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Relaxed conditions for norm-controllability of nonlinear systems

Matthias A. Müller, Daniel Liberzon, and Frank Allgöwer

Abstract— **In this paper, we further study the recently introduced notion of norm-controllability, which captures the responsiveness of a nonlinear system with respect to applied inputs in terms of the norm of an output map. We give sufficient conditions for this property based on higher-order lower directional derivatives, which generalize the results obtained in our earlier work and help to establish norm-controllability for systems with outputs having relative degree greater than one. Furthermore, we illustrate the obtained results by means of a chemical reaction example.**

I. INTRODUCTION

When considering dynamical systems with inputs and outputs, a key question is how the applied inputs affect the behavior of the states and outputs of the system. In this respect, various properties are of interest. For example, one fundamental system property is controllability, which is usually formulated as the ability to reach any state from any other state by choosing an appropriate control input (see, e.g., [1, 2]). Another important question is whether bounded inputs lead to bounded system states or outputs. This question is dealt with in the context of input-to-state stability (ISS) [3] and $\mathcal{L}_\infty$ stability (see, e.g., [4]), respectively, where an upper bound on the system state and the infinity norm of the output, respectively, are considered in terms of the infinity norm of the input.

In other settings, a problem complementary to the above is of interest. Namely, one would like to obtain a *lower* bound on the system state or the output in terms of the norm of the applied inputs. This could, e.g., be the case in the process industry, where one wants to determine whether and how an increasing amount of reagent yields an increasing amount of product, or in economics, where certain inputs such as the price of a product or the number of advertisements influence the profit of a company. Furthermore, in case of systems subject to bounded disturbances, it is interesting to obtain a lower bound for the effect of the worst case disturbance on the system states or the output.

In order to deal with the above questions, in our earlier work [5] we introduced the notion of norm-controllability. As the wording suggests, in contrast to point-to-point controllability, we consider the *norm* of the system state (or, more generally, of an output), and ask how it is affected by the applied inputs. In particular, we are interested in whether this norm can be made large by applying large enough inputs for sufficiently long time. The definition of norm-controllability (see Section II) is such that the reachable set of the system projected on the output space can be lower bounded in terms of the norm of the applied inputs and the time horizon over which they were applied. In this respect, norm-controllability can be seen as complementary to the concepts of ISS and $\mathcal{L}_\infty$ stability, respectively.

Besides introducing the concept of norm-controllability, the main contribution of [5] was to provide a sufficient Lyapunov-like condition for this property, which requires the existence of a function V whose derivative can be lower-bounded in terms of the input u. This condition, however, can be restrictive and is in general not satisfied for a system whose output y has a relative degree greater than one. In this paper, we resolve this issue and provide relaxed sufficient conditions for a system to be norm-controllable in terms of higher-order derivatives of V (see Theorems 1 and 2 in Section III). This generalization is nontrivial and the involved higher-order derivatives are not classical ones but higher-order lower directional derivatives (see, e.g., [6, 7]). A further generalization compared to the results in [5] is that the obtained sufficient conditions only have to hold on a subset of $\mathbb{R}^n$ satisfying a suitable control-invariance condition. In this case, norm-controllability can be established on this set. Finally, we illustrate the obtained results by means of a chemical reaction example, where we examine the influence of the concentration of the reagent in the inlet stream on the amount of the obtained product.

II. PRELIMINARIES AND SETUP

Let $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$ denote the set of nonnegative real numbers. Let $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function, i.e., $\mathrm{id}(x) = x$ for all $x \in \mathbb{R}^n$. For a set $S \subseteq \mathbb{R}^n$, let $\mathrm{int}(S)$ denote its interior and ∂S its boundary.

We consider nonlinear control systems of the type

$$\dot{x} = f(x, u), \qquad y = h(x), \qquad x(0) = x_0 \tag{1}$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^k$, and input $u \in U \subseteq \mathbb{R}^m$, where the set U of admissible input values can be any closed subset of $\mathbb{R}^m$ (or the whole $\mathbb{R}^m$). Suppose that $f \in C^{\bar{k}-1}$ for some $\bar{k} \geq 1$ and $\partial^{\bar{k}-1} f / \partial x^{\bar{k}-1}$ is locally Lipschitz in x and u. Input signals $u(\cdot)$ to the system (1) satisfy $u(\cdot) \in L^\infty_{loc}(\mathbb{R}_{\geq 0}, U)$, where $L^\infty_{loc}(\mathbb{R}_{\geq 0}, U)$ denotes the set of

Matthias A. Müller and Frank Allgöwer are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, 70550 Stuttgart, Germany. Their work was supported by the German Research Foundation (DFG) within the Priority Programme 1305 “Control Theory of Digitally Networked Dynamical Systems” and within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart. {mueller,allgower}@ist.uni-stuttgart.de

Daniel Liberzon is with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. His work was supported by the NSF ECCS-0821153 grant and by a National Research Foundation Grant funded by the Korean Government (NRF-2011-220-D00043). liberzon@illinois.edu

all measurable and locally bounded functions from $\mathbb{R}_{\geq 0}$ to U. We say that a set $\mathcal{B} \subseteq \mathbb{R}^n$ is rendered *control-invariant* for system (1) by a set $\bar{U} \subseteq U$, if for every $x_0 \in \mathcal{B}$ and every $u(\cdot)$ satisfying $u(\cdot) \in L^\infty_{loc}(\mathbb{R}_{\geq 0}, \bar{U})$ the corresponding state trajectory satisfies $x(t) \in \mathcal{B}$ for all $t \geq 0$. We assume that the system (1) exhibits the *unboundedness observability* property (see [8] and the references therein), which means that for every trajectory of the system (1) with finite escape time t_{esc}, also the corresponding output becomes unbounded for $t \to t_{esc}$. This is a very reasonable assumption as one cannot expect to measure responsiveness of the system in terms of an output map h (as we will later do) if a finite escape time cannot be detected by this output map.

