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On almost Lyapunov functions

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Abstract— **We study asymptotic stability properties of nonlinear systems in the presence of "almost Lyapunov" functions which decrease along solutions in a given region not everywhere but rather on the complement of a set of small volume. Nothing specific about the structure of this set is assumed besides an upper bound on its volume. We show that solutions starting inside the region approach a small set around the origin whose volume depends on the volume of the set where the Lyapunov function does not decrease, as well as on other system parameters. The result is established by a perturbation argument which compares a given system trajectory with nearby trajectories that lie entirely in the set where the Lyapunov function is known to decrease, and trades off convergence speed of these trajectories against the expansion rate of the distance to them from the given trajectory.**

I. INTRODUCTION

Consider a general dynamical system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{1}$$

Suppose that we are working on a compact region $D \subset \mathbb{R}^n$ which contains an equilibrium of interest for the system (1), say the origin. What we want to know is whether all solutions of (1) that lie in D converge to the origin or at least to a small neighborhood of the origin.

A well-known method for showing convergence of all solutions of (1) to the origin consists in finding a Lyapunov function (see, e.g., [1]). This is a positive definite smooth function $V : \mathbb{R}^n \to [0, \infty)$ which decays along all non-zero solutions, i.e., its derivative along solutions satisfies

$$\dot{V}(x) := \frac{\partial V}{\partial x}(x) \cdot f(x) < 0 \qquad \forall\, x \neq 0. \tag{2}$$

Given a candidate Lyapunov function V, it is usually not difficult to analytically compute the expression for $\dot{V}$ on the left-hand side of the inequality in (2). However, verifying the inequality itself—i.e., checking that $\dot{V}$ is negative definite—may be quite challenging. For example, in case both V and f are polynomial functions, $\dot{V}$ is also a polynomial and we need to check that this polynomial is negative definite. This problem is related to Hilbert's 17th problem [2] and is an important subject of current research (see, e.g., [3], [4]).

As a workaround to trying to establish the inequality in (2) by deterministic methods, one can consider techniques based on random sampling, in the context of so-called randomized algorithms [5]. The basic idea is to generate at random a sequence of points x_i in D and check whether the inequality in (2) holds for all of them. If it does, then one can use the Chernoff bound (see, e.g., [5], [6]) to characterize the number of such sample points needed to obtain a reliable upper bound on the measure of points in D for which the inequality can possibly fail. In this way, instead of (2) we would verify, with some level of confidence, the property

$$\dot{V}(x) := \frac{\partial V}{\partial x}(x) \cdot f(x) < 0 \qquad \forall\, x \in D \setminus \Omega \tag{3}$$

where Ω is an unknown subset of D (containing the origin) whose relative measure in D does not exceed some known small number $\varepsilon > 0$.

Going back to our original objective of studying the asymptotic behavior of solutions of the system (1), we are now faced with the following question: under what circumstances does an inequality of the form (3) allow us to still prove that all solutions of (1) converge to some small neighborhood of 0? Standard Lyapunov stability methods are no longer applicable, since inside Ω the "almost Lyapunov" function V can in general increase along solutions. However, if the measure of Ω is small, we expect that this increase can be dominated by the decrease of V in the complement of Ω.

In this paper we present what we believe to be the first known result along these lines. Namely, we show that if the measure of the set Ω where V is not known to decrease is less than a sufficiently small number ε, then all solutions starting inside D (at some ε-dependent minimal distance from its boundary) approach a sublevel set of V whose volume depends on ε and continuously approaches 0 as ε tends to 0. We accomplish this by a perturbation argument which compares a given system trajectory with a nearby trajectory that lies, at least for some time, entirely in the set $D \backslash \Omega$ where the Lyapunov function is known to decrease. This nearby trajectory, which converges towards 0 (at least on some interval of time), is shown to eventually "pull" the given trajectory towards 0 as well, even though V initially might not decrease along the given trajectory. Suitable relations between system constants and the volume of Ω must hold in order for the comparison trajectory to converge faster than the given trajectory can deviate from it. The argument is completed by iterating over time.

Section II contains the necessary definitions. Our main result (Theorem 1) is stated in Section III. Its proof is given in Section IV. Section V contains a numerical example, and Section VII concludes the paper.

