The Bang-Bang Funnel Controller (long version)

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***Abstract*—A bang-bang controller is proposed which is able to ensure reference signal tracking with prespecified time-varying error bounds (the funnel) for nonlinear systems with relative degree one or two. For the design of the controller only the knowledge of the relative degree is needed. The controller is guaranteed to work when certain feasibility assumptions are fulfilled, which are explicitly given in the main results. Linear systems with relative degree one or two are feasible if the system is minimum phase and the control values are large enough.**

I. Introduction

Consider the single-input, single-output nonlinear system

$$\begin{aligned} \dot{x} &= F(x,u), \quad x(0) = x^0 \in \mathbb{R}^n, \\ y &= H(x), \end{aligned} \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{N}$, is locally Lipschitz continuous and $H : \mathbb{R}^n \to \mathbb{R}$ is continuous. For a given reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we would like to achieve approximate reference signal tracking with a bang-bang feedback controller, i.e. $u(t) \in \{U_-, U_+\}$ for all $t \geq 0$ and some $U_- < U_+$. Furthermore, we are aiming for guaranteed transient behavior of the error signal $e := y - y_{\text{ref}}$ in the sense that the controller guarantees strict time-varying error bounds given by a so-called *funnel*

$$\mathcal{F} := \{\ (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \ |\ \varphi_-(t) \leq e \leq \varphi_+(t)\ \}, \tag{2}$$

where $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the prespecified (time-varying) error bound with $\varphi_-(t) < 0 < \varphi_+(t)$ for all $t \geq 0$, see also Figure 1.

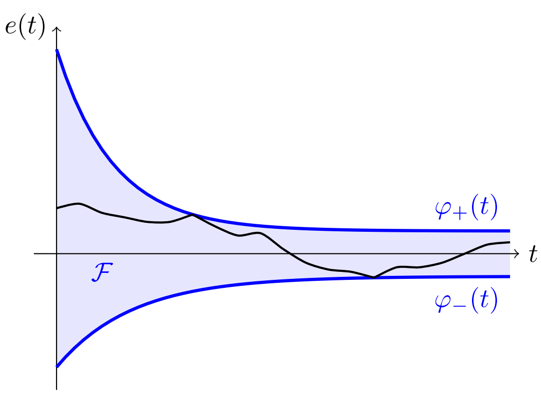

Fig. 1. The funnel $\mathcal{F}$: A time-varying error bound.

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The controller is governed by a *switching logic* whose output is a boolean variable $q : \mathbb{R}_{\geq 0} \to \{\texttt{true}, \texttt{false}\}$ which yields the control law

$$u(t) = \begin{cases} U_-, & \text{if } q(t) = \texttt{true}, \\ U_+, & \text{if } q(t) = \texttt{false}. \end{cases} \tag{3}$$

The overall feedback system is illustrated in Figure 2.

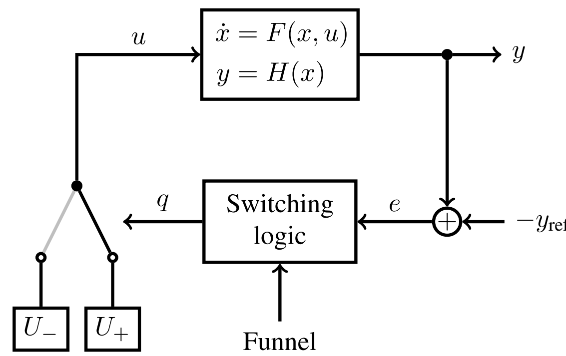

Fig. 2. Overall system structure.

The main purpose of this paper is twofold: On the one hand, we would like to find *feasibility assumptions* which take into account that, in general, an input signal with only two values cannot achieve arbitrary control objectives. On the other hand, we would like to find a *switching logic* which achieves the control objectives for feasible systems. We are aiming for a universal controller in the sense that the definition of the controller (given by the switching logic) does not involve the systems dynamics at all.

The feasibility assumption can be further distinguished into qualitative properties versus quantitative bounds of the systems dynamics. A qualitative property like the *relative degree* (see Definitions 2.1 and 3.1) yields different controller designs, while quantitative bounds do not influence the controller design but restrict the applicability of the controller. In this paper we will present controller designs for the relative degree one and two case.

We assume that the switching logic can also use derivatives of the error signal; in fact, for the relative degree two case the error e and its derivative $\dot{e}$ is used to obtain the switching signal. For the relative degree one case only the error itself is needed.

Tracking control with prespecified strict time-varying error bounds has been studied first in [1] where the *funnel controller* was introduced. Funnel control itself is based on ideas from high-gain control and λ-tracking (see for example the survey [2]). There have been several extensions of the funnel controller, e.g. more general gain functions [3] and higher relative degree via backstepping and filters [4], [5]. A similar approach with a switched controller was proposed earlier in [6]; however, there the freedom to choose the time-varying bounds was more restricted and the gain was monotone. These results yielded universal controllers which were able to control all systems with some qualitative property (for example, relative degree one with stable zero dynamics). However, the price of the generality is that the input must be allowed to become arbitrarily large, which is problematic in applications. The first result concerning the funnel controller with input saturation was presented in [7] which was based on a model of an exothermic reactor. We will later use this model for illustrating the behavior of our bang-bang controller. Until now input saturation was only considered for the relative degree one case [8], [9]; input saturation for relative degree two systems is work in progress [10]. The design of the bang-bang funnel controller is inspired by the above results on funnel control with input saturation and the feasibility assumptions are very similar when $U_\pm$ are the bounds from the input saturation.

The key advantages of the bang-bang funnel controller in comparison to classical controllers are, firstly, the same advantages the continuous funnel controller from [1] has: no knowledge of the systems parameters is necessary for the controller design and prespecified transient behavior can be guaranteed. Secondly, in comparison to the continuous funnel controller, the bang-bang funnel controller is much simpler because it only uses two control values and doesn't need a time varying gain function. Furthermore, the bang-bang funnel controller is a piecewise linear system and certain aspects are therefore easier to analyze. Finally, the switching logic is still well defined when the error leaves the funnel; therefore, we believe the bang-bang funnel controller is also more robust to time delays (this is ongoing research). However, in applications where a continuous controller signal is desired or where the input should not be saturated all the time, the bang-bang funnel controller is not applicable.

Using the bang-bang controller governed by a discrete switching logic together with a continuous system yields a *hybrid system* (see e.g. [11] and the references therein for an overview). The prespecified error bounds given by the funnel can be seen as a time-varying generalization of the invariance problem for hybrid systems as studied e.g. in [12]. In general, the coupling of continuous and discrete dynamics could lead to problems concerning the existence of solutions; however, by implementing the switching logic with some hysteresis effect this solvability problem can be avoided. For switched systems (average) dwell times are important, because in practical applications arbitrarily fast switching is often not possible and from a mathematical point of view dwell times also exclude so-called Zeno-behavior (i.e. infinitely many switches in finite time). The main results in this paper also give conditions when the switching times of the control input have an (average) dwell time.

