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Brief paper

# Generalized switching signals for input-to-state stability of switched systems[☆]

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ABSTRACT

This article deals with input-to-state stability (ISS) of continuous-time switched nonlinear systems. Given a family of systems with exogenous inputs such that not all systems in the family are ISS, we characterize a new and general class of switching signals under which the resulting switched system is ISS. Our stabilizing switching signals allow the number of switches to grow faster than an affine function of the length of a time interval, unlike in the case of average dwell time switching. We also recast a subclass of average dwell time switching signals in our setting and establish analogs of two representative prior results.



## 1. Introduction

A *switched system* comprises of two components—a family of systems and a switching signal. The *switching signal* selects an active subsystem at every instant of time, i.e., the system from the family that is currently being followed (Liberzon, 2003, Section 1.1.2). Stability of switched systems is broadly classified into two categories—*stability under arbitrary switching* (Liberzon, 2003, Chapter 2) and *stability under constrained switching* (Liberzon, 2003, Chapter 3). In the former category, conditions on the family of systems are identified such that the resulting switched system is stable under all admissible switching signals; in the latter category, given a family of systems, conditions on the switching signals are identified such that the resulting switched system is stable. In this article our focus is on stability of switched systems with exogenous inputs under constrained switching.

Prior study in the direction of stability under constrained switching primarily utilizes the concept of *slow switching* vis-a-vis *(average) dwell time switching*. Exponential stability of a switched linear system under *dwell time switching* was studied in Morse (1996). In Xie, Wen, and Li (2001) the authors showed that a switched nonlinear system is ISS under dwell time switching if all subsystems are ISS. A class of state-dependent switching signals obeying dwell time property under which a switched nonlinear system is integral input-to-state stable (iISS) was proposed in De Persis, De Santis, Morse (2003). The dwell time requirement for stability was relaxed to *average dwell time switching* to switched linear systems with inputs and switched nonlinear systems without inputs in Hespanha and Morse (1999). ISS of switched nonlinear systems under average dwell time was studied in Vu, Chatterjee, and Liberzon (2007). It was shown that if the individual subsystems are ISS and their ISS-Lyapunov functions satisfy suitable conditions, then the switched system has the ISS, exponentially-weighted ISS, and exponentially-weighted iISS properties under switching signals obeying sufficiently large average dwell time. Given a family of systems such that not all systems in the family are ISS, it was shown in the recent work (Yang & Liberzon, 2014) that it is possible to construct a class of hybrid Lyapunov functions to guarantee ISS of the switched system provided that the switching signal neither switches too frequently nor activates the non-ISS subsystems for too long. In Müller and Liberzon (2012) input/output-to-state stability (IOSS) of switched nonlinear systems with families in which not all subsystems are IOSS, was studied. It was shown that the switched system is IOSS under a class of switching signals obeying *average dwell time* property and constrained point-wise activation of unstable subsystems.

Given a family of systems, possibly containing non-ISS dynamics, in this article we study ISS of switched systems under

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switching signals that transcend beyond the average dwell time regime in the sense that the number of switches on every interval of time can grow faster than an affine function of the length of the interval. Our characterization of stabilizing switching signals involves pointwise constraints on the duration of activation of the ISS and non-ISS systems, and the number of occurrences of the admissible switches, certain pointwise properties of the quantities defining the above constraints, and a summability condition. In particular, our contributions are:

- We allow non-ISS systems in the family and identify a class of switching signals under which the resulting switched system is ISS.
- Our class of stabilizing switching signals encompasses the average dwell time regime in the sense that on every interval of time the number of switches is allowed to grow faster than an affine function of the length of the interval. Earlier in Kundu and Chatterjee (2015) we proposed a class of switching signals beyond the average dwell time regime for global asymptotic stability (GAS) of continuous-time switched nonlinear systems.
- Although this is not the first instance when non-ISS subsystems are considered (see e.g., Müller & Liberzon, 2012 and Yang & Liberzon, 2014), to the best of our knowledge, this is the first instance when non-ISS subsystems are considered *and* the proposed class of stabilizing switching signals goes beyond the average dwell time condition.
- We recast a subclass of average dwell time switching signals in our setting and establish analogs of an ISS version of Müller and Liberzon (2012, Theorem 2), and Vu et al. (2007, Theorem 3.1) as two corollaries of our main result.

The remainder of this article is organized as follows: In Section 2 we formulate the problem under consideration and catalog certain properties of the family of systems and the switching signal. Our main results appear in Section 3, and we provide a numerical example illustrating our main result in Section 4. In Section 5 we recast prior results in our setting. The proofs of our main results are presented in a consolidated manner in Section 7.

**Notations**: Let $\mathbb{R}$ denote the set of real numbers, $\|\cdot\|$ denote the Euclidean norm, and for any interval $I \subset [0,+\infty[$ we denote by $\|\cdot\|_I$ the essential supremum norm of a map from $I$ into some Euclidean space. For measurable sets $A \subset \mathbb{R}$ we let $|A|$ denote the Lebesgue measure of $A$.

## 2. Preliminaries

We consider the *switched system*

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t), v(t)\big), \qquad x(0) = x_0 \text{ (given)}, \quad t \geq 0 \tag{1}$$

generated by

- a family of continuous-time systems with exogenous inputs
$$\dot{x}(t) = f_i\big(x(t), v(t)\big), \qquad x(0) = x_0 \text{ (given)}, \quad i \in \mathcal{P},\ t \geq 0, \tag{2}$$
where $x(t) \in \mathbb{R}^d$ is the vector of states and $v(t) \in \mathbb{R}^m$ is the vector of inputs at time $t$, $\mathcal{P} = \{1, 2, \ldots, N\}$ is a finite index set, and
- a piecewise constant function $\sigma : [0,+\infty[ \longrightarrow \mathcal{P}$ that selects, at each time $t$, the index of the active system from the family (2); this function $\sigma$ is called a *switching signal*. By convention, $\sigma$ is assumed to be continuous from right and having limits from the left everywhere, and we call such switching signals admissible. We let $\mathcal{S}$ denote the set of all such admissible switching signals.

We assume that for each $i \in \mathcal{P}$, $f_i$ is locally Lipschitz, and $f_i(0, 0) = 0$. Let the exogenous inputs $t \mapsto v(t)$ be Lebesgue measurable and essentially bounded; therefore, a solution to the switched system (1) exists in the Carathéodory sense (Filippov, 1988, Chapter 2) for some non-trivial time interval containing 0. Given a family of systems (2), our focus is on identifying a class of switching signals $\sigma \in \mathcal{S}$ under which the switched system (1) is ISS. Recall that

**Definition 1** (*Vu et al., 2007, Section 2*)**.** The switched system (1) is input-to-state stable (ISS) for a given $\sigma$ if there exist class $\mathcal{K}_\infty$ functions $\alpha$, $\chi$ and a class $\mathcal{KL}$ function $\beta$ such that for all inputs $v$ and initial states $x_0$, we have[2]

$$\alpha(\|x(t)\|) \leq \beta(\|x_0\|, t) + \chi(\|v\|_{[0,t]}) \quad \text{for all } t \geq 0. \tag{3}$$

If one can find $\alpha$, $\beta$ and $\chi$ such that (3) holds over a class $\mathcal{S}'$ of $\sigma$, then we say that (1) is uniformly ISS over $\mathcal{S}'$.

Note that when the input is set to 0, i.e., $v \equiv 0$, then (3) reduces to GAS of (1). We next catalog certain properties of the family of systems (2) and the switching signal $\sigma$. These properties will be required for our analysis towards deriving the class of stabilizing switching signals.

### 2.1. Properties of the family of systems

Let $\mathcal{P}_S$ and $\mathcal{P}_U \subset \mathcal{P}$ denote the sets of indices of ISS and non-ISS systems in the family (2), respectively, $\mathcal{P} = \mathcal{P}_S \sqcup \mathcal{P}_U$. Let $E(\mathcal{P})$ be the set of all ordered pairs $(i, j)$ such that it is admissible to switch from system $i$ to system $j$, $i, j \in \mathcal{P}$.

