Further Results on Stability of Linear Systems with Slow and Fast Time Variation and Switching

Hyungbo Shim
Seoul National University
Seoul, Korea
hshim@snu.ac.kr

Daniel Liberzon
University of Illinois Urbana-Champaign
Urbana, USA
liberzon@uiuc.edu

ABSTRACT

This paper studies exponential stability of linear systems with slow and fast time variation and switching. We use averaging to eliminate the fast dynamics and only retain the slow dynamics. We then use a recent stability criterion for slowly time-varying and switched systems, combined with perturbation analysis, to prove stability of the original system. The analysis involves working with an impulsive system in new coordinates, which enables us to treat a more general class of systems compared to previous work.

CCS CONCEPTS

• **Mathematics of computing → Ordinary differential equations**; • **Computing methodologies → Computational control theory**.

KEYWORDS

Switched systems, Lyapunov stability, Averaging

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1 INTRODUCTION

In this paper, we study stability of a class of linear time-varying systems of the form

$$\dot{x} = F(t, t/\varepsilon)x \tag{1}$$

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where $x \in \mathbb{R}^n$ is the state, and $F : [0, \infty) \times [0, \infty) \to \mathbb{R}^{n \times n}$ is periodic in its second argument, i.e., $F(t, \cdot)$ is periodic with a period T, for each $t \geq 0$. We allow F to be discontinuous, and will soon impose some further assumptions on the structure of the function F. Here, ε is a small positive number, which is introduced in order to represent fast variation of F in its second argument. Periodicity of F allows us to define an averaged version of (1): an *average system* given by

$$\dot{x} = A(t)x \tag{2}$$

where

$$A(t) := \frac{1}{T} \int_0^T F(t, s) ds. \tag{3}$$

Regarding stability of the time-varying linear system (2), it is well-known (see, e.g., [8, Example 4.22]) that the following assumption is *not* a sufficient condition.

Assumption 1. The matrices $A(t)$ are uniformly Hurwitz, i.e., $\exists \kappa > 0$ such that the real parts of their eigenvalues satisfy

$$Re\, \lambda_i(A(t)) \leq -\kappa \quad \forall\, t \geq 0,\ i = 1, 2, \ldots, n.$$

As a matter of fact, it is also well-known that Assumption 1 can be made sufficient by imposing additional assumptions on the system (2). One of them is slow variation of $A(t)$ with respect to time, as appearing in the classical textbooks such as [7, Section 3.4] and [8, Section 9.6]. In these classical results, the way to quantify slowness is to place some type of upper bound on the time derivative of $A(\cdot)$, and hence, $A(\cdot)$ should be continuously differentiable. This restriction is relaxed in the recent work [5], in which *total variation* is introduced as a new quantification of slow variation of $A(\cdot)$. The total variation is the quantity obtained, loosely speaking, by integrating the norm of the derivative of $A(\cdot)$ and adding, at each discontinuous instant, the norm of the jump. It is then shown in [5] that exponential stability of (2) is ensured if the total variation is suitably small. The contribution of [5] can also be regarded as an extension of well-known stability criteria of switched systems, which are formulated in terms of stability of (2) for individual modes and a slow-switching condition, typically in terms of sufficiently large (average) dwell time; see, e.g., [9] for an introduction to this class of systems, while the result of [5] goes beyond the basic ones appearing in [9, Section 3.2]. An extension of the approach of [5] to nonlinear systems was presented in [4].

Regarding stability of (1), there are several results asserting that it is inherited from the stability of its average system (2) if the variation is sufficiently fast, i.e., if ε is sufficiently small. In particular, the classical formulation (e.g., [8, Section 10.4],[12]) assumes that the average system is time-invariant, which is still the case (when the external input is zero) in the more recent work [15] that considers switched systems. This restriction is relaxed in the works [1, 11]. In [11], a time-varying average system is considered under the assumption that the equilibrium of interest is not affected by the time-varying parameter, which is then relaxed in [1].

