Relaxed conditions for norm-controllability of nonlinear systems

Matthias A. Müller, Daniel Liberzon, and Frank Allgöwer

Abstract— **In this paper, we further study the recently introduced notion of norm-controllability, which captures the responsiveness of a nonlinear system with respect to applied inputs in terms of the norm of an output map. We give sufficient conditions for this property based on higher-order lower directional derivatives, which generalize the results obtained in our earlier work and help to establish norm-controllability for systems with outputs having relative degree greater than one. Furthermore, we illustrate the obtained results by means of a chemical reaction example.**

I. Introduction

When considering dynamical systems with inputs and outputs, a key question is how the applied inputs affect the behavior of the states and outputs of the system. In this respect, various properties are of interest. For example, one fundamental system property is controllability, which is usually formulated as the ability to reach any state from any other state by choosing an appropriate control input (see, e.g., [1, 2]). Another important question is whether bounded inputs lead to bounded system states or outputs. This question is dealt with in the context of input-to-state stability (ISS) [3] and $\mathcal{L}_\infty$ stability (see, e.g., [4]), respectively, where an upper bound on the system state and the infinity norm of the output, respectively, are considered in terms of the infinity norm of the input.

In other settings, a problem complementary to the above is of interest. Namely, one would like to obtain a *lower* bound on the system state or the output in terms of the norm of the applied inputs. This could, e.g., be the case in the process industry, where one wants to determine whether and how an increasing amount of reagent yields an increasing amount of product, or in economics, where certain inputs such as the price of a product or the number of advertisements influence the profit of a company. Furthermore, in case of systems subject to bounded disturbances, it is interesting to obtain a lower bound for the effect of the worst case disturbance on the system states or the output.

In order to deal with the above questions, in our earlier work [5] we introduced the notion of norm-controllability. As the wording suggests, in contrast to point-to-point controllability, we consider the *norm* of the system state (or, more generally, of an output), and ask how it is affected by the applied inputs. In particular, we are interested in whether this norm can be made large by applying large enough inputs for sufficiently long time. The definition of norm-controllability (see Section II) is such that the reachable set of the system projected on the output space can be lower bounded in terms of the norm of the applied inputs and the time horizon over which they were applied. In this respect, norm-controllability can be seen as complementary to the concepts of ISS and $\mathcal{L}_\infty$ stability, respectively.

Besides introducing the concept of norm-controllability, the main contribution of [5] was to provide a sufficient Lyapunov-like condition for this property, which requires the existence of a function V whose derivative can be lower-bounded in terms of the input u. This condition, however, can be restrictive and is in general not satisfied for a system whose output y has a relative degree greater than one. In this paper, we resolve this issue and provide relaxed sufficient conditions for a system to be norm-controllable in terms of higher-order derivatives of V (see Theorems 1 and 2 in Section III). This generalization is nontrivial and the involved higher-order derivatives are not classical ones but higher-order lower directional derivatives (see, e.g., [6, 7]). A further generalization compared to the results in [5] is that the obtained sufficient conditions only have to hold on a subset of $\mathbb{R}^n$ satisfying a suitable control-invariance condition. In this case, norm-controllability can be established on this set. Finally, we illustrate the obtained results by means of a chemical reaction example, where we examine the influence of the concentration of the reagent in the inlet stream on the amount of the obtained product.

II. Preliminaries and setup

Let $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$ denote the set of nonnegative real numbers. Let $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function, i.e., $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^n$. For a set $S \subseteq \mathbb{R}^n$, let $\overline{S}$ denote its closure, $\mathrm{int}(S)$ its interior, and ∂S its boundary. A function α: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$.

We consider nonlinear control systems of the type

$$\dot{x} = f(x,u), \qquad x(0) = x_0, \\ y = h(x) \tag{1}$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^k$, and input $u \in U \subseteq \mathbb{R}^m$, where the set U of admissible input values can be any closed subset of $\mathbb{R}^m$ (or the whole $\mathbb{R}^m$). Suppose that $f \in C^{\bar{k}-1}$

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for some $\bar{k} \geq 1$ and $\partial^{\bar{k}-1} f / \partial x^{\bar{k}-1}$ is locally Lipschitz in x and u. Input signals $u(\cdot)$ to the system (1) satisfy $u(\cdot) \in L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U)$, where $L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U)$ denotes the set of all measurable and locally bounded functions from $\mathbb{R}_{\geq 0}$ to U. We say that a set $\mathcal{B} \subseteq \mathbb{R}^n$ is rendered *control-invariant* for system (1) by a set $\bar{U} \subseteq U$, if for every $x_0 \in \mathcal{B}$ and every $u(\cdot)$ satisfying $u(\cdot) \in L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, \bar{U})$ the corresponding state trajectory satisfies $x(t) \in \mathcal{B}$ for all $t \geq 0$. We assume that the system (1) exhibits the *unboundedness observability* property (see [8] and the references therein), which means that for every trajectory of the system (1) with finite escape time t_{esc}, also the corresponding output becomes unbounded for $t \to t_{esc}$. This is a very reasonable assumption as one cannot expect to measure responsiveness of the system in terms of an output map h (as we will later do) if a finite escape time cannot be detected by this output map. All linear systems satisfy this assumption, as do all nonlinear systems with radially unbounded output maps.

Let $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$. For every $b > 0$, denote by

$$U_b := \{u \in U : b \leq |u| \leq \mu(b)\} \tag{2}$$

the set of all admissible input values with norm in the interval $[b, \mu(b)]$, which we assume to be nonempty. Furthermore, for every $a, b > 0$, denote by

$$\mathcal{U}_{a,b} := \{u(\cdot) : u(t) \in U_b,\ \forall t \in [0, a]\} \subseteq L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U) \tag{3}$$

the set of all measurable and locally bounded input signals whose norm takes values in the interval $[b, \mu(b)]$ on the time interval $[0, a]$. Let $\mathcal{R}^{\tau}\{x_0, \mathcal{U}\} \subseteq \mathbb{R}^n \cup \{\infty\}$ be the reachable set of the system (1) at time $\tau \geq 0$, starting at the initial condition $x(0) = x_0$ and applying input signals $u(\cdot)$ in some set $\mathcal{U} \subseteq L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U)$. The reachable set $\mathcal{R}^{\tau}\{x_0, \mathcal{U}\}$ contains ∞ if for some $u(\cdot) \in \mathcal{U}$ a finite escape time $t_{esc} \leq \tau$ exists. Furthermore, let $\mathcal{R}^{\leq \tau}\{x_0, \mathcal{U}\} := \bigcup_{0 \leq t \leq \tau} \mathcal{R}^{t}\{x_0, \mathcal{U}\}$. Define $R^{\tau}_h(x_0, \mathcal{U})$ as the radius of the smallest ball in the output space centered at $y = 0$ which contains the image of the reachable set $\mathcal{R}^{\tau}\{x_0, \mathcal{U}\}$ under the output map $h(\cdot)$, or ∞ if this image is unbounded.

Definition 1 ([5]): The system (1) is *norm-controllable* from x_0 with scaling function μ and gain function γ, if there exist a function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$ and a function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is non-decreasing in the first argument and a $\mathcal{K}_{\infty}$-function in the second argument, such that for all $a > 0$ and $b > 0$

$$R^{a}_h(x_0, \mathcal{U}_{a,b}) \geq \gamma(a, b), \tag{4}$$

where $\mathcal{U}_{a,b}$ is defined in (3). □

Loosely speaking, the above definition means that for each fixed time horizon a, the output y can be made large by applying an input with large magnitude b. On the other hand, for fixed b, an increasing time horizon a should lead to a nondecreasing magnitude of the output. For a more detailed discussion regarding the concept of norm-controllability as well as illustrating examples (which also further explain the role of the scaling function μ), the reader is referred to [5].

III. Norm-controllability: relaxed sufficient conditions

In our earlier work [5], we obtained a Lyapunov-like sufficient condition for norm-controllability involving first-order lower directional derivatives. In this section, we show how this condition can be relaxed. First, we consider the situation where not the first-order directional derivative as in [5], but some k-th order lower directional derivative, can be lower bounded in terms of $|u|$ for some fixed $k \geq 1$. After that, we will consider the more general situation where for different regions in the state space, lower directional derivatives of different order can be lower bounded in terms of $|u|$. Furthermore, while in [5] we established norm-controllability on $\mathbb{R}^n$, in this paper we also treat the more general case where the sufficient condition is only satisfied in a subset of $\mathbb{R}^n$ satisfying a suitable control-invariance condition, and hence norm-controllability can only be established there.