The structure of the paper is as follows. The main results for the relative degree one case are given in Sections II and the relative degree two case is considered in Section III. In both cases, first the precise meaning of relative degree is defined. Afterwards, the switching logic is given and some general properties of the closed loop are highlighted. Afterwards, the two main results, Theorem 2.4 and Theorem 3.4, are given. For the relative degree one case we apply the bang-bang funnel controller to an exothermic reactor model. In the Appendix, we formulate Lemma A.1 which is a general statement ensuring well-posedness of the closed loop illustrated in Figure 2.

Throughout the paper we use $\|f\|$ for the supremum norm of the function $f : \mathbb{R} \to \mathbb{R}$; by $|x|$ for $x \in \mathbb{R}^n$ we denote the Euclidean vector norm and $|A|$ denotes the induced norm of the matrix $A \in \mathbb{R}^{m\times n}$. For defining predicates (i.e. for functions with values in the set $\{\texttt{true}, \texttt{false}\}$) we use the notation $[$*statement*$] \in \{\texttt{true}, \texttt{false}\}$. Solutions of differential equations are considered in the sense of Carathéodory, i.e. solutions are assumed to be absolutely continuous and fulfill the differential equation almost everywhere.

II. Relative degree one case

Definition 2.1 (Relative degree one): The system (1) is said to have *(global) relative degree one (with positive gain)* when there exist locally Lipschitz continuous functions $f : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$, $h : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, continuous $g : \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}_{>0}$ and a diffeomorphism $\Phi : \mathbb{R}^n \to \mathbb{R}\times\mathbb{R}^{n-1}$, $x \mapsto (y, z)$ which equivalently transforms (1) to

$$\dot y = f(y, z) + g(y, z)u, \qquad y(0) = y^0, \tag{4a}$$
$$\dot z = h(y, z), \qquad z(0) = z^0, \tag{4b}$$

where $(y^0, z^0) = \Phi(x^0)$.

For the relative degree one case we propose the following simple switching logic:

$$\begin{aligned} q(0-) &= [e(0) \geq 0],\\ q(t) &= \mathfrak{S}(e(t), \varphi_+(t), \varphi_-(t), q(t-)), \end{aligned} \tag{5}$$

where $\mathfrak{S} : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\{\texttt{true}, \texttt{false}\} \to \{\texttt{true}, \texttt{false}\}$ is the switching predicate given by

$$\mathfrak{S}(e, \overline{e}, \underline{e}, q_{\text{old}}) := [e \geq \overline{e} \ \vee \ (e > \underline{e} \ \wedge \ q_{\text{old}})]. \tag{6}$$

Since $\varphi_+(t) > \varphi_-(t)$ for all $t \geq 0$ the switching logic (5) together with (3) can also be described by a state diagram as shown in Figure 3.

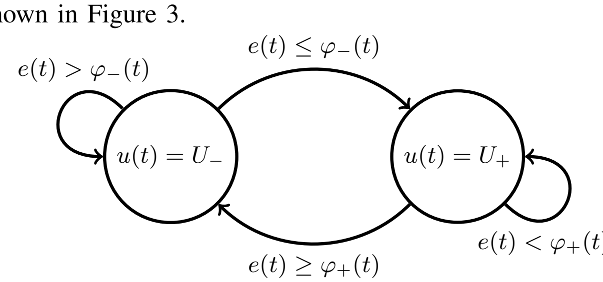

Fig. 3. The switching logic for the relative degree one case.

Note that for $\overline{e} > \underline{e}$, the following equivalence holds

$$\mathfrak{S}(e, \overline{e}, \underline{e}, q_{\text{old}}) \ \Leftrightarrow\ \neg\mathfrak{S}(-e, -\underline{e}, -\overline{e}, \neg q_{\text{old}}),$$

which explains the symmetry in Figure 3.

The following lemma is essential to prove existence of solutions of the closed loop.

Lemma 2.2 (Well-defined causal switching logic): For every continuous error function $e : [0, T) \to \mathbb{R}$, $0 < T \leq \infty$, there exists a unique causal right-continuous switching signal $q : [0, T) \to \{\texttt{true}, \texttt{false}\}$ fulfilling (5). Furthermore, if e is absolutely continuous with right-continuous bounded derivative $\dot{e}$ and

$$\inf_{t\geq 0} \varphi_+(t) + \inf_{t\geq 0} -\varphi_-(t) := \lambda_+ + \lambda_- > 0 \tag{7}$$

then the switching signal has a positive *dwell time* $\tau_d > 0$, i.e. two switches of q are at least τ_d apart. In fact

$$\tau_d \geq \frac{\lambda_+ + \lambda_-}{\|\dot{e}\|}.$$

Proof: The claimed properties all follow easily from the observation that $\varphi_+(t) > 0 > \varphi_-(t)$ for all $t \geq 0$ which implies that after a switch of q (say triggered by $e(t_1) \geq \varphi_+(t_1) > 0$) the continuous error function has to evolve for some time until the next switch is triggered (by $e(t_2) \leq \varphi_-(t_2) < 0$). ∎

Note that in Lemma 2.2 we did not assume that the error evolves within the funnel. A direct consequence of Lemma 2.2 and Lemma A.1 is the following result for the closed loop.

Corollary 2.3 (Closed loop well posed): Consider the system (1) in closed loop with the bang-bang controller given by (3) and (5), where $e := y - y_{\text{ref}}$ for some continuous reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then for every initial value $x^0 \in \mathbb{R}^n$ there exists a unique maximal solution $(x, q) : [0, \omega) \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\}$, $0 < \omega \leq \infty$ of the closed loop.

Proof: Under the above assumptions the error signal is continuous, hence Lemma 2.2 ensures existence and uniqueness of a right-continuous switching signal, therefore Lemma A.1 yields the assertion. ∎

Note that Corollary 2.3 neither claims nor assumes that the error signal evolves within the funnel, even finite escape time is not excluded at this point. To prove that the error evolves within the funnel, we need some additional feasibility assumptions which are formulated in the following theorem.