II. PRELIMINARIES

The system is given by (1), where we assume that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. We consider

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a candidate Lyapunov function $V : \mathbb{R}^n \to [0,\infty)$ which is positive definite and C^1 with locally Lipschitz gradient V_x. For a given $c > 0$, consider the region

$$D := V^{-1}([0,c]) \tag{4}$$

We need D to be compact and connected. Compactness of D is automatic if V is radially unbounded.[1] In case D is not connected, redefine it to be the connected component of $V^{-1}([0,c])$ containing 0; we can then modify V outside D so that (4) becomes true again.

Denoting by $|\cdot|$ the standard Euclidean norm in $\mathbb{R}^n$, we define the following constants:

$$L_0 := \max_{x\in D} |f(x)|, \tag{5}$$

L_1 is a Lipschitz constant of f over D:

$$|f(x_1) - f(x_2)| \le L_1|x_1 - x_2| \qquad \forall\, x_1, x_2 \in D, \tag{6}$$

$$M_1 := \max_{x\in D} |V_x(x)|, \tag{7}$$

M_2 is a Lipschitz constant of V_x over D:

$$|V_x(x_1) - V_x(x_2)| \le M_2|x_1 - x_2| \qquad \forall\, x_1, x_2 \in D \tag{8}$$

Define $\dot V(x) = V_x(x)\cdot f(x)$. We assume that there exist a constant $a > 0$ and a subset $\Omega \subset D$ such that

$$\dot V(x) \le -aV(x) \qquad \forall\, x \in D \setminus \Omega \tag{9}$$

Here a is known, while Ω is not known but later we'll impose an upper bound on its volume vol(Ω). For every $q \in [0,a]$, we define the set

$$G_q := \{x \in D : \dot V(x) \le -qV(x)\} \tag{10}$$

III. Main result

Let $B_r(z)$ denote the closed ball in $\mathbb{R}^n$ with radius r and center z (when only the volume of the ball is important, we will sometimes omit the center). Define the function $\rho : (0,\infty) \to (0,\infty)$ by the relation

$$\mathrm{vol}(B_{\rho(\varepsilon)}) = \varepsilon \qquad \forall\, \varepsilon > 0 \tag{11}$$

where "vol" is the standard volume in $\mathbb{R}^n$. The following is our main result.

Theorem 1 *Consider the system* (1) *with a locally Lipschitz right-hand side f, and a function V which is positive definite and C^1 with locally Lipschitz gradient. Let the region D be defined via* (4) *and assume that it is compact and connected. Assume that* (9) *holds. Then there exist a constant $\bar\varepsilon > 0$ and a function $\bar R : [0,\bar\varepsilon] \to [0,\infty)$ which is 0 at 0, continuous, and increasing,*[2] *such that for every $\varepsilon \in (0,\bar\varepsilon]$, if vol$(\Omega) < \varepsilon$ then for every initial condition $x_0 \in D$ with*

$$V(x_0) < c - 2M_1\rho(\varepsilon) \tag{12}$$

where M_1 is defined by (7)*, the corresponding solution $x(\cdot)$ of* (1) *(with $x(0) = x_0$) has the following properties:*

- *$V(x(t)) \le V(x_0) + 2M_1\rho(\varepsilon)$ for all $t \ge 0$ (and hence $x(t) \in D$ for all $t \ge 0$).*
- *$V(x(T)) \le \bar R(\varepsilon)$ for some $T \ge 0$.*
- *$V(x(t)) \le \bar R(\varepsilon) + 2M_1\rho(\varepsilon)$ for all $t \ge T$.*

Obviously, for the result to be meaningful, $\bar\varepsilon$ must be small enough so that $2M_1\rho(\bar\varepsilon) < c$ and $\bar R(\varepsilon) < c - 2M_1\rho(\varepsilon)$.

Remark 1 A consequence of the conditions (11) and vol$(\Omega) < \varepsilon$ is that Ω cannot contain a ball of radius $\rho(\varepsilon)$. It will be clear from the proof of Theorem 1 that this latter property (which is less restrictive than the condition that the total volume of Ω be smaller than ε) is enough for the theorem to hold. In other words, if every point $x \in D$ is known to be within a distance less than $\rho(\varepsilon)$ from the set G_a, then the conclusions of the theorem are valid. To verify this condition in practice, we can in principle just sample sufficiently many points from D (for example, points on a uniform lattice) and check if (9) holds for all of them.