**Assumption 1.** There exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, $\gamma$, continuously differentiable functions $V_i : \mathbb{R}^d \to [0,+\infty[$, $i \in \mathcal{P}$, and constants $\lambda_i \in \mathbb{R}$ with $\lambda_i > 0$ for $i \in \mathcal{P}_S$ and $\lambda_i < 0$ for $i \in \mathcal{P}_U$, such that for all $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^m$, we have

$$\underline{\alpha}(\|\xi\|) \leq V_i(\xi) \leq \overline{\alpha}(\|\xi\|), \tag{4}$$

$$\left\langle \frac{\partial V_i}{\partial \xi}(\xi), f_i(\xi, \eta) \right\rangle \leq -\lambda_i V_i(\xi) + \gamma(\|\eta\|). \tag{5}$$

**Remark 1.** Conditions (4) and (5) are equivalent to an ISS version of Müller and Liberzon (2012, (7) and (18)). The functions $V_i$'s are called the *ISS-Lyapunov-like functions*, see Angeli and Sontag (1999), Krichman, Sontag, and Wang (2001) and Sontag and Wang (1995) for detailed discussion regarding the existence of such functions and their properties. In particular, condition (5) is equivalent to the ISS property for ISS subsystems (Sontag & Wang, 1995) and the unboundedness observability property for the non-ISS subsystems (Krichman et al., 2001).

**Assumption 2.** For each pair $(i, j) \in E(\mathcal{P})$ there exist $\mu_{ij} > 0$ such that the ISS-Lyapunov-like functions are related as follows:

$$V_j(\xi) \leq \mu_{ij} V_i(\xi) \quad \text{for all } \xi \in \mathbb{R}^d. \tag{6}$$

**Remark 2.** The assumption of linearly comparable Lyapunov-like functions, i.e., there exists $\mu \geq 1$ such that

$$V_j(\xi) \leq \mu V_i(\xi) \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } i, j \in \mathcal{P} \tag{7}$$

is standard in the theory of stability under average dwell time switching (Liberzon, 2003, Theorem 3.2); (6) affords sharper estimates compared to (7).

[2] $\mathcal{K} := \{\phi : [0,+\infty[ \to [0,+\infty[ \,|\, \phi \text{ is continuous, strictly increasing}, \phi(0) = 0\}$, $\mathcal{KL} := \{\phi : [0,+\infty[^2 \to [0,+\infty[ \,|\, \phi(\cdot, s) \in \mathcal{K} \text{ for each } s \text{ and } \phi(r, \cdot) \searrow 0 \text{ as } s \nearrow +\infty \text{ for each } r\}$, $\mathcal{K}_\infty := \{\phi \in \mathcal{K} \,|\, \phi(r) \to +\infty \text{ as } r \to +\infty\}$.

### 2.2. Properties of the switching signal

Fix $t > 0$. For a switching signal $\sigma$ we let $N_\sigma(0,t)$ denote the number of switches on the interval $]0,t]$, and $0 =: \tau_0 < \tau_1 < \cdots < \tau_{N_\sigma(0,t)}$ denote the corresponding switching instants before (and including) $t$. We denote the *i-th holding time* of $\sigma$ by

$$S_{i+1} := \tau_{i+1} - \tau_i, \quad i = 0, 1, \ldots. \tag{8}$$

On an interval $]s,t] \subset [0,+\infty[$ of time, let

$$\mathrm{T}_j(s,t) := \left| ]s,t] \cap \left( \bigcup_{\substack{i=0\\ \sigma(\tau_i)=j}}^{+\infty} ]\tau_i, \tau_{i+1}] \right) \right| \tag{9}$$

denote the *duration of activation* of a system $j \in \mathcal{P}$. Clearly, $\sum_{j\in\mathcal{P}} \mathrm{T}_j(s,t) = t - s$ for all $0 \le s < t < +\infty$. For a pair $(m,n) \in E(\mathcal{P})$ let

$$\mathrm{N}_{mn}(s,t) := \#\{m \to n\}_s^t \tag{10}$$

be the *number of switches* from system $m$ to system $n$ on the interval $]s,t] \subset [0,+\infty[$ of time. We have the immediate identity: $\mathrm{N}_\sigma(0,t) = \sum_{(m,n)\in E(\mathcal{P})} \mathrm{N}_{mn}(0,t)$, $t > 0$.

In the sequel we require the following class of functions:

**Definition 2.** A function $\varrho : [0,+\infty[^2 \to [0,+\infty[$ belongs to class $\mathcal{FK}_\infty$ if $\varrho$ is continuous, and for every fixed first argument, $\varrho$ is in class $\mathcal{K}_\infty$ in the second argument.

**Assumption 3.** There exist class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}, j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}$, $k \in \mathcal{P}_U$, $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$, and positive constants $\overline{\mathrm{T}}_j^{\mathrm{S}}, j \in \mathcal{P}_S$, $\overline{\mathrm{T}}_k^{\mathrm{U}}$, $k \in \mathcal{P}_U$, $\overline{\mathrm{N}}_{mn}$, $(m,n) \in E(\mathcal{P})$, such that on every interval $]s,t] \subset [0,+\infty[$ of time, the functions $\mathrm{T}_j(s,t), j \in \mathcal{P}_S$, $\mathrm{T}_k(s,t)$, $k \in \mathcal{P}_U$, and $\mathrm{N}_{mn}(s,t)$, $(m,n) \in E(\mathcal{P})$, defined in (9) and (10), respectively, satisfy the following inequalities:

$$\mathrm{T}_j(s,t) \ge -\overline{\mathrm{T}}_j^{\mathrm{S}} + \rho_j^{\mathrm{S}}(s, t-s), \tag{11}$$

$$\mathrm{T}_k(s,t) \le \overline{\mathrm{T}}_k^{\mathrm{U}} + \rho_k^{\mathrm{U}}(s, t-s), \tag{12}$$

$$\mathrm{N}_{mn}(s,t) \le \overline{\mathrm{N}}_{mn} + \rho_{mn}(s, t-s). \tag{13}$$

**Remark 3.** On every interval $]s,t] \subset [0,+\infty[$ of time, conditions (11) and (12) constrain the duration of activation of a system $j \in \mathcal{P}_S$ and $k \in \mathcal{P}_U$, respectively, and condition (13) constrains the number of occurrences of an admissible switch $(m,n) \in E(\mathcal{P})$. Each bound is provided in terms of a class $\mathcal{FK}_\infty$ function (from $\rho_j^{\mathrm{S}}, j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}, k \in \mathcal{P}_U$, $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$) and a positive offset (from $\overline{\mathrm{T}}_j^{\mathrm{S}}, j \in \mathcal{P}_S$, $\overline{\mathrm{T}}_k^{\mathrm{U}}, k \in \mathcal{P}_U$, $\overline{\mathrm{N}}_{mn}$, $(m,n) \in E(\mathcal{P})$). We consider point-wise lower bounds on the duration of activation of ISS subsystems, and upper bounds on the duration of activation of non-ISS subsystems and the number of occurrences of admissible switches on every interval $]s,t] \subset [0,+\infty[$ of time. In the analysis towards identifying switching signals for ISS of switched systems, such bounds are standard assumptions. For example, in Müller and Liberzon (2012) and Vu et al. (2007) the number of switches on every interval of time is allowed to grow at most as an affine function of the length of the interval. In the presence of non-ISS subsystems, the duration of activation of such systems is also constrained on every interval of time in Müller and Liberzon (2012). We use the class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}(s, t-s), j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}(s, t-s), k \in \mathcal{P}_U$, and $\rho_{mn}(s, t-s)$, $(m,n) \in E(\mathcal{P})$ with two arguments—the initial value of the interval $s \in [0,+\infty[$ and the length of the interval $(t-s)$, with the objective to allow the number of switches on any interval of time to grow faster than the case of average dwell time switching as we shall see momentarily.