Let $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$. For every $b > 0$, denote by $U_b := \{u \in U : b \leq |u| \leq \mu(b)\}$ the set of all admissible input values with norm in the interval $[b, \mu(b)]$, which we assume to be nonempty. Furthermore, for every $a, b > 0$, denote by

$$\mathcal{U}_{a,b} := \{u(\cdot) : u(t) \in U_b,\ \forall t \in [0, a]\} \subseteq L^\infty_{loc}(\mathbb{R}_{\geq 0}, U) \quad (2)$$

the set of all measurable and locally bounded input signals whose norm takes values in the interval $[b, \mu(b)]$ on the time interval $[0, a]$. Let $\mathcal{R}^\tau\{x_0, \mathcal{U}\} \subseteq \mathbb{R}^n \cup \{\infty\}$ be the reachable set of the system (1) at time $\tau \geq 0$, starting at the initial condition $x(0) = x_0$ and applying input signals $u(\cdot)$ in some set $\mathcal{U} \subseteq L^\infty_{loc}(\mathbb{R}_{\geq 0}, U)$. The reachable set $\mathcal{R}^\tau\{x_0, \mathcal{U}\}$ contains ∞ if for some $u(\cdot) \in \mathcal{U}$ a finite escape time $t_{esc} \leq \tau$ exists. Furthermore, let $\mathcal{R}^{\leq \tau}\{x_0, \mathcal{U}\} := \bigcup_{0 \leq t \leq \tau} \mathcal{R}^t\{x_0, \mathcal{U}\}$. Define $R^\tau_h(x_0, \mathcal{U})$ as the radius of the smallest ball in the output space centered at $y = 0$ which contains the image of the reachable set $\mathcal{R}^\tau\{x_0, \mathcal{U}\}$ under the output map $h(\cdot)$, or ∞ if this image is unbounded.

Definition 1 ([5]): The system (1) is *norm-controllable* from x_0 with scaling function μ and gain function γ, if there exist a function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$ and a function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ which is non-decreasing in the first argument and a $\mathcal{K}_\infty$-function in the second argument, such that for all $a > 0$ and $b > 0$

$$R^a_h(x_0, \mathcal{U}_{a,b}) \geq \gamma(a, b), \quad (3)$$

where $\mathcal{U}_{a,b}$ is defined in (2). □

Loosely speaking, the above definition means that for each fixed time horizon a, the output y can be made large by applying an input with large magnitude b. On the other hand, for fixed b, an increasing time horizon a should lead to a nondecreasing magnitude of the output. For a more detailed discussion regarding the concept of norm-controllability as well as illustrating examples (which also further explain the role of the scaling function μ), the reader is referred to [5].

III. Norm-controllability: relaxed sufficient conditions

In our earlier work [5], we obtained a Lyapunov-like sufficient condition for norm-controllability involving first-order lower directional derivatives. In this section, we show how this condition can be relaxed. First, we consider the situation where not the first-order directional derivative as in [5], but some k-th order lower directional derivative, can be lower bounded in terms of $|u|$ for some fixed $k \geq 1$. This helps to establish norm-controllability not only for systems with output maps of relative degree $r = 1$, but also for systems with higher relative degree outputs. After that, we will consider the more general situation where for different regions in the state space, lower directional derivatives of different order can be lower bounded in terms of $|u|$. Furthermore, while in [5] we established norm-controllability on $\mathbb{R}^n$, in this paper we also treat the more general case where the sufficient condition is only satisfied in a subset of $\mathbb{R}^n$ satisfying a suitable control-invariance condition, and hence norm-controllability can only be established there.

A. Higher-order lower directional derivatives

For a function $V : \mathbb{R}^n \to \mathbb{R}$, define as in [6, 9] the lower directional derivative of V at a point $x \in \mathbb{R}^n$ in the direction of a vector $h_1 \in \mathbb{R}^n$ as

$$V^{(1)}(x; h_1) := \liminf_{t \searrow 0, \bar{h}_1 \to h_1} (1/t)\big(V(x + t\bar{h}_1) - V(x)\big).$$

Note that at each point $x \in \mathbb{R}^n$ where V is continuously differentiable, it holds that $V^{(1)}(x; h_1) = L_{h_1}V = (\partial V/\partial x)h_1$. Furthermore, if $V^{(1)}(x; h_1)$ exists, define as in [6, Equation (3.4a)] (compare also the earlier work [7]) the second-order lower directional derivative of V at a point x in the directions h_1 and h_2 as[1]

$$V^{(2)}(x; h_1, h_2) := \liminf_{t \searrow 0, \bar{h}_2 \to h_2} (2/t^2)\Big(V(x + th_1 + t^2\bar{h}_2) - V(x) - tV^{(1)}(x; h_1)\Big).$$