Theorem 2.4 (Relative degree one main result): Assume that (1) has relative degree one, i.e. (1) is equivalent to (4). Consider a funnel $\mathcal{F}$ as given by (2) and assume additionally that the funnel boundaries $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as well as the reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are absolutely continuous with right-continuous derivatives. Assume that the initial conditions for (4) fulfill

$$y^0 - y_{\text{ref}}(0) \in [\varphi_-(0), \varphi_+(0)], \quad z^0 \in Z_0 \subseteq \mathbb{R}^{n-1}$$

and assume that for every continuous $y : [0, \infty) \to \mathbb{R}$ with $\varphi_-(t) \leq y(t) - y_{\text{ref}}(t) \leq \varphi_+(t)$ for all $t \geq 0$ and all initial values $z^0 \in Z_0$ there exist a unique (global) solution $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-1}$ of the zero dynamics (4b); for $t > 0$ let

$$Z_t := \left\{ z(t) \;\middle|\; \begin{array}{l} z : [0,t] \to \mathbb{R}^{n-1} \text{ solves (4b) for some} \\ z^0 \in Z_0 \text{ and for some } y : [0,t] \to \mathbb{R} \\ \text{with } \varphi_-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi_+(\tau) \\ \forall \tau \in [0,t] \end{array} \right\}.$$

If the feasibility conditions

$$\begin{aligned} U_- &< \frac{\dot{\varphi}_+(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_+(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_+(t), z_t)} \\ U_+ &> \frac{\dot{\varphi}_-(t) + \dot{y}_{\text{ref}}(t) - f(y_{\text{ref}}(t) + \varphi_-(t), z_t)}{g(y_{\text{ref}}(t) + \varphi_-(t), z_t)} \end{aligned} \tag{8}$$

hold for all $t \geq 0$ and all $z_t \in Z_t$ then the closed loop composed of the system (1) or, equivalently, (4) and the bang-bang controller (3) governed by the switching logic (5) has the following properties:

1) There exists a unique (global) solution $(x, q) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\}$.
2) The error $e := y - y_{\text{ref}}$ evolves within the funnel, i.e. $(t, e(t)) \in \mathcal{F}$ for all $t \geq 0$.
3) If f and g are uniformly bounded on $\bigcup_{t\geq 0}[y_{\text{ref}}(t) + \varphi_-(t), y_{\text{ref}}(t) + \varphi_+(t)] \times Z_t$, $\dot{y}_{\text{ref}}$ is bounded and (7) holds then the jumping times of u or, equivalently, the switches of q have a positive dwell time $\tau_d > 0$.

Before proving Theorem 2.4 we give some remarks.

Remarks 2.5:
1) The feasibility conditions (8) can be simplified by using upper bounds for the funnel boundaries (and their derivatives), the zero dynamics, and the reference signal (and its derivative):

$$\begin{aligned} U_- &< -\frac{\|\dot{\varphi}_+\| + \|\dot{y}_{\text{ref}}\| + F_{\max}}{G_{\min}}, \\ U_+ &> \frac{\|\dot{\varphi}_-\| + \|\dot{y}_{\text{ref}}\| + F_{\max}}{G_{\min}}, \end{aligned} \tag{9}$$

where $F_{\max} := \max_{|y|\leq Y_{\max}, |z|\leq Z_{\max}} |f(y,z)|$, $G_{\min} := \min_{|y|\leq Y_{\max}, |z|\leq Z_{\max}} g(y,z) > 0$, $Y_{\max} := \|y_{\text{ref}}\| + \max\{\|\varphi_+\|, \|\varphi_-\|\}$ and $Z_{\max}$ is an upper bound for the zero dynamics, i.e. all solutions of (4b) with $|y(t)| \leq Y_{\max}$, for all $t \geq 0$, fulfill $z(t) \leq Z_{\max}$ for all $t \geq 0$ (in particular, the initial value z_0 must be bounded by $Z_{\max}$). A consequence of considering this more conservative feasibility assumption is that $U_- < 0$ and $U_+ > 0$ has to hold which is often too restrictive especially for nonlinear systems, see Example 2.6.
2) Consider a linear system with relative degree one in normal form [13] (see also [5, Lem. 3.5])

$$\begin{aligned} \dot{y} &= \alpha y + s^\top z + \gamma u \qquad & y(0) &= y^0, \\ \dot{z} &= py + Qz & z(0) &= z^0, \end{aligned}$$

where $\alpha \in \mathbb{R}$, $s, p \in \mathbb{R}^{n-1}$, $Q \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\gamma > 0$. Assume that the initial value for the zero dynamics is bounded say by $M > 0$. If the linear system is minimum phase, i.e. Q is Hurwitz with $|e^{Qt}| \leq Ce^{-\lambda t}$, $C, \lambda > 0$, then boundedness of y implies

$$\begin{aligned}|z(t)| &\leq Ce^{-\lambda t}|z_0| + \int_0^t Ce^{-\lambda(t-s)}|p||y(s)| \ \mathrm{d}s \\ &\leq CM + \tfrac{C}{\lambda}Y_{\max} =: Z_{\max}.\end{aligned}$$

Hence with $F_{\max} = |\alpha|Y_{\max} + |s^\top|Z_{\max}$ and $G_{\min} = \gamma$ the condition (9) is always fulfilled when $U_- < 0$ and $U_+ > 0$ are large enough.

3) The sets $Z_t \subseteq \mathbb{R}^{n-1}$, $t \geq 0$, are defined by considering $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as an input to the system governed by (4b). For the definition of Z_t it is not assumed that y solves the closed loop, it is merely assumed that y evolved within the funnel on the interval $[0, t]$. For the feasibility assumptions (8), it is not needed that the sets Z_t are uniformly bounded as long as f and $1/g$ do not get unbounded for unbounded $t \mapsto z_t \in Z_t$. In particular, it is therefore possible to apply the result also to time-varying systems by the common trick of including time as an additional differential equation $\dot{t} = 1$.
4) The bang-bang controller works also when the funnel boundaries are not bounded away from zero; however, then the length of the switching intervals will converge to zero. The corresponding behavior for the continuous funnel controller from [1] is that the gain $k(t)$ grows unbounded (however, all continuous funnel controller results are only formulated for the case that the funnel boundaries are bounded away from zero). In contrast to the continuous funnel controller, this undesired behavior can already be excluded by assuming (7) which allows that one of the two funnel boundaries approaches zero. In fact, (7) can be further weakened (cf. (16)) such that both funnel boundaries can approach zeros, as long as they don't do it simultaneously.

Proof of Theorem 2.4: Corollary 2.3 already shows existence and uniqueness of a maximal solution $(x, q) : [0, \omega) \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\}$. We first show that $(t, e(t)) \in \mathcal{F}$ for all $t \in [0, \omega)$. For this we show that the funnel $\mathcal{F}$ is positively invariant for e by showing that the following implications hold for all $t \in [0, \omega)$:

$$\begin{aligned} e(t) = \varphi_+(t) &\ \Rightarrow\ u(t) = U_-, \\ e(t) = \varphi_-(t) &\ \Rightarrow\ u(t) = U_+ \end{aligned}$$

and

$$\begin{aligned} e(t) = \varphi_+(t) &\ \Rightarrow\ \dot{e}(t) < \dot{\varphi}_+(t), \\ e(t) = \varphi_-(t) &\ \Rightarrow\ \dot{e}(t) > \dot{\varphi}_-(t). \end{aligned}$$

The first two implications follow directly from the switching logic (5). The last two implications follow from

$$\dot{e}(t) = f(y_{\text{ref}} + e(t), z(t)) + g(y_{\text{ref}} + e(t), z(t))u(t) - \dot{y}_{\text{ref}} \quad (10)$$

together with $u(t) = U_\pm$ and the corresponding feasibility assumption.