## 3. Main results

We are now in a position to present our main results.

**Theorem 1.** *Consider the family of systems (2). Let $\mathcal{P}_S, \mathcal{P}_U \subset \mathcal{P}$ and $E(\mathcal{P})$ be as described in Section 2.1. Suppose that Assumptions 1 and 2 hold. Let there exist constants $c_1$ and $c_2$, and a class $\mathcal{FK}_\infty$ function $\rho : [0,+\infty[^2 \to [0,+\infty[$ satisfying $\rho(0,0) = 0$ such that the following conditions hold:*

$$-\sum_{j\in\mathcal{P}_S} |\lambda_j|\, \rho_j^{\mathrm{S}}(r,s) + \sum_{k\in\mathcal{P}_U} |\lambda_k|\, \rho_k^{\mathrm{U}}(r,s) + \sum_{(m,n)\in E(\mathcal{P})} (\ln \mu_{mn}) \rho_{mn}(r,s) \le c_1 - \rho(r,s) \tag{14}$$

*for every interval $]r, r+s] \subset [0,+\infty[$ of time, and*

$$\lim_{t\to+\infty} \sum_{i=0}^{\mathrm{N}_\sigma(0,t)} \exp\big(-\rho(\tau_i, t-\tau_i)\big) \le c_2. \tag{15}$$

*Here $\lambda_j, j \in \mathcal{P}_S$, $\lambda_k, k \in \mathcal{P}_U$, and $\mu_{mn}$, $(m,n) \in E(\mathcal{P})$ are as in (5) and (6), respectively, and class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}, j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}, k \in \mathcal{P}_U$, $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ are as in Assumption 3. Then the switched system (1) is uniformly input-to-state stable (ISS) for every $\sigma \in \mathcal{S}$ satisfying (11), (12), and (13) for every interval $]r, r+s] \subset [0,+\infty[$ of time.*

See Section 7 for a detailed proof of the above theorem.

**Remark 4.** The condition (14) provides a point-wise upper bound on the difference between the weighted class $\mathcal{FK}_\infty$ functions $(r,s) \mapsto \sum_{k\in\mathcal{P}_U} |\lambda_k|\, \rho_k^{\mathrm{U}}(r,s) + \sum_{(m,n)\in E(\mathcal{P})} (\ln \mu_{mn}) \rho_{mn}(r,s)$, and $(r,s) \mapsto \sum_{j\in\mathcal{P}_S} |\lambda_j|\, \rho_j^{\mathrm{S}}(r,s)$ in terms of a constant $c_1$ and another class $\mathcal{FK}_\infty$ function $\rho$ satisfying $\rho(0,0) = 0$, where the class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}, j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}, k \in \mathcal{P}_U$, and $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ constrain the duration of activation of ISS subsystems and non-ISS subsystems, and the number of occurrences of the admissible switches, respectively on every interval $]r, r+s] \subset [0,+\infty[$ of time.

**Remark 5.** The condition (15) deals with summability of a series with non-negative terms involving the class $\mathcal{FK}_\infty$ function $\rho$ satisfying $\rho(0,0) = 0$, the number of switches $\mathrm{N}_\sigma(0,t)$ before (and including) $t > 0$, and the corresponding switching instants $0 =: \tau_0 < \tau_1 < \cdots < \tau_{\mathrm{N}_\sigma(0,t)}$.

**Remark 6.** The constants $c_1$ and $c_2$ on the right-hand sides of (14) and (15) ensure *uniform* ISS of the switched system (1) over all switching signals $\sigma$ satisfying (11)–(15).

**Remark 7.** If $v \equiv 0$, condition (14) guarantees GAS of switched system (1). Earlier in Kundu and Chatterjee (2015) we proposed a class of switching signals that transcends beyond the average dwell time regime and ensures GAS of continuous-time switched systems. On the one hand, the stability condition in Kundu and Chatterjee (2015, Theorem 5) *solely* involved certain asymptotic properties of the switching signals: the asymptotic frequency of switching, the asymptotic fraction of activation of the constituent systems, and the asymptotic densities of the admissible transitions among them. As a natural deficiency, the members of the above class of switching signals do not accommodate uniform transient behavior. On the other hand, Theorem 1 involves pointwise constraints on the duration of activation of ISS and non-ISS systems, and the occurrence of admissible transitions, but guarantees uniform ISS. As regard to synthesis, it is computationally more challenging to design a switching signal that satisfies conditions (14), (15) as certain conditions need to be verified on *every* interval of time than verifying asymptotic conditions proposed in Kundu and Chatterjee (2015, Theorem 5).

**Remark 8.** Our class of stabilizing switching signals goes beyond the average dwell time regime in the sense that on every interval of time the number of switches is allowed to grow faster than an affine function of the length of the interval. We elaborate on this feature with the aid of the following example: Fix $t > 0$. Let us study how close to $t$ can the $\tau_i$'s be placed under condition (13). We have $\mathrm{N}_\sigma(0,t) \le \mathrm{N}_0 + \lfloor \rho_{\mathrm{N}}(0,t)\rfloor$, where $\mathrm{N}_0 := \sum_{(m,n)\in E(\mathcal{P})} \overline{\mathrm{N}}_{mn}$ and $\rho_{\mathrm{N}}(0,t) := \sum_{(m,n)\in E(\mathcal{P})} \rho_{mn}(0,t)$. Consequently, $\sum_{i=0}^{\mathrm{N}_\sigma(0,t)} \exp\bigl(-\rho(\tau_i, t-\tau_i)\bigr)$ is at most equal to $\sum_{i=0}^{\mathrm{N}_0+\lfloor\rho_{\mathrm{N}}(0,t)\rfloor} \exp\bigl(-\rho(\tau_i, t-\tau_i)\bigr)$. However small a time interval may be, at most $\mathrm{N}_0$ switches are allowed. So these many switches can be placed arbitrarily close to $t$. For the rest of the $\lfloor \rho_{\mathrm{N}}(0,t)\rfloor = n$ (say) switches that can be placed on $]0,t]$, we have that on the interval $]\tau_n, t]$ at most $\mathrm{N}_0+1$ switches are allowed, on the interval $]\tau_{n-1}, t]$ at most $\mathrm{N}_0+2$ switches are allowed, and so on. Joint validity of the above conditions leads to $\tau_n = t - \inf\{r > 0 \mid \rho_{\mathrm{N}}(r,s) > 1 \text{ with } s = t-r\}$, $\tau_{n-1} = t - \inf\{r > 0 \mid \rho_{\mathrm{N}}(r,s) > 2 \text{ with } s = t-r\}, \ldots, \tau_1 = t - \inf\{r > 0 \mid \rho_{\mathrm{N}}(r,s) > n \text{ with } s = t-r\}$, i.e., $\tau_n = t - \rho_{\mathrm{N}}^{-1}(\cdot, t-\cdot)(1)$, $\tau_{n-1} = t - \rho_{\mathrm{N}}^{-1}(\cdot, t-\cdot)(2), \ldots, \tau_1 = t - \rho_{\mathrm{N}}^{-1}(\cdot, t-\cdot)(n)$.

Now let us study the above phenomenon under average dwell time switching. Recall that (Liberzon, 2003, p. 58) a switching signal $\sigma$ has average dwell time $\tau_a$ if there exist two positive numbers $\mathrm{N}_0$ and $\tau_a$ such that $\mathrm{N}_\sigma(s,t) \le \mathrm{N}_0 + \frac{t-s}{\tau_a}$ for all $0 \le s < t$. Let the $\mathrm{N}_0$ switches be placed arbitrarily close to $t$ as already explained. As regard to the remaining $\lfloor \frac{t}{\tau_a} \rfloor$ switches, we have that on every interval of length $t-(t-n\tau_a)$, at most $\mathrm{N}_0+n$ switches are allowed, on every interval of length $(t-(n-1)\tau_a)-(t-n\tau_a)$, at most $\mathrm{N}_0+1$ switches are allowed, and so on. Consequently, we have $\tau_n = t-\tau_a$, $\tau_{n-1} = t-2\tau_a, \ldots, \tau_1 = t-n\tau_a$.