In contrast with these existing results, we want to explicitly handle a time-varying average system (2) with discontinuous $A(\cdot)$. This topic was pursued in our recent paper [10] where, through a novel combination of the total variation and averaging techniques reviewed above, exponential stability of (1) was established for small total variation of $A(\cdot)$ and small ε. However, in [10] there is a restriction that

$$B(t,s) := F(t,s) - A(t) \tag{4}$$

is a function of s only; in other words, the system considered in [10] takes the form

$$\dot{x} = \big(A(t) + B(t/\varepsilon)\big)x. \tag{5}$$

This restriction potentially limits applicability of the approach, as seen in a practical example that appears shortly.

The goal of this paper is to extend the method of [10] in order to assert the same conclusion for the more general class of systems in (1). The key to the result in [10] is to apply to the system (1) a change of coordinates which brings it to the form of (2) with a perturbation of size $O(\varepsilon)$. For the case of (5), this coordinate transformation was $x = y + \varepsilon \int_0^{t/\varepsilon} B(s)ds \cdot y$ and had the feature that y is continuous in spite of the discontinuities of $A(\cdot)$ and $B(\cdot)$. For the more general case of (1) treated in this paper, we will consider a similar coordinate transformation, but the new state variable y will experience jumps at the discontinuities of B with respect to t. Consequently, in the y-coordinates the system will be an impulsive one. We overcome this challenge by conducting a Lyapunov analysis of this impulsive system. This generalization is the main new contribution compared to [10].

Among other works that address slow and fast—and possibly discontinuous—time variation, it is relevant to mention [13], [14], and [6, Section 7.4]. The tools employed in these references and the spirit of the results are quite different from ours. In particular, [13] proves robustness results, proceeding from the assumption that the slow and the fast dynamics are separately stable (before they are coupled), while we develop explicit stability conditions by starting with appropriate restrictions on the system data and the slow variation. The paper [14] considered averaging for hybrid systems, but the averaging was of restricted kind in that it was only applied to continuous dynamics. Corollary 7.28 in [6] addresses hybrid systems with slow average dwell time, which are also covered as a special case by the approach based on the total variation, as discussed in [5]. On the other hand, the classes of systems considered in [6, 13] are much larger, which suggests some potential generalizations of our results.

1.1 Motivating example

Let us consider an inverted pendulum on a cart depicted in Fig. 1. The system has a rigid tube on top of the cart (instead of a rigid rod that is typically used), and the tube holds a moving ball of mass m as seen in Fig. 1. The location of the ball inside the tube is controlled by air (green arrow in the figure) blown from the bottom of the tube, and we suppose that the air lifts the ball linearly from the height 0.1 to 0.6 (as seen in the first plot of Fig. 2), and when the ball reaches the height 0.6, the air blowing stops so that the ball falls down to the ground quickly. The falling down is so quick that we model it as a discrete jump as in the first plot of Fig. 2. By assuming that the weight of the tube is negligible, the system can be modeled as a time-varying linear system:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(I(t)+mL^2(t))b}{p(t)} & \frac{m^2gL^2(t)}{p(t)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mL(t)b}{p(t)} & \frac{mg(M+m)L(t)}{p(t)} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ \frac{(I(t)+mL^2(t))}{p} \\ 0 \\ \frac{mL(t)}{p} \end{bmatrix} u =: A_{\mathrm{p}}(t)X(t) + B_{\mathrm{p}}(t)u(t)$$

where x is the horizontal position of the cart, ϕ is the angle of the pendulum as seen in Fig. 1, u is the force exerted on the cart, $p(t) := (M+m)I(t) + MmL^2(t)$, and the parameters are mass of the cart $M = 0.5$, mass of the ball $m = 0.2$, friction coefficient of the cart $b = 0.1$, and acceleration of gravity $g = 9.8$. The inertia is given by $I(t) = mL^2(t)$ where the time-varying distance of the ball from the pivot is $L(t) = 0.1 + \mathtt{mod}(t, 0.5)$.[1] The system (which is slightly modified from [2]) is clearly a time-varying one, and the first plot of Fig. 2 depicts $I(t)$, which is not only time-varying but also exhibits periodic jumps.

