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On stability of linear switched differential algebraic equations

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Abstract— **This paper studies linear switched differential algebraic equations (DAEs), i.e., systems defined by a finite family of linear DAE subsystems and a switching signal that governs the switching between them. We show by examples that switching between stable subsystems may lead to instability, and that the presence of algebraic constraints leads to a larger variety of possible instability mechanisms compared to those observed in switched systems described by ordinary differential equations (ODEs). We prove two sufficient conditions for stability of switched DAEs based on the existence of suitable Lyapunov functions. The first result states that a common Lyapunov function guarantees stability under arbitrary switching when an additional condition involving consistency projectors holds (this extra condition is not needed when there are no jumps, as in the case of switched ODEs). The second result shows that stability is preserved under switching with sufficiently large dwell time.**

I. Introduction

We consider linear *switched differential algebraic equations (switched DAEs)* of the form

$$E_\sigma \dot{x} = A_\sigma x, \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$, is some switching signal, and $E_p, A_p \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, are constant matrices for each parameter $p \in \{1, 2, \ldots, N\}$. The goal of this paper is to develop sufficient conditions for stability of such switched DAEs, based on the existence of appropriate Lyapunov functions.

Classical linear DAEs (i.e. without switching) naturally appear when modeling electrical circuits because Kirchhoff's circuit laws add algebraic equations to the differential equations stemming from capacitors and inductances. They also occur when modeling simple (i.e. linear) mechanical systems with (linear) constraints. For more details and further motivation for studying DAEs the reader is referred to [1]. Adding, for example, (ideal) switches to an electrical circuit yields a switched DAE as in (1). When studying the zero dynamics of an ordinary differential equation (ODE) one arrives at a DAE because of the additional algebraic constraint $0 = y = Cx$. In particular, using a switched controller to stabilize the zero dynamics (as was done in [2]) yields a switched DAE (1) even if one starts with an ODE.

When each matrix E_p is invertible, (1) reduces to a more familiar switched ordinary differential equation (switched ODE), or switched system. The stability theory of switched ODEs has received considerable attention in the last couple of decades, and is now relatively mature. In particular, it is well known that switching among stable subsystems may lead to instability; a switched system is asymptotically stable under arbitrary switching if (and only if) the subsystems share a common Lyapunov function; and stability is preserved under sufficiently slow switching, as can be shown using multiple Lyapunov functions (one for each subsystem). We refer the reader to the book [3] for these and other results on switched systems and for an extensive literature overview.

On the other hand, an investigation of stability questions for switched DAEs by similar methods has not yet appeared in the literature. In this paper, we begin such an investigation by establishing Lyapunov-based sufficient conditions for stability of switched DAEs. In the special case of switched ODEs, our results reduce to the known results mentioned above. However, we will demonstrate by means of examples that the presence of algebraic constraints leads to new types of instability mechanisms. It also poses additional technical challenges related to interpreting system solutions and defining suitable Lyapunov functions.

It is well known that the solution of each individual DAE $E_p\dot{x} = A_p x$, $p \in \{1, \ldots N\}$, evolve within a so-called consistency space, which is a subspace of $\mathbb{R}^n$. In general, at a switching time $t \in \mathbb{R}$ there does not exist a continuous extension of the solution into the future, because the value $x(t-)$ immediately before the switch need not be within the consistency space corresponding to the DAE after the switch. Therefore, it is necessary to allow for solutions with jumps. However, this leads to difficulties in evaluating the derivative of the solutions. To resolve this problem we adopt the distributional framework introduced in [4], [5], i.e. as solutions of the switched DAE (1) distributions (generalized functions), in particular Dirac impulses, are considered. For this, we have to assume that the switching signal has only a locally finite set of switching times. Furthermore, to ensure existence and uniqueness of solutions we have to assume that each matrix pair (E_p, A_p) is regular, i.e. the polynomial $\det(E_p s - A_p)$ is not identically zero. Finally, we will make one more assumption which ensures that no impulses occur in the solutions of the switched DAE (1); for details see Section IV and Theorem 8. A consequence of these assumptions is that although a distributional solution framework is necessary as a theoretical basis for treating switched DAEs (1), the only solutions that arise in this paper are normal (piecewise-smooth) functions.

It should be noted that so called linear complementarity problems, see [6] and the references therein, were success-

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fully applied to study passive linear electrical circuits with switches; however the results are not directly comparable because in that framework only switches are allowed which do not change the structure of the circuit, i.e. the underlying equations for the dynamics do not change, which is not the case for systems of the form (1); on the other hand [6] also considers state-depending switching as well as inequality constraints which are not considered here.