A. Higher-order lower directional derivatives

For a function $V : \mathbb{R}^n \to \mathbb{R}$, define as in [6, 9] the lower directional derivative of V at a point $x \in \mathbb{R}^n$ in the direction of a vector $h_1 \in \mathbb{R}^n$ as

$$V^{(1)}(x; h_1) := \liminf_{t \searrow 0, \bar{h}_1 \to h_1} \frac{V(x + t\bar{h}_1) - V(x)}{t}.$$

Note that at each point $x \in \mathbb{R}^n$ where V is continuously differentiable, it holds that $V^{(1)}(x; h_1) = L_{h_1} V = (\partial V / \partial x) h_1$. Furthermore, if $V^{(1)}(x; h_1)$ exists, define as in [6, Equation (3.4a)] (compare also the earlier work [7]) the second-order lower directional derivative of V at a point x in the directions h_1 and h_2 as[1]

$$V^{(2)}(x; h_1, h_2) := \liminf_{t \searrow 0, \bar{h}_2 \to h_2} (2/t^2) \Big(V(x + t h_1 + t^2 \bar{h}_2) - V(x) - t V^{(1)}(x; h_1) \Big).$$

In general, if the corresponding lower-order lower directional derivatives exist, for $k \geq 1$ define the kth-order lower directional derivative as

$$\begin{aligned} V^{(k)}(x; h_1, \ldots, h_k) &:= \liminf_{t \searrow 0, \bar{h}_k \to h_k} (k!/t^k) \\ &\times \Big(V(x + t h_1 + \cdots + t^k \bar{h}_k) - V(x) - t V^{(1)}(x; h_1) \\ &- \cdots - (1/(k-1)!) t^{k-1} V^{(k-1)}(x; h_1, \ldots, h_{k-1}) \Big). \end{aligned} \tag{5}$$

We will later consider lower directional derivatives of a function V along the solution $x(\cdot)$ of system (1). For $k \leq \bar{k}$, we obtain the following expansion for the solution $x(\cdot)$ of system (1), starting at time t' at the point $x := x(t')$ and applying some *constant* input u:

$$x(t) = x + (\Delta t) h_1 + \cdots + (\Delta t)^k h_k + o((\Delta t)^{k+1}) \tag{6}$$

[1] In contrast to [6, 7], here we include the factor 2 (and later, in (5), the factor $k!$) into the definition of higher-order lower directional derivatives. We take this slightly different approach such that later, at each point where V is sufficiently smooth, these lower directional derivatives reduce to classical directional derivatives (without any extra factors as in [6, 7]).

with $\Delta t := t - t'$ and

$$\begin{aligned} h_1 &:= \dot{x}(t') = f(x,u), \\ h_2 &:= (1/2)\ddot{x}(t') = (1/2)\partial f/\partial x|_{(x,u)} f(x,u), \\ &\dots \\ h_k &:= (1/k!)x^{(k)}(t'). \end{aligned} \tag{7}$$

In order to facilitate notation, in the following we write

$$V^{(k)}(x; f(x,u)) := V^{(k)}(x; h_1, \dots, h_k) \tag{8}$$

for the kth-order lower directional derivative of V at the point x along the solution of (1) when a *constant* input u is applied, i.e., with $h_1, \dots, h_k$ given in (7). It is straightforward to verify that at every point where V is sufficiently smooth, $V^{(k)}(x; f(x,u))$ reduces to $L_f^k V|_{(x,u)}$ (compare [6, Section 3] and [7, p.73]), where $L_f^k V|_{(x,u)}$ is the kth-order Lie derivative of V along the vector field f. Namely, if V is sufficiently smooth, the kth-order lower directional derivative reduces to the kth-order directional derivative (i.e., $\liminf$ in (5) can be replaced by $\lim$ without varying the direction h_k ([6, Proposition 3.4])) which in this case exists and is equal to $L_f^k V$ ([7, p.73]).

B. Sufficient condition involving higher-order directional derivatives

Theorem 1: Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which for some $1 \leq k \leq \bar{k}$ is k times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial^k V/\partial x^k$ is locally Lipschitz there, functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$, a function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$, and for each $x \in \mathcal{B}$ a set $U(x) \subseteq \bar{U}$, such that the following holds:

- For all $x \in \mathcal{B}$,

$$\nu(|\omega(x)|) \leq |h(x)|, \tag{9}$$

$$\alpha_1(|\omega(x)|) \leq V(x) \leq \alpha_2(|\omega(x)|). \tag{10}$$

- For each $b > 0$ and $x \in \mathcal{B}$,

$$U(x) \cap U_b \neq \emptyset. \tag{11}$$

- For all $(x,u) \in \mathcal{B} \times \mathbb{R}^m$ such that $u \in U(x)$ and $|\omega(x)| \leq \rho(|u|)$, we have

$$V^{(j)}(x; f(x,u)) \geq 0 \qquad j = 1, \dots, k-1, \tag{12}$$

$$V^{(k)}(x; f(x,u)) \geq \chi(|u|). \tag{13}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with scaling function μ and gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\frac{1}{k!}a^k\chi(b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big). \tag{14}$$

Remark 1: For the special case of $k = 1$, $\bar{U} = U$ and $\mathcal{B} = \mathbb{R}^n$, Theorem 1 in [5] is recovered.

Remark 2: We allow V to be not continuously differentiable for all x where $\omega(x) = 0$ in order to be able to fulfill the assumptions of the Theorem. In particular, for $k = 1$, $V \in C^1$ together with (10) would imply that the gradient of V vanishes for all x where $\omega(x) = 0$, and thus it would be impossible to satisfy (13) there. Also for $k \geq 2$, allowing V to be not continuously differentiable for all x where $\omega(x) = 0$ helps in finding V satisfying (12)–(13); this is in particular true later for the situation of Theorem 2, where we generalize the results of Theorem 1 such that (12)–(13) can hold for flexible k. In the two examples we consider later on, we will choose $V(x) = |\omega(x)|$. □

Proof of Theorem 1: In the following, we will develop two technical lemmas and then obtain the proof of Theorem 1 by combining them. Let $a, b > 0$ be arbitrary but fixed, and assume in the sequel that the hypotheses of Theorem 1 are satisfied. Furthermore, we assume that no $u(\cdot) \in \mathcal{U}_{a,b}$ leads to a finite escape time $t_{esc} \leq a$, for otherwise, by the unboundedness observability property, also $R_h^a(x_0, \mathcal{U}_{a,b}) = \infty$, and thus (4) is satisfied with γ as in (14) and we are done.

The idea of the proof is to construct a piecewise constant input signal $u(\cdot) \in \mathcal{U}_{a,b}$ such that when applying this input signal, the corresponding output trajectory satisfies $y(a) = h(x(a)) \geq \gamma(a,b)$. The first lemma considers the initial phase and proves that V can be increased, and is in particular needed for the case where $\omega(x_0) = 0$, i.e., $x_0 \in W$. To this end, define the set

$$X_{b,\kappa} := \{x \in \mathcal{B} : \kappa \leq |\omega(x)| \leq \rho(b)\} \tag{15}$$

with κ satisfying $0 \leq \kappa \leq \rho(b)$.

Lemma 1: Let $u_0 \in U(x_0) \cap U_b$ and assume that $x_0 \in X_{b,0}$. There exists some $\tau > 0$ such that for all $t \in (0, \tau]$, it holds that $V(x(t)) > V(x_0)$ and hence in particular $V(x(t)) > 0$, where $x(\cdot)$ is the trajectory of the system (1) that results from applying the constant input u_0 during this time interval.

Proof: See appendix. □

Next, we consider the situation where the state x is already away from the set W. Then, according to our assumptions, V is k times continuously differentiable and $\partial^k V/\partial x^k$ is locally Lipschitz, which we can use to show that if some input u_i is "good" at some point x_i, it is also "good" for nearby x_i.

