d_1, c_1\tau_a)$, $L_2 \in (-d_2, c_2\tau_a)$ such that (7) *holds for γ_1, γ_2 defined by* (24).

From Corollary 4 and the analysis above it follows that:

Theorem 6. *Consider the interconnection* (6)*. Suppose that Assumption 2 holds with $c_1, c_2 > 0 \geq d_1, d_2$, and that there is an $\varepsilon > 0$ such that* (25) *holds. For each ADT τ_a and integer $N_0 \geq 1$ satisfying* (23) *and* (26) *for all $i \in \{1,2\}$, there exist $L_1 \in (-d_1, c_1\tau_a)$ and $L_2 \in (-d_2, c_2\tau_a)$ such that* (7) *holds for γ_1, γ_2 defined by* (24)*. Then the function $W : \mathcal{X} \times [0, N_0] \to \mathbb{R}_{\geq 0}$ defined by $W(x,\tau) := \max\{\rho(e^{L_1\tau}V_1(x_1)), e^{L_2\tau}V_2(x_2)\}$ with ρ in* (8) *is a Lyapunov function w.r.t. $\mathcal{A} := \{0\} \times [0, N_0]$ for the augmented interconnection* (20)*, and hence the pre-GAS estimate* (5) *holds for all solutions satisfying* (19)*.*

For each $c_1, c_2 > 0 \geq d_1, d_2$ and each $\varepsilon > 0$ satisfying (25), there exists an ADT τ_a large enough that (23) holds for all $i \in \{1,2\}$. (Unlike (18) which depends on τ_a^*, (26) is independent to τ_a.) Meanwhile, if $d_i \geq 0$ then (23) and (26) always hold; and if $d_1, d_2 \geq 0$ then the conclusions of Theorem 6 hold for arbitrary $\tau_a > 0$ and $N_0 \geq 1$.

Compared with those in (17), the coefficients in (25) satisfy $(1+\varepsilon)e^{-d_1} > e^{-d_1} > 1$ and $(1+\varepsilon)e^{-d_2} > e^{-d_2} > 1$. Consider the following small-gain condition:

(SG3) There is an $\varepsilon > 0$ such that (25) holds.

Unlike (SG2) in Section IV-B, (SG3) is clearly not generic in (SG1) in the same sense. Hence the ADT modification leads to substantially larger feedback gains.

D. Destabilizing jumps: an alternate construction

Consider the case that Assumption 2 holds with $c_1, c_2 > 0 \geq d_1, d_2$, and γ_1, γ_2 defined by (9) are linear, that is,

$$\gamma_1(r) = \xi_1 r, \ \ \gamma_2(r) = \xi_2 r \qquad \forall\, r \geq 0 \tag{27}$$

with $\xi_1, \xi_2 > 0$. Then pre-GAS can be established under the less restrictive small-gain condition (SG1) instead of (SG3), by applying the ADT modification to (6) instead of its subsystems [19]. In this case (SG1) becomes $\xi_1\xi_2 < 1$, and (8) holds for all $\rho(r) := \mu r$ with $\mu \in (\xi_2, 1/\xi_1)$. Following Corollary 4, the function V defined by (10) is a candidate exponential Lyapunov function for (6) with rate coefficients $c := \min\{c_1, c_2\}$ and $d := \min\{d_1, d_2\}$. Then the ADT modification yields the following result.

Proposition 7. *Consider the interconnection* (6)*. Suppose that Assumption 2 holds with $c_1, c_2 > 0 \geq d_1, d_2$, and γ_1, γ_2 defined by* (9) *satisfy* (27) *with $\xi_1\xi_2 < 1$. For each ADT $\tau_a > -\min\{d_1, d_2\}/\min\{c_1, c_2\}$ and integer $N_0 \geq 1$, the function $W : \mathcal{X} \times [0, N_0] \to \mathbb{R}_{\geq 0}$ defined by $W(x,\tau) := e^{L\tau}\max\{\mu V_1(x_1), V_2(x_2)\}$ with $-\min\{d_1, d_2\} < L < \min\{c_1, c_2\}\tau_a$ and $\mu \in (\xi_2, 1/\xi_1)$ is a Lyapunov function for the augmented interconnection* (20)*, and hence the pre-GAS estimate* (5) *holds for all solutions satisfying* (19)*.*

Remark 4. The lower bound on the ADT τ_a in Proposition 7 is clearly greater than or equal to that in Theorem 6, that is,

$$\frac{\max\{-d_1, -d_2\}}{\min\{c_1, c_2\}} \geq \max\left\{\frac{-d_1}{c_1}, \frac{-d_2}{c_2}\right\}.$$

However, if the assumptions in Proposition 7 hold with $d_1 + d_2 \leq \ln(\xi_1\xi_2)$ then there is no $\varepsilon > 0$ such that (25) holds, and hence Theorem 6 cannot be applied.