In general, if the corresponding lower-order lower directional derivatives exist, for $k \geq 1$ define the kth-order lower directional derivative as

$$V^{(k)}(x; h_1, \ldots, h_k) := \liminf_{t \searrow 0, \bar{h}_k \to h_k} (k!/t^k) \times \Big(V(x + th_1 + \cdots + t^k\bar{h}_k) - V(x) - tV^{(1)}(x; h_1) - \cdots - (1/(k-1)!)t^{k-1}V^{(k-1)}(x; h_1, \ldots, h_{k-1})\Big). \quad (4)$$

We will later consider lower directional derivatives of a function V along the solution $x(\cdot)$ of system (1). For $k \leq \bar{k}$, we obtain the following expansion for the solution $x(\cdot)$ of system (1), starting at time t' at the point $x := x(t')$ and applying some *constant* input u:

$$x(t) = x + (\Delta t)h_1 + \cdots + (\Delta t)^k h_k + o((\Delta t)^{k+1}) \quad (5)$$

[1] In contrast to [6, 7], here we include the factor 2 (and later, in (4), the factor $k!$) into the definition of higher-order lower directional derivatives. We take this slightly different approach such that later, at each point where V is sufficiently smooth, these lower directional derivatives reduce to classical directional derivatives (without any extra factors as in [6, 7]).

with $\Delta t := t - t'$ and

$$h_1 := \dot{x}(t') = f(x,u),$$
$$h_2 := (1/2)\ddot{x}(t') = (1/2)\partial f/\partial x|_{(x,u)} f(x,u),$$
$$\dots$$
$$h_k := (1/k!)x^{(k)}(t'). \tag{6}$$

In order to facilitate notation, in the following we write $V^{(k)}(x; f(x,u)) := V^{(k)}(x; h_1, \dots, h_k)$ for the kth-order lower directional derivative of V at the point x along the solution of (1) when a *constant* input u is applied, i.e., with $h_1, \dots, h_k$ given in (6). It is straightforward to verify that at every point where V is sufficiently smooth, $V^{(k)}(x; f(x,u))$ reduces to $L_f^k V|_{(x,u)}$ (compare [6, Section 3] and [7, p.73]), where $L_f^k V|_{(x,u)}$ is the kth-order Lie derivative of V along the vector field f. Namely, if V is sufficiently smooth, the kth-order lower directional derivative reduces to the kth-order directional derivative (i.e., $\liminf$ in (4) can be replaced by $\lim$ without varying h_k ([6, Proposition 3.4])) which in this case exists and is equal to $L_f^k V$ ([7, p.73]).

B. Sufficient condition involving higher-order directional derivatives

Theorem 1: Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which for some $1 \leq k \leq \bar{k}$ is k times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial^k V/\partial x^k$ is locally Lipschitz there, functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$, a function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$, and for each $x \in \mathcal{B}$ a set $U(x) \subseteq \bar{U}$, such that the following holds:

- For all $x \in \mathcal{B}$,
$$\nu(|\omega(x)|) \leq |h(x)|, \tag{7}$$
$$\alpha_1(|\omega(x)|) \leq V(x) \leq \alpha_2(|\omega(x)|). \tag{8}$$
- For each $b > 0$ and $x \in \mathcal{B}$,
$$U(x) \cap U_b \neq \emptyset. \tag{9}$$
- For all $(x,u) \in \mathcal{B} \times \mathbb{R}^m$ such that $u \in U(x)$ and $|\omega(x)| \leq \rho(|u|)$, we have
$$V^{(j)}(x; f(x,u)) \geq 0 \qquad j = 1, \dots, k-1, \tag{10}$$
$$V^{(k)}(x; f(x,u)) \geq \chi(|u|). \tag{11}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with scaling function μ and gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\frac{1}{k!}a^k\chi(b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big). \tag{12}$$

Remark 1: For the special case of $k = 1$, $\bar{U} = U$ and $\mathcal{B} = \mathbb{R}^n$, Theorem 1 in [5] is recovered. □

Remark 2: We allow V to be not continuously differentiable for all x where $\omega(x) = 0$ in order to be able to fulfill the assumptions of the Theorem. In particular, for $k = 1$, $V \in C^1$ together with (8) would imply that the gradient of V vanishes for all x where $\omega(x) = 0$, and thus it would be impossible to satisfy (11) there. Also for $k \geq 2$, allowing V to be not continuously differentiable for all x where $\omega(x) = 0$ helps in finding V satisfying (10)–(11); this is in particular true later for the situation of Theorem 2, where we generalize the results of Theorem 1 such that (10)–(11) can hold for flexible k. In the two examples we consider later on, we will choose $V(x) = |\omega(x)|$. □

Proof of Theorem 1: In the following, we will develop two technical lemmas and then obtain the proof of Theorem 1 by combining them. Let $a, b > 0$ be arbitrary but fixed, and assume in the sequel that the hypotheses of Theorem 1 are satisfied. Furthermore, we assume that no $u(\cdot) \in \mathcal{U}_{a,b}$ leads to a finite escape time $t_{esc} \leq a$, for otherwise, by the unboundedness observability property, also $R_h^a(x_0, \mathcal{U}_{a,b}) = \infty$, and thus (3) is satisfied with γ as in (12) and we are done. The idea of the proof is to construct a piecewise constant input signal $u(\cdot) \in \mathcal{U}_{a,b}$ such that when applying this input signal, the corresponding output trajectory satisfies $y(a) = h(x(a)) \geq \gamma(a,b)$. The first lemma considers the initial phase and proves that V can be increased, and is in particular needed for the case where $\omega(x_0) = 0$, i.e., $x_0 \in W$. To this end, define the set

$$X_{b,\kappa} := \{x \in \mathcal{B} : \kappa \leq |\omega(x)| \leq \rho(b)\} \tag{13}$$

with κ satisfying $0 \leq \kappa \leq \rho(b)$.