As is evident from the above discussion, our class of switching signals allows number of switches on every interval of time to grow faster than an affine function of the length of the interval.

We next consider two simple cases where both the functions $\rho$ and $\rho_N$ are such that for all $r_1, r_2 \ge 0$ and all $s > 0$, $\rho(r_1,s) = \rho(r_2,s)$ and $\rho_{\mathrm{N}}(r_1,s) = \rho_{\mathrm{N}}(r_2,s)$, and discuss boundedness of the quantity $\sum_{i=0}^{\mathrm{N}_\sigma(0,t)} \exp\bigl(-\rho(\tau_i, t-\tau_i)\bigr)$ with $t$.

**Lemma 1.** *Let $\rho(r,s) = k_1 s + k_2$ for some $k_1, k_2 > 0$, $s \ge 0$ and let $\rho_N$ be such that the switches be equispaced in time, i.e., they satisfy $\tau_n = t - \rho_{\mathrm{N}}^{-1}(\cdot, t-\cdot)(1)$, $\tau_{n-1} = t - 2\rho_{\mathrm{N}}^{-1}(\cdot, t-\cdot)(1), \ldots, \tau_1 = t - n\rho_{\mathrm{N}}^{-1}(\cdot, t-\cdot)(1)$. Then $\lim_{t\to+\infty} \sum_{i=0}^{\mathrm{N}_\sigma(0,t)} \exp\bigl(-\rho(\tau_i, t-\tau_i)\bigr) < +\infty$.*

**Lemma 2.** *Let $\rho(r,s) = k_1 s^{3/2} + k_2$ for some $k_1, k_2 > 0$, $s \ge 0$, and let $\rho_N$ be such that the switches be equispaced in time as explained in Lemma 1. Then $\lim_{t\to+\infty} \sum_{i=0}^{\mathrm{N}_\sigma(0,t)} \exp\bigl(-\rho(\tau_i, t-\tau_i)\bigr) < +\infty$.*

The proofs of Lemmas 1 and 2 are presented in Section 7.

## 4. Numerical example

*A. The family of systems*: We consider a family with $\mathcal{P} = \{1,2,3,4\}$ with

$$f_1(x,v) = \begin{pmatrix} -x_1 + \sin(x_1 - x_2) \\ -x_2 + 0.8\sin(x_2 - x_1) + 0.5v \end{pmatrix},$$

$$f_2(x,v) = \begin{pmatrix} x_2 \\ -x_1 - x_2 + v \end{pmatrix},$$

$$f_3(x,v) = \begin{pmatrix} x_1 + \sin(x_1 - x_2) \\ x_2 + \sin(x_2 - x_1) + 0.5v \end{pmatrix},$$

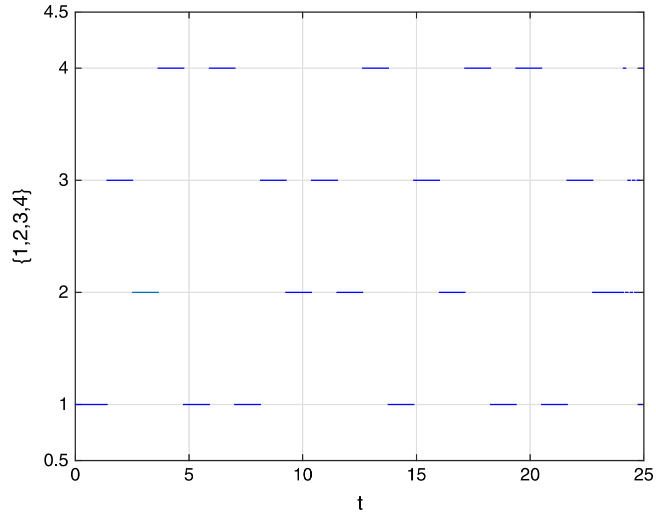


**Fig. 1.** An execution of $\sigma$ described in section 4B.

$$f_4(x,v) = \begin{pmatrix} x_2 \\ -x_2 + v \end{pmatrix}.$$

Clearly, $\mathcal{P}_S = \{1,2\}$ and $\mathcal{P}_U = \{3,4\}$. Let $E(\mathcal{P}) = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1), (4,2)\}$. We choose $V_1(x) = 0.5(x_1^2 + 1.25x_2^2)$, $V_2(x) = x_1^2 + x_1x_2 + x_2^2$, $V_3(x) = 0.5(x_1^2 + x_2^2)$, $V_4(x) = x_1^2 + x_2^2$, and obtain the following estimates: $\lambda_1 = 1.75$, $\lambda_2 = 0.5$, $\lambda_3 = -2.1667$, $\lambda_4 = -0.6378$, $\mu_{12} = 6$, $\mu_{13} = 1$, $\mu_{14} = 2$, $\mu_{21} = 4$, $\mu_{23} = 1$, $\mu_{24} = 2$, $\mu_{31} = 1.25$, $\mu_{32} = 6$, $\mu_{41} = 1$, $\mu_{42} = 2$.

*B. The switching signal*: Let a switching signal $\sigma$ satisfy (11)–(13) with $\overline{\mathrm{T}}_j^{\mathrm{S}} = 0.0010$, $j \in \mathcal{P}_S$, $\overline{\mathrm{T}}_k^{\mathrm{U}} = 0.0100$, $k \in \mathcal{P}_U$, $\overline{\mathrm{N}}_{mn} = 1$, $(m,n) \in E(\mathcal{P})$, and $\rho_1^{\mathrm{S}}(r,s) = 0.0350\, s^{3/2} + 0.2690s$, $\rho_2^{\mathrm{S}}(r,s) = 0.0350s^{3/2} + 0.1345s$, $\rho_3^{\mathrm{U}}(r,s) = 0.0100s$, $\rho_4^{\mathrm{U}}(r,s) = 0.0500s$, $\rho_{mn}(r,s) = 0.0100s^{3/2} + 0.0435s$, $(m,n) \in E(\mathcal{P})$.

*C. The verification*: We verify that for the family of systems and the switching signal described above, (14) holds with $\rho(r,s) = k_1 s^{3/2} + k_2$, where $k_1 = 0.0061$ and $c_1 - k_2 = 0$. Selecting the switching instants as demonstrated in Fig. 1, we observe that (15) holds with $c_2 = 124.1898$.

Fig. 1 illustrates an execution of a switching signal $\sigma$ described in Section 4B till $t = 25$ units of time. In Fig. 2 we study the process $(\|x(t)\|)_{t\ge 0}$ under (i) five different initial conditions $x_0$ (selected uniformly at random from the interval $[-1000, 1000]^2$) and input $v$ (selected uniformly at random from the interval $[-10, 10]$), and (ii) the switching signal $\sigma$ demonstrated in Fig. 1.[3] We observe that the bounds on $\|x(t)\|$ in the simulation are much smaller than those obtained from the analysis in the proof of Theorem 1.

The class $\mathcal{KL}$ function $\beta$ is visible from Fig. 2. To study the class $\mathcal{K}_\infty$ function $\chi$, we fix $T = 1000$ units of time, $x_0 = [-100, 100]$, and simulate $x(t)$ with (i) input $v$ selected uniformly at random from the interval $[-k, k]$, where $k$ ranges from 1 to 100, and (ii) the switching signal $\sigma$ demonstrated in Fig. 1. In Fig. 3 we plot $\|x(T)\|$ versus $k$.