To stabilize the system, we design a feedback control $u(t) = K(t)X(t)$, where $X \in \mathbb{R}^4$ denotes the state vector,

[1] $\mathtt{mod}(t, a)$ denotes the remainder in the division of t by a.

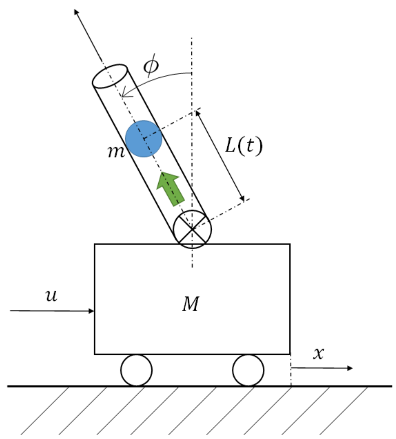

Figure 1: An inverted pendulum on a cart that has a tube instead of a rigid rod.

such that, for every t,

$$\begin{aligned}&\text{spectrum of } A_\text{p}(t) + B_\text{p}(t)K(t)\\&\quad = \{-8.5 \pm 7.9i, -4.8 \pm 0.8i\},\end{aligned} \tag{6}$$

in which the eigenvalues are taken from [2]. The strategy (6) is to enforce the matrix $A_\text{p}(t)+B_\text{p}(t)K(t)$ to remain Hurwitz for every t, and this is possible because the pair $(A_\text{p}(t), B_\text{p}(t))$ is controllable for all t. While this is not enough for stability, the closed-loop system becomes exponentially stable as long as the total variation of $A_\text{p}(t) + B_\text{p}(t)K(t)$, determined by the distance $L(t)$ of the ball from the pivot, is small [5]. This is indeed the case for $L(t)$ given above, as can be seen from the second plot of Fig. 2.

We now consider an unhappy situation where the power amplifier is interfered by a power line disturbance which is sinusoidal at 20Hz frequency, so that the actual gains applied are not $K_i(t)$ but $K_i(t) + 20\sin(2\pi \cdot 20t)$, $i = 1, 2, 3, 4$. (As an example, $K_1(t)$ is plotted in the third plot of Fig. 2.) Then the closed-loop system is written as

$$\dot{X} = \Big(A_\text{p}(t) + B_\text{p}(t)K(t) + B_\text{p}(t)[1\ 1\ 1\ 1]20\sin(2\pi \cdot 20t)\Big)X. \tag{7}$$

While simulations show the closed-loop system is still stable (see the fourth plot of Fig. 2), we are not aware of an off-the-shelf theorem that ensures the stability for the above system. The work of [10] does not apply because (4) becomes $B(t,s) = B_\text{p}(t)20\sin(s)$ for the case of (7), and B depends on t. We also note that B has discontinuity in its first argument as B_p does. On the other hand, the tools developed in this paper will assert that, if 20Hz is fast enough compared to the system dynamics, then the closed-loop system (7) is exponentially stable.

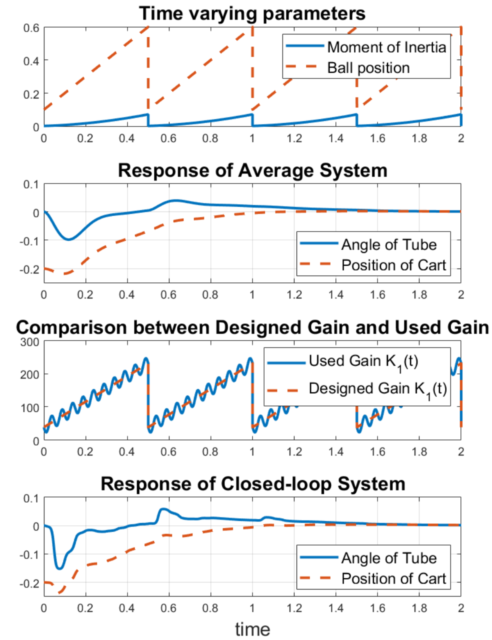

Figure 2: Simulation results: the second plot is for the system $\dot{X} = (A_\text{p}(t) + B_\text{p}(t)K(t))X$ without the disturbance, and the fourth plot is for the actual system (7) **with the disturbance.**