Lemma 1: Let $u_0 \in U(x_0) \cap U_b$ and assume that $x_0 \in X_{b,0}$. There exists some $\tau > 0$ such that for all $t \in (0, \tau]$, it holds that $V(x(t)) > V(x_0)$ and hence in particular $V(x(t)) > 0$, where $x(\cdot)$ is the trajectory of the system (1) that results from applying the constant input u_0 during this time interval.

Proof: See appendix. □

Next, we consider the situation where the state x is already away from the set W. Then, according to our assumptions, V is k times continuously differentiable and $\partial^k V/\partial x^k$ is locally Lipschitz, which we can use to show that if some input u_i is "good" at some point x_i in the sense that $V^{(k)}$ is positive, it is also "good" for nearby x_i.

Lemma 2: Consider some time instant $0 \leq s < a$ with $x(s) \in \mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}$, and assume that $x(s) \in X_{b,\delta}$ for some $\delta > 0$; furthermore, pick an arbitrary $u_s \in U(x(s)) \cap U_b$. Then, for each $0 < \varepsilon \leq 1$, there exists a number $\Delta(\varepsilon, \delta) > 0$ such that $V^{(k)}(x(t)) \geq (1-\varepsilon)\chi(b)$ and

$$V^{(j)}(x(t)) - V^{(j)}(x(s)) \geq \frac{1-\varepsilon}{(k-j)!}\chi(b)(t-s)^{k-j} \tag{14}$$

for all $j = 0, \dots, k-1$ and all $t \in [s, s + \Delta(\varepsilon, \delta)] \cap [0, a]$, where $x(\cdot)$ is the trajectory that results from applying the constant input u_s during this time interval, and $V^{(j)}(x(t)) := V^{(j)}(x(t); f(x(t), u_s)) = L_f^j V|_{(x(t), u_s)}$ for $j = 1, \dots, k-1$ and $V^{(0)}(x(t)) := V(x(t))$.

Proof: The proof of Lemma 2 is omitted in this conference paper due to space restrictions, but can be found in an online version of this paper [10]. □

Combining Lemmas 1 and 2, we are now able to prove Theorem 1. Fix an arbitrary $0 < \bar{\varepsilon} < 1$. Denote by Λ_b the sublevel set

$$\Lambda_b := \{x \in \mathbb{R}^n : V(x) \le \alpha_1(\rho(b))\}. \tag{15}$$

We construct a desired input signal in a recursive fashion using the following algorithm. This input signal will by construction satisfy $u(t) \in \bar{U}$ for all $t \in [0, a]$; hence, the resulting state trajectory $x(\cdot)$ will remain in the set $\mathcal{B}$ in this time interval if $x_0 \in \mathcal{B}$.

Step 0: Consider $x_0 \in \mathcal{B}$. If $x_0 \in \text{int}(\Lambda_b)$, then by (8) we have $|\omega(x_0)| < \rho(b)$ and so $x_0 \in X_{b,0}$ according to (13). We can then pick some $u_0 \in U(x_0) \cap U_b$, which exists by (9), and apply Lemma 1 to find a time $\tau > 0$ such that the trajectory corresponding to the constant control $u \equiv u_0$ satisfies $V(x(t)) > V(x_0)$ and hence in particular also $V(x(t)) > 0$ for all $0 < t \le \tau$. Pick some $t_1 \in (0, \min\{\tau, a\}]$; note that t_1 can be chosen arbitrarily small. Apply the constant input $u \equiv u_0$ on the interval $[0, t_1)$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (0, t_1)$, then denote this time t by $\check{t}_1$ and skip to Step 2, otherwise proceed to Step 1. If $x_0 \in \partial\Lambda_b$, let $t_1 = \check{t}_1 := 0$ and skip to Step 2. If already $x_0 \notin \Lambda_b$, then let $t_1 := 0$, pick some $u_1 \in U(x_0) \cap U_b$ which exists by (9), and apply the constant input $u \equiv u_1$ on the interval $[0, a)$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If $x(t) \in \partial\Lambda_b$ for some $t \in [0, a)$, then denote this time t by $\check{t}_1$ and skip to Step 2, otherwise skip to Step 3.