E. Destabilizing flows and jumps

Consider the case that the jumps in one subsystem and the flows in the other one have destabilizing effects. Without loss of generality, suppose Assumption 2 holds with $c_1, d_2 \leq 0 < c_2, d_1$. Pre-GAS can be established for solutions that jump neither too fast nor too slow, in the sense of combined ADT τ_a and RADT τ_a^*. Consider the augmented interconnection

$$\begin{aligned} &\dot{x}_1 = f_1(x), \ \ \dot{\tau}_1 = 1/\tau_a^*, \\ &\dot{x}_2 = f_2(x), \ \ \dot{\tau}_2 \in [0, 1/\tau_a], \end{aligned} \qquad (x, \tau_1, \tau_2) \in \tilde{\mathcal{C}},$$
$$\begin{aligned} &x_1^+ = g_1(x), \ \ \tau_1^+ = \max\{0, \tau_1 - 1\}, \\ &x_2^+ = g_2(x), \ \ \tau_2^+ = \tau_2 - 1, \end{aligned} \qquad (x, \tau_1, \tau_2) \in \tilde{\mathcal{D}}, \tag{28}$$

where $\tilde{\mathcal{C}} = \mathcal{C} \times [0, N_0^*] \times [0, N_0]$ and $\tilde{\mathcal{D}} = \mathcal{D} \times [0, N_0^*] \times [1, N_0]$. The function W_1 defined by (13) with $L_1 > 0$ is a candidate exponential ISS Lyapunov function for the (x_1, τ_1)-subsystem of (28) (with input x_2) with rate coefficients $\bar{c}_1^*, \bar{d}_1^*$ defined by (14); and an exponential ISS Lyapunov function if $-c_1\tau_a^* < L_1 < d_1$. Meanwhile, the function W_2 defined by (21) with $L_2 > 0$ is a candidate exponential ISS Lyapunov function for the (x_2, τ_2)-subsystem of (28) (with input x_1) with rate coefficients $\bar{c}_2, \bar{d}_2$ defined by (22); and an exponential ISS Lyapunov function if $-d_2 < L_2 < c_2\tau_a$. Suppose that there is an $\varepsilon > 0$ so that

$$(1+\varepsilon)\chi_1\big(\psi_{12}^{-1}\big((1+\varepsilon)e^{-d_2}\chi_2(\psi_{11}^{-1}(r))\big)\big) < r \qquad \forall\, r > 0, \tag{29}$$

and that τ_a, τ_a^* and N_0^*, N_0 satisfy (15), (18) for $i = 1$ and (23), (26) for $i = 2$. Then there exist $L_1 \in (-c_1\tau_a^*, d_1)$, $L_2 \in (-d_2, c_2\tau_a)$ such that (7) holds for γ_1, γ_2 defined by

$$\gamma_1(r) = \chi_1(\psi_{12}^{-1}(r)), \quad \gamma_2(r) = e^{L_2 N_0}\chi_2(\psi_{11}^{-1}(e^{L_1 N_0^*}r)). \tag{30}$$

From Corollary 4 and the analysis above it follows that:

Theorem 8. *Consider the interconnection* (6)*. Suppose that Assumption 2 holds with $c_1, d_2 \leq 0 < c_2, d_1$, and that there is an $\varepsilon > 0$ such that* (29) *holds. For each ADT τ_a, RADT τ_a^* and integers $N_0^*, N_0 \geq 1$ satisfying* (15), (18) *for $i = 1$ and* (23), (26) *for $i = 2$, there exist $L_1 \in (-c_1\tau_a^*, d_1)$ and $L_2 \in (-d_2, c_2\tau_a)$ such that* (7) *holds for γ_1, γ_2 defined by* (30)*. Then the function $W : \mathcal{X} \times [0, N_0^*] \times [0, N_0] \to \mathbb{R}_{\geq 0}$ defined by $W(x, \tau_1, \tau_2) := \max\{\rho(e^{-L_1\tau_1}V_1(x_1)), e^{L_2\tau_2}V_2(x_2)\}$ with ρ in* (8) *is a Lyapunov function w.r.t. $\mathcal{A} := \{0\} \times [0, N_0^*] \times [0, N_0]$ for the augmented interconnection* (28)*,* *and hence the pre-GAS estimate* (5) *holds for all solutions satisfying* (11) *and* (19)*.*