2 SET-UP AND STATEMENT OF THE MAIN RESULT

For convenience, we rewrite the system (1) as

$$\dot{x} = (A(t) + B(t, t/\varepsilon))x \tag{8}$$

where A and B are defined in (3) and (4), respectively, and it follows that $B(t,\cdot)$ is periodic with period T and

$$\frac{1}{T}\int_0^T B(t,s)ds = 0, \quad \forall t \geq 0.$$

We impose the following assumptions.

Assumption 2. The function B takes the separable form

$$B(t,s) = B_\text{s}(t)B_\text{f}(s) \tag{9}$$

with matrix-valued functions B_s and B_f of compatible dimensions.

The above properties of $B(t,\cdot)$ imply that B_f is periodic with period T and has zero average, i.e.,

$$\frac{1}{T}\int_0^T B_\text{f}(s)ds = 0. \tag{10}$$

Here the subscripts 's' and 'f' stand for "slow" and "fast", respectively. Assumption 2 is satisfied by the system (7) in the above example, where we have

$$B(t,s) = B_{\rm p}(t)[1\,1\,1\,1]20\sin(s).$$

Assumption 2 is made for convenience and to make the developments more accessible. It can be lifted without too much difficulty; the more general case will be reported elsewhere.

Assumption 3. *$A(\cdot)$, $B_{\rm s}(\cdot)$, and $B_{\rm f}(\cdot)$ are piecewise continuous, càdlàg[2], and uniformly bounded.*

For $0 \le t_1 < t_2$, define the set of jump times of $A(\cdot)$ as[3]

$$J_A(t_1,t_2) = \left\{\tau \in (t_1,t_2] : A(\tau^+) \neq A(\tau^-)\right\}.$$

Similarly, define the set of jump times of $B_{\rm s}$ as

$$J_B(t_1,t_2) = \{\tau \in (t_1,t_2] : B_{\rm s}(\tau^+) \neq B_{\rm s}(\tau^-)\}.$$

Piecewise continuity of A and B_s means, by definition, that for each finite pair of times $t_1 < t_2$, both $J_A(t_1,t_2)$ and $J_B(t_1,t_2)$ are finite.

Assumption 4. *Between two consecutive jump times in the set $J_A(0,\infty)$, $A(\cdot)$ is C^1, and $\dot{A}(\cdot)$ and $\|\dot{A}(\cdot)\|$ are Riemann integrable. Similarly, between two consecutive jump times of $J_B(0,\infty)$, $B_{\rm s}(\cdot)$ is C^1, and $\dot{B}_{\rm s}(\cdot)$ is uniformly bounded on $[0,\infty) \setminus J_B(0,\infty)$.*

Regularity assumptions imposed on $A(\cdot)$ are the same as in [5, Assumption 2], which is for ensuring that the results from [5] can be applied. In particular, Assumptions 1 and 3 together imply that for each $\lambda \in (0,\kappa)$ there exists $c > 0$ such that

$$\|e^{A(t)s}\| \le ce^{-\lambda s} \quad \forall\, t \ge 0,\ s \ge 0 \tag{11}$$

(see [8, Section 9.6, proof of Lemma 9.9]).

Next, we consider the *total variation* of $A(\cdot)$ as defined in [5]. For an arbitrary time interval $[t_1,t_2]$, this is given by

$$\int_{t_1}^{t_2} \|dA\| := \sum_{i=0}^{m} \int_{\tau_i}^{\tau_{i+1}} \|\dot{A}(t)\|dt + \sum_{i=1}^{m} \|A(\tau_i^+) - A(\tau_i^-)\| \tag{12}$$

where τ_i, $i = 1,\cdots,m$ are the jump times in $J_A(t_1,t_2)$, with $t_1 =: \tau_0 < \tau_1 < \cdots < \tau_m \le \tau_{m+1} := t_2$, and m being the cardinality of $J_A(t_1,t_2)$. We refer the reader to [5] for a more intrinsic but equivalent definition of the total variation and for further discussion. The next assumption places an upper bound on the total variation.