By rewriting a classical DAE $E\dot{x} = Ax$ as $\dot{x} \in E^{-1}(Ax)$ and defining a jump map with the help of the consistency projectors (see Definition 5 and Theorem 8) it seems possible to study switched DAEs (1) as a special hybrid system in the framework of [7]. It is not clear yet whether this has advantages because one loses the special structure of (1). Furthermore, it should be highlighted that in contrast to the hybrid systems framework the jump map must not be given additionally for the switched DAE (1), it is an intrinsic property of the matrix pairs (E_p, A_p), $p \in \{1, \ldots, N\}$.

The structure of the paper is as follows. Section II summarizes relevant results for classical DAEs (i.e. DAEs with constant coefficients), most of which can be found in, or follow from, [8]. In particular, Lyapunov functions for classical DAEs are introduced. The notion of a consistency projector, needed for studying switched DAEs, is also defined there. Section III gives several examples which motivate the stability problem for switched DAEs by illustrating some stable as well as unstable behaviors that can occur. In Section IV the stability of switched DAEs is studied. The main result is Theorem 9, where sufficient conditions for asymptotic stability of the switched DAE (1) under arbitrary switching are given. Theorem 11 shows that a switching signal is not destabilizing if it has a large enough dwell time, which is a generalization of the same result for switched ODEs to switched DAEs.

The following notation is used throughout the paper. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the natural numbers, integers, real and complex numbers, respectively. For a matrix $M \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, the kernel (null space) of M is $\ker M$, the image (range, column space) of M is $\operatorname{im} M$, and the transpose of M is $M^\top \in \mathbb{R}^{m\times n}$. For a matrix $M \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subset \mathbb{R}^n$, the image of $\mathcal{S}$ under M is $M\mathcal{S} := \{ Mx \in \mathbb{R}^n \mid x \in \mathcal{S} \}$ and the pre-image of $\mathcal{S}$ under M is $M^{-1}\mathcal{S} := \{ x \in \mathbb{R}^n \mid \exists y \in \mathcal{S} : Mx = y \}$. The identity matrix is denoted by I. For a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ the left-sided evaluation $\lim_{\varepsilon\searrow 0} f(t-\varepsilon)$ at $t \in \mathbb{R}$ is denoted by $f(t-)$.

II. Lyapunov functions for differential algebraic equations

Consider the classical DAE

$$E\dot{x} = Ax, \tag{2}$$

where the matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular, i.e. $\det(Es - A)$ is not the zero polynomial. A (classical) solution of (2) is any differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ such that (2) is fulfilled.

Definition 1 (Consistency space): Let the *consistency space* of (2) be given by

$$\mathfrak{C}_{(E,A)} := \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } x \text{ of } (2) \\ \text{with } x(0) = x^0 \end{array} \right\}.$$

It is well known that for regular matrix pairs each solution of (2) is uniquely determined by any consistent initial condition $x(0) = x^0 \in \mathfrak{C}_{(E,A)}$. Since (2) is time invariant, all solutions x evolve within the consistency space, i.e. $x(t) \in \mathfrak{C}_{(E,A)}$ for all $t \in \mathbb{R}$. Furthermore, if (2) is an ordinary differential equation, i.e. $E \in \mathbb{R}^{n\times n}$ is an invertible matrix, then $\mathfrak{C}_{(E,A)} = \mathbb{R}^n$.

The following lemma gives a nice characterization of the consistency space in terms of the matrices E, A.

Lemma 2 ([8]): Consider the DAE (2) with regular matrix pair (E, A). Let $\mathcal{V}^0 = \mathbb{R}^n$ and $\mathcal{V}^{k+1} := A^{-1}(E\mathcal{V}^k)$ for $k \in \mathbb{N}$. Then

$$\exists k^* \in \mathbb{N} : \quad \mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_{k^*} = \mathcal{V}_{k^*+1} = \cdots$$

and $\mathfrak{C}_{(E,A)} = \mathcal{V}_{k^*}$. Furthermore, $\ker E \cap \mathfrak{C}_{(E,A)} = \{0\}$.

Definition 3 (Lyapunov function): Consider the DAE (2) with regular matrix pair (E, A) and corresponding consistency space $\mathfrak{C}_{(E,A)} \subseteq \mathbb{R}^n$. Assume there exist a positive definite matrix $P = \overline{P}^\top \in \mathbb{C}^{n\times n}$ and a matrix $Q = \overline{Q}^\top \in \mathbb{C}^{n\times n}$ which is positive definite on $\mathfrak{C}_{(E,A)}$ such that the *generalized Lyapunov equation*

$$A^\top PE + E^\top PA = -Q$$

is fulfilled. Then

$$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$$

is called a *Lyapunov function* for the DAE (2).