**Remark 9.** Consider an ISS version of Müller and Liberzon (2012, Theorem 2). For the family of systems under consideration, we have $\lambda_{\mathrm{S}} = 0.5$, $\lambda_{\mathrm{U}} = 2.1667$, $\mu = 6$, $\overline{\rho} \in ]0, 0.1875[$. We fix

[3] By selecting the input $v$ uniformly at random from an interval $[-a,a]$, we mean that at every Euler step we plug in the random variable as indicated, which corresponds to a 'sample and hold' continuous-time process with uniform random variables at the jump times.

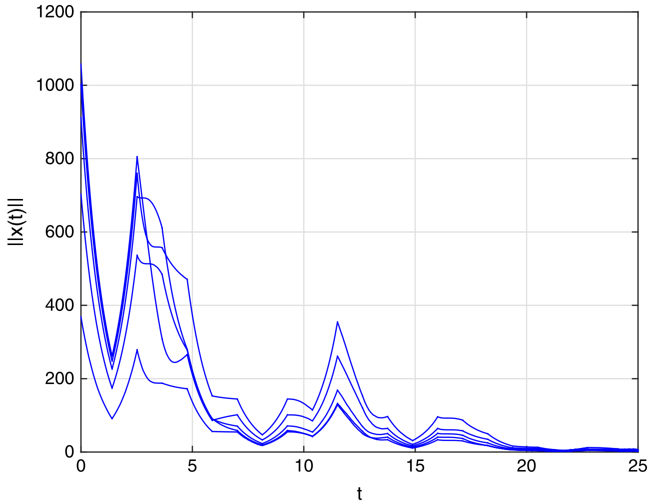


**Fig. 2.** Plot for $(\|x(t)\|)_{t\geq 0}$.

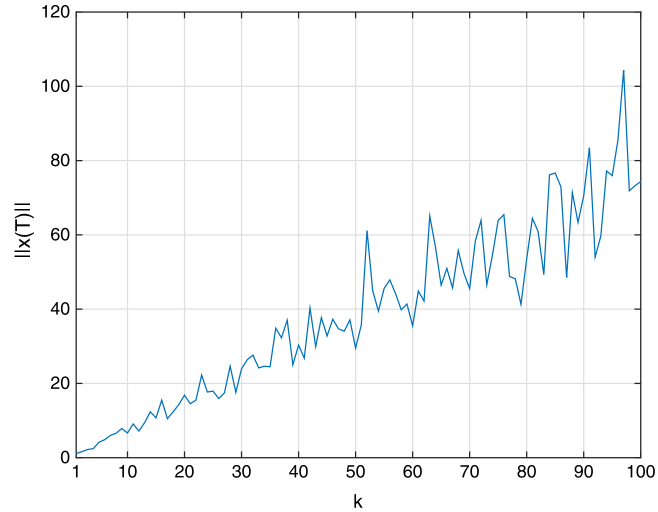


**Fig. 3.** Plot for $\|x(T)\|$ against $k$ with $T = 1000$ units of time and input $v \in [-k, k]$.

$\overline{\rho} = 0.1874$ and $\tau_a = 2.3$.[4] Fix $\mathrm{N}_0 = \sum_{(m,n)\in E(\mathcal{P})} \overline{\mathrm{N}}_{mn}$. By definition of average dwell time switching, $\mathrm{N}_\sigma(s,t) \leq 10 + \lfloor 0.4348(t-s)\rfloor$ for every interval $]s,t] \subset [0,+\infty[$ of time. In contrast, we have $\mathrm{N}_\sigma(s,t) \leq 10 + \sum_{(m,n)\in E(\mathcal{P})} \lfloor 0.0100(t-s)^{3/2} + 0.0435(t-s)\rfloor$ for every interval $]s,t] \subset [0,+\infty[$ of time.

**Remark 10.** The design of a switching signal $\sigma$ satisfying the proposed conditions involves selecting the constants $\overline{\mathrm{T}}_j^{\mathrm{S}}$, $j \in \mathcal{P}_S$, $\overline{\mathrm{T}}_k^{\mathrm{U}}$, $k \in \mathcal{P}_U$, $\overline{\mathrm{N}}_{mn}$, $(m,n) \in E(\mathcal{P})$, and the class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}$, $j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}$, $k \in \mathcal{P}_U$, $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ such that there exist constants $c_1, c_2$ and a class $\mathcal{FK}_\infty$ function $\rho$ with $\rho(0,0) = 0$ satisfying (14) and (15). For a fixed $t$, validation of condition (15) involves the number of switches $\mathrm{N}_\sigma(0,t)$ and the switching instants $\tau_0, \tau_1, \ldots, \tau_{\mathrm{N}_\sigma(0,t)}$. The quantity $\mathrm{N}_\sigma(0,t)$ can be obtained from the already chosen $\overline{\mathrm{N}}_{mn}$ and $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ as explained in Remark 8. In Lemmas 1 and 2 we have discussed two structures of the class $\mathcal{FK}_\infty$ function $\rho$ and the switching instants for which (15) is satisfied. A next natural question in the context of Theorem 1 is: given a family of systems (2), how do we algorithmically detect/design a switching signal $\sigma$ that satisfies (11)–(13) such that (14) and (15) hold? This feature is currently under investigation.

## 5. Discussion

In this section we recast a subclass of average dwell time switching signals in our setting and establish analogs of the prior results: an ISS version of Müller and Liberzon (2012, Theorem 2), and Vu et al. (2007, Theorem 3.1), with the aid of our main result. Our first result of this section is:

**Proposition 1.** *Consider the family of systems (2). Suppose that Assumption 1 holds with $|\lambda_j| = \lambda_S$ for all $j \in \mathcal{P}_S$ and $|\lambda_k| = \lambda_U$ for all $k \in \mathcal{P}_U$, and Assumption 2 holds with $\mu_{mn} = \mu$ for all $(m,n) \in E(\mathcal{P})$. Let $\overline{\rho}$ and $\tau_a$ be constants satisfying $\overline{\rho} \in \left]0, \frac{\lambda_S}{\lambda_S+\lambda_U}\right[$ and*

$$\tau_a \in \left]\frac{\ln\mu}{\lambda_S\cdot(1-\overline{\rho}) - \lambda_U\cdot\overline{\rho}}, +\infty\right[. \tag{16}$$

*Let the class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}$, $j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}$, $k \in \mathcal{P}_U$, $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ described in Assumption 3, be such that for all $r_1, r_2 \geq 0$ and all $s > 0$*

$$\rho_j^{\mathrm{S}}(r_1,s) = \rho_j^{\mathrm{S}}(r_2,s), \tag{17}$$

$$\rho_k^{\mathrm{U}}(r_1,s) = \rho_k^{\mathrm{U}}(r_2,s), \tag{18}$$

$$\text{and}\quad \rho_{mn}(r_1,s) = \rho_{mn}(r_2,s). \tag{19}$$

*Moreover, let for every interval $]r, r+s] \subset [0,+\infty[$ of time*

$$\sum_{j\in\mathcal{P}_S} \rho_j^{\mathrm{S}}(r,s) + \sum_{k\in\mathcal{P}_U} \rho_k^{\mathrm{U}}(r,s) \geq s, \tag{20}$$

$$\sum_{k\in\mathcal{P}_U} \rho_k^{\mathrm{U}}(r,s) \leq \overline{\rho}\cdot s, \tag{21}$$

$$\text{and}\quad \sum_{(m,n)\in E(\mathcal{P})} \rho_{mn}(r,s) \leq \frac{s}{\tau_a}. \tag{22}$$

*Then the switched system (1) is ISS for every switching signal $\sigma \in \mathcal{S}$ satisfying (11), (12), and (13) for every interval $]r, r+s] \subset [0,+\infty[$ of time.*

**Remark 11.** Given a family of systems in which not all subsystems are input/output-to-state stable (IOSS), in Müller and Liberzon (2012, Theorem 2) the authors identified a class of switching signals obeying the average dwell time property under which the resulting switched system is IOSS. Our Proposition 1 is an analog of an ISS version of Müller and Liberzon (2012, Theorem 2) obtained as a corollary of our main result Theorem 1.