Since the error e evolves within the (bounded) funnel, finite escape time for y is not possible and hence, by the assumption on the zero dynamics, also z cannot escape in finite time. In particular y and z are bounded on $[0, \omega)$. Hence $\omega < \infty$ can only occur if the switching times accumulate for $t \to \omega$.

Seeking a contradiction assume $\omega < \infty$. Then there exist increasing sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ with $s_n < t_n < s_{n+1}$ for all $n \in \mathbb{N}$ and $s_n \to \omega$ (hence also $t_n \to \omega$) such that $q_{[s_n, t_n)} = \texttt{true}$ and $q_{[t_n, s_{n+1})} = \texttt{false}$. By the definition of the switching logic it follows that $e(s_n) = \varphi_+(s_n)$ and $e(t_n) = \varphi_-(t_n)$. By compactness of $[0, \omega]$ and continuity of $\varphi_\pm$ it follows that $\lambda := \min_{t\in[0,\omega]} \varphi_+(t) - \max_{t\in[0,\omega]} \varphi_-(t) > 0$, hence $e(s_n) - e(t_n) > \lambda$ for all $n \in \mathbb{N}$. Invoking the Mean Value Theorem, choose a sequence $(\tau_n)_{n\in\mathbb{N}}$ in $[0, \omega)$ with $\dot{e}(\tau_n) = \frac{e(t_n) - e(s_n)}{t_n - s_n} < -\frac{\lambda}{t_n - s_n} \to -\infty$ as $n \to \infty$. This unboundedness of $\dot{e}$ contradicts the observation that (10) for bounded y, z, u and y_{ref} yields a bounded $\dot{e}$ on $[0, \omega)$. Hence $\omega = \infty$ is shown.

Finally, the boundedness assumption for f, g and $\dot{y}_{\text{ref}}$ together with (10) implies that $\dot{e}$ is (globally) bounded. Hence Lemma 2.2 shows the last assertion of the theorem. ∎

Example 2.6: We consider a model of exothermic chemical reactions which was used in [7] to study the funnel control with input saturation. In the notation of the present paper the model with one reactant and one product reads as

$$\begin{aligned} \dot{y} &= br(z_1, z_2, y) - qy + u, & y(0) &= y^0 > 0, \\ \dot{z}_1 &= c_1 r(z_1, z_2, y) + d(z_1^{\text{in}} - z_1), & z_1(0) &= z_1^0 \geq 0, \\ \dot{z}_2 &= c_2 r(z_1, z_2, y) + d(z_2^{\text{in}} - z_2), & z_2(0) &= z_2^0 \geq 0, \end{aligned}$$

where $b \geq 0$, $q > 0$, $c_1 < 0$, $c_2 \in \mathbb{R}$, $d > 0$, $z_{1/2}^{\text{in}} \geq 0$ and $r : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{> 0} \to \mathbb{R}_{\geq 0}$ is assumed to be locally Lipschitz continuous with $r(0, T) = 0$ for all $T > 0$. The reference signal is $y_{\text{ref}}(t) = y^* > 0$ for all $t \geq 0$. In [7] the input is saturated to some interval $[U_-, U_+]$ with $U_- < U_+$, i.e. $u(t) \in [U_-, U_+]$ for all $t \geq 0$, and the feasibility assumption in [7] is that there exists $\gamma \in \mathbb{R}^2_{>0}$ with $(c_1, c_2)\gamma \leq 0$ and

$$\begin{aligned} &\exists \rho_-, \rho_+ > 0\ \exists \overline{y} > y^*\ \forall y \in [y^*, \overline{y}]\ \forall z_1, z_2 \in Z_0 : \\ &0 < U_- + \rho_- < qy - br(z_1, z_2, y) < U_+ - \rho_+. \end{aligned}$$

where $Z_0 := \left\{ (z_1, z_2) \in \mathbb{R}^2_{\geq 0} \;\middle|\; (z_1, z_2)\gamma < (z_1^{\text{in}}, z_2^{\text{in}})\gamma \right\}$. It can be shown that Z_0 is positively invariant for every $y : \mathbb{R} \to \mathbb{R}_{>0}$. Hence, in the notation of Theorem 2.4, $Z_t \subseteq Z_0$ for all $t \geq 0$ if $(z_1^0, z_2^0) \in Z_0$. Now [7, Rem. 2] shows that for every funnel $\mathcal{F}$ whose funnel boundaries $\varphi_\pm$ fulfill

$$\begin{aligned} &\varphi_+(t) \in (0, \overline{y} - y^*], & &\varphi_-(t) \in (-y^*, 0), \\ &\dot{\varphi}_+(t) > -\rho_-, & &\dot{\varphi}_-(t) < \rho_+, \end{aligned}$$

the feasibility assumption (8) holds. A simulation of the bang-bang controller applied to the above model with parameters as given in [7, Sec. 3.3] is shown in Figure 4.

III. Relative degree two case

Definition 3.1 (Relative degree two): The system (1) is said to have *(global) relative degree two (with positive gain)* when there exist locally Lipschitz continuous functions $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{R}$, $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$, continuous $g : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} \to \mathbb{R}_{>0}$ and a diffeomorphism

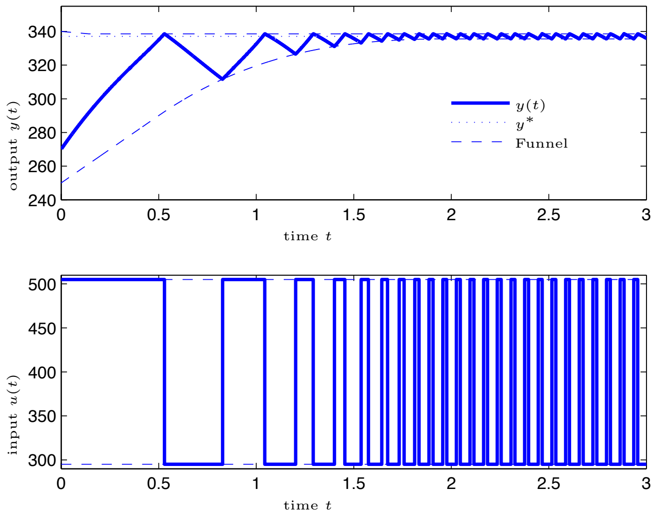

Fig. 4. The bang-bang funnel controller applied to an exothermic reactor model.

$\Phi : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$, $x \mapsto (y, \dot{y}, z)$ which equivalently transforms (1) to

$$\ddot{y} = f(y, \dot{y}, z) + g(y, \dot{y}, z)u \quad y(0) = y^0,\ \dot{y}(0) = \dot{y}^0, \tag{11a}$$
$$\dot{z} = h(y, \dot{y}, z) \quad z(0) = z^0, \tag{11b}$$

where $(y^0, \dot{y}^0, z^0) = \Phi(x^0)$.