Assumption 5. *The total variation of $A(\cdot)$ satisfies the bound*

$$\int_{t_1}^{t_2} \|dA\| \le \mu_A(t_2 - t_1) + \alpha_A \quad \forall\, t_2 > t_1 \ge 0 \tag{13}$$

with some $\alpha_A > 0$ and

$$0 < \mu_A < \frac{\beta_1}{2\beta_2^3}, \tag{14}$$

where

$$\beta_1 := \frac{1}{2L}, \quad \beta_2 := \frac{c^2}{2\lambda}, \tag{15}$$

c and λ come from (11)*, and L is such that*

$$\|A(t)\| \le L \quad \forall\, t \ge 0 \tag{16}$$

which exists by Assumption 3.

Our last assumption asks a linear growth in the accumulation of jumps in $B(\cdot,s)$.

Assumption 6. *There are $\mu_B > 0$ and $\alpha_B > 0$ such that*

$$\sum_{\tau \in J_B(t_1,t_2)} \left\|B_s(\tau^+) - B_s(\tau^-)\right\| \le \mu_B(t_2 - t_1) + \alpha_B$$

for all $t_2 > t_1 \ge 0$.

Our main result states that, under the above assumptions, the system (1) is globally exponentially stable (in the classical sense, with respect to the equilibrium at the origin) for sufficiently small ε.

Theorem 1. *Let Assumptions 1–6 hold. Then there exists an $\varepsilon^* > 0$ such that the system* (1) *is globally exponentially stable for all $\varepsilon \in (0,\varepsilon^*)$.*

As will be clear from the proof given next, the exponential stability is uniform over ε in the indicated range, in the sense that there exist constants $\bar{\gamma}$ and $\bar{\theta}$ (independent of ε) such that the solutions satisfy $|x(t)| \le \bar{\gamma}e^{-\bar{\theta}t}|x(0)|$.

3 PROOF OF THEOREM 1

The proof proceeds by, first, deriving an impulsive system that equivalently describes the behavior of (1) in new coordinates. Then, by observing that this system is a perturbation of the average system (2), and after showing that the average system is exponentially stable, a perturbation analysis verifies exponential stability of the original system.

3.1 Derivation of an impulsive system in new coordinates

To approximate the original system (1) by the average system (2), we consider the change of variables

$$y = \left(I - \varepsilon B_{\rm s}(t)\int_0^{t/\varepsilon} B_{\rm f}(s)ds\right)x. \tag{17}$$

The matrix in the parenthesis is invertible for all $t \ge 0$ as long as $\varepsilon \in (0,\varepsilon_1^*]$ where ε_1^* is chosen such that

$$\varepsilon_1^* < \left(T\sup_{t\ge0}\|B_{\rm s}(t)\| \sup_{0\le s\le T}\|B_{\rm f}(s)\|\right)^{-1} \tag{18}$$

[2] Continuous from the right, has limits from the left; this assumption is made for notational convenience.
[3] $A(\tau^-)$ denotes the left limit of $A(\cdot)$ at τ, and the right limit $A(\tau^+)$ equals $A(\tau)$. We often use τ^+ instead of τ for convenience.

which is well-defined by Assumption 3. Invertibility of the matrix in (17) then follows by virtue of T-periodicity of $B_f(\cdot)$ and (10). This change of variables is a variation on the one considered for the more general nonlinear case in [1] and, modulo time rescaling, in [8, Section 10.4], when specialized to the linear system (1). A similar coordinate transformation was also considered in [15]. The paper [3] considers a different coordinate change which transforms the system to a time-delay one.