**Remark 12.** Since under average dwell time switching, the bounds on every interval $]r, r+s] \subset [0,+\infty[$ of time are independent of the initial point $r \in [0,+\infty[$ of the interval, the assumption that the class $\mathcal{FK}_\infty$ functions $\rho_j^{\mathrm{S}}$, $j \in \mathcal{P}_S$, $\rho_k^{\mathrm{U}}$, $k \in \mathcal{P}_U$, $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ satisfy (17)–(19) is natural.

**Remark 13.** The bound on $\overline{\rho}$ ensures that $0 < \overline{\rho} < 1$. Consequently, the activation of unstable systems on every interval of time is restricted. A switching signal $\sigma$ that satisfies (13) on every interval $]r, r+s] \subset [0,+\infty[$ of time such that hypothesis (22) holds with $\rho_{mn}(r,s)$, $(m,n) \in E(\mathcal{P})$ satisfying (19), implies that the switching signal satisfies the average dwell time property (Liberzon, 2003, p. 58). We have $\mathrm{N}_\sigma(s,t) = \sum_{(m,n)\in E(\mathcal{P})} \mathrm{N}_{mn}(s,t) \leq \sum_{(m,n)\in E(\mathcal{P})} \overline{\mathrm{N}}_{mn} + \sum_{(m,n)\in E(\mathcal{P})} \rho_{mn}(s, t-s)$. Pick $\mathrm{N}_0$ such that $\sum_{(m,n)\in E(\mathcal{P})} \overline{\mathrm{N}}_{mn} \leq \mathrm{N}_0$. By hypothesis (22), we

[4] We call the scalar $\rho$ in Müller and Liberzon (2012, Theorem 2) as $\overline{\rho}$ here to avoid notational overlap. The choice of $\overline{\rho}$ and $\tau_a$ is motivated by the objective to maximize $\frac{1}{\tau_a}$ in the given setting.

have $\sum_{(m,n)\in E(\mathcal{P})} \rho_{mn}(s, t-s) \le \frac{t-s}{\tau_a}$. Consequently, $\mathrm{N}_\sigma(s,t) \le \mathrm{N}_0 + \frac{t-s}{\tau_a}$ for positive constants $\mathrm{N}_0$ and $\tau_a$.

A special case of Müller and Liberzon (2012, Theorem 2) where all subsystems are ISS was treated in Vu et al. (2007, Theorem 3.1). A subclass of average dwell time switching signals was proposed under which the resulting switched system is ISS. We recast an analog of Vu et al. (2007, Theorem 3.1) as a corollary of our main result:

**Proposition 2.** *Consider the family of systems (2). Let $\mathcal{P}_U = \emptyset$. Suppose that Assumption 1 holds with $|\lambda_j| = \lambda_0$ for all $j \in \mathcal{P}_S$ and Assumption 2 holds with $\mu_{mn} = \mu$ for all $(m,n) \in E(\mathcal{P})$. Let $\tau_a$ be a constant satisfying*

$$\tau_a \in \left]\frac{\ln \mu}{\lambda_0}, +\infty\right[. \tag{23}$$

*Let the class $\mathcal{FK}_\infty$ functions $\rho_j^S$, $j \in \mathcal{P}_S$ and $\rho_{mn}$, $(m,n) \in E(\mathcal{P})$ described in Assumption 3 be such that for all $r_1, r_2 \ge 0$ and all $s > 0$*

$$\rho_j^S(r_1, s) = \rho_j^S(r_2, s), \tag{24}$$

$$\text{and} \quad \rho_{mn}(r_1, s) = \rho_{mn}(r_2, s). \tag{25}$$

*Moreover, let for every interval $]r, r+s] \subset [0, +\infty[$ of time*

$$\sum_{j\in\mathcal{P}_S} \rho_j^S(r, s) \ge s, \tag{26}$$

$$\text{and} \quad \sum_{(m,n)\in E(\mathcal{P})} \rho_{mn}(r, s) \le \frac{s}{\tau_a}. \tag{27}$$

*Then the switched system (1) is ISS for every switching signal $\sigma \in \mathcal{S}$ that for every interval $]r, r+s] \subset [0, +\infty[$ of time, satisfies (11) and (13).*

**Remark 14.** Since $\mathcal{P}_U = \emptyset$, condition (12) is automatically satisfied. A switching signal that satisfies (13) such that (27) holds implies that the switching signal satisfies the average dwell time property as explained in Remark 13.

## 6. Concluding remarks

In this article we presented a class of switching signals under which a continuous-time switched system is uniformly ISS. We utilized multiple ISS-Lyapunov-like functions for our analysis and our characterization of stabilizing switching signals allowed the number of switches on any interval of time to grow faster than an affine function of the length of the interval unlike in the case of average dwell time switching. We also discussed two representative prior results: an ISS version of Müller and Liberzon (2012, Theorem 2), and Vu et al. (2007, Theorem 2) in our setting. Our results extend readily to the discrete-time setting. This matter has been reported in Kundu, Mishra, and Chatterjee (in press).

## 7. Proofs

**Proof of Theorem 1.** Fix $t > 0$. Let $0 =: \tau_0 < \tau_1 < \cdots < \tau_{\mathrm{N}_\sigma(0,t)}$ be the switching instants before (and including) $t$. In view of (5),

$$V_{\sigma(t)}(x(t)) \le \exp\big(-\lambda_{\sigma(\tau_{\mathrm{N}_\sigma(0,t)})}(t - \tau_{\mathrm{N}_\sigma(0,t)})\big)V_{\sigma(t)}(x(\tau_{\mathrm{N}_\sigma(0,t)})) + \gamma\big(\|v\|_{[0,t]}\big)\int_{\tau_{\mathrm{N}_\sigma(0,t)}}^{t} \exp\big(-\lambda_{\sigma(\tau_{\mathrm{N}_\sigma(0,t)})}(t-s)\big)ds.$$

Applying (6) and iterating the above, we obtain the estimate

$$V_{\sigma(t)}(x(t)) \le \psi_1(t)V_{\sigma(0)}(x_0) + \gamma(\|v\|_{[0,t]})\psi_2(t), \tag{28}$$

where

$$\psi_1(t) := \exp\left(-\sum_{\substack{i=0\\ \tau_{\mathrm{N}_\sigma(0,t)+1}:=t}}^{\mathrm{N}_\sigma(0,t)} \lambda_{\sigma(\tau_i)}S_{i+1} + \sum_{i=0}^{\mathrm{N}_\sigma(0,t)-1} \ln \mu_{\sigma(\tau_i)\sigma(\tau_{i+1})}\right),$$

$$\psi_2(t) := \sum_{\substack{i=0\\ \tau_{\mathrm{N}_\sigma(0,t)+1}:=t}}^{\mathrm{N}_\sigma(0,t)} \left(\exp\left(-\sum_{\substack{k=i+1\\ \tau_{\mathrm{N}_\sigma(0,t)+1}:=t}}^{\mathrm{N}_\sigma(0,t)} \lambda_{\sigma(\tau_k)}S_{k+1} + \sum_{k=i+1}^{\mathrm{N}_\sigma(0,t)-1} \ln \mu_{\sigma(\tau_k)\sigma(\tau_{k+1})}\right) \times \frac{1}{\lambda_{\sigma(\tau_i)}}\big(1 - \exp(-\lambda_{\sigma(\tau_i)}S_{i+1})\big)\right).$$

Here $S_{i+1}$ is as defined in (8).

In view of (4) we rewrite the estimate (28) as

$$\underline{\alpha}(\|x(t)\|) \le \psi_1(t)\overline{\alpha}(\|x_0\|) + \gamma\big(\|v\|_{[0,t]}\big)\psi_2(t).$$

By Definition 1, for ISS of (1), we need to show the following: (i) $\overline{\alpha}(*)\psi_1(\cdot)$ can be bounded above by a class $\mathcal{KL}$ function, and (ii) $\psi_2(\cdot)$ is bounded by a constant, say $\overline{\psi}_2$.