Remark 2 In the special case when $\dot V(x) \le -aV(x)$ for all $x \in D$, Theorem 1 reduces to Lyapunov's classical asymptotic stability theorem. Indeed, $\Omega = \emptyset$ means that the condition vol$(\Omega) < \varepsilon$ holds for arbitrarily small $\varepsilon > 0$. Since $\bar R(\cdot)$ is 0 at 0 and continuous, the conclusions of Theorem 1 then imply that $V(x(t)) \to 0$ as $t \to \infty$ and the system is asymptotically stable. (Note, though, that V is not necessarily strictly decreasing along the trajectories.)

IV. Proof of Theorem 1

We suppose that an $\varepsilon > \mathrm{vol}(\Omega)$ is given, and we write simply ρ for $\rho(\varepsilon)$ defined via (11). We start by noting the following simple property of V.

Lemma 1 *For all $x_1, x_2 \in D$, we have*

$$V(x_1) - M_1|x_2 - x_1| \le V(x_2) \le V(x_1) + M_1|x_2 - x_1|$$

Proof. Applying the mean-value theorem, we have $V(x_2) = V(x_1) + V_x(x_3)\cdot(x_2 - x_1)$, where x_3 is a point on the segment connecting x_1 and x_2. If this segment is contained in D (which is guaranteed when D is convex), then using (7) we have $|V_x(x_3)\cdot(x_2 - x_1)| \le M_1|x_2 - x_1|$ and the claim follows. On the other hand, if the segment connecting x_1 and x_2 gets outside of D, then by (4) each subsegment lying outside of D connects two points of the level set $\{x : V(x) = c\}$. "Collapsing" such subsegments and applying the mean-value theorem on those subsegments that lie in D, it is easy to see that the claim still holds. $\square$

Remark 3 If V were a norm (not necessarily the Euclidean norm) then the above analysis via the mean-value theorem could be replaced with one based on the triangle inequality. We would then have to work with this norm throughout. However, norms on $\mathbb{R}^n$ are not differentiable at 0 (because they are equivalent to the Euclidean norm which is not differentiable at 0). Although we could allow V to not be differentiable around 0, we do not pursue this option here.

[1] V is *radially unbounded* if $V(x) \to \infty$ whenever $|x| \to \infty$ or, equivalently, all sublevel sets of V are compact.

[2] Such functions are known as *class $\mathcal{K}$* functions.

A. An approximating trajectory $\bar{x}$

Lemma 2 *For every x_0 satisfying* (12)*, the ball $B_\rho(x_0)$ is contained in D and is not contained in Ω.*

PROOF. We already know from the definition (11) of ρ and the bound vol$(\Omega) < \varepsilon$ that $B_\rho(x_0)$ is not contained in Ω (see Remark 1). To show that $B_\rho(x_0) \subseteq D$, note that for each x with $|x - x_0| \le \rho$ we have from Lemma 1 that $V(x) \le V(x_0)+M_1\rho$, which in view of (12) implies $V(x) < c$ hence $x \in D$ by the definition (4) of D. □

It follows from Lemma 2 that there exists a point $\bar{x}_0 \in B_\rho(x_0)$ which is in $D\backslash\Omega$ and hence is in G_a by (9) and (10).

Lemma 3 *Two solutions $x(\cdot)$ and $\bar{x}(\cdot)$ of* (1) *with initial conditions x_0 and $\bar{x}_0$ satisfy, as long as they are both in D,*

$$|x(t) - \bar{x}(t)| \le |x_0 - \bar{x}_0|e^{L_1 t}$$

PROOF. This is a standard consequence of (6) and the Bellman-Gronwall lemma (see, e.g., [1]).