Differentiating the right-hand side of (17) with respect to time around $t \notin J_B(0,\infty)$, we easily obtain

$$A(t)x - \varepsilon \dot{B}_s(t)\int_0^{t/\varepsilon} B_f(s)ds \cdot x - \varepsilon B_s(t)\int_0^{t/\varepsilon} B_f(s)ds \cdot \dot{x}.$$

Replacing x and $\dot{x}$ using (17), the system (1) in the y-coordinates has the form

$$\dot{y} = A(t)y + \varepsilon C(t,\varepsilon)y, \quad t \notin J_B(0,\infty) \tag{19}$$

where

$$\begin{aligned} C(t,\varepsilon) := \Big(& A(t)B_s(t)\int_0^{t/\varepsilon} B_f(s)ds \\ &- B_s(t)\int_0^{t/\varepsilon} B_f(s)ds \cdot A(t) - \dot{B}_s(t)\int_0^{t/\varepsilon} B_f(s)ds \\ &- B_s(t)\int_0^{t/\varepsilon} B_f(s)ds \cdot B_s(t)B_f(t/\varepsilon)\Big) \\ &\times \left(I - \varepsilon B_s(t)\int_0^{t/\varepsilon} B_f(s)ds\right)^{-1}. \end{aligned}$$

It is seen that $C(t,\varepsilon)$ is uniformly bounded for all $\varepsilon \in (0,\varepsilon_1^*]$ and for all $t \notin J_B(0,\infty)$ because of Assumptions 3 and 4, periodicity of $B_f(\cdot)$, (10), and (18).

On the other hand, at the times $t \in J_B(0,\infty)$, the variable y has discontinuity[4] because of (17), in which x is continuous for all t by (1). The size of the jump in y can be computed again by (17). That is, with $x(t^+) = x(t^-)$ at $t \in J_B(0,\infty)$, equation (17) implies that

$$\begin{aligned} y(t^+) &= \left(I - \varepsilon B_s(t^+)\int_0^{t/\varepsilon} B_f(s)ds\right) \\ &\quad \times \left(I - \varepsilon B_s(t^-)\int_0^{t/\varepsilon} B_f(s)ds\right)^{-1} y(t^-) \\ &=: D(t,\varepsilon)y(t^-). \end{aligned} \tag{20}$$

Lemma 2. *There are $\beta_* > 0$ and $\beta_o > 0$ such that*

$$\|D(t,\varepsilon)\| \le \beta_* \tag{21}$$

$$\|D(t,\varepsilon) - I\| \le \varepsilon \beta_o \Delta_B(t) \tag{22}$$

for all $t \ge 0$ and all $\varepsilon \in (0,\varepsilon_1^*]$, where

$$\Delta_B(t) := \left\|B_s(t^-) - B_s(t^+)\right\| \cdot \sup_{0\le\tau\le T}\left\|\int_0^\tau B_f(s)ds\right\|.$$

[4]This is where the analysis of this paper differs from that of [10].

Proof: Let

$$\beta_o := \sup_{t\ge 0}\left\|\left(I - \varepsilon B_s(t^-)\int_0^{t/\varepsilon} B_f(s)ds\right)^{-1}\right\|,$$

$$\beta_* := \beta_o \cdot \sup_{t\ge 0}\left\|I - \varepsilon B_s(t^+)\int_0^{t/\varepsilon} B_f(s)ds\right\|,$$

both of which are well-defined by (18) and periodicity of $B_f(\cdot)$ with (10) under Assumption 3. Then, the claim follows by noting that

$$\begin{aligned} \|D(t,\varepsilon) - I\| = \Big\|&\varepsilon(B_s(t^-) - B_s(t^+))\int_0^{t/\varepsilon} B_f(s)ds \\ &\times \left(I - \varepsilon B_s(t^-)\int_0^{t/\varepsilon} B_f(s)ds\right)^{-1}\Big\|. \end{aligned}$$ □