Note that this definition ensures that V is not increasing along solutions, i.e., for any solution $x : \mathbb{R} \to \mathbb{R}^n$ and all $t \in \mathbb{R}$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V\big(x(t)\big) = -x(t)^\top Qx(t) \leq 0$$

and equality only holds for $x(t) = 0$. Furthermore, the property $\ker E \cap \mathfrak{C}_{(E,A)} = \{0\}$ ensures that V is positive definite on $\mathfrak{C}_{(E,A)}$.

With some abuse of terminology, we call (2) *asymptotically stable* if, and only if, $x(t) \to 0$ as $t \to \infty$ for all solutions x of (2). Note that attractivity of the zero solution already implies attractivity and stability in the sense of Lyapunov for all solutions of (2), [9]. The following theorem shows the equivalence between asymptotic stability of (2) and the existence of a Lyapunov function.

Theorem 4 ([8], [9]): The DAE (2) with regular matrix pair (E, A) is asymptotically stable if, and only if, there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for (2).

For the switched DAE (1) so-called consistency projectors will play an important role; these projectors describe how an inconsistent initial value jumps to a consistent one in the event of a switch.

Definition 5 (Consistency projector): For a regular matrix pair (E,A) choose invertible $S,T\in\mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0\\ 0 & N\end{bmatrix}, \begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}\right), \qquad (3)$$

where $J\in\mathbb{R}^{n_1\times n_1}$, $0\leq n_1\leq n$, is some matrix and $N\in\mathbb{R}^{n_2\times n_2}$, $n_2 := n-n_1$, is a *nilpotent* matrix, i.e. $N^{n_2}=0$. The consistency projector is then defined as

$$\Pi_{(E,A)} := T\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}T^{-1},$$

where $I\in\mathbb{R}^{n_2\times n_2}$.

Remark 6: It is well known that any regular matrix pair allows the decomposition (3) which is called Weierstrass normal form (or Kronecker form in the non-regular case) and it is also easy to see that the definition of $\Pi_{(E,A)}$ does not depend on the specific choice of T. However, for the Weierstrass form it is normally assumed that J and N are in Jordan canonical form, but for the definition of the conistency projectors this is not necessary. Therefore, (3) might be called quasi-Weierstrass form. The quasi-Weierstrass form can easily be obtained by using the Wong sequences [10], [11] given by $\mathbb{R}^n=\mathcal{V}_0\supset\mathcal{V}_1\supset\ldots\supset\mathcal{V}_{k^*}$ as in Lemma 2 and $\{0\}=:\mathcal{W}_0\subset\mathcal{W}_1\subset\ldots\subset\mathcal{W}_{k^*}=\mathcal{W}_{k^*+1}=\ldots$ with $\mathcal{W}_{i+1}:=E^{-1}(A\mathcal{W}_i)$. Choose full rank matrices V,W such that $\operatorname{im}V=\mathcal{V}_{k^*}$ and $\operatorname{im}W=\mathcal{W}_{k^*}$, then $T:=[V,W]$ and $S:=[EV,AW]^{-1}$ yield the decomposition (3), for details see [11].

Remark 7: Note that from the decomposition (3) the structure of the (consistent) solutions of $E\dot{x}=Ax$ can be read off very easily: It is not difficult to see that the so called pure DAE $N\dot{w}=w$ only has the trivial solution $w=0$, hence all consistent solutions x of $E\dot{x}=Ax$ can be expressed as $x=T\left[\begin{smallmatrix} v\\ 0\end{smallmatrix}\right]$ where v is a solution of the *underlying ODE* $\dot{v}=Jv$. Although the structure of N does not play a role when considering consistent solutions, it is important when studying inconsistent intial values as is necessary for switched DAEs (1).

III. SWITCHED DAES: MOTIVATING EXAMPLES

For switched ODEs there exist several well known examples of destabilizing switching. Of course, these examples are also examples for switched DAEs (because every ODE is a special DAE), but in the following we will give examples which are specific to switched DAEs. For all examples we consider a switching signal $\sigma:\mathbb{R}\to\{1,2\}$ with a constant interval $\Delta t>0$ between switching times, i.e. $\sigma(t)=1$ for all $t\in[2k\Delta t,(2k+1)\Delta t)$ and $\sigma(t)=2$ for all $t\in[(2k+1)\Delta t,(2k+2)\Delta t)$, $k\in\mathbb{Z}$.