Lemma 2: Consider some time instant $0 \leq s < a$ with $x(s) \in \mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}$, and assume that $x(s) \in X_{b,\delta}$ for some $\delta > 0$; furthermore, pick an arbitrary $u_s \in U(x(s)) \cap U_b$. Then, for each $0 < \varepsilon \leq 1$, there exists a number $\Delta(\varepsilon, \delta) > 0$ such that $V^{(k)}(x(t)) \geq (1-\varepsilon)\chi(b)$ and

$$V^{(j)}(x(t)) - V^{(j)}(x(s)) \geq \frac{1-\varepsilon}{(k-j)!}\chi(b)(t-s)^{k-j} \tag{16}$$

for all $j = 0, \dots, k-1$ and all $t \in [s, s + \Delta(\varepsilon, \delta)] \cap [0, a]$, where $x(\cdot)$ is the trajectory that results from applying the constant input u_s during this time interval, and $V^{(j)}(x(t)) := V^{(j)}(x(t); f(x(t), u_s)) = L_f^j V|_{(x(t), u_s)}$ for $j = 1, \dots, k-1$ and $V^{(0)}(x(t)) := V(x(t))$.

Proof: See appendix. □

Combining Lemmas 1 and 2, we are now able to prove Theorem 1. Fix an arbitrary $0 < \bar{\varepsilon} < 1$. Denote by Λ_b the sublevel set

$$\Lambda_b := \{x \in \mathbb{R}^n : V(x) \le \alpha_1(\rho(b))\}. \quad (17)$$

We construct a desired input signal in a recursive fashion using the following algorithm. This input signal will by construction satisfy $u(t) \in \bar{U}$ for all $t \in [0, a]$; hence, the resulting state trajectory $x(\cdot)$ will remain in the set $\mathcal{B}$ in this time interval if $x_0 \in \mathcal{B}$.

Step 0: Consider $x_0 \in \mathcal{B}$. If $x_0 \in \text{int}(\Lambda_b)$, then by (10) we have $|\omega(x_0)| < \rho(b)$ and so $x_0 \in X_{b,0}$ according to (15). We can then pick some $u_0 \in U(x_0) \cap U_b$, which exists by (11), and apply Lemma 1 to find a time $\tau > 0$ such that the trajectory corresponding to the constant control $u \equiv u_0$ satisfies $V(x(t)) > V(x_0)$ and hence in particular also $V(x(t)) > 0$ for all $0 < t \le \tau$. Pick some $t_1 \in (0, \min\{\tau, a\}]$; note that t_1 can be chosen arbitrarily small. Apply the constant input $u \equiv u_0$ on the interval $[0, t_1)$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (0, t_1)$, then denote this time t by $\check{t}_1$ and skip to Step 2, otherwise proceed to Step 1. If $x_0 \in \partial\Lambda_b$, let $t_1 = \check{t}_1 := 0$ and skip to Step 2. If already $x_0 \notin \Lambda_b$, then let $t_1 := 0$, pick some $u_1 \in U(x_0) \cap U_b$ which exists by (11), and apply the constant input $u \equiv u_1$ on the interval $[0, a)$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If $x(t) \in \partial\Lambda_b$ for some $t \in [0, a)$, then denote this time t by $\check{t}_1$ and skip to Step 2, otherwise skip to Step 3.

Step 1: If[2] $x(t_1) \in \partial\Lambda_b$, then let $\check{t}_1 := t_1$ and skip to Step 2. Otherwise, $x(t_1) \in \text{int}(\Lambda_b)$. Let $\bar{\delta} := \alpha_2^{-1}(V(x(t_1)))$ and note that $\bar{\delta} > 0$ according to the definition of t_1 in Step 0. From (10) and the definition of Λ_b we have $\bar{\delta} \le |\omega(x(t_1))| < \rho(b)$, hence $x(t_1) \in X_{b,\bar{\delta}}$ by (15). We can thus pick some $u_1 \in U(x(t_1)) \cap U_b$ and apply Lemma 2 with $s = t_1$, $u_s = u_1$, $\varepsilon = \bar{\varepsilon}$ and $\delta = \bar{\delta}$ to find a $\Delta(\bar{\varepsilon}, \bar{\delta})$ such that the trajectory corresponding to the constant control $u \equiv u_1$ on the interval $[t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$ satisfies (16) with $s = t_1$ and $\varepsilon = \bar{\varepsilon}$ on this interval. Apply the constant input $u \equiv u_1$ on this interval for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$, then denote this time t by $\check{t}_1$ and skip to Step 2. If this does not happen but $t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}) \ge a$, then skip to Step 3. Otherwise, let $t_2 := t_1 + \Delta(\bar{\varepsilon}, \bar{\delta})$. In this case, $x(t_2) \in \Lambda_b$ and, by Lemma 2, $V(x(t_2)) > V(x(t_1))$ according to (16) with $j = 0$ and $s = t_1$. So, we can check that $x(t_2) \in X_{b,\bar{\delta}}$ in the same way as we did earlier for $x(t_1)$. Therefore, we can repeat Step 1 for the times $t_2, t_3, \ldots$ (but without changing the value of $\bar{\delta}$).

Step 2: We have $x(\check{t}_1) \in \partial\Lambda_b$, i.e., $V(x(\check{t}_1)) = \alpha_1(\rho(b))$. If $\check{t}_1 = a$ then skip to Step 3. Otherwise, pick some $\check{u}_1 \in U(x(\check{t}_1)) \cap U_b$, which exists by (11). Apply the constant input $u \equiv \check{u}_1$ on the interval $[\check{t}_1, \check{t}_2)$ where $\check{t}_2 := \min\{\inf\{t : t > \check{t}_1, x(t) \in \partial\Lambda_b\}, a\}$. This interval is non-empty; in fact, $\check{t}_2 \ge \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}$ where $\check{\delta} := \alpha_2^{-1}(\alpha_1(\rho(b)))$ and $\Delta(\cdot, \cdot)$ comes from Lemma 2. To see why this is true, note that $\check{\delta} \le |\omega(x(\check{t}_1))| \le \rho(b)$ according to (10) and the definition of $\check{t}_1$. Hence we can apply Lemma 2 with $s = \check{t}_1$, $u_s = \check{u}_1$, $\varepsilon = 1/2$ (or any other constant $0 < \varepsilon < 1$), $\delta = \check{\delta}$ and $j = 0$ in order to conclude that $V(x(t)) - V(x(\check{t}_1)) > 0$ for all $t \in (\check{t}_1, \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}]$, which implies that indeed $\check{t}_2 \ge \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}$. Moreover, if $\check{t}_2 < a$ then $x(\check{t}_2) \in \partial\Lambda_b$ and we can repeat Step 2 for the times $\check{t}_2$, $\check{t}_3$, and so on.

Step 3: We have now reached the time $t = a$ and we have constructed the following control input defined on the interval $[0, a)$, with the control values $u_i, \check{u}_j$ and the times $t_i, \check{t}_j$ as specified above (those times that are never defined are treated as ∞):

$$u(t) = \begin{cases} u_0 & 0 \le t < \min\{t_1, \check{t}_1\} \\ u_i & t_i \le t < \min\{t_{i+1}, \check{t}_1, a\}, \quad i = 1, 2, \ldots \\ \check{u}_j & \check{t}_j \le t < \min\{\check{t}_{j+1}, a\}, \quad j = 1, 2, \ldots \end{cases}$$