In summary, we have an impulsive system that is an equivalent representation of (1) in the coordinates (17), given by

$$\dot{y} = A(t)y + \varepsilon C(t,\varepsilon)y \qquad t \notin J_B(0,\infty) \tag{23}$$

$$y^+ = D(t,\varepsilon)y \qquad t \in J_B(0,\infty). \tag{24}$$

3.2 Stability of the average system

The system (23) can be regarded as a perturbation of the average system (2). And, Theorem 3 from [5] establishes that, under Assumptions 1, 3, 4, and 5, the system (2) is exponentially stable. Here, instead of reproducing the analysis in [5], we borrow the main ingredients as follows. For each $t \ge 0$ we let $P(t)$ be the unique symmetric positive definite solution to the Lyapunov equation

$$P(t)A(t) + A^T(t)P(t) = -I. \tag{25}$$

Then, with β_1 and β_2 from (15), it can be shown that

$$\beta_1 \le \|P(t)\| \le \beta_2, \qquad \forall t \ge 0, \tag{26}$$

$$\|\dot{P}(t)\| \le 2\beta_2^2\|\dot{A}(t)\|, \quad \forall t \notin J_A(0,\infty), \tag{27}$$

$$\|P(t^+) - P(t^-)\| \le 2\beta_2^2\|A(t^+) - A(t^-)\|, \quad \forall t \in J_A(0,\infty). \tag{28}$$

Indeed, (26) follows from [8, Lemma 9.9], (27) from [8, proof of Lemma 9.9] or [7, Theorem 3.4.11], and (28) from [5, Proposition 1 and Lemma 3].

3.3 Stability of original system by perturbation analysis

Consider the candidate Lyapunov function

$$V(t,y) := y^T P(t)y \tag{29}$$

whose time derivative along (23) around $t \notin J := J_A(0,\infty) \cup J_B(0,\infty)$ is given by

$$\begin{aligned}\dot V &= y^T(PA + A^TP)y + 2\varepsilon y^T PC(t,\varepsilon)y + y^T\dot P y\\ &\le -|y|^2 + 2\varepsilon\bar c\|P\||y|^2 + \|\dot P\||y|^2\\ &\le -\left(\beta_2^{-1} - 2\beta_2\beta_1^{-1}\varepsilon\bar c - 2\beta_2^2\beta_1^{-1}\|\dot A\|\right)V\end{aligned}$$

where

$$\bar c := \sup_{t\notin J,\, \varepsilon\in(0,\varepsilon_1^*]} \|C(t,\varepsilon)\|,$$

in which (26) and (27) are used. Let τ_1 and τ_2 be any consecutive elements in J such that $\tau_2 > \tau_1$ and $(\tau_1,\tau_2)\cap J = \emptyset$. By the comparison lemma (e.g., [8, Lemma 3.4]) we obtain, with $V(t, y(t)) =: V(t)$,

$$V(\tau_2^-) \le \exp\Big(-\left(\beta_2^{-1} - 2\beta_2\beta_1^{-1}\varepsilon\bar c\right)(\tau_2-\tau_1) + 2\beta_2^2\beta_1^{-1}\int_{\tau_1^+}^{\tau_2^-}\|\dot A(s)\|ds\Big)V(\tau_1^+). \quad (30)$$

Now, at a jump time $\tau\in J$, $P(t)$ jumps if $A(t)$ jumps and $y(t)$ jumps if $B_{\rm s}(t)$ jumps at $t=\tau$. To inspect the variation of $V(t)$ passing through the jumps, let us denote $y_+ = y(\tau^+)$, $y_- = y(\tau^-)$, $P^+ = P(\tau^+)$, and $P^- = P(\tau^-)$. Then,

$$V(\tau^+) - V(\tau^-) = \left(y_+^TP^+y_+ - y_-^TP^+y_-\right) + \left(y_-^TP^+y_- - y_-^TP^-y_-\right). \quad (31)$$