Example 1

Let

$$(E_1,A_1)=\left(\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&-1\\1&-1\end{smallmatrix}\right]\right),(E_2,A_2)=\left(\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}-1&-1\\1&0\end{smallmatrix}\right]\right).$$

The solutions of the corresponding switched DAE (1) are shown in Figure 1. For small enough Δt all solutions grow unbounded and for large enough Δt the solutions converge to zero. Furthermore, there exists a value of Δt for which all solutions are periodic.

The consistency spaces are given by

$$\mathfrak{C}_1:=\mathfrak{C}_{(E_1,A_1)}=\operatorname{im}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],\quad \mathfrak{C}_2:=\mathfrak{C}_{(E_2,A_2)}=\operatorname{im}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right].$$

In view of Definition 5, $T_1=\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$, $T_2=\left[\begin{smallmatrix}0&1\\1&-1\end{smallmatrix}\right]$ and the corresponding consistency projectors are

$$\Pi_1=\left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right],\quad \Pi_2=\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right].$$

It follows that both DAEs are governed by the same underlying scalar ODE $\dot{y}=-y$; in particular, both DAEs are asymptotically stable. Furthermore, it is easy to see that

$$V:\mathbb{R}^2\to\mathbb{R}_{\geq 0},\quad x\mapsto x^\top x$$

restricted to the corresponding consistency space is a Lyapunov function for both subsystems. In spite of this, the switched system is not stable under arbitrary switching.

Example 2

Let

$$(E_1,A_1)=\left(\left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right],\left[\begin{smallmatrix}-1&8\pi&0\\\pi/2&-1&0\\0&0&-1\end{smallmatrix}\right]\right),$$
$$(E_2,A_2)=\left(\left[\begin{smallmatrix}0&4&0\\1&0&0\\0&0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}-4\pi&-4&0\\-1&4\pi&0\\-1&-4&4\end{smallmatrix}\right]\right).$$

The first system is an ODE, hence $\mathfrak{C}_1=\mathbb{R}^3$, $S_1=T_1=\Pi_1=I$. For the second DAE,

$$\mathfrak{C}_2=\operatorname{im}\left[\begin{smallmatrix}0&4\\1&0\\1&1\end{smallmatrix}\right],\quad T_2=\left[\begin{smallmatrix}0&4&0\\1&0&0\\1&1&1\end{smallmatrix}\right],\quad \Pi_2=\left[\begin{smallmatrix}1&0&0\\0&1&0\\1/4&1&0\end{smallmatrix}\right].$$

Choosing $S_2=\frac{1}{4}I$ it follows that the underlying two dimensional ODE of subsystem 2 is given by

$$\dot{y}=\left[\begin{smallmatrix}-1&-4\pi\\\pi&-1\end{smallmatrix}\right]y.$$

We select $\Delta t=1/4$, which ensures that the switching only occurs at that moment when the solution is located in the intersection of the consistency spaces (i.e. in $\mathfrak{C}_2$). Hence the solution of the switched DAE exhibits no jumps. The solutions of the unswitched DAEs are shown in the left part of Figure 2 and the unstable solutions of the switched DAE are illustrated in the right part of Figure 2.

Example 3

Let

$$(E_1,A_1)=\left(\left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}-1&2\pi&0\\-2\pi&-1&0\\0&0&1\end{smallmatrix}\right]\right),$$
$$(E_2,A_2)=\left(\left[\begin{smallmatrix}0&1&0\\1&0&1\\0&0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}4\pi&-1&4\pi\\-1&\pi&-1\\1&0&0\end{smallmatrix}\right]\right).$$

The consistency spaces are

$$\mathfrak{C}_1=\operatorname{im}\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right],\quad \mathfrak{C}_2=\operatorname{im}\left[\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right].$$

With $T_1=\left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right]$, $T_2=\left[\begin{smallmatrix}0&0&1\\1&0&0\\0&1&-1\end{smallmatrix}\right]$ the corresponding consistency projectors are given by

$$\Pi_1=\left[\begin{smallmatrix}1&0&0\\0&1&0\\0&0&0\end{smallmatrix}\right],\quad \Pi_2=\left[\begin{smallmatrix}0&0&0\\0&1&0\\1&0&1\end{smallmatrix}\right].$$