This input, extended with the last value (u_0, u_i or $\check{u}_j$) at $t = a$ (and arbitrarily for $t > a$), satisfies $u \in \mathcal{U}_{a,b}$, as by construction, $u(t) \in U_b$ for all $t \in [0, a]$. For each $0 < \bar{\varepsilon} < 1$, this input signal is piecewise constant in the interval $[0, a]$ with only finitely many different values u_i and $\check{u}_j$; this follows from the construction in Step 1 and the argument given in Step 2. The state trajectory $x(\cdot)$ resulting from the application of the control input $u(\cdot)$ to the system (1) has the following properties. First, consider the case where $t_1 > 0$ (recall from Step 0 that this corresponds to $x_0 \in \text{int}(\Lambda_b)$). By recursively applying (16) with $j = k - 1$ and $\varepsilon = \bar{\varepsilon}$, for $t_1 \le t \le \min\{\check{t}_1, a\}$ we have $V^{(k-1)}(x(t)) \ge (1 - \bar{\varepsilon})(t - t_1)\chi(b) + V^{(k-1)}(x(t_1)) \ge (1 - \bar{\varepsilon})(t - t_1)\chi(b)$, where the second inequality follows from (12) with $j = k - 1$, $x = x(t_1)$ and $u = u_1$. Integrating this inequality from t_1 to t, we obtain $V^{(k-2)}(x(t)) \ge (1/2)(1 - \bar{\varepsilon})(t - t_1)^2\chi(b) + V^{(k-2)}(x(t_1))$. Repeating the above $k - 2$ times results in $V(x(t)) \ge (1/k!)(1 - \bar{\varepsilon})(t - t_1)^k\chi(b) + V(x(t_1))$ for all for $t_1 \le t \le \min\{\check{t}_1, a\}$; furthermore, recall from Step 0 that $V(x(t_1)) > V(x_0)$ due to Lemma 1. Next, if $\check{t}_1 < a$, then for $\check{t}_1 \le t \le a$ the construction guarantees that $V(x(t)) \ge V(x(\check{t}_1)) = \alpha_1(\rho(b))$. Finally, if $t_1 = 0$ (recall from Step 0 that this corresponds to $x_0 \notin \text{int}(\Lambda_b)$), then the preceding inequality $V(x(t)) \ge \alpha_1(\rho(b))$ is satisfied for all $0 \le t \le a$. Combining the above yields

$$V(x(a)) \ge \min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}. \quad (18)$$

Hence, using (10), we have

$$|\omega(x(a))| \ge \alpha_2^{-1}\big(\min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}\big).$$

Finally, using (9), we obtain

$$|h(x(a))| \ge \nu\big(\alpha_2^{-1}\big(\min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}\big)\big).$$

[2]Even though $u(t_1)$ is not yet defined, $x(t_1)$ is defined by continuity as $\lim_{t \nearrow t_1} x(t)$.

As $u(\cdot)$ is contained in $\mathcal{U}_{a,b}$ and as the above calculations hold for arbitrary $x_0 \in \mathcal{B}$, it follows that

$$R_h^a(x_0, \mathcal{U}_{a,b}) \geq \nu\big(\alpha_2^{-1}\big(\min\{(1/k!)(1-\bar{\varepsilon})(a-t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}\big)\big) \quad (19)$$

for all $x_0 \in \mathcal{B}$. Note that (19) holds for every $0 < \bar{\varepsilon} \leq 1$, and according to Step 0, either $t_1 = 0$ or t_1 can be chosen arbitrarily small. Thus, as the left-hand side of (19) is independent of $\bar{\varepsilon}$ and t_1, we can let $\bar{\varepsilon} \to 0$ and $t_1 \to 0$ (in case t_1 is not 0) and arrive at the desired bound (4) with γ as defined in (14). The function γ satisfies the required properties of Definition 1, i.e., $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot) \in \mathcal{K}_\infty$ for each fixed $a > 0$. This concludes the proof of Theorem 1. □

With the help of Theorem 1, one can establish norm-controllability for systems with output maps that have a relative degree $r > 1$. This is illustrated with the following simple example.

Example 1: Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2^2 + u \quad (20)$$

with output $h(x) = x_1$. The relative degree of this output is $r = 2$. Consider the set $\mathcal{B}_1 := \{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$ which is rendered control-invariant by $\bar{U} := \mathbb{R}_{\geq 0}$. Let $\omega(x) := x_1$ and $V(x) := |x_1|$. For all $x \in \mathcal{B}_1$ and $u \in \bar{U}$, we obtain $V^{(1)}(x; f(x,u)) = x_2$ and $V^{(2)}(x; f(x,u)) = u - x_2^2$. From this we obtain that $V^{(2)}(x; f(x,u)) \geq (1-\varepsilon)u =: \chi(|u|)$ for some $0 < \varepsilon < 1$ if $x_2 \leq \sqrt{\varepsilon u} =: \rho(|u|)$. We can now apply Theorem 1 with $k = 2$, $\mu = Id$, $\chi = Id$ and $U(x) = \bar{U} = \mathbb{R}_{\geq 0}$ to conclude that the system (20) is norm-controllable from all $x_0 \in \mathcal{B}_1$ with scaling function $\mu = Id$ and gain function γ given by (14). Similarly, consider the set $\mathcal{B}_2 := \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0\}$ which is rendered control-invariant by $\bar{U} := \mathbb{R}_{\leq 0}$. Again, one can use Theorem 1 with $k = 2$, $\mu = Id$, $\chi = Id$ and $U(x) = \bar{U} = \mathbb{R}_{\leq 0}$ to conclude that the system (20) is norm-controllable from all $x_0 \in \mathcal{B}_2 := \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 \leq 0\}$. □

In the following, we show that in Theorem 1, the assumption that the set $\mathcal{B}$ is control-invariant under $\bar{U}$ can be relaxed. Namely, for each $x \in \mathcal{B}$, let a set $U(x) \subseteq U$ be given such that the hypotheses of Theorem 1 expressed by (11)–(13) are satisfied, and define $\widetilde{U} := \cup_{x \in \mathcal{B}} U(x)$. If $\mathcal{B}$ is control-invariant under $\widetilde{U}$, we have the situation of Theorem 1 (with $\bar{U} = \widetilde{U}$); if not, consider the following.

Proposition 1: In Theorem 1, the assumption that a set $\bar{U}$ exists such that the set $\mathcal{B}$ is control-invariant under $\bar{U}$ can be replaced by the following. For each $b > 0$, there exists a set $H_b \subseteq \mathbb{R}^n$ with $H_b \cap \Lambda_b = \emptyset$ and Λ_b defined by (17) such that if $x_0 \in \mathcal{B}$ and $u(t) \in U_b \cap \widetilde{U}$ for all $t \geq 0$, then $x(t) \in \mathcal{B} \cup H_b$ for all $t \geq 0$. □

Remark 3: The condition in Proposition 1 means that each trajectory $x(\cdot)$ cannot exit $\mathcal{B}$ before exiting Λ_b. In other words, when at some time instant t we have $x(t) \in \Lambda_b$, then also $x(t) \in \mathcal{B}$. □

Proof of Proposition 1: Let $a, b > 0$ be arbitrary but fixed. We want to show that the construction of the piecewise constant input signal in the proof of Theorem 1 is still valid. First, note that as $x_0 \in \mathcal{B}$, the control value u_0 defined in Step 0 exists by (11) and satisfies $u_0 \in U_b \cap \widetilde{U}$. According to Step 1, subsequent control values u_i at time instants t_i are defined at points where $x(t_i) \in \mathrm{int}(\Lambda_b)$. Recursively, as up to time t_i only input values in $U_b \cap \widetilde{U}$ have been applied to the system, by assumption we have that $x(t_i) \in \mathcal{B} \cup H_b$. But as $H_b \cap \Lambda_b = \emptyset$, it holds that $x(t_i) \in \mathcal{B}$. Hence the input values u_i given in Step 1 are well defined and satisfy $u_i \in U_b \cap \widetilde{U}$. The same argumentation holds for the input values $\check{u}_j$ defined in Step 2 at time instants $\check{t}_j$ where $x(\check{t}_j) \in \partial\Lambda_b$. Recursively, as up to time $\check{t}_j$ only input values in $U_b \cap \widetilde{U}$ have been applied to the system, by assumption we have that $x(\check{t}_j) \in \mathcal{B} \cup H_b$. But as $H_b \cap \Lambda_b = \emptyset$, it holds that $x(\check{t}_j) \in \mathcal{B}$. Hence the input values $\check{u}_j$ given in Step 2 are well defined and satisfy $\check{u}_j \in U_b \cap \widetilde{U}$. Thus we conclude that under the new hypotheses of Proposition 1, the piecewise constant input signal constructed in the proof of Theorem 1 is still well defined and along the corresponding trajectory $x(\cdot)$ again (18) is satisfied. The rest of the proof follows along the lines of the proof of Theorem 1. □

C. Sufficient condition involving lower directional derivatives of different order

In this section, we generalize the previous results to the case where the control-invariant set $\mathcal{B}$ can be partitioned into several regions where (12)–(13) holds for different k. To this end, for a set $\mathbb{X} \subseteq \mathbb{R}^n$ and $\ell \geq 1$, denote by $\mathcal{R}^\ell(\mathbb{X})$ a partition of $\mathbb{X}$ such that $\mathbb{X} = \cup_{i=1}^{\ell} \mathcal{R}_i$ and $\mathcal{R}_i$ is closed for all $1 \leq i \leq \ell$. Note that $\mathcal{R}^1(\mathbb{X}) = \mathbb{X}$.