Step 1: If $x(t_1) \in \partial\Lambda_b$, then let $\check{t}_1 := t_1$ and skip to Step 2. Otherwise, $x(t_1) \in \text{int}(\Lambda_b)$. Let $\bar{\delta} := \alpha_2^{-1}(V(x(t_1)))$ and note that $\bar{\delta} > 0$ according to the definition of t_1 in Step 0. From (8) and the definition of Λ_b we have $\bar{\delta} \le |\omega(x(t_1))| < \rho(b)$, hence $x(t_1) \in X_{b,\bar{\delta}}$ by (13). We can thus pick some $u_1 \in U(x(t_1)) \cap U_b$ and apply Lemma 2 with $s = t_1$, $u_s = u_1$, $\varepsilon = \bar{\varepsilon}$ and $\delta = \bar{\delta}$ to find a $\Delta(\bar{\varepsilon}, \bar{\delta})$ such that the trajectory corresponding to the constant control $u \equiv u_1$ on the interval $[t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$ satisfies (14) with $s = t_1$ and $\varepsilon = \bar{\varepsilon}$ on this interval. Apply the constant input $u \equiv u_1$ on this interval for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$, then denote this time t by $\check{t}_1$ and skip to Step 2. If this does not happen but $t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}) \ge a$, then skip to Step 3. Otherwise, let $t_2 := t_1 + \Delta(\bar{\varepsilon}, \bar{\delta})$. In this case, $x(t_2) \in \Lambda_b$ and, by Lemma 2, $V(x(t_2)) > V(x(t_1))$ according to (14) with $j = 0$ and $s = t_1$. So, we can check that $x(t_2) \in X_{b,\bar{\delta}}$ in the same way as we did earlier for $x(t_1)$. Therefore, we can repeat Step 1 for $t_2, t_3, \ldots$ (but without changing the value of $\bar{\delta}$).

Step 2: We have $x(\check{t}_1) \in \partial\Lambda_b$, i.e., $V(x(\check{t}_1)) = \alpha_1(\rho(b))$. If $\check{t}_1 = a$ then skip to Step 3. Otherwise, pick some $\check{u}_1 \in U(x(\check{t}_1)) \cap U_b$, which exists by (9). Apply the constant input $u \equiv \check{u}_1$ on the interval $[\check{t}_1, \check{t}_2)$ where $\check{t}_2 := \min\{\inf\{t : t > \check{t}_1, x(t) \in \partial\Lambda_b\}, a\}$. This interval is non-empty; in fact, $\check{t}_2 \ge \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}$ where $\check{\delta} := \alpha_2^{-1}(\alpha_1(\rho(b)))$ and $\Delta(\cdot, \cdot)$ comes from Lemma 2. To see why this is true, note that $\check{\delta} \le |\omega(x(\check{t}_1))| \le \rho(b)$ according to (8) and the definition of $\check{t}_1$. Hence we can apply Lemma 2 with $s = \check{t}_1$, $u_s = \check{u}_1$, $\varepsilon = 1/2$ (or any other constant $0 < \varepsilon < 1$), $\delta = \check{\delta}$ and $j = 0$ in order to conclude that $V(x(t)) - V(x(\check{t}_1)) > 0$ for all $t \in (\check{t}_1, \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}]$, which implies that indeed $\check{t}_2 \ge \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}$. Moreover, if $\check{t}_2 < a$ then $x(\check{t}_2) \in \partial\Lambda_b$ and we can repeat Step 2 for $\check{t}_2$, $\check{t}_3, \ldots$

Step 3: We have now reached the time $t = a$ and we have constructed the following control input defined on the interval $[0, a)$, with the control values $u_i, \check{u}_j$ and the times $t_i, \check{t}_j$ as specified above (those times that are never defined are treated as ∞):

$$u(t) = \begin{cases} u_0 & 0 \le t < \min\{t_1, \check{t}_1\} \\ u_i & t_i \le t < \min\{t_{i+1}, \check{t}_1, a\}, \quad i = 1, 2, \ldots \\ \check{u}_j & \check{t}_j \le t < \min\{\check{t}_{j+1}, a\}, \quad j = 1, 2, \ldots \end{cases}$$

This input, extended with the last value (u_0, u_i or $\check{u}_j$) at $t = a$ (and arbitrarily for $t > a$), satisfies $u \in \mathcal{U}_{a,b}$, as by construction, $u(t) \in U_b$ for all $t \in [0, a]$. For each $0 < \bar{\varepsilon} < 1$, this input signal is piecewise constant in the interval $[0, a]$ with only finitely many different values u_i and $\check{u}_j$; this follows from the construction in Step 1 and the argument given in Step 2. The state trajectory $x(\cdot)$ resulting from the application of the control input $u(\cdot)$ to the system (1) has the following properties. First, consider the case where $t_1 > 0$ (recall from Step 0 that this corresponds to $x_0 \in \text{int}(\Lambda_b)$). By recursively applying (14) with $j = k - 1$ and $\varepsilon = \bar{\varepsilon}$, for $t_1 \le t \le \min\{\check{t}_1, a\}$ we have $V^{(k-1)}(x(t)) \ge (1 - \bar{\varepsilon})(t - t_1)\chi(b) + V^{(k-1)}(x(t_1)) \ge (1 - \bar{\varepsilon})(t - t_1)\chi(b)$, where the second inequality follows from (10) with $j = k - 1$, $x = x(t_1)$ and $u = u_1$. Integrating this inequality from t_1 to t, we obtain $V^{(k-2)}(x(t)) \ge (1/2)(1 - \bar{\varepsilon})(t - t_1)^2\chi(b) + V^{(k-2)}(x(t_1))$. Repeating the above $k - 2$ times results in $V(x(t)) \ge (1/k!)(1 - \bar{\varepsilon})(t - t_1)^k\chi(b) + V(x(t_1))$ for all for $t_1 \le t \le \min\{\check{t}_1, a\}$; furthermore, recall from Step 0 that $V(x(t_1)) > V(x_0)$ due to Lemma 1. Next, if $\check{t}_1 < a$, then for $\check{t}_1 \le t \le a$ the construction guarantees that $V(x(t)) \ge V(x(\check{t}_1)) = \alpha_1(\rho(b))$. Finally, if $t_1 = 0$ (recall from Step 0 that this corresponds to $x_0 \notin \text{int}(\Lambda_b)$), then the preceding inequality $V(x(t)) \ge \alpha_1(\rho(b))$ is satisfied for all $0 \le t \le a$. Combining the above yields $V(x(a)) \ge \min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}$. Hence, using (8), we have

$$|\omega(x(a))| \ge \alpha_2^{-1}\big(\min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}\big).$$