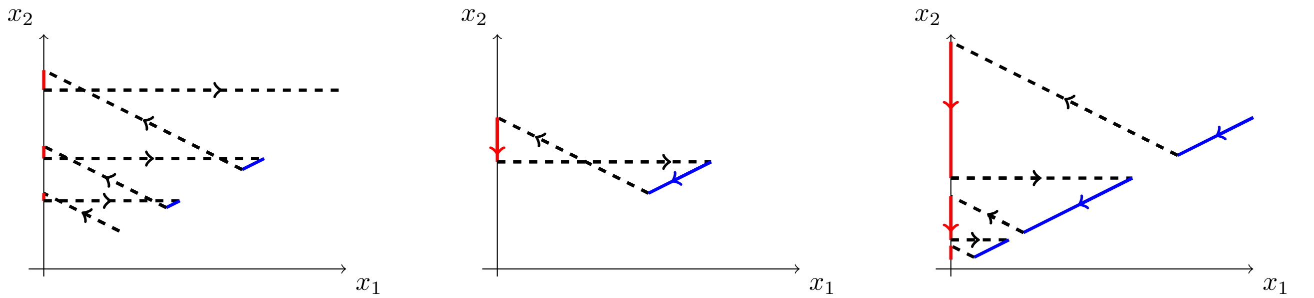

Fig. 1. Solutions for Example 1 for different switching signals (dashed lines mean jumps induced by the switching), left: $\Delta t < \frac{1}{2}\ln 2$, all nontrivial solutions grow unbounded; middle: $\Delta t = \frac{1}{2}\ln 2$, all solutions are periodic on $[0,\infty)$; right: $\Delta t > \frac{1}{2}\ln 2$, all solution tend to zero.

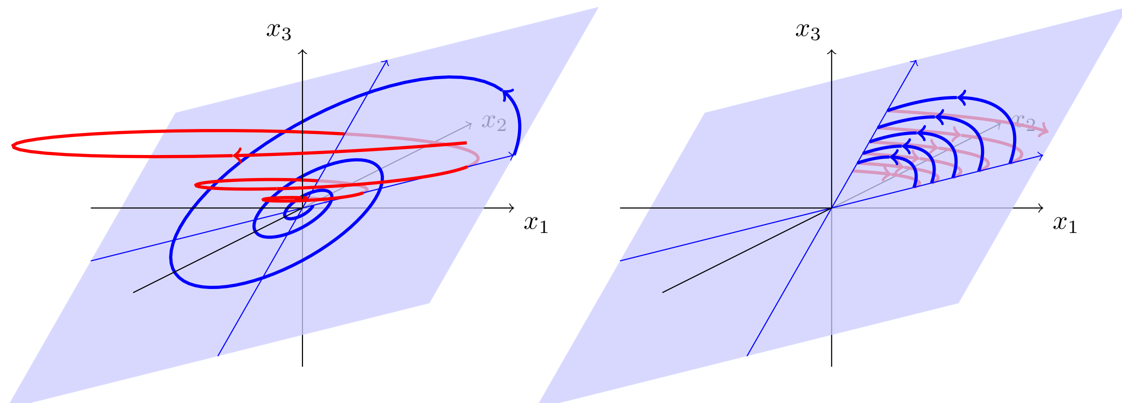

Fig. 2. Solutions for the Example 2. Left: Without switching, red: solution of subsystem 1, a three dimensional spiral converging to zero, blue: solution of subsystem 2, a two dimensional spiral converging to zero, Right: with switching, the solutions grow unbounded and exhibit no jumps.

The underlying ODE for the first DAE can be read off directly from the matrix pair (E_1, A_1), with $S_2 = I$ the underlying ODE for the second DAE is given by

$$\dot{y} = \begin{bmatrix} -1 & 4\pi \\ -\pi & -1 \end{bmatrix} y.$$

The solutions of the unswitched subsystems are illustrated in the left part of Figure 3. For $\Delta t = 1/2$ and an initial value at $t = 0$ which is located on the x_2 axis, the switching does not induce jumps and all solutions converge to zero. However, the choice $\Delta t = 1/4$ induces jumps and destabilizes the system, see the right part of Figure 3. Note that $V(x) = x^\top x$ is a common Lyapunov function on the intersection of the consistency spaces.

IV. Stability of switched DAEs

As already mentioned and motivated in the introduction, we will make the following assumptions for the switched DAE (1).

A1 The switching signal $\sigma : \mathbb{R} \to \{1,\dots,N\}$ is piecewise constant with a locally finite set of jump points and right-continuous.

A2 Each matrix pair (E_p, A_p), $p \in \{1,\dots,N\}$ is regular, i.e. $\det(sE_p - A_p)$ is not the zero polynomial.

A3 For the consistency projectors $\Pi_p := \Pi_{(E_p,A_p)}$, $p \in \{1,\dots,N\}$ corresponding to the regular matrix pairs (E_p, A_p) from (1), it holds that

$$\forall p, q \in \{1,\dots,N\} : E_p(I - \Pi_p)\Pi_q = 0.$$

Under these assumptions, the following result is known to hold.