Theorem 2: Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a partition $\mathcal{R}^\ell(\mathcal{B})$ for some $\ell \geq 1$ with corresponding integer constants $1 \leq k_1 < k_2 < \ldots < k_\ell \leq \bar{k}$, a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which is k_ℓ times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial^{k_\ell} V / \partial x^{k_\ell}$ is locally Lipschitz there, functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$, a function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$, and for each $x \in \mathcal{B}$ a set $U(x) \subseteq \bar{U}$, such that the hypotheses of Theorem 1 expressed by (9)–(11) are satisfied and the following holds: For all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $x \in \mathcal{R}_i$ for some $1 \leq i \leq \ell$, $u \in U(x)$, and $|\omega(x)| \leq \rho(|u|)$, we have

$$V^{(j)}(x; f(x,u)) \geq 0 \qquad j = 1, \ldots, k_i - 1, \quad (21)$$

$$V^{(k_i)}(x; f(x,u)) \geq \chi_i(|u|). \quad (22)$$

Then the system (1) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with scaling function μ and gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\Psi(a,b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big),$$

where

$$\Psi(a,b) = \min_{i \in \{1,\ldots,\ell\}} \frac{(k_\ell - k_i)!}{k_\ell!} a^{k_i} \chi_i(b). \quad (23)$$

Remark 4: In the special case of $\ell = 1$, Theorem 1 is recovered. □

In order to prove Theorem 2, we need the following auxiliary result.

Lemma 3: Let $a_0, a_1, \ldots, a_\ell \geq 0$ for some $\ell \geq 1$ with $a_\ell > 0$. Define $\mathcal{P} := \{p : 0 \leq p \leq \ell, a_p > 0\}$ and $b_p(s) := a_p(s - t_1)^p$ for all $p \in \mathcal{P}$ and some $t_1 \geq 0$. Then for each $t \geq t_1$ it holds that

$$\int_{t_1}^{t} \min_{p\in\mathcal{P}} b_p(s)ds \geq \frac{1}{\ell+1} \min_{p\in\mathcal{P}} \Big\{(p+1)\int_{t_1}^{t} b_p(s)ds\Big\}. \quad (24)$$

Proof: See appendix. □

Proof of Theorem 2: Let $a, b > 0$ be arbitrary but fixed, and assume in the sequel that the hypotheses of Theorem 2 are satisfied. Similar to the proof of Theorem 1, we construct a piecewise constant input signal $u(\cdot) \in \mathcal{U}_{a,b}$ such that when applying this input signal, the corresponding output trajectory satisfies $y(a) = h(x(a)) \geq \gamma(a, b)$. Hereby, the times instances t_1 and $\check{t}_j$ are defined as in the proof of Theorem 1, i.e., both Step 0 and Step 2 remain unchanged. However, the definition of the time instances t_i (in case $t_1 > 0$) slightly changes. Namely, for each $x \in \mathcal{B}$, define $\ell_{\max}(x) := \max\{1 \leq i \leq \ell : x \in \mathcal{R}_i\}$, and replace Step 1 of the proof of Theorem 1 by the following:

Step 1: If $x(t_1) \in \partial\Lambda_b$, then let $\check{t}_1 := t_1$ and skip to Step 2. Otherwise, $x(t_1) \in \text{int}(\Lambda_b)$. Let $\bar{\delta} := \alpha_2^{-1}\big(V(x(t_1))\big)$ and note that $\bar{\delta} > 0$ according to the definition of t_1 in Step 0. From (10) and the definition of Λ_b we have $\bar{\delta} \leq |\omega(x(t_1))| < \rho(b)$, hence $x(t_1) \in X_{b,\bar{\delta}}$ by (15). We can thus pick some $u_1 \in U(x(t_1)) \cap U_b$ and apply Lemma 2 with $s = t_1$, $u_s = u_1$, $\varepsilon = \bar{\varepsilon}$, $\delta = \bar{\delta}$ and $k = k_{\ell_{\max}(x(t_1))}$ to find a $\Delta(\bar{\varepsilon}, \bar{\delta})$ such that the trajectory corresponding to the constant control $u \equiv u_1$ on the interval $[t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$ satisfies (16) with $s = t_1$, $\varepsilon = \bar{\varepsilon}$, $k = k_{\ell_{\max}(x(t_1))}$ and $\chi = \chi_{\ell_{\max}(x(t_1))}$ on this interval. Apply the constant input $u \equiv u_1$ on this interval for as long as one of the following three things does not happen. First, if we have $x(t) \in \partial\Lambda_b$ for some $t \in (t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$, then denote this time t by $\check{t}_1$ and skip to Step 2. Second, if we have $x(t) \in \partial\mathcal{R}_{\ell_{\max}(x(t_1))}$ for some $t \in (t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$ and $l_{\max}(x(t)) > l_{\max}(x(t_1))$, then denote this time by t_2. If none of the above two cases happens but $t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}) \geq a$, then skip to Step 3. Otherwise, let $t_2 := t_1 + \Delta(\bar{\varepsilon}, \bar{\delta})$. In case that t_2 is defined, $x(t_2) \in \Lambda_b$ and, by Lemma 2, $V(x(t_2)) > V(x(t_1))$ according to (16) with $s = t_1$, $\varepsilon = \bar{\varepsilon}$, $k = k_{\ell_{\max}(x(t_1))}$, $\chi = \chi_{\ell_{\max}(x(t_1))}$ and $j = 0$. So we can check that $x(t_2) \in X_{b,\bar{\delta}}$ in the same way as we did earlier for $x(t_1)$. Therefore, we can repeat Step 1 for the times t_2, t_3, … (but without changing the value of $\bar{\delta}$).

Using this modified Step 1, the constructed input signal $u(\cdot)$ extended with the last value (u_0, u_i or $\check{u}_j$) at $t = a$ (and arbitrarily for $t > a$), again satisfies $u \in \mathcal{U}_{a,b}$, as by construction, $u(t) \in U_b$ for all $t \in [0, a]$. Furthermore, for each $0 < \bar{\varepsilon} < 1$, this input signal is piecewise constant in the interval $[0, a]$ with only finitely many different values u_i and $\check{u}_j$, where the latter follows from the argument given in Step 2. To see why only finitely many different values u_i are used, consider the following. While it is true that some of the time intervals $t_{i+1} - t_i$ can be arbitrarily small (or at least $t_{i+1} - t_i < \Delta(\bar{\varepsilon}, \bar{\delta})$), after at most $\ell - 1$ such time intervals one time interval follows which satisfies $t_{i+1} - t_i = \Delta(\bar{\varepsilon}, \bar{\delta})$ (at the latest when the trajectory enters $\mathcal{R}_\ell$), and hence indeed only finitely many different values u_i are used. The state trajectory $x(\cdot)$ resulting from application of the control input $u(\cdot)$ satisfies the following properties. First, we consider the situation where $t_1 > 0$ (recall from Step 0 that this corresponds to $x_0 \in \text{int}(\Lambda_b)$) and $x(t_1) \in \mathcal{R}_\ell$. Due to the definition of the time instances t_i and the input values u_i, we obtain that (16) with $s = t_1$, $\varepsilon = \bar{\varepsilon}$, $k = k_\ell$ and $\chi = \chi_\ell$ is satisfied for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$ for as long as $x(\cdot)$ stays inside $\mathcal{R}_\ell$. If for some t_i, $x(\cdot)$ has left $\mathcal{R}_\ell$ and entered $\mathcal{R}_{\ell-1}$, i.e., $k_{\ell_{\max}(x(t_i))} = k_{\ell-1}$, then from Lemma 2 with $s = t_i$, $\varepsilon = \bar{\varepsilon}$, $k = k_{\ell-1}$ and $\chi = \chi_{\ell-1}$ we obtain $V^{(k_{\ell-1})}(x(t)) \geq (1 - \bar{\varepsilon})\chi_{\ell-1}(b)$ for all $t_i \leq t \leq \min\{\check{t}_1, a\}$ for as long as $x(\cdot)$ stays inside $\mathcal{R}_{\ell-1}$. On the other hand, if $x(\cdot)$ enters $\mathcal{R}_\ell$ again, this is "detected" by some time instant $t_r > t_i$ due to the definition of the time instances t_i in the modified Step 1. As all the regions $\mathcal{R}_i$ are closed, this means that both $x(t_r) \in \mathcal{R}_{\ell-1}$ and $x(t_r) \in \mathcal{R}_\ell$. Hence (16) with $s = t_r$, $\varepsilon = \bar{\varepsilon}$, $k = k_\ell$, $j = k_{\ell-1}$ and $\chi = \chi_\ell$ holds for all $t_r \leq t \leq \min\{\check{t}_1, a\}$ for as long as $x(\cdot)$ stays inside $\mathcal{R}_\ell$, and hence in particular $V^{(k_{\ell-1})}(x(t)) - V^{(k_{\ell-1})}(x(t_r)) \geq 0$ for all such t. But this implies that $V^{(k_{\ell-1})}(x(t)) \geq V^{(k_{\ell-1})}(x(t_r)) \geq \chi_{\ell-1}(b) \geq (1 - \bar{\varepsilon})\chi_{\ell-1}(b)$ for all such t, where the second-to-last inequality follows from the already established fact that also $x(t_r) \in \mathcal{R}_{\ell-1}$. Summarizing the above, we obtain that