Finally, using (7), we obtain

$$|h(x(a))| \ge \nu\big(\alpha_2^{-1}\big(\min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}\big)\big).$$

As $u(\cdot)$ is contained in $\mathcal{U}_{a,b}$ and as the above calculations hold for arbitrary $x_0 \in \mathcal{B}$, it follows that

$$R_h^a(x_0, \mathcal{U}_{a,b}) \ge \nu\big(\alpha_2^{-1}\big(\min\{(1/k!)(1 - \bar{\varepsilon})(a - t_1)^k\chi(b) + V(x_0), \alpha_1(\rho(b))\}\big)\big) \tag{16}$$

for all $x_0 \in \mathcal{B}$. Note that (16) holds for every $0 < \bar{\varepsilon} \leq 1$, and according to Step 0, either $t_1 = 0$ or t_1 can be chosen arbitrarily small. Thus, as the left-hand side of (16) is independent of $\bar{\varepsilon}$ and t_1, we can let $\bar{\varepsilon} \to 0$ and $t_1 \to 0$ (in case t_1 is not 0) and arrive at the desired bound (3) with γ as defined in (12). The function γ satisfies the required properties of Definition 1, i.e., $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot) \in \mathcal{K}_\infty$ for each fixed $a > 0$. This concludes the proof of Theorem 1. □

In the following, we show that in Theorem 1, the assumption that the set $\mathcal{B}$ is control-invariant under $\bar{U}$ can be relaxed. Namely, for each $x \in \mathcal{B}$, let a set $U(x) \subseteq U$ be given such that the hypotheses of Theorem 1 expressed by (9)–(11) are satisfied, and define $\widetilde{U} := \cup_{x \in \mathcal{B}} U(x)$. If $\mathcal{B}$ is control-invariant under $\widetilde{U}$, we have the situation of Theorem 1 (with $\bar{U} = \widetilde{U}$); if not, consider the following.

Proposition 1: In Theorem 1, the assumption that a set $\bar{U}$ exists such that the set $\mathcal{B}$ is control-invariant under $\bar{U}$ can be replaced by the following. For each $b > 0$, there exists a set $H_b \subseteq \mathbb{R}^n$ with $H_b \cap \Lambda_b = \emptyset$ and Λ_b defined by (15) such that if $x_0 \in \mathcal{B}$ and $u(t) \in U_b \cap \widetilde{U}$ for all $t \geq 0$, then $x(t) \in \mathcal{B} \cup H_b$ for all $t \geq 0$. □

Proof: The proof of Proposition 1, which shows that the construction of the piecewise constant input signal in the proof of Theorem 1 is still valid under the new hypotheses, is omitted in this conference paper due to space restrictions, but can be found in an online version of this paper [10]. □

Remark 3: The condition in Proposition 1 means that each trajectory $x(\cdot)$ cannot exit $\mathcal{B}$ before exiting Λ_b. In other words, when at some time instant t we have $x(t) \in \Lambda_b$, then also $x(t) \in \mathcal{B}$. □

C. Sufficient condition involving lower directional derivatives of different order

In this section, we generalize the previous results to the case where the control-invariant set $\mathcal{B}$ can be partitioned into several regions where (10)–(11) holds for different k. To this end, for a set $\mathbb{X} \subseteq \mathbb{R}^n$ and $\ell \geq 1$, denote by $\mathcal{R}^\ell(\mathbb{X})$ a partition of $\mathbb{X}$ such that $\mathbb{X} = \cup_{i=1}^{\ell} \mathcal{R}_i$ and $\mathcal{R}_i$ is closed for all $1 \leq i \leq \ell$. Note that $\mathcal{R}^1(\mathbb{X}) = \mathbb{X}$.

Theorem 2: Suppose there exist a set $\bar{U} \subseteq U$ and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is rendered control-invariant by $\bar{U}$ for system (1). Furthermore, suppose there exist a partition $\mathcal{R}^\ell(\mathcal{B})$ for some $\ell \geq 1$ with corresponding integer constants $1 \leq k_1 < k_2 < ... < k_\ell \leq \bar{k}$, a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$, which is k_ℓ times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and $\partial^{k_\ell} V / \partial x^{k_\ell}$ is locally Lipschitz there, functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$, a function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\mu(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$, and for each $x \in \mathcal{B}$ a set $U(x) \subseteq \bar{U}$, such that the hypotheses of Theorem 1 expressed by (7)–(9) are satisfied as well as the following: For all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $x \in \mathcal{R}_i$ for some $1 \leq i \leq \ell$, $u \in U(x)$, and $|\omega(x)| \leq \rho(|u|)$, we have

$$V^{(j)}(x; f(x,u)) \geq 0 \qquad j = 1, \ldots, k_i - 1, \qquad (17)$$

$$V^{(k_i)}(x; f(x,u)) \geq \chi_i(|u|). \qquad (18)$$

Then the system (1) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with scaling function μ and gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\Psi(a,b) + V(x_0), \alpha_1(\rho(b))\Big\}\Big)\Big),$$

where

$$\Psi(a,b) = \min_{i \in \{1,\ldots,\ell\}} \frac{(k_\ell - k_i)!}{k_\ell!} a^{k_i} \chi_i(b). \qquad (19)$$