Consider the small-gain condition in Theorem 8:
(SG4) There is an $\varepsilon > 0$ such that (29) holds.
Similarly to (SG3) in Section IV-C, (SG4) is clearly not generic in (SG1) because of the substantially larger gain for the (x_2, τ_2)-subsystem due to the ADT modification.

F. Destabilizing flows and jumps: an alternate construction

Consider the case that Assumption 2 holds with $c_1, d_2 \leq 0 < c_2, d_1$, and γ_1, γ_2 defined by (9) are linear, that is, (27) holds with $\xi_1, \xi_2 > 0$. Then pre-GAS can be established under the less restrictive small-gain condition (SG1) instead of (SG4), by applying first the RADT modification to the (x_1, τ_1)-subsystem in (28) (with input x_2), and then the ADT one to the (x_1, x_2, τ_1)-interconnection. The function W_1 defined by (13) with $L_1 > 0$ is a candidate exponential ISS Lyapunov function for the (x_1, τ_1)-subsystem of (28) (with input x_2) with rate coefficients $\bar{c}_1^*, \bar{d}_1^*$ defined by (14). To invoke Corollary 4 for the (x_1, x_2, τ_1)-interconnection it requires (7) with γ_1 defined by (9) and γ_2 by (16), that is,

$$L_1 < -\ln(\xi_1 \xi_2)/N_0^* =: \bar{L}. \tag{31}$$

With $\mu \in (\xi_2 e^{L_1 N_0^*}, 1/\xi_1)$, the function $\bar{W} : \mathcal{X} \times [0, N_0^*] \to \mathbb{R}_{\geq 0}$ defined by $\bar{W}(x, \tau_1) := \max\{\mu e^{-L_1 \tau_1} V_1(x_1), V_2(x_2)\}$ is a candidate exponential Lyapunov function w.r.t. $\bar{A} := \{0\} \times [0, N_0^*]$ with rate coefficients $\bar{c} := \min\{c_1 + L_1/\tau_a^*, c_2\}$ and $\bar{d} := \min\{d_1 - L_1, d_2\}$, where $\bar{c} > 0$ if

$$L_1 > -c_1 \tau_a^*, \tag{32}$$

and $\bar{d} \leq d_2 \leq 0$ (hence, unlike in the case in Section IV-B, $L_1 < d_1$ is not needed). Moreover, from Section IV-C we see the ADT modification requires $\bar{c}\tau_a > L_2 > -\bar{d}$, that is,

$$\tau_a \min\{c_1 + L_1/\tau_a^*, c_2\} > L_2 > \max\{L_1 - d_1, -d_2\}. \tag{33}$$

Following the proof of Lemmas 2, 3, if τ_a, τ_a^* and N_0^* satisfy

$$\begin{aligned} &c_2 \tau_a > -d_2, \\ &(c_1 - \bar{L}/\tau_a^*)\tau_a > \max\{\bar{L} - d_1, -d_2\}, \\ &\max\{c_2(\tau_a - \tau_a^*), (\tau_a^*/\tau_a - 1)d_2\} > -c_1 \tau_a^* - d_1, \end{aligned} \tag{34}$$

then there exist $L_1, L_2 > 0$ such that (31)–(33) hold. The ADT modification yields the following result:

Proposition 9. *Consider the interconnection* (6)*. Suppose that Assumption 2 holds with $c_1, d_2 \leq 0 < c_2, d_1$, and that γ_1, γ_2 defined by* (9) *satisfy* (27) *with $\xi_1 \xi_2 < 1$. For each ADT τ_a, RADT τ_a^* and integer $N_0^* \geq 1$ satisfying* (34)*, there exists L_1, L_2 such that* (31)–(33) *hold. Then the function $W : \mathcal{X} \times [0, N_0^*] \times [0, N_0] \to \mathbb{R}_{\geq 0}$ defined by $W(x, \tau_1, \tau_2) := e^{L_2 \tau_2} \max\{\mu e^{-L_1 \tau_1} V_1(x_1), V_2(x_2)\}$ with $\mu \in (\xi_2 e^{L_1 N_0^*}, 1/\xi_1)$ is a Lyapunov function for the augmented interconnection* (28)*, and hence the pre-GAS estimate* (5) *holds for all solutions satisfying* (11) *and* (19)*.*

See [25, Sec. 4.3] for an example of this construction.