$$V^{(k_{\ell-1})}(x(t)) - V^{(k_{\ell-1})}(x(t_r)) \geq (1 - \bar{\varepsilon})\chi_{\ell-1}(b)$$

for all $t_r \leq t \leq \min\{\check{t}_1, a\}$ for as long as $x(\cdot)$ stays inside $\mathcal{R}_\ell \cup \mathcal{R}_{\ell-1}$. Together with the fact established above that (16) with $s = t_1$, $\varepsilon = \bar{\varepsilon}$, $k = k_\ell$, $j = k_{\ell-1}$ and $\chi = \chi_\ell$ is satisfied for all $t_1 \leq t \leq t_r$, this yields

$$\begin{aligned} &V^{(k_{\ell-1})}(x(t)) - V^{(k_{\ell-1})}(x(t_1)) \\ &\geq \min_{i\in\{\ell-1,\ell\}} \frac{(1-\bar{\varepsilon})(k_\ell - k_i)!}{(k_\ell - k_{\ell-1})!}(\tau - t_1)^{k_i - k_{\ell-1}}\chi_i(b) \\ &=: \bar{\varphi}(\tau, t_1) \end{aligned} \quad (25)$$

for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$ for as long as $x(\cdot)$ stays inside $\mathcal{R}_\ell \cup \mathcal{R}_{\ell-1}$. Furthermore, the above inequality (25) is not only valid in case that $x(t_1) \in \mathcal{R}_\ell$, but also in case that $x(t_1) \in \mathcal{R}_{\ell-1}$.

Using the same argumentation as above, by integrating (25) $r := k_{\ell-2} - k_{\ell-1}$ times from t_1 to t and using (21) with $x = x(t_1)$ and $u = u_1$, one obtains that if $x(t_1) \in \mathcal{R}_\ell \cup \mathcal{R}_{\ell-1} \cup \mathcal{R}_{\ell-2}$, then

$$\begin{aligned} &V^{(k_{\ell-2})}(x(t)) - V^{(k_{\ell-2})}(x(t_1)) \\ &\geq \min\Big\{\int_{t_1}^{t}\cdots\int_{t_1}^{s_2}\bar{\varphi}(s_1, t_1)ds_1\ldots ds_r, (1-\bar{\varepsilon})\chi_{\ell-2}(b)\Big\} \\ &\geq \min_{i\in\{\ell-2,\ldots,\ell\}} \frac{(1-\bar{\varepsilon})(k_\ell - k_i)!}{(k_\ell - k_{\ell-2})!}(\tau - t_1)^{k_i - k_{\ell-2}}\chi_i(b) \end{aligned} \quad (26)$$

for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$, as long as $x(\cdot)$ stays inside $\mathcal{R}_\ell \cup \mathcal{R}_{\ell-1} \cup \mathcal{R}_{\ell-2}$. Note that in order to obtain the second inequality in (26), we have used Lemma 3 $k_{\ell-2} - k_{\ell-1}$ times. Applying the above argument repeatedly yields that (26) with $\ell - 2$ replaced by 1 is satisfied for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$. Integrating this inequality another k_1 times while using Lemma 3 and (12) with $x = x(t_1)$ and $u = u(t_1)$, we obtain that $V(x(t)) \geq (1-\bar{\varepsilon})\Psi(t - t_1, b) + V(t_1)$ for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$ and for all $x_0 \in \text{int}(\Lambda_b)$, with Ψ given in (23). From here, the rest of the proof follows along the lines of the proof of Theorem 1. □

Remark 5: It is straightforward to verify that Proposition 1 also can be applied to Theorem 2, i.e., the assumption that a set $\bar{U}$ exists which renders $\mathcal{B}$ control-invariant can be relaxed as described in Proposition 1. Furthermore, the results of Theorem 2 can also be extended in a straightforward way to the case where the sets $\mathcal{R}_i$ of the partition $\mathcal{R}^\ell(\mathcal{B})$ depend on the magnitude of the applied input, i.e., $\mathcal{R}_i = \mathcal{R}_i(b)$. Namely, in this case, the condition of Theorem 2 expressed by (21)–(22) is modified as follows. *For each* $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ *and each* $b > 0$ *such that* $x \in \mathcal{R}_i(b)$, $u \in U(x) \cap U_b$, *and* $|\omega(x)| \leq \rho(b)$, *it holds that* (21) *is satisfied and* $V^{(k_i)}(x; f(x,u)) \geq \chi_i(b)$. With this modification, the proof works exactly the same, as for each fixed $b > 0$, the same fixed partition (depending on b) is considered. □

Example 2: Consider an isothermal continuous stirred tank reactor (CSTR) in which an irreversible, second-order reaction from reagent A to product B takes place [10]:

$$\frac{dC_A}{dt} = \frac{q}{V}(C_{A_i} - C_A) - kC_A^2$$
$$\frac{dC_B}{dt} = -\frac{q}{V}C_B + kC_A^2, \tag{27}$$

where C_A and C_B denote the concentrations of species A and B (in $[mol/m^3]$), respectively, V is the volume of the reactor (in $[m^3]$), q is the flow rate of the inlet and outlet stream (in $[m^3/s]$), k is the reaction rate (in $[1/s]$), and C_{A_i} is the concentration of A in the inlet stream, which can be interpreted as the input. Using $x_1 := C_A$, $x_2 := C_B$, $c := q/V$ and $u := C_{A_i}$, one obtains the system

$$\dot{x}_1 = -cx_1 - kx_1^2 + cu =: f_1(x,u)$$
$$\dot{x}_2 = kx_1^2 - cx_2 =: f_2(x). \tag{28}$$

The physically meaningful states and inputs are $x_1 \geq 0, x_2 \geq 0, u \geq 0$, i.e., nonnegative concentrations of the two species. We are interested in the amount of product B per time unit, i.e. in the output

$$y = h(x) = qx_2. \tag{29}$$

Consider the region $\mathcal{B} := \{x : 0 \leq x_2 \leq (k/c)x_1^2\}$, and for each $x \in \mathcal{B}$, let $U(x) := \mathbb{R}_{\geq 0}$. Hence also $\widetilde{U} := \cup_{x \in \mathcal{B}} U(x) = \mathbb{R}_{\geq 0}$. The set $\mathcal{B}$ is not rendered control-invariant by $\widetilde{U}$; however, for each $b > 0$ we can find a set H_b such that Proposition 1 applies. To this end, note that for all x such that $x_2 = 0$ and $x_1 \geq 0$, $f(x,u)$ points inside $\mathcal{B}$ for all $u \in \widetilde{U}$, and hence no trajectory can leave the set $\mathcal{B}$ there. At the other boundary, i.e., for all x such that $x_2 = (k/c)x_1^2$, $f(x,u)$ points outside $\mathcal{B}$ only if $x_1 \geq (-c + \sqrt{c^2 + 4cku})/(2k) =: \delta(u)$. However, for each $b > 0$, if $x_1(\tau) \geq \delta(b)$ for some $\tau \geq 0$, then it follows from (28) that also $x_1(t) \geq \delta(b)$ for all $t \geq \tau$ in case that $u(t) \geq b$ for all $t \geq \tau$. Hence for each $b > 0$, we define the set $H_b := \{x : x_1 \geq \delta(b), x_2 \geq (k/c)x_1^2\}$.