Proof: The proof of Theorem 2 is omitted in this conference paper due to space restrictions, but can be found in an online version of this paper [10]. □

Remark 4: In the special case of $\ell = 1$, Theorem 1 is recovered. □

Remark 5: It is straightforward to verify that Proposition 1 also can be applied to Theorem 2, i.e., the assumption that a set $\bar{U}$ exists which renders $\mathcal{B}$ control-invariant can be relaxed as described in Proposition 1. Furthermore, the results of Theorem 2 can also be extended in a straightforward way to the case where the sets $\mathcal{R}_i$ of the partition $\mathcal{R}^\ell(\mathcal{B})$ depend on the magnitude of the applied input, i.e., $\mathcal{R}_i = \mathcal{R}_i(b)$. Namely, in this case, the condition of Theorem 2 expressed by (17)–(18) is modified as follows. *For each* $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ *and each* $b > 0$ *such that* $x \in \mathcal{R}_i(b)$, $u \in U(x) \cap U_b$, *and* $|\omega(x)| \leq \rho(b)$, *it holds that* (17) *is satisfied and* $V^{(k_i)}(x; f(x,u)) \geq \chi_i(b)$. With this modification, the proof works exactly the same, as for each fixed $b > 0$, the same fixed partition (depending on b) is considered. □

Example 1: Consider an isothermal continuous stirred tank reactor (CSTR) in which an irreversible, second-order reaction from reagent A to product B takes place [11]:

$$dC_A/dt = \frac{q}{V}(C_{A_i} - C_A) - kC_A^2$$
$$dC_B/dt = -\frac{q}{V}C_B + kC_A^2,$$

where C_A and C_B denote the concentrations of species A and B (in $[mol/m^3]$), respectively, V is the volume of the reactor (in $[m^3]$), q is the flow rate of the inlet and outlet stream (in $[m^3/s]$), k is the reaction rate (in $[1/s]$), and C_{A_i} is the concentration of A in the inlet stream, which can be interpreted as the input. Using $x_1 := C_A$, $x_2 := C_B$, $c := q/V$ and $u := C_{A_i}$, one obtains the system

$$\dot{x}_1 = -cx_1 - kx_1^2 + cu =: f_1(x,u)$$
$$\dot{x}_2 = kx_1^2 - cx_2 =: f_2(x). \qquad (20)$$

The physically meaningful states and inputs are $x_1 \geq 0, x_2 \geq 0, u \geq 0$, i.e., nonnegative concentrations of the two species. We are interested in the amount of product B per time unit, i.e. in the output $y = h(x) = qx_2$.

Consider the region $\mathcal{B} := \{x : 0 \leq x_2 \leq (k/c)x_1^2\}$, and for each $x \in \mathcal{B}$, let $U(x) := \mathbb{R}_{\geq 0}$. Hence also $\widetilde{U} := \cup_{x \in \mathcal{B}} U(x) = \mathbb{R}_{\geq 0}$. The set $\mathcal{B}$ is not rendered control-invariant by $\widetilde{U}$; however, one can show that for each $b > 0$, the set $H_b := \{x : x_1 \geq \delta(b), x_2 \geq (k/c)x_1^2\}$ with $\delta(b) := (-c + \sqrt{c^2 + 4ckb})/(2k)$ satisfies the conditions of Proposition 1. Let $0 < \varepsilon, \theta < 1$, and for $b \geq 0$ define $\varphi_1(b; \varepsilon) := (-(3-\varepsilon)c + \sqrt{(3-\varepsilon)^2c^2 + 16ck\theta b})/(4k)$,

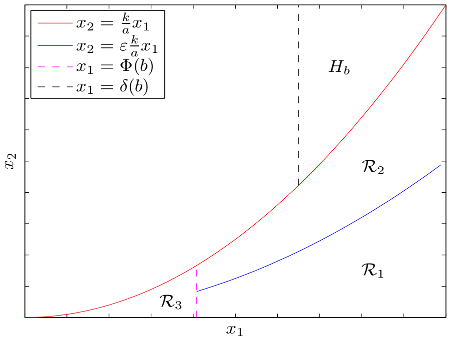

Fig. 1. Partition of the control-invariant region $\mathcal{B}$ and the set H_b in Example 1.