Theorem 8 ([5, Thms. 4.2.13&4.2.8]): Consider the switched DAE (1) satisfying Assumptions A1, A2 and A3. Then every distributional solution of (1) is impulse free and is represented by a piecewise-smooth function $x : \mathbb{R} \to \mathbb{R}^n$. Furthermore, for all solutions $x : \mathbb{R} \to \mathbb{R}^n$,

$$\forall t \in \mathbb{R} : \quad x(t) = \Pi_{\sigma(t)} x(t-).$$

In the following we call the switched DAE (1) *asymptotically stable* if, and only if, all distributional solutions are impulse free and each solution $x : \mathbb{R} \to \mathbb{R}^n$ fulfills $x(t) \to 0$ as $t \to \infty$.

Theorem 9: Consider the switched DAE (1) satisfying A1, A2 and A3. Let $\mathfrak{C}_p := \mathfrak{C}_{(E_p,A_p)} \subseteq \mathbb{R}^n$, $p = 1,\dots,N$, and $\Pi_p := \Pi_{(E_p,A_p)} \in \mathbb{R}^{n\times n}$ be the consistency spaces and projectors corresponding to the matrix pairs (E_p, A_p). Assume the classical DAE $E_p\dot{x} = A_p x$ is for every $p = 1,\dots,N$ asymptotically stable with Lyapunov function $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. If

$$\forall p, q = 1,\dots,N \;\forall x \in \mathfrak{C}_q : \quad V_p(\Pi_p x) \leq V_q(x) \tag{4}$$

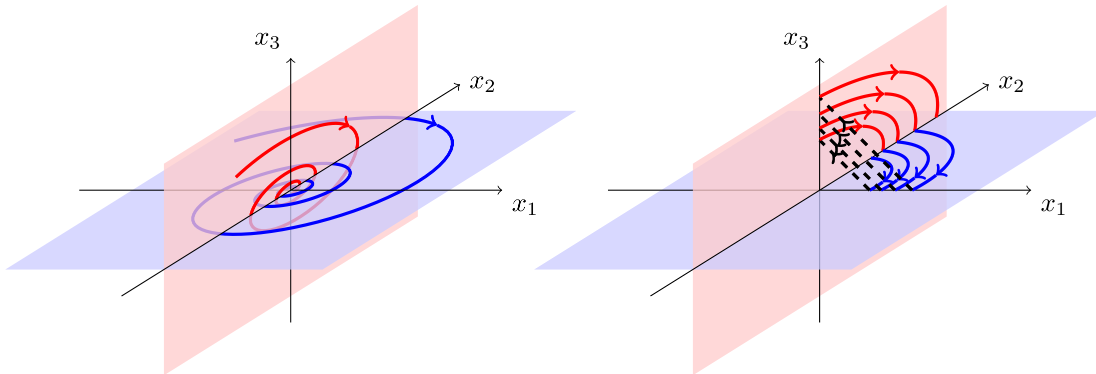

Fig. 3. Solutions for the Example 3. Left: Solutions of the individual subsystems, Right: Solutions of the switched system (dashed lines are jumps induced by the switches), the solutions grow unbounded.

then the switched DAE (1) is asymptotically stable for every switching signal.

Proof: Theorem 8 already shows that all (distributional) solutions of (1) are impulse free, hence it remains to show the convergence to zero.

Step 1: Definition of a common Lyapunov function candidate.
If $x \in \mathfrak{C}_p \cap \mathfrak{C}_q$ for some $p, q \in \{1, \ldots, N\}$ then $x = \Pi_p x = \Pi_q x$ hence (4) implies $V_p(x) = V_q(x)$, therefore

$$V : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \begin{cases} V_p(x), & x \in \mathfrak{C}_p, \\ 0, & \text{otherwise}, \end{cases}$$

is well defined.

Step 2: $V\big(x(t)\big) \to 0$ as $t \to \infty$.
For $p = \{1, \ldots, N\}$ let $P_p, Q_p \in \mathbb{C}^{n\times n}$ be the matrices as in Definition 3 corresponding to the DAE $E_p \dot{x} = A_p x$. Let furthermore

$$\lambda_p := \min_{x \in \mathfrak{C}_p \setminus \{0\}} \frac{x^T Q_p x}{V_p(x)} = \min_{\substack{x \in \mathfrak{C}_p \\ V_p(x)=1}} x^\top Q_p x > 0,$$

where positivity follows from positive definiteness of V_p and Q_p on $\mathfrak{C}_p$. Consider a solution $x : \mathbb{R} \to \mathbb{R}^n$ of (1), then from [5, Lemma 4.2.6] it follows that on each open interval (s, t) which does not contain a switching time of σ the function x is smooth and a local solution of $E_p \dot{x} = A_p x$, where $p = \sigma(\tau)$, $\tau \in (s, t)$. From $x(\tau) \in \mathfrak{C}_p$ for all $\tau \in (s, t)$ it follows that $V(x(\tau)) = V_p(x(\tau))$ for all $\tau \in (s, t)$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_p\big(x(\tau)\big) = x(\tau)^\top Q_p x(\tau) \le -\lambda_p V_p\big(x(\tau)\big).$$