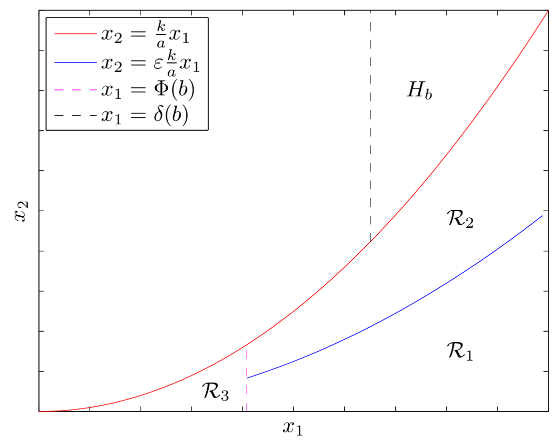

Fig. 1. Partition of the control-invariant region $\mathcal{B}$ and the set H_b in Example 9.

Next, taking $\omega(x) = x_2$ and $V(x) = |\omega(x)|$, one obtains that for all $x \in \mathcal{B}$ and $u \in \widetilde{U}$

$$V^{(1)}(x; f(x,u)) = kx_1^2 - cx_2, \tag{30}$$
$$V^{(2)}(x; f(x,u)) = -kx_1^2(3c + 2kx_1) + c^2x_2 + 2kcx_1u, \tag{31}$$
$$V^{(3)}(x; f(x,u)) = (-6kcx_1 - 6k^2x_1^2 + 2kcu)f_1(x,u) + c^2f_2(x). \tag{32}$$

Note that $V^{(1)}(x; f(x,u)) \geq 0$ for all $x \in \mathcal{B}$. Let $0 < \varepsilon, \theta < 1$, and for $b \geq 0$ define $\varphi_1(b;\varepsilon) := (-(3-\varepsilon)c + \sqrt{(3-\varepsilon)^2c^2 + 16ck\theta b})/(4k)$, $\varphi_2(b) := \min\{cb/(8(c+k)), \sqrt{cb/(8(c+k))}\}$ and $\Phi(b) := \min\{\varphi_1(b;\varepsilon), \varphi(b)\}$. Now consider the following partition of $\mathcal{B}$, which is also exemplarily depicted in Figure 1:

$$\mathcal{R}_1(b) := \{x \in \mathcal{B} : 0 \leq x_2 \leq \varepsilon(k/c)x_1^2, x_1 \geq \Phi(b)\},$$
$$\mathcal{R}_2(b) := \{x \in \mathcal{B} : (\varepsilon k/c)x_1^2 \leq x_2 \leq (k/c)x_1^2, x_1 \geq \Phi(b)\},$$
$$\mathcal{R}_3(b) := \{x \in \mathcal{B} : 0 \leq x_2 \leq (k/c)x_1^2, x_1 \leq \Phi(b)\}.$$

For all $x \in \mathcal{R}_1$, we obtain from (30) that

$$V^{(1)}(x; f(x,u)) \geq (1-\varepsilon)kx_1^2 \geq (1-\varepsilon)k\Phi^2(b) =: \chi_1(b).$$

For all $x \in \mathcal{R}_2$ and $u \in U(x) \cap U_b$, (31) yields

$$V^{(2)}(x; f(x,u)) \geq 2(1-\theta)kcb\Phi(b) =: \chi_2(b),$$

for all $x_1 \leq \varphi_1(b;\varepsilon)$, which holds if $x_2 \leq (\varepsilon k/c)\varphi_1^2(b;\varepsilon) =: \rho(b)$. Finally, for all $x \in \mathcal{R}_3$ and $u \in U(x) \cap U_b$, we obtain from (31) that $V^{(2)}(x; f(x,u)) \geq 0$ and from (32) that

$$V^{(3)}(x; f(x,u)) \geq kc^2b^2 =: \chi_3(b).$$

Furthermore, it is straightforward to verify that for each $b > 0$, $H_b \cap \Lambda_b = \emptyset$, where $\Lambda_b = \{x : |x_2| \leq \rho(b)\}$.

Summarizing the above, we can apply Theorem 2 with $\ell = 3$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $\alpha_1 = \alpha_2 = \mu = \text{id}$ and $\nu = q\text{id}$ together with Remark 5 and Proposition 1 to conclude that the system (28) is norm-controllable from all $x_0 \in \mathcal{B}$ with scaling function $\mu = \text{id}$ and gain function

$$\gamma(r,s) = q \min\left\{\Psi(r,s) + V(x_0), \rho(s)\right\}$$

with Ψ defined in (23). An interpretation of this fact is as follows. If $x_2 \leq (k/c)x_1^2$, then a sufficiently large amount of reagent A compared to the amount of product B is present in the reactor in order that the amount of product B can be increased. On the other hand, if $x_2 > (k/c)x_1^2$, then already too much product B is inside the reactor such that its amount will first decrease (due to the outlet stream), no matter how large the concentration of A in the inlet stream (i.e., the input u) is, and hence the system is not norm-controllable for such initial conditions.

IV. Conclusions

In this paper, we obtained two sufficient conditions for a system to be norm-controllable. The first one involves kth-order lower directional derivatives for some fixed k, while the second involves higher-order lower directional derivatives of different order. These conditions allow us to establish norm-controllability for output maps with relative degree $r > 1$. Furthermore, we illustrated the obtained results through a chemical reaction example.

Appendix

Proof of Lemma 1: First, note that an input u_0 as defined in the lemma exists, as according to (11), $U(x_0) \cap U_b \neq \emptyset$. Let $h(t) := 1/t^k\big(x(t) - x_0 - th_1 - \cdots - t^{k-1}h_{k-1}\big)$ for $t > 0$, where $h_1, \ldots, h_{k-1}$ are defined as in (7) with $t' = 0$, $x = x_0$, and $u = u_0$. Note that h varies continuously in t and $\lim_{t\searrow 0} h(t) =: h_k$ according to (6). Furthermore, for $t > 0$, define the function

$$\begin{aligned} g(t) := k!/t^k \Big(& V(x_0 + th_1 + \cdots + t^k h(t)) - V(x) \\ & - tV^{(1)}(x_0; h_1) - \cdots - t^{k-1}V^{(k-1)}(x_0; h_1, \ldots, h_{k-1})\Big) \end{aligned}$$

Consider $g_- := \liminf_{t\searrow 0} g(t)$. By the definitions of g and $V^{(k)}$, it holds that

$$g_- = \liminf_{t\searrow 0} g(t) \geq V^{(k)}(x_0; f(x_0, u_0)) \geq \chi(|u_0|) \geq \chi(b).$$

The first inequality holds because in the definition of $V^{(k)}$ in (5), the infimum over all $\bar{h}_k$ with $\bar{h}_k \to h_k$ is taken, while in g_- the specific choice $\bar{h}_k = h(t) \to h_k$ is used. Thus, by definition of the (one-sided) limit inferior, for every $\varepsilon > 0$ there exists a $\tau > 0$ such that for all $0 < t \leq \tau$, it holds that

$$g(t) \geq g_- - \varepsilon \geq \chi(b) - \varepsilon, \tag{33}$$

and thus

$$\begin{aligned} V(x(t)) &= V(x_0 + th_1 + \cdots + t^k h(t)) \\ &= (1/k!)g(t)t^k + V(x_0) + tV^{(1)}(x_0; h_1) \\ &\quad + \cdots + t^{k-1}V^{(k-1)}(x_0; h_1, \ldots, h_{k-1}) \\ &\overset{(33),(12)}{\geq} \frac{1}{k!}(\chi(b) - \varepsilon)t^k + V(x_0) > V(x_0) \geq 0. \end{aligned}$$

The second but last inequality is due to the fact that ε can be made arbitrarily small such that $\chi(b) > \varepsilon$. ☐

In order to prove Lemma 2, we first need a simple auxiliary result.

Lemma 4: For each $\kappa > 0$ there exist constants M and N such that

$$|f(x,u)| \leq M, \tag{34}$$

$$\left|V^{(k)}(x'; f(x',u)) - V^{(k)}(x; f(x,u))\right| \leq N|x - x'| \tag{35}$$

for all $x, x' \in \overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$ and $u \in U_b$, where $X_{\infty,\kappa} := \{x \in \mathcal{B} : \kappa \leq |\omega(x)|\}$.