With $D := D(\tau,\varepsilon)$, the first parenthesis can be spelled out, using (20), Lemma 2 and (26), as

$$\begin{aligned}&y_+^TP^+y_+ - y_-^TP^+y_-\\ &= \left(y_+^TP^+y_+ - y_+^TP^+y_-\right) + \left(y_+^TP^+y_- - y_-^TP^+y_-\right)\\ &\le |P^+y_+||y_+-y_-| + |P^+y_-||y_+-y_-|\\ &\le \|P^+\|\|D\||y_-||y_+-y_-| + \|P^+\||y_-||y_+-y_-|\\ &\le \|P^+\|\|D\|\|D-I\||y_-|^2 + \|P^+\|\|D-I\||y_-|^2\\ &\le \beta_2(\beta_*+1)\varepsilon\beta_o\Delta_B(\tau)\beta_1^{-1}V(\tau^-).\end{aligned}$$

The second parenthesis in (31) becomes, using (26) and (28),

$$\begin{aligned}y_-^T(P^+-P^-)y_- &\le \|P^+-P^-\||y_-|^2\\ &\le 2\beta_2^2\|A(\tau^+)-A(\tau^-)\|\beta_1^{-1}V(\tau^-).\end{aligned}$$

Hence, we have

$$V(\tau^+) - V(\tau^-) \le \Big(2\beta_2^2\beta_1^{-1}\|A(\tau^+)-A(\tau^-)\| + \varepsilon(\beta_*+1)\beta_2\beta_o\beta_1^{-1}\Delta_B(\tau)\Big)V(\tau^-)$$

and, using $1+z\le e^z$,

$$V(\tau^+) \le \exp\Big(2\beta_2^2\beta_1^{-1}\|A(\tau^+)-A(\tau^-)\| + \varepsilon(\beta_*+1)\beta_2\beta_o\beta_1^{-1}\Delta_B(\tau)\Big)V(\tau^-) \quad (32)$$

at the jump time $\tau\in J$.

Combining (30) and (32), it can be shown that, for any $t_2 > t_1$,

$$V(t_2) \le \exp\Big(-(\beta_2^{-1} - 2\beta_2\beta_1^{-1}\varepsilon\bar c)(t_2-t_1) + 2\beta_2^2\beta_1^{-1}\int_{t_1}^{t_2}\|dA\| + \varepsilon(\beta_*+1)\beta_2\beta_o\beta_1^{-1}\sum_{\tau\in J_B(t_1,t_2)}\Delta_B(\tau)\Big)V(t_1).$$

From this, Assumptions 5 and 6 along with the definition of Δ_B in Lemma 2 bring us to

$$V(t_2) \le \gamma e^{-\theta(t_2-t_1)}V(t_1)$$

where

$$\begin{aligned}\gamma &:= e^{2\beta_2^2\beta_1^{-1}\alpha_A + \varepsilon(\beta_*+1)\beta_2\beta_o\beta_1^{-1}\alpha_B\nu},\\ \theta &:= \beta_2^{-1} - 2\varepsilon\beta_2\beta_1^{-1}\bar c\\ &\qquad - 2\beta_2^2\beta_1^{-1}\mu_A - \varepsilon(\beta_*+1)\beta_2\beta_o\beta_1^{-1}\mu_B\nu\end{aligned}$$

with $\nu := \sup_{0\le\tau\le T}\|\int_0^\tau B_{\rm f}(s)ds\|$. Since $\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\mu_A > 0$ by (14), take

$$\varepsilon^* := \min\left\{\varepsilon_1^*, \frac{\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\mu_A}{2\beta_2\beta_1^{-1}\bar c + (\beta_*+1)\beta_2\beta_o\beta_1^{-1}\mu_B}\right\}.$$

Then the system (23)–(24) is exponentially stable. In the x-coordinates, the same conclusion then holds for the original system (1) by (17) because the norm of the transformation $x \leftrightarrow y$ is uniformly bounded.

4 CONCLUSIONS

We studied stability of a class of linear systems with slow and fast time variation and switching. This was accomplished by combining the averaging method as used in [1] with the result from [5] on stability of linear systems with slow time variation and switching. Compared with the recent paper [10], we were able to handle a more general class of linear systems, at the expense of more involved analysis of an impulsive system arising from a coordinate transformation. Ongoing work is focused on extending this approach to nonlinear systems. We also envision applications in domains such as PWM (pulse-width-modulation) as well as dose control in medical drug delivery (see [1]).

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