The switching logic for the relative degree two case requires a second funnel for $\dot{e}$ given by

$$\mathcal{F}^d := \left\{ (t, \dot{e}) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \varphi_-^d(t) \leq \dot{e} \leq \varphi_+^d(t) \right\}, \tag{12}$$

where $\varphi_-^d(t) < 0 < \varphi_+^d(t)$ for all $t \geq 0$. The idea to use a derivative funnel originates from the recent work [10]. This funnel might reflect real physical bounds for $\dot{e}$ or might be used as a design parameter for the controller. Anyway, the derivative funnel $\mathcal{F}_d$ cannot restrict $\dot{e}$ in such a way that e cannot decrease or increase fast enough to follow the boundaries of the original funnel $\mathcal{F}$; in fact it must hold that

$$\forall t \geq 0: \quad \varphi_+^d(t) > \dot{\varphi}_-(t) \quad \text{and} \quad \varphi_-^d(t) < \dot{\varphi}_+(t), \tag{13}$$

where we assumed that the funnel boundary functions $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are absolutely continuous with right-continuous derivatives. In addition to the derivative funnel a "safety distance" $\varepsilon_\pm > 0$ from the corresponding funnel boundary $\varphi_\pm$ is needed to prevent the error e from leaving the funnel $\mathcal{F}$. This distance will play an essential role in the feasibility assumptions later; at this point we already make the following assumption:

$$\forall t \geq 0: \quad \varphi_+(t) - \varepsilon_+ > 0 \quad \text{and} \quad \varphi_-(t) + \varepsilon_- < 0. \tag{14}$$

The switching logic is now given by $q_0(0-) = [e(0) \geq 0]$, $q(0-) = q_0(0-)$ and

$$q_0(t) = \mathfrak{S}\big(e(t), \varphi_+(t) - \varepsilon_+, \varphi_-(t) + \varepsilon_-, q_0(t-)\big)$$
$$q(t) = \begin{cases} \mathfrak{S}\big(\dot{e}(t), \min\{\dot{\varphi}_+(t), 0\}, \varphi_-^d(t), q(t-)\big), & \text{if } q_0(t), \\ \mathfrak{S}\big(\dot{e}(t), \varphi_+^d(t), \max\{\dot{\varphi}_-(t), 0\}, q(t-)\big), & \text{else,} \end{cases} \tag{15}$$

where $\mathfrak{S}$ is the switching predicate as given in (6). The switching logic (applied to the control law (3)) is illustrated as a simplified state diagram in Figure 5.

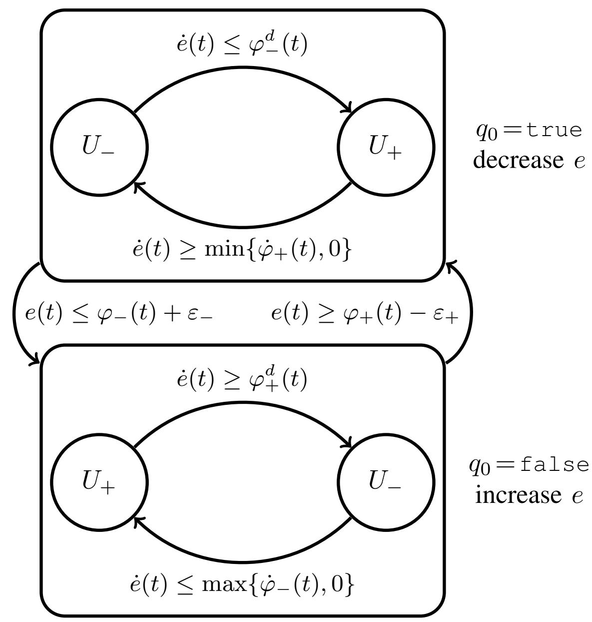

Fig. 5. The switching logic for the relative degree two case.

The reasoning behind the switching logic (15) is as follows (see also the schematic illustration in Figure 6): Whenever the error gets close to the upper funnel boundary, i.e. $e(t) \geq \varphi_+(t) - \varepsilon_+$, we would like to decrease $e(t)$, a task which is encoded by $q_0(t) = \texttt{true}$. To do this we have to decrease $\dot{e}$ which under certain feasibility assumptions is possible by applying $u(t) = U_-$. It will take some time until $\dot{e}$ is small enough, which is the case when $\dot{e}(t) \leq \dot{\varphi}_+(t)$ because then the distance to the upper funnel boundary starts increasing. At this point we could keep $u(t) = U_-$ until the error gets close to the lower funnel boundary. However, this would unnecessarily decrease $\dot{e}(t)$ further, which implies that when we want to increase the error later (when we got close the lower funnel boundary) it will take longer until $\dot{e}(t)$ is big enough so that the distance of $e(t)$ to the lower funnel boundary is increasing. That is why we stop decreasing $\dot{e}$ by setting $u(t) = U_+$ when the lower derivative funnel boundary is hit, i.e. when $\dot{e}(t) \leq \varphi_-^d(t)$. If we still want to decrease the error, we have to stop increasing the derivative of the error when $\dot{e}(t) \geq \dot{\varphi}_+(t)$ or $\dot{e}(t) \geq 0$.

A similar result as Lemma 2.2 holds also for the relative degree two case.

Lemma 3.2 (Well-defined causal switching logic): For every continuously differentiable error function $e : [0, T) \to \mathbb{R}$, $0 < T \leq \infty$ there exists a unique causal right-continuous switching signal $q : [0, T) \to \{\texttt{true}, \texttt{false}\}$ fulfilling (15). If additionally $\dot{e}$ is bounded and absolutely continuous with

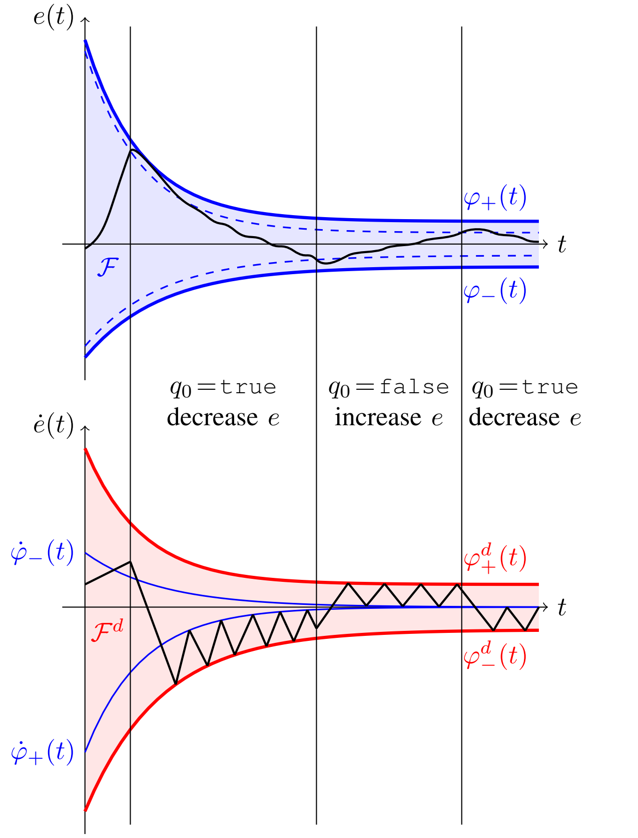

Fig. 6. A schematic illustration how the error and its derivative evolve under the switching logic (15), the vertical lines indicate the switching of the predicate q_0 which indicates wether we would like to decrease or increase the error.