Let $t \in \mathbb{R}$ be a jump of σ, then $x(t) = \Pi_{\sigma(t)} x(t-)$ and $x(t-) \in \mathfrak{C}_{\sigma(t-)}$, hence, by (4),

$$\begin{aligned} V(x(t)) = V_{\sigma(t)}\big(x(t)\big) &= V_{\sigma(t)}\big(\Pi_{\sigma(t)} x(t-)\big) \\ &\le V_{\sigma(t-)}\big(x(t-)\big) = V\big(x(t-)\big) \end{aligned}$$

For $\lambda := \min_p \lambda_p$ and any $t_0 \in \mathbb{R}$ it therefore follows

$$\forall t \in \mathbb{R} : \quad V\big(x(t)\big) \le e^{-\lambda(t-t_0)} V\big(x(t_0)\big),$$

which implies that $V\big(x(t)\big) \to 0$ for all solutions x of (1).

Step 3: Solutions tend to zero.
Seeking a contradiction, assume $x(t) \not\to 0$. Then there exists $\varepsilon > 0$ and a sequence $(s_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $s_i \to \infty$ as $i \to \infty$ such that $\|x(s_i)\| > \varepsilon$ for all $i \in \mathbb{N}$. There is at least one $p \in \{1, \ldots, N\}$ such that the set $\{\, i \in \mathbb{N} \mid \sigma(s_i) = p \,\}$ has infinitely many elements, therefore assume that $\sigma(s_i) = p$ for some p and all $i \in \mathbb{N}$. Then $x(s_i) \in \mathfrak{C}_p \setminus \{\, \xi \in \mathfrak{C}_p \mid \|\xi\| < \varepsilon \,\}$ for all $i \in \mathbb{N}$ and since V_p is positive definite on $\mathfrak{C}_p$ there exists $\delta > 0$ such that $V\big(x(s_i)\big) > \delta$ for all $i \in \mathbb{N}$. This is a contradiction to $V\big(x(t)\big) \to 0$ as $t \to \infty$. Therefore $x(t) \to 0$ as $t \to \infty$. ■

Condition (4) implies that any two Lyapunov functions V_p and V_q coincide on the intersection $\mathfrak{C}_p \cap \mathfrak{C}_q$, hence Theorem 9 is a generalization of the switched ODE case where the existence of a common Lyapunov function is sufficient to ensure stability under arbitrary switching. However, the existence of a common Lyapunov function is not enough in the DAE case, as becomes clear from Example 1 in Section III. Under arbitrary switching, solutions will in general exhibit jumps; these jumps are described by the consistency projectors, and these projectors must “fit together” with the Lyapunov functions in the sense of (4) to ensure stability of the switched DAE under arbitrary switching. If one assumes that the switching signal is chosen in such a way that no jumps occur, then the conditions on the consistency projectors are not needed and we get the following corollary.

Corollary 10: Consider the switched DAE (1) satisfying A1, A2, A3 and assume each DAE $E_p \dot{x} = A_p x$, $p = 1, \ldots, N$ is asymptotically stable with Lyapunov function V_p. Let

$$\Sigma_{x^0} := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \sigma \text{ fulfills A1 and} \\ \exists \text{ solution } x \text{ of (1)} \\ \text{with } x(0) = x^0 \text{ and} \\ x \text{ has no jumps} \end{array} \right\}.$$

If

$$\forall p, q = 1, \ldots, N \;\; \forall x \in \mathfrak{C}_p \cap \mathfrak{C}_q : \quad V_p(x) = V_q(x) \tag{5}$$

then all solutions x of (1) with $x(0) = x^0 \in \mathbb{R}^n$ and $\sigma \in \Sigma_{x^0}$ converge to zero as $t \to \infty$.

Example 3 from Section III fulfills the assumptions of Corollary 10, hence if no jumps occur all solutions tend to zero. In contrast to this, in Example 1 only the constant switching signals yield non-jumping non-trivial solutions, hence Corollary 10 is not very useful in this case. For Example 2 it is not possible to find Lyapunov functions for both subsystems such that condition (5) is fulfilled.

For switched ODEs it is well known that switching between stable subsystems always yields a stable system provided the so-called dwell time (i.e. the minimal time between any two switches) is large enough. Denote by Σ^{τ_d} the set of all switching times having a dell time not smaller than $\tau_d > 0$.