Proof of Lemma 4: As stated earlier, we assume that for each $u(\cdot) \in \mathcal{U}_{a,b}$ there is no finite escape time $t_{esc} \leq a$. But this implies that the set $\overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}}$ is compact [11, Proposition 5.1]. As $\mathcal{B}$ is closed, also $X_{\infty,\kappa}$ is closed due to continuity of ω, and hence $\overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$ is compact, for each $\kappa > 0$. Furthermore, also U_b is compact. From the fact that $V \in C^k$ for all $x \in \mathbb{R}^n \setminus W$ and $\partial^k V/\partial x^k$ is locally Lipschitz there, and furthermore $f \in C^{k-1}$ and $\partial^{k-1} f/\partial x^{k-1}$ is locally Lipschitz in x, it follows that $V^{(k)}(x; f(x,u)) = L_f^k V|_{(x,u)}$ is Lipschitz in x on $\overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$, for each $u \in U_b$ and each $\kappa > 0$. This follows from the fact that $L_f^k V|_{(x,u)}$ consists of sums of products of locally Lipschitz functions, and sums and products of Lipschitz functions on a compact set are Lipschitz on that set. Hence there exist constants M and N satisfying (34)–(35) for each $x, x' \in \overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$ and each $u \in U_b$. ☐

Proof of Lemma 2: First, note that an input u_s as defined in the lemma exists according to (11). Let $\tau_0 := \inf\{\tau : \tau \geq s, V(x(\tau)) = (1/2)V(x(s))\} \in (s, \infty)$, where $x(\cdot)$ is the trajectory resulting from the application of the constant control input u_s for all $\tau \geq s$. Note that due to continuity of V and $x(\cdot)$, $\tau_0 > s$ and hence by (10) $x(t) \in X_{\infty,\delta'}$ with $\delta' = \alpha_2^{-1}((1/2)\alpha_1(\delta))$ for all $t \in [s, \tau_0)$ (respectively for all $t \in [s, \tau_0]$ if $\tau_0 < \infty$). Now consider some $s \leq s' \leq \tau_0$. For all $t \in [s, s'] \cap [0, a]$, the trajectory $x(\cdot)$ satisfies

$$x(t) = x(s) + \int_s^t f(x(\tau), u_s) d\tau. \tag{36}$$

Furthermore, Lemma 4 (with $\kappa = \delta'$) provides a constant M such that from (36) it follows that for each $t \in [s, s'] \cap [0, a]$

$$|x(t) - x(s)| \leq M(t - s). \tag{37}$$

With this we obtain that for each $t \in [s, s'] \cap [0, a]$,

$$\begin{aligned} V^{(k)}(x(t)) &\overset{(35)}{\geq} V^{(k)}(x(s)) - N|x(t) - x(s)| \\ &\overset{(37)}{\geq} V^{(k)}(x(s)) - NM(t - s) \\ &\overset{(13)}{\geq} \chi(|u_s|) - NM(t - s) \geq (1 - \varepsilon)\chi(|b|) \end{aligned} \quad (38)$$

for every $0 < \varepsilon \leq 1$ if $s' \leq \min\{\tau_0, s + \Delta(\varepsilon, \delta)\}$ with

$$\Delta(\varepsilon, \delta) := \varepsilon\chi(b)/(NM).$$

The dependence of Δ on δ is due to the fact that the constants M and N possibly depends on δ' and hence on δ. Integrating (38) from s to t yields

$$V^{(k-1)}(x(t)) - V^{(k-1)}(x(s)) \geq (1 - \varepsilon)\chi(b)(t - s) \quad (39)$$

for all $t \in [s, \min\{\tau_0, s+\Delta(\varepsilon, \delta)\}] \cap [0, a]$. Furthermore, for $j = k - 2, \dots, 0$, integrating (39) $k - j - 1$ times from s to t while using (12) with $x = x(s)$ and $u = u_s$ in order to get rid of the terms $V^{(j+1)}(x(s)), \dots, V^{(k-1)}(x(s))$ yields

$$V^{(j)}(x(t)) - V^{(j)}(x(s)) \geq \frac{1 - \varepsilon}{(k - j)!}\chi(b)(t - s)^{k-j} \quad (40)$$

for all $t \in [s, \min\{\tau_0, s+\Delta(\varepsilon, \delta)\}] \cap [0, a]$. Hence Lemma 2 is established if we can show that $\tau_0 \geq s + \Delta(\varepsilon, \delta)$, which will be done by contradiction. Namely, suppose that $\tau_0 < s + \Delta(\varepsilon, \delta)$. But then (40) is valid for all $t \in [s, \tau_0]$, and from there it follows (with $j = 0$) that $V(x(\tau_0)) > V(x(s))$, which contradicts the definition of τ_0. □

Proof of Lemma 3: Let $p_{min} := \min\{p : p \in \mathcal{P}\}$. In case that $p_{\min} = \ell$, inequality (24) is trivially satisfied with equality. Hence in the following we assume that $p_{\min} < \ell$ and thus $\mathcal{P}$ contains at least two elements. According to the definition of the functions b_p, for each pair of elements $p, q \in \mathcal{P}$ with $p > q$ there exists an $\bar{s} > t_1$ such that $b_p(s) < b_q(s)$ for all $t_1 < s < \bar{s}$, $b_p(\bar{s}) = b_q(\bar{s})$, and $b_p(s) > b_q(s)$ for all $s > \bar{s}$. Hence

$$\begin{aligned} \int_{t_1}^{\bar{s}} b_p(s)ds &= \frac{1}{p+1}a_p(\bar{s} - t_1)^{p+1} \\ &= \frac{1}{p+1}a_q(\bar{s} - t_1)^{q+1} = \frac{q+1}{p+1}\int_{t_1}^{\bar{s}} b_q(s)ds. \end{aligned} \quad (41)$$

Furthermore, the above implies that there exist some $s'' \geq s' > 0$ such that $\min_{p\in\mathcal{P}} b_p(s) = b_\ell(s)$ for all $0 \leq s \leq s'$ and $\min_{p\in\mathcal{P}} b_p(s) = b_{p_{\min}}(s)$ for all $s \geq s''$. Let $\tau_0 := t_1$ and $p_0 := \ell$. Define recursively

$$\begin{aligned} \tau_i &:= \min\{s : s > \tau_{i-1}, \exists p \in \mathcal{P}, 0 \leq p \leq p_{i-1} - 1, \\ &\qquad\qquad b_{p_{i-1}}(s) = b_p(s)\} \\ p_i &:= \min\{p : 0 \leq p \leq p_{i-1} - 1, b_p(\tau_i) = b_{p_{i-1}}(\tau_i)\} \end{aligned}$$

for all $i = 1, \dots, i_{\max}$, where $i_{\max} := \min\{i \geq 1 : p_i = p_{\min}\}$. Note that according to the above considerations, the time instances τ_i are well defined and $i_{\max} \leq |\mathcal{P}|$, where $|\mathcal{P}|$ denotes the number of elements in $\mathcal{P}$. With this, for each $t \geq t_1$, define $i'(t) := \max\{i \geq 0 : \tau_i \leq t\}$. Then, by using the fact that the sequence $\{p_i\}$ is decreasing by construction, the left hand side of (24) is equal to

$$\begin{aligned} &\int_{t_1}^{\tau_1} b_{p_0}(s)ds + \int_{\tau_1}^{\tau_2} b_{p_1}(s)ds + \dots + \int_{\tau_{i'(t)}}^{t} b_{p_{i'(t)}}(s)ds \\ &\overset{(41)}{\geq} \frac{p_1 + 1}{p_0 + 1}\int_{t_1}^{\tau_2} b_{p_1}(s)ds + \dots + \int_{\tau_{i'(t)}}^{t} b_{p_{i'(t)}}(s)ds \\ &\overset{(41)}{\geq} \cdots \overset{(41)}{\geq} \frac{p_{i'(t)} + 1}{p_0 + 1}\int_{t_1}^{t} b_{p_{i'(t)}}(s)ds \\ &\geq \frac{1}{\ell + 1}\min_{p\in\mathcal{P}}\Big\{(p + 1)\int_{t_1}^{t} b_p(s)ds\Big\}, \end{aligned}$$

which concludes the proof of Lemma 3. □

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