right-continuous bounded derivative $\ddot{e}$ and, for $\delta := \|\ddot{e}\|$,

$$\begin{aligned} &0 < \lambda := \inf_{t\geq 0} \varphi_+(t) - \varepsilon_+ - \sup_{t\geq 0} \varphi_-(t) - \varepsilon_-, \\ &0 < \tau_+(\delta) := \\ &\quad \inf_{t\geq 0} \inf \left\{ |\tau| \;\middle|\; \varphi_+^d(t) - \max\{\dot{\varphi}_-(t+\tau), 0\} \leq \tau\delta \right\} \\ &0 < \tau_-(\delta) := \\ &\quad \inf_{t\geq 0} \inf \left\{ |\tau| \;\middle|\; \min\{\dot{\varphi}_+(t), 0\} - \varphi_-^d(t+\tau) \leq \tau\delta \right\} \end{aligned} \tag{16}$$

then the switching signal has an average dwell time [14]

$$\tau_a \geq \frac{1}{\frac{\|\dot{e}\|}{\lambda} + \min\{\tau_+(\|\ddot{e}\|), \tau_-(\|\ddot{e}\|)\}}$$

with chattering bound two, i.e. the number of switches $N(t,T)$ in every time interval $[t,T)$ is bounded by $2 + \frac{T-t}{\tau_a}$.

Proof: Identically to Lemma 2.2 it follows that the map $e \mapsto q_0$ is well defined and q_0 is right-continuous, i.e. there exists a family of disjoint intervals $[s_n, s_{n+1})$, $n \in \mathbb{N}$, whose union is the whole interval $[0,T)$. Similar ideas as in Lemma 2.2 applied to each interval $[s_n, s_{n+1})$ yields that also $\dot{e} \mapsto q$ is well defined and causal for a given initial value $q(s_n-)$. Hence the overall mapping $e \mapsto q$ is well defined and causal and q is right-continuous.

With the given properties it follows easily that q_0 has a dwell time $\tau_0 \geq \lambda/\|\dot{e}\|$ and on each interval where q_0 is constant, the dwell time of q is at least either $\tau_+ := \tau_+(\|\ddot{e}\|)$ or $\tau_- := \tau_-(\|\ddot{e}\|)$. Hence in any interval $[t,T)$ there can be at most $2 + (T-t)/\tau_0 + (T-t)/\min\{\tau_+, \tau_-\}$ switches in q which yields that the switching signal q has an average dwell time of at least $1/(1/\tau_0 + 1/\min\{\tau_+, \tau_-\})$. ∎

Corollary 3.3 (Closed loop well posed): Consider system (1) with relative degree two in closed loop with the bang-bang controller given by (15) and (3) where $e := y - y_{\text{ref}}$ for some continuously differentiable reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Then for every initial value $x^0 \in \mathbb{R}^n$ there exists a unique maximal solution $(x, q_0, q) : [0,\omega) \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\} \times \{\texttt{true}, \texttt{false}\}$, $0 < \omega \leq \infty$.

Proof: The relative degree two assumption ensures that the output y is continuously differentiable (for any locally integrable input u), hence e is also continuously differentiable and Lemma 3.2 together with Lemma A.1 ensure existence and uniqueness of solutions in the closed loop. ∎

As in the relative degree one case, Corollary 3.3 does not assume or claim that the error evolves within the funnel. For this some additional feasibility assumptions are needed.

Theorem 3.4 (Relative degree two case main result): Assume that (1) has relative degree two, i.e. (1) is equivalent to (11). Consider a funnel $\mathcal{F}$ as given by (2) whose differentiable boundary functions $\varphi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$ have absolutely continuous derivatives with right-continuous second derivatives and fulfill (14) for some $\varepsilon_\pm > 0$. Choose a derivative funnel $\mathcal{F}^d$ as in (12) whose funnel boundaries $\varphi_\pm^d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are absolutely continuous with right-continuous derivative and fulfill assumption (13). Let $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a differentiable reference signal whose derivative is absolutely continuous with right-continuous second derivative. Assume that the initial conditions for (11) fulfill

$$\begin{aligned} &y^0 - y_{\text{ref}}(0) \in [\varphi_-(0) + \varepsilon_-, \varphi_+(0) - \varepsilon_+], \\ &\dot{y}^0 - \dot{y}_{\text{ref}}(0) \in [\varphi_-^d(0), \varphi_+^d(0)], \quad z^0 \in Z_0 \subseteq \mathbb{R}^{n-2} \end{aligned}$$

and assume that for every differentiable $y : [0,\infty) \to \mathbb{R}$ with $\varphi_-(t) \leq y(t) - y_{\text{ref}}(t) \leq \varphi_+(t)$, $\varphi_-^d(t) \leq \dot{y}(t) - \dot{y}_{\text{ref}}(t) \leq \varphi_+^d(t)$ and for every initial value $z^0 \in Z_0$ there exists a unique (global) solution $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-2}$ of the zero dynamics (11b); for $t > 0$ let

$$Z_t := \left\{ z(t) \;\middle|\; \begin{array}{l} z : [0,t] \to \mathbb{R}^{n-1} \text{ solves (11b) for some} \\ z^0 \in Z_0 \text{ and for some } y : [0,t] \to \mathbb{R} \text{ with} \\ \varphi_-(\tau) \leq y(\tau) - y_{\text{ref}}(\tau) \leq \varphi_+(\tau) \text{ and} \\ \varphi_-^d(\tau) \leq \dot{y}(\tau) - \dot{y}_{\text{ref}}(\tau) \leq \varphi_+^d(\tau) \\ \forall \tau \in [0,t] \end{array} \right\}.$$