Observe that

$$\begin{aligned}\psi_1(t) &= \exp\left(-\sum_{j\in\mathcal{P}_S}|\lambda_j|\left|\,]0,t] \cap \left(\bigcup_{\substack{i=0\\ \sigma(\tau_i)=j}}^{+\infty} ]\tau_i, \tau_{i+1}]\right)\right| + \sum_{k\in\mathcal{P}_U}|\lambda_k|\left|\,]0,t] \cap \left(\bigcup_{\substack{i=0\\ \sigma(\tau_i)=k}}^{+\infty} ]\tau_i, \tau_{i+1}]\right)\right| + \sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\#\{m\to n\}_0^{\tau_{\mathrm{N}_\sigma(0,t)-1}}\right)\\ &= \exp\Big(-\sum_{j\in\mathcal{P}_S}|\lambda_j|\,\mathrm{T}_j(0,t) + \sum_{k\in\mathcal{P}_U}|\lambda_k|\,\mathrm{T}_k(0,t) + \sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\mathrm{N}_{mn}(0, \tau_{\mathrm{N}_\sigma(0,t)-1})\Big).\end{aligned} \tag{29}$$

Similarly,

$$\begin{aligned}\psi_2(t) = &\sum_{j\in\mathcal{P}_S}\frac{1}{|\lambda_j|}\sum_{\substack{i:\sigma(\tau_i)=j\\ i=0,\ldots,\mathrm{N}_\sigma(0,t)\\ \tau_{\mathrm{N}_\sigma(0,t)+1}:=t}}\Bigg(\exp\Big(-\sum_{p\in\mathcal{P}_S}|\lambda_p|\,\mathrm{T}_p(\tau_{i+1},t) + \sum_{q\in\mathcal{P}_U}|\lambda_q|\,\mathrm{T}_q(\tau_{i+1},t) + \sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\mathrm{N}_{mn}(\tau_{i+1}, \tau_{\mathrm{N}_\sigma(0,t)-1})\Big)\\ &\times\Big(1 - \exp\big(-|\lambda_j|\,S_{i+1}\big)\Big)\Bigg)\\ &+ \sum_{k\in\mathcal{P}_U}\frac{1}{|\lambda_k|}\sum_{\substack{i:\sigma(\tau_i)=k\\ i=0,\ldots,\mathrm{N}_\sigma(0,t)\\ \tau_{\mathrm{N}_\sigma(0,t)+1}:=t}}\Bigg(\exp\Big(-\sum_{p\in\mathcal{P}_S}|\lambda_p|\,\mathrm{T}_p(\tau_{i+1},t) + \sum_{q\in\mathcal{P}_U}|\lambda_q|\,\mathrm{T}_q(\tau_{i+1},t) + \sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\mathrm{N}_{mn}(\tau_{i+1}, \tau_{\mathrm{N}_\sigma(0,t)-1})\Big)\\ &\times\Big(1 - \exp(|\lambda_k|\,S_{i+1})\Big)\Bigg)\\ \le &\sum_{j\in\mathcal{P}_S}\frac{1}{|\lambda_j|}\sum_{\substack{i:\sigma(\tau_i)=j\\ i=0,\ldots,\mathrm{N}_\sigma(0,t)\\ \tau_{\mathrm{N}_\sigma(0,t)+1}:=t}}\Bigg(\exp\Big(-\sum_{p\in\mathcal{P}_S}|\lambda_p|\,\mathrm{T}_p(\tau_{i+1},t) + \sum_{q\in\mathcal{P}_U}|\lambda_q|\,\mathrm{T}_q(\tau_{i+1},t) + \sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\mathrm{N}_{mn}(\tau_{i+1}, t)\Big)\Bigg)\end{aligned}$$

$$+\sum_{k\in\mathcal{P}_U}\frac{1}{|\lambda_k|}\sum_{\substack{i:\sigma(\tau_i)=k\\ i=0,\ldots,N_\sigma(0,t)\\ \tau_{N_\sigma(0,t)+1}:=t}}\Bigg(\exp\Big(-\sum_{p\in\mathcal{P}_S}|\lambda_p|\,T_p(\tau_i,t) + \sum_{q\in\mathcal{P}_U}|\lambda_q|\,T_q(\tau_i,t)+\sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})N_{mn}(\tau_i,t)\Big)\Bigg). \quad (30)$$

We first verify (ii). By hypotheses (11)–(13), we have that the right-hand side of (30) is at most equal to

$$\sum_{j\in\mathcal{P}_S}\frac{1}{|\lambda_j|}\sum_{\substack{i:\sigma(\tau_i)=j\\ i=0,\ldots,N_\sigma(0,t)\\ \tau_{N_\sigma(0,t)+1}:=t}}\exp\Bigg(\sum_{p\in\mathcal{P}_S}|\lambda_p|\left(\overline{T}_p^S-\rho_p^S(\tau_{i+1},t-\tau_{i+1})\right)$$
$$+\sum_{q\in\mathcal{P}_U}|\lambda_q|\left(\overline{T}_q^U+\rho_q^U(\tau_{i+1},t-\tau_{i+1})\right)$$
$$+\sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\left(\overline{N}_{mn}+\rho_{mn}(\tau_{i+1},t-\tau_{i+1})\right)\Bigg)$$
$$+\sum_{k\in\mathcal{P}_U}\frac{1}{|\lambda_k|}\sum_{\substack{i:\sigma(\tau_i)=k\\ i=0,\ldots,N_\sigma(0,t)\\ \tau_{N_\sigma(0,t)+1}:=t}}\exp\Bigg(\sum_{p\in\mathcal{P}_S}|\lambda_p|\left(\overline{T}_p^S-\rho_p^S(\tau_i,t-\tau_i)\right)$$
$$+\sum_{q\in\mathcal{P}_U}|\lambda_q|\left(\overline{T}_q^U+\rho_q^U(\tau_i,t-\tau_i)\right)$$
$$+\sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\left(\overline{N}_{mn}+\rho_{mn}(\tau_i,t-\tau_i)\right)\Bigg).$$

From (14), the above expression is bounded above by

$$\sum_{j\in\mathcal{P}_S}\frac{1}{|\lambda_j|}\sum_{\substack{i:\sigma(\tau_i)=j\\ i=0,\ldots,N_\sigma(0,t)\\ \tau_{N_\sigma(0,t)+1}:=t}}\exp\left(c+c_1-\rho(\tau_{i+1},t-\tau_{i+1})\right)$$
$$+\sum_{k\in\mathcal{P}_U}\frac{1}{|\lambda_k|}\sum_{\substack{i:\sigma(\tau_i)=k\\ i=0,\ldots,N_\sigma(0,t)\\ \tau_{N_\sigma(0,t)+1}:=t}}\exp\left(c+c_1-\rho(\tau_i,t-\tau_i)\right)$$

for some $c$ satisfying $\sum_{j\in\mathcal{P}_S}\overline{T}_j^S+\sum_{k\in\mathcal{P}_U}\overline{T}_k^U+\sum_{(m,n)\in E(\mathcal{P})}\overline{N}_{mn}\le c$. In view of (15) and the fact that $\mathcal{P}$ is finite, both the terms $\sum_{j\in\mathcal{P}_S}\frac{1}{|\lambda_j|}\sum_{i=0}^{N_\sigma(0,t)}\exp\left(c+c_1-\rho(\tau_{i+1},t-\tau_{i+1})\right)$, and $\sum_{k\in\mathcal{P}_U}\frac{1}{|\lambda_k|}\sum_{i=0}^{N_\sigma(0,t)}\exp\left(c+c_1-\rho(\tau_i,t-\tau_i)\right)$ are bounded. Consequently, (ii) holds.