Theorem 11: Consider the switched DAE (1) satisfying A1, A2, A3 and assume that each DAE $E_p\dot{x} = A_p x$, $p = 1, \ldots, N$, is asymptotically stable with Lyapunov functions V_p and corresponding matrices $Q_p \in \mathbb{C}^{n\times n}$. Let

$$\lambda := \min_p \min_{x\in\mathfrak{C}_p\setminus\{0\}} \frac{x^T Q_p x}{V_p(x)}.$$

Let $\mu \geq 1$ be such that

$$\forall p,q = 1,\ldots,N\ \forall x \in \mathfrak{C}_q: \quad V_p(\Pi_p x) \leq \mu V_q(x). \tag{6}$$

Then the switched DAE (1) with $\sigma \in \Sigma^{\tau_d}$ is asymptotically stable whenever

$$\tau_d > \frac{\ln\mu}{\lambda}.$$

Proof: First note that all solutions of (1) by Theorem 8 are impulse free. Fix a solution $x \in \mathbb{R} \to \mathbb{R}^n$ of (1) with a fixed switching signal $\sigma \in \Sigma^{\tau_d}$. If σ has only finitely many switching times then asymptotic stability of (1) is obvious, therefore assume that the set of switching times $\{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}$ of σ is infinite. Let $v : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $t \mapsto V_{\sigma(t)}\big(x(t)\big)$ and $0 < \varepsilon := \tau_d - \frac{\ln\mu}{\lambda}$. Then as in the proof of Theorem 9 it follows that

$$v(t_{i+1}-) \leq e^{-\lambda(t_{i+1}-t_i)} v(t_i) \leq \frac{e^{-\lambda\varepsilon}}{\mu} v(t_i).$$

Furthermore, condition (6) yields

$$\begin{aligned} v(t_i) &= V_{\sigma(t_i)}\big(\Pi_{\sigma(t_i)} x(t_i-)\big) \\ &\leq \mu V_{\sigma(t_i-)}\big(x(t_i-)\big) = \mu v(t_i-). \end{aligned}$$

All together this yields for all $i \in \mathbb{Z}$,

$$v(t_{i+1}-) \leq e^{-\lambda\varepsilon} v(t_i-),$$

hence $v(t_i-) \to 0$ as $i \to \infty$. Since $v(t) \leq e^{-\lambda(t-t_i)} v(t_i) \leq \mu v(t_i-)$ for all $t \in [t_i, t_{i+1})$, $i \in \mathbb{Z}$, it also follows that $v(t) \to 0$ as $t \to \infty$. As in the proof of Theorem 9 it now follows that $x(t) \to 0$ as $t \to 0$. ■

Remark 12: Since each Lyapunov function V_q, $q \in \{1, \ldots, N\}$, is a quadratic function which is positive definite on $\mathfrak{C}_q$ if follows that for all $p, q \in \{1, \ldots, N\}$

$$\mu_{q,p} := \min_{x\in\mathfrak{C}_q\setminus\{0\}} \frac{V_p\big(\Pi_p x\big)}{V_q(x)} = \min_{x\in\mathfrak{C}_q : V_q(x)=1} V_p\big(\Pi_p x\big) > 0,$$

hence (6) is always fulfilled for $\mu \geq \max_{q,p} \mu_{q,p}$. Therefore, Theorem 11 states that switching between asymptotically stable subsystems yields asymptotic stability provided the dwell time of the switching signal is large enough.

For Example 1 from Section III condition (6) is fulfilled for the (common) Lyapunov function $x \mapsto V(x) = x^\top x$ with $\lambda = 2$ and $\mu = 2$. Hence for dwell times larger than $\frac{1}{2}\ln 2$ the switched system (1) is asymptotically stable, see also Figure 1.

V. Conclusions

This paper studied linear switched differential algebraic equations (DAEs). We constructed several examples showing that switching between stable subsystems may lead to instability, and illustrating a large variety of possible instability mechanisms caused by the presence of algebraic constraints. We presented two sufficient conditions for stability of switched DAEs based on the existence of suitable Lyapunov functions. The first result (Theorem 9) says that a common Lyapunov function guarantees stability under arbitrary switching when an additional condition involving consistency projectors holds; this extra condition is not needed when the switching signal is chosen in such a way that no jumps occur (Corollary 10). The second result (Theorem 11) shows that stability is preserved under switching with sufficiently large dwell time.

The work reported here is just an initial step in the investigation of stability of switched DAEs by Lyapunov-based methods. Some avenues for future research include: stability under average dwell-time and other useful classes of switching signals; converse Lyapunov theorems; and switched DAEs with inputs and/or outputs.

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