If there exist $\delta_\pm > 0$ with

$$\begin{aligned} \delta_+ &> \max\{\dot{\varphi}_-^d(t), \ddot{\varphi}_-(t), 0\} \quad \text{and} \\ -\delta_- &< \min\{\dot{\varphi}_+^d(t), \ddot{\varphi}_+(t), 0\} \quad \forall t \geq 0 \end{aligned}$$

such that the first set of feasibility conditions

$$
\begin{aligned}
U_- &< \frac{-\delta_- + \ddot{y}_{\text{ref}}(t) + f(y_t, \dot{y}_t, z_t)}{g(y_t, \dot{y}_t, z_t)}, \\
U_+ &> \frac{\delta_+ + \ddot{y}_{\text{ref}}(t) + f(y_t, \dot{y}_t, z_t)}{g(y_t, \dot{y}_t, z_t)},
\end{aligned} \tag{17}
$$

hold for all $t \geq 0$ and all $(y_t, \dot{y}_t, z_t) \in [y_{\text{ref}}(t)+\varphi_-(t), y_{\text{ref}}(t)+\varphi_+(t)] \times [\dot{y}_{\text{ref}}(t)+\varphi_-^d(t), \dot{y}_{\text{ref}}(t)+\varphi_+^d(t)] \times Z_t$, and if the second set of feasibility conditions

$$
\begin{aligned}
\varepsilon_+ &\geq \frac{(\|\varphi_-^d\| + \|\min\{\dot{\varphi}_+, 0\}\|)^2}{2\delta_-} \\
\varepsilon_- &\geq \frac{(\|\varphi_+^d\| + \|\max\{\dot{\varphi}_-, 0\}\|)^2}{2\delta_+}
\end{aligned} \tag{18}
$$

hold then the bang-bang controller (3) governed by the switching logic (15) applied to (1) or, equivalently, (11) achieves the control objectives, i.e. the closed loop has the following properties:

1) There exists a (global) unique solution $(x, q_0, q) : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\} \times \{\texttt{true}, \texttt{false}\}$.
2) The error $e := y - y_{\text{ref}}$ evolves within the funnel $\mathcal{F}$ and the derivative of the error $\dot{e}$ evolves within the derivative funnel $\mathcal{F}^d$, i.e. $(t, e(t)) \in \mathcal{F}$ and $(t, \dot{e}(t)) \in \mathcal{F}^d$ for all $t \geq 0$.
3) If f and g are uniformly bounded on $\bigcup_{t\geq 0}[y_{\text{ref}}(t) - \varphi_i(t), y_{\text{ref}}(t) + \varphi_+] \times [\dot{y}_{\text{ref}} + \varphi_-^d, \dot{y}_{\text{ref}} + \varphi_+^d] \times Z_t$, $\ddot{y}_{\text{ref}}$ is bounded, $\varphi_\pm^d$ are bounded and (16) holds for all $\delta > 0$ then the switching signal q has a positive average dwell time $\tau_a > 0$.

Proof: Existence and uniqueness of a maximal solution $(x, q_0, q) : [0, \omega) \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\}^2$ for $0 < \omega \leq \infty$ follows from Corollary 3.3.

If $e(t)$ leaves the funnel $\mathcal{F}$ then let $\omega_1 > 0$ be the first time the error crosses the funnel boundary, otherwise let $\omega_1 = \omega$.

Step 1: We show that $\dot{e}$ evolves within $\mathcal{F}^d$ on $[0, \omega_1)$.

The switching logic ensures, for all $t \in [0, \omega_1)$,

$$
\begin{aligned}
\dot{e}(t) = \varphi_+^d(t) \;&\Rightarrow\; u(t) = U_- \quad \text{and} \\
\dot{e}(t) = \varphi_-^d(t) \;&\Rightarrow\; u(t) = U_+
\end{aligned}
$$

and the first feasibility assumption (17) together with

$$
\begin{aligned}
\ddot{e}(t) = f(&y_{\text{ref}}(t) + e(t), \dot{y}_{\text{ref}}(t) + \dot{e}(t), z(t)) \\
&+ g(y_{\text{ref}}(t) + e(t), \dot{y}_{\text{ref}}(t) + \dot{e}(t), z(t))u(t) - \ddot{y}_{\text{ref}}(t)
\end{aligned} \tag{19}
$$

yields

$$
\begin{aligned}
\dot{e}(t) = \varphi_+^d(t) \;&\Rightarrow\; \ddot{e}(t) < -\delta_- < \dot{\varphi}_+^d(t) \quad \text{and} \\
\dot{e}(t) = \varphi_-^d(t) \;&\Rightarrow\; \ddot{e}(t) > \delta_+ > \dot{\varphi}_-^d(t)
\end{aligned}
$$

for all $t \in [0, \omega_1)$, hence the derivative funnel $\mathcal{F}^d$ is positively invariant for $\dot{e}$ on the interval $[0, \omega_1)$.

Step 2: We show that $\omega_1 = \omega$.

Let $t_0 \in [0, \omega_1)$ be such that $e(t_0) = \varphi_+(t_0) - \varepsilon_+$. The switching logic ensures $q_0(t) = \texttt{true}$ for all $t \in [t_0, t_1)$ where $t_1 > t_0$ is the smallest time when $e(t_1) = \varphi_-(t_1) + \varepsilon_-$ or $t_1 = \omega_1$. Choose a maximal $s_0 \in [t_0, t_1]$ such that $q(t) = \texttt{true}$ for all $t \in [t_0, s_0)$, i.e. $u(t) = U_-$ for all $t \in [t_0, s_0)$. From the first feasibility assumption (17) and (19) it follows that $\ddot{e}(t) < -\delta_-$. Hence, for all $t \in [t_0, s_0)$,

$$
\begin{aligned}
e(t) \;&<\; e(t_0) + \dot{e}(t_0)(t - t_0) - \tfrac{1}{2}\delta_-(t - t_0)^2 \\
&\leq\; \varphi_+(t_0) - \varepsilon_+ + \|\varphi_+^d\|(t - t_0) - \tfrac{1}{2}\delta_-(t - t_0)^2 \\
&=\; \varphi_+(t_0) - \|\min\{\dot{\varphi}_+, 0\}\|(t - t_0) - \varepsilon_+ \\
&\quad\; + (\|\varphi_+^d\| + \|\min\{\dot{\varphi}_+, 0\}\|)(t - t_0) - \tfrac{1}{2}\delta_-(t - t_0)^2 \\
&\overset{(18)}{\leq}\; \varphi_+(t_0) - \|\min\{\dot{\varphi}_+, 0\}\|(t - t_0) \\
&\leq\; \varphi_+(t).
\end{aligned}
$$

As long as $q_0(t) = \texttt{true}$ the switching logic ensures that the set $\left\{ (t, \dot{e}) \;\middle|\; \varphi_-^d(t) \leq \dot{e} \leq \min\{\dot{\varphi}_+(t), 0\} \right\}$ is positively invariant and $\dot{e}(s_0) = \varphi_-^d(s_0)$ if $s_0 < t_1$. Therefore, $\dot{e}(t) \leq \dot{\varphi}_+(t)$ for all $t \in [s_0, t_1)$. Altogether this yields $e(t) < \varphi_+(t)$ for all $t \in [t_0, t_1)$.

For $t_1 \in [0, \omega_1)$ with $e(t_1) = \varphi_-(t_1) + \varepsilon_-$ an analogous argument shows $e(t) > \varphi_-(t)$ for all $t \in [t_1, t_2)$ where $t_2 > t_1$ is the smallest time when $e(t_2) = \varphi_+(t_2) - \varepsilon_+$ or $t_2 = \omega_1$. Hence an inductive argument yields that the error cannot leave the funnel and $\omega_1 = \omega$.