It remains to verify (i). Towards this end, we already see that $\overline{\alpha}\in\mathcal{K}_\infty$ from Assumption 1. Therefore, it remains to show that $\psi_1(\cdot)$ is bounded above by a function in class $\mathcal{L}$ to complete the proof of (i).[5] By hypotheses (11), (12), and (13), we have $\psi_1(t)$ is bounded above by

$$\exp\Bigg(\sum_{j\in\mathcal{P}_S}|\lambda_j|\left(\overline{T}_j^S-\rho_j^S(0,t)\right)+\sum_{k\in\mathcal{P}_U}|\lambda_k|\left(\overline{T}_k^U+\rho_k^U(0,t)\right)$$
$$+\sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\left(\overline{N}_{mn}+\rho_{mn}(0,t)\right)\Bigg).$$

By (14) the above quantity is at most $\exp\left(c+c_1-\rho(0,t)\right)$, which decreases as $t$ increases, and tends to 0 as $t\to+\infty$.

To summarize, $\alpha(\|x(t)\|)\le\beta(\|x_0\|,t)+\chi(\|v\|_{[0,t]})$ for all $t\ge 0$ holds with $\alpha(r):=r$, $\beta(r,s)=\overline{\alpha}(r)\exp\left(c+c_1-\rho(0,s)\right)$ and $\chi(r):=\gamma(r)\overline{\psi}_2$, where

$$\overline{\psi}_2=\Bigg(\sum_{j\in\mathcal{P}_S}\frac{1}{|\lambda_j|}\sup_t\sum_{i=0}^{N_\sigma(0,t)}\exp\left(c+c_1-\rho(\tau_{i+1},t-\tau_{i+1})\right)$$
$$+\sum_{k\in\mathcal{P}_U}\frac{1}{|\lambda_k|}\sup_t\sum_{i=0}^{N_\sigma(0,t)}\exp\left(c+c_1-\rho(\tau_i,t-\tau_i)\right)\Bigg).$$

This completes our proof for ISS. For uniformity over $\sigma$, we note that the functions $\beta$ and $\chi$ do not depend on the specific switching signal $\sigma$ satisfying (11)–(13) under our assumptions. This completes our proof of Theorem 1. □

**Proof of Lemma 1.** We express $\rho_N^{-1}(\cdot,t-\cdot)(1)$ by $\rho_N^{-1}(1)$ for notational simplicity. We have

$$\sum_{i=0}^{N_\sigma(0,t)}\exp\left(-\rho(\tau_i,t-\tau_i)\right)\le\sum_{i=0}^{N_0+\lfloor\rho_N(0,t)\rfloor}\exp\left(-\rho(\tau_i,t-\tau_i)\right)$$
$$\le\exp(-k_2)\sum_{i=0}^{N_0+\lfloor\rho_N(0,t)\rfloor}\exp\left(-k_1\cdot(t-\tau_i)\right)$$
$$=\exp(-k_2)\Big(1+N_0+\exp(-nk_1\rho_N^{-1}(1))+\exp(-(n-1)k_1\rho_N^{-1}(1))$$
$$+\cdots+\exp(-2k_1\rho_N^{-1}(1))+\exp(-k_1\rho_N^{-1}(1))\Big)$$
$$\le\exp(-k_2)\left(1+N_0+\frac{1}{\exp(-k_1\rho_N^{-1}(1))-1}\right).\quad\square$$

**Proof of Lemma 2.** We have

$$\sum_{i=0}^{N_\sigma(0,t)}\exp\left(-\rho(\tau_i,t-\tau_i)\right)\le\sum_{i=0}^{N_0+\lfloor\rho_N(0,t)\rfloor}\exp\left(-\rho(\tau_i,t-\tau_i)\right)$$
$$\le\exp(-k_2)\Big((N_0+1)+\exp\left(-k_1(\rho_N^{-1}(1))^{3/2}n^{3/2}\right)+\cdots$$
$$+\exp\left(-k_1(\rho_N^{-1}(1))^{3/2}2^{3/2}\right)+\exp\left(-k_1(\rho_N^{-1}(1))^{3/2}\right)\Big).$$

We apply the integral test (Kaczor & Nowak, 2000, Section 3.3); we define a new variable $y^2:=x^3$, and compute $\int_0^{+\infty}\exp\left(-k_1(\rho_N^{-1}(1))^{3/2}\right)dx=\frac{2}{3}\int_0^{+\infty}y^{-1/3}\exp\left(-k_1(\rho_N^{-1}(1))^{3/2}y\right)dy=\frac{2}{3k_1(\rho_N^{-1}(1))^{3/2}}\Gamma\left(\frac{2}{3}\right)$, which is finite, showing thereby that $\sum_{i=0}^{N_\sigma(0,t)}\exp\left(-\rho(\tau_i,t-\tau_i)\right)$ is bounded. □

**Proof of Proposition 1.** By the hypotheses $|\lambda_j|=\lambda_S$ for all $j\in\mathcal{P}_S$, $|\lambda_k|=\lambda_U$ for all $k\in\mathcal{P}_U$, $\mu_{mn}=\mu$ for all $(m,n)\in E(\mathcal{P})$, and (20)–(22), we obtain the following upper bound on left-hand side of (14):

$$-\lambda_S\cdot(1-\overline{\rho})\cdot(t-s)+\lambda_U\cdot\overline{\rho}\cdot(t-s)+(\ln\mu)\cdot\frac{t-s}{\tau_a}.\quad(31)$$

By (16), we have $\frac{1}{\tau_a}\le\frac{\lambda_S\cdot(1-\overline{\rho})-\lambda_U\cdot\overline{\rho}}{\ln\mu}-\varepsilon$ for some $\varepsilon>0$. The above estimate results in (31) to be bounded above by $-(\ln\mu)\cdot\varepsilon\cdot(t-s)$, which is equivalent to $c_1-\rho(s,t-s)$ with $c_1=0$ and $\rho$ linear in the second argument.

*Claim*: The series $\sum_{i=0}^{N_\sigma(0,t)}\exp\left(-\varepsilon\cdot(\ln\mu)\cdot(t-\tau_i)\right)$ for some $\varepsilon>0$, is bounded with respect to $t$ under average dwell time switching.

Let $\varepsilon'=\varepsilon\cdot(\ln\mu)$. We consider the situation identified in

[5] $\mathcal{L}:=\{\gamma:[0,+\infty[\longrightarrow[0,+\infty[\,|\,\gamma$ is continuous and $\gamma(s)\searrow 0$ as $s\nearrow+\infty\}$.

Remark 8.

$$\sum_{i=0}^{N_0+\left\lfloor \frac{t}{\tau_a} \right\rfloor} \exp\big(-\varepsilon' \cdot (t-\tau_{i+1})\big) \le \exp(-\varepsilon' t) + \exp\left(-\varepsilon'\left(\left\lfloor \frac{t}{\tau_a}\right\rfloor - 1\right)\tau_a\right) + \cdots + \exp(-2\varepsilon'\tau_a) + \exp(-\varepsilon'\tau_a) + N_0$$

$$\le 1 + N_0 + \exp(-\varepsilon'\tau_a)\frac{1-(\exp(-\varepsilon'\tau_a))^{\left\lfloor \frac{t}{\tau_a}+1\right\rfloor}}{1-\exp(-\varepsilon'\tau_a)}$$

$$\le 1 + N_0 + \frac{1}{\exp(\varepsilon'\tau_a)-1}.$$

This proves our claim and the assertion of Proposition 1 follows at once. □

**Proof of Proposition 2 (Sketch).** Observe that under the hypothesis $\mathcal{P}_U = \emptyset$, for every interval $]s,t] \subset [0,+\infty[$ of time, the left-hand side of (14) becomes $-\sum_{j\in\mathcal{P}_S}|\lambda_j|\,\rho_j^S(s,t-s) + \sum_{(m,n)\in E(\mathcal{P})}(\ln\mu_{mn})\rho_{mn}(s,t-s)$. The rest of the proof for Proposition 2 follows under the same set of arguments as in the proof of Proposition 1. □

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