Remark 5. Similarly to Remark 4, if the assumptions in Proposition 9 hold with $d_2 \leq \ln(\xi_1 \xi_2)$ then there is no $\varepsilon > 0$ such that (29) holds, and hence Theorem 8 cannot be applied.

References

[1] C. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. SIAM, 2009.

[2] D. J. Hill, "A generalization of the small-gain theorem for nonlinear feedback systems," *Automatica*, vol. 27, pp. 1043–1045, 1991.

[3] I. M. Y. Mareels and D. J. Hill, "Monotone stability of nonlinear feedback systems," *Journal of Mathematical Systems, Estimation, and Control*, vol. 2, pp. 275–291, 1992.

[4] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, pp. 435–443, 1989.

[5] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals, and Systems*, vol. 7, pp. 95–120, 1994.

[6] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An ISS small gain theorem for general networks," *Mathematics of Control, Signals, and Systems*, vol. 19, pp. 93–122, 2007.

[7] I. Karafyllis and Z.-P. Jiang, "A vector small-gain theorem for general non-linear control systems," *IMA Journal of Mathematical Control and Information*, vol. 28, pp. 309–344, 2011.

[8] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, pp. 1211–1215, 1996.

[9] D. S. Laila and D. Nešić, "Discrete-time Lyapunov-based small-gain theorem for parameterized interconnected ISS systems," *IEEE Transactions on Automatic Control*, vol. 48, pp. 1783–1788, 2003.

[10] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[11] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Systems & Control Letters*, vol. 58, pp. 47–53, 2009.

[12] D. Nešić and D. Liberzon, "A small-gain approach to stability analysis of hybrid systems," in *44th IEEE Conference on Decision and Control*, 2005, pp. 5409–5414.

[13] I. Karafyllis and Z.-P. Jiang, "A small-gain theorem for a wide class of feedback systems with control applications," *SIAM Journal on Control and Optimization*, vol. 46, pp. 1483–1517, 2007.

[14] D. Liberzon and D. Nešić, "Stability analysis of hybrid systems via small-gain theorems," in *Hybrid Systems: Computation and Control*. Springer Berlin Heidelberg, 2006, pp. 421–435.

[15] D. Liberzon, D. Nešić, and A. R. Teel, "Lyapunov-based small-gain theorems for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 59, pp. 1395–1410, 2014.

[16] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *38th IEEE Conference on Decision and Control*, vol. 3, 1999, pp. 2655–2660.

[17] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, pp. 2735–2744, 2008.

[18] A. Mironchenko, G. Yang, and D. Liberzon, "Lyapunov small-gain theorems for not necessarily ISS hybrid systems," in *21st International Symposium on Mathematical Theory of Networks and Systems*, 2014, pp. 1001–1008.

[19] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, "Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods," *Nonlinear Analysis: Hybrid Systems*, vol. 6, pp. 899–915, 2012.

[20] F. H. Clarke, *Optimization and Nonsmooth Analysis*. SIAM, 1990.

[21] D. Angeli and E. D. Sontag, "Forward completeness, unboundedness observability, and their Lyapunov characterizations," *Systems & Control Letters*, vol. 38, pp. 209–217, 1999.

[22] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems Part II: (Pre)Asymptotically stable compact sets," *IEEE Transactions on Automatic Control*, vol. 53, pp. 734–748, 2008.

[23] A. S. Morse, "Supervisory control of families of linear set-point controllers—Part I. Exact matching," *IEEE Transactions on Automatic Control*, vol. 41, pp. 1413–1431, 1996.

[24] S. Mitra, D. Liberzon, and N. Lynch, "Verifying average dwell time of hybrid systems," *ACM Transactions on Embedded Computing Systems*, vol. 8, pp. 1–37, 2008.

[25] A. Mironchenko, G. Yang, and D. Liberzon, "Lyapunov small-gain theorems for networks of not necessarily ISS hybrid systems," submitted to *Automatica*.