Step 3: We show $\omega = \infty$.

Since e and $\dot{e}$ evolve within the funnel, finite escape time for y and $\dot{y}$ is not possible. By the property of the zero dynamics this also precludes finite escape time for z. In particular $y, \dot{y}, z$ are bounded on $[0, \omega)$, therefore $\omega < \infty$ is only possible when the switching times accumulate for $t \to \omega$. A similar idea as in the proof of Theorem 2.4 yields that this accumulation contradicts boundedness of $\dot{e}$ and $\ddot{e}$ on the compact interval $[0, \omega]$, hence $\omega = \infty$.

Step 4: The average dwell time condition is shown.

The boundedness assumption on f, g and $\ddot{y}_{\text{ref}}$ together with (19) ensures that $\ddot{e}$ is bounded. Since $\dot{e}$ evolves within the bounded funnel $\mathcal{F}^d$ it is also bounded, hence Lemma 2.2 yields the average-dwell time property. ■

Remarks 3.5: 1) The two main results, Theorem 2.4 and Theorem 3.4, do not depend on the initialization $q(0-)$ and $q_0(0-)$ for the switching logic. However, the choice in (5) and (15) intuitively improves performance, because the control action is in the "right" direction just from the start and not only after the first boundary is hit.

2) The second feasibility assumption (18) might be in contradiction with the assumption (14). However, increasing/decreasing $U_\pm$ (without changing anything else) allows for bigger $\delta_\pm$ so that (18) yields arbitrarily small lower bounds for $\varepsilon_\pm$ and (14) is not in contradiction with (18) anymore.

3) As for the relative degree one case it is possible to simplify the feasibility assumption (17) by considering upper bounds for the funnel boundaries (and their derivatives), the zero dynamics, and the reference signal (and its derivatives). In particular, for minimum phase linear systems with relative degree two it follows then that (17) holds whenever $U_- < 0$ and $U_+ > 0$ are large enough.

4) The feasibility assumptions could possibly be made less conservative by introducing time-varying safety distances $\varepsilon_\pm(t)$. Typically the funnels are large with large derivatives at the beginning, hence require larger safety distances by (18), and on the other hand tighter funnels with small derivatives later in time restrict the size of the safety distance by (14) although, at least locally, (18) does not require big safety distances anymore.
5) The first feasibility assumption (17) looks very similar to the feasibility assumption in Theorem 2.4 applied to $\dot{e}$ and $\mathcal{F}^d$. The two main differences are that, firstly, $\dot{\varphi}^d_\pm$ are replaced by uniform lower/upper bounds $\delta_\mp$ and, secondly, (17) has to hold on the whole funnel region and not only on the boundary. The reason for both is that we need a certain minimum decrease/increase of $\dot{e}$ in the whole funnel (and not only on the boundary) to ensure that we can quantify the overshoot of e (in fact, condition (18) is this quantification).
6) The switching logic for the relative degree two case is hierarchically composed, where the outer switching logic is identical (apart from the safety distance) to the switching logic of the relative degree one case. The authors were already able to define a switching logic for the relative degree three case based on a hierarchical composition similar to the one presented here, but due to space limitations this result is not included here. In fact, it seems much more interesting to come up with a general solution for an arbitrary relative degree; this is ongoing research.

IV. Conclusions

A universal controller was proposed which only uses two input values and is governed by a simple switching logic. This switching logic depends on the relative degree of the system, otherwise no knowledge of the system is necessary to design the controller. Feasibility assumptions are given which ensure that approximate reference signal tracking with strict time-varying error bounds is achieved. We assumed that the gain function g in the relative degree normal forms is positive; however, it should be possible to extend the results to an unknown (but definite) sign of the gain function by slightly changing the switching logic to first detect the sign of the gain function.

The nature of the controller seems to make it more "robust" than the continuous funnel controller because, in contrast to the latter, the bang-bang funnel controller is still well defined when the error leaves the funnel, for example when a time delay is present. A precise robustness result is a future research topic.

The switching logic for the relative degree two case already hints to switching logics for higher relative degrees; this is a topic of ongoing research.

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Appendix

The closed loop as illustrated in Figure 2 leads to a switched system with state dependent switching and it is well known that local existence of (Carathéodory) solutions is not guaranteed in general. Consider for example the following closed loop

$$\begin{aligned}\dot{y}(t) &= u(t), \quad y(0) = y^0 \in \mathbb{R}, \\ q(t) &= [e(t) \geq 0], \ \forall t \geq 0, \quad \text{and (3)}\end{aligned}$$

with $U_- = -1$, $U_+ = 1$ and $y_{\text{ref}} \equiv 0$. It is easy to see that this closed loop does not have a differentiable solution for the initial value $y(0) = 0$. Hence not all switching logics are suitable. It turns out that the underlying problem of this example is that for a given continuous function e the switching signal is in general not *right-continuous*. The next result shows that right-continuity of the switching signal is sufficient for existence of local solutions of the closed loop.

Lemma A.1 (Well posedness of closed loop): Consider system (1) with the controller (3) governed by some switching law q which is generated by some causal switching logic $\mathfrak{L} : y \mapsto q$ (here we include the reference signal y_{ref} into the switching logic). Let $\mathcal{Y} \subseteq \{ \ y : [0,\omega) \to \mathbb{R} \ | \ 0 < \omega \leq \infty \ \}$ be a function space which contains all possible outputs of (1) for arbitrary locally integrable inputs (we do not exclude finite escape time at this point). If for every $y \in \mathcal{Y}$ the resulting switching signal q is *right-continuous* then the closed loop as illustrated in Figure 2 is well posed, i.e. for every initial value $x^0 \in \mathbb{R}^n$ there exists a maximally extended solution $(x,q) : [0,\omega) \to \mathbb{R}^n \times \{\texttt{true}, \texttt{false}\}$, $0 < \omega \leq \infty$.

Proof: The initial value of (1) yields the value $y(0)$ and causality of the switching logic yields a unique value for $q(0)$. Let $x : [0,\omega)$ be the unique local solution of the differential equation (1) with the constant input $u(t) = u(0)$ for all $t \geq 0$. This results in an (open loop) output $y : [0,\omega) \to \mathbb{R}$ contained in $\mathcal{Y}$. The corresponding (open loop) switching signal q is right-continuous and there exists a maximal $\omega_1 \in (0,\omega]$ such that $q(t) = q(0)$ for all $t \in [0,\omega_1)$. Hence (x,q) is also a solution of the closed loop on $[0,\omega_1)$. If $\omega_1 = \omega$ then the solution is maximal and cannot be extended and we are done. Therefore assume $\omega_1 < \omega$. We can now inductively repeat the argument with the new initial value $x(\omega_1)$ and $q(\omega_1)$. ■

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