$\varphi_2(b) := \min\{cb/(8(c+k)), \sqrt{cb/(8(c+k))}\}$ and $\Phi(b) := \min\{\varphi_1(b;\varepsilon), \varphi_2(b)\}$. Now consider the following partition of $\mathcal{B}$, which is also exemplarily depicted in Figure 1:

$$\mathcal{R}_1(b) := \{x \in \mathcal{B} : 0 \le x_2 \le \varepsilon(k/c)x_1^2, x_1 \ge \Phi(b)\},$$
$$\mathcal{R}_2(b) := \{x \in \mathcal{B} : (\varepsilon k/c)x_1^2 \le x_2 \le (k/c)x_1^2, x_1 \ge \Phi(b)\},$$
$$\mathcal{R}_3(b) := \{x \in \mathcal{B} : 0 \le x_2 \le (k/c)x_1^2, x_1 \le \Phi(b)\}.$$

Taking $\omega(x) = x_2$ and $V(x) = |\omega(x)|$, one obtains that $V^{(1)}(x; f(x,u)) \ge 0$ for all $x \in \mathcal{B}$ and

$$V^{(1)}(x; f(x,u)) \ge (1-\varepsilon)kx_1^2 \ge (1-\varepsilon)k\Phi^2(b) =: \chi_1(b)$$

for all $x \in \mathcal{R}_1$. For all $x \in \mathcal{R}_2$ and $u \in U(x) \cap U_b$, we get

$$V^{(2)}(x; f(x,u)) \ge 2(1-\theta)kcb\Phi(b) =: \chi_2(b),$$

for all $x_1 \le \varphi_1(b;\varepsilon)$, which holds if $x_2 \le (\varepsilon k/c)\varphi_1^2(b;\varepsilon) =: \rho(b)$. Finally, for all $x \in \mathcal{R}_3$ and $u \in U(x) \cap U_b$, we obtain that $V^{(2)}(x; f(x,u)) \ge 0$ and $V^{(3)}(x; f(x,u)) \ge kc^2b^2 =: \chi_3(b)$. Furthermore, it is straightforward to verify that for each $b > 0$, $H_b \cap \Lambda_b = \emptyset$, where $\Lambda_b = \{x : |x_2| \le \rho(b)\}$.

Summarizing the above, we can apply Theorem 2 with $\ell = 3$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $\alpha_1 = \alpha_2 = \mu = \text{id}$ and $\nu = q\text{id}$ together with Remark 5 and Proposition 1 to conclude that the system (20) is norm-controllable from all $x_0 \in \mathcal{B}$ with scaling function $\mu = \text{id}$ and gain function $\gamma(r,s) = q\min\{\Psi(r,s) + V(x_0), \rho(s)\}$ with Ψ defined in (19). An interpretation of this fact is as follows. If $x_2 \le (k/c)x_1^2$, then a sufficiently large amount of reagent A compared to the amount of product B is present in the reactor in order that the amount of product B can be increased. On the other hand, if $x_2 > (k/c)x_1^2$, then already too much product B is inside the reactor such that its amount will first decrease (due to the outlet stream), no matter how large the concentration of A in the inlet stream (i.e., the input u) is, and hence the system is not norm-controllable for such initial conditions.

IV. Conclusions

In this paper, we obtained two sufficient conditions for a system to be norm-controllable. The first one involves kth-order lower directional derivatives for some fixed k, while the second involves higher-order lower directional derivatives of different order. These conditions allow us to establish norm-controllability for output maps with arbitrary relative degree. Furthermore, we illustrated the obtained results through a chemical reaction example.

Appendix

Proof of Lemma 1: First, note that an input u_0 as defined in the lemma exists, as according to (9), $U(x_0) \cap U_b \ne \emptyset$. Let $h(t) := 1/t^k\big(x(t) - x_0 - th_1 - \cdots - t^{k-1}h_{k-1}\big)$ for $t > 0$, where $h_1, \dots, h_{k-1}$ are defined as in (6) with $t' = 0$, $x = x_0$, and $u = u_0$. Note that h varies continuously in t and $\lim_{t\searrow 0} h(t) =: h_k$ according to (5). Furthermore, for $t > 0$, define the function

$$g(t) := k!/t^k\Big(V(x_0 + th_1 + \cdots + t^k h(t)) - V(x) - tV^{(1)}(x_0; h_1) - \cdots - t^{k-1}V^{(k-1)}(x_0; h_1, \dots, h_{k-1})\Big)$$

Consider $g_- := \liminf_{t\searrow 0} g(t)$. By the definitions of g and $V^{(k)}$, it holds that

$$g_- = \liminf_{t\searrow 0} g(t) \ge V^{(k)}(x_0; f(x_0,u_0)) \ge \chi(|u_0|) \ge \chi(b).$$

The first inequality holds because in the definition of $V^{(k)}$ in (4), the infimum over all $\bar{h}_k$ with $\bar{h}_k \to h_k$ is taken, while in g_- the specific choice $\bar{h}_k = h(t) \to h_k$ is used. Thus, by definition of the (one-sided) limit inferior, for every $\varepsilon > 0$ there exists a $\tau > 0$ such that for all $0 < t \le \tau$, it holds that

$$g(t) \ge g_- - \varepsilon \ge \chi(b) - \varepsilon, \tag{21}$$

and thus

$$\begin{aligned} V(x(t)) &= V(x_0 + th_1 + \cdots + t^k h(t)) \\ &= (1/k!)g(t)t^k + V(x_0) + tV^{(1)}(x_0; h_1) \\ &\quad + \cdots + t^{k-1}V^{(k-1)}(x_0; h_1, \dots, h_{k-1}) \\ &\overset{(21),(10)}{\ge} \frac{1}{k!}(\chi(b) - \varepsilon)t^k + V(x_0) > V(x_0) \ge 0. \end{aligned}$$

The second but last inequality is due to the fact that ε can be made arbitrarily small such that $\chi(b) > \varepsilon$. □

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