Lemma 4 *For every initial condition $\bar{x}_0 \in G_a$ and every constant $\xi \in (0, 1)$, the corresponding solution $\bar{x}(\cdot)$ of* (1) *satisfies $\bar{x}(t) \in G_{\xi a}$ for all $t \in [0, T^+_{\bar{x}_0}]$, where*

$$T^+_{\bar{x}_0} := \frac{(1-\xi)aV(\bar{x}_0)}{(M_1L_1 + L_0M_2 + \xi aM_1)L_0} \tag{13}$$

PROOF. Define the function $h(x) := \dot{V}(x) + \xi aV(x) = V_x(x)\cdot f(x)+\xi aV(x)$. Let us calculate its Lipschitz constant over D. For arbitrary $x_1, x_2 \in D$, we have

$$\begin{aligned}
&|h(x_1) - h(x_2)| \\
&= |V_x(x_1)\cdot f(x_1) + \xi aV(x_1) - V_x(x_2)\cdot f(x_2) - \xi aV(x_2)| \\
&\le |V_x(x_1)\cdot f(x_1) - V_x(x_1)\cdot f(x_2)| \\
&+ |V_x(x_1)\cdot f(x_2) - V_x(x_2)\cdot f(x_2)| + \xi a|V(x_1) - V(x_2)| \\
&\le \max_{x\in D} |V_x(x)|\cdot|f(x_1) - f(x_2)| \\
&+ \max_{x\in D} |f(x)|\cdot|V_x(x_1) - V_x(x_2)| + \xi a|V(x_1) - V(x_2)| \\
&\le (M_1L_1 + L_0M_2 + \xi aM_1)|x_1 - x_2|
\end{aligned}$$

Now, let $x_1 := \bar{x}(t)$ for some t, and let $x_2 := \bar{x}_0$. As long as $\bar{x}(\cdot)$ remains in D, we have $|\bar{x}(t) - \bar{x}_0| \le L_0t$, and so the above calculation implies $|h(\bar{x}(t)) - h(\bar{x}_0)| \le (M_1L_1 + L_0M_2 + \xi aM_1)L_0t$. Since $\bar{x}_0 \in G_a$, we know that $h(\bar{x}_0) \le -(1-\xi)aV(\bar{x}_0)$. Noting that $(M_1L_1 + L_0M_2 + \xi aM_1)L_0T^+_{\bar{x}_0} = (1-\xi)aV(\bar{x}_0)$, we see that $h(\bar{x}(t)) \le |h(\bar{x}(t)) - h(\bar{x}_0)| + h(\bar{x}_0) \le 0$, which is equivalent to $\bar{x}(t) \in G_{\xi a}$, for $t \le T^+_{\bar{x}_0}$. This proves the lemma because, as long as V is not increasing, $\bar{x}(\cdot)$ remains in D and the above estimates are valid. □

Remark 4 The calculations in the proof of Lemma 4 also tell us the following: if $\bar{x} \in G_a$ and if another point x satisfies

$$|x - \bar{x}| \le \frac{(1-\xi)aV(\bar{x})}{M_1L_1 + L_0M_2 + \xi aM_1}$$

then $x \in G_{\xi a}$. In particular, letting $\xi \to 0$, we obtain the "robustness radius" for stability, i.e., the radius of a ball around a point $\bar{x} \in G_a$ which consists only of points x such that $\dot{V}(x) < 0$. In case D is completely covered by such balls, V is actually decreasing everywhere along solutions; otherwise, Theorem 1 allows the possibility that V temporarily increases.

B. The bounding function $W_{\rho,\gamma,R}$

Let us pick a number $\gamma \in (0, 1)$. We want to look for a time T at which $V(x(t))$ becomes smaller than its original value $V(x_0)$ by the factor of γ:

$$V(x(T)) \le \gamma V(x_0) \tag{14}$$

To this end, we employ an auxiliary trajectory $\bar{x}(\cdot)$ which starts in G_a and use its properties established in Lemmas 3 and 4 to try to show that $\bar{x}(\cdot)$ at least temporarily "pulls" $x(\cdot)$ towards the origin.[3] For $R > 0$ and $\xi \in (0, 1)$, define the function

$$W_{\rho,\gamma,R}(t) := M_1\rho\big(e^{L_1t} + e^{-\xi at}\big) + R\big(e^{-\xi at} - \gamma\big) \tag{15}$$

where $M_1 \ge 0$, $\rho > 0$, $L_1 \ge 0$, and $a > 0$ come from (7), (11), (6), and (9), respectively. We treat ρ, R, and γ as parameters which we will eventually allow to vary, while ξ is also a design parameter but, once chosen, we keep it fixed. For $R \ge M_1\rho$, define also

$$T^+_\rho(R) := \frac{(1-\xi)a(R - M_1\rho)}{(M_1L_1 + L_0M_2 + \xi aM_1)L_0} \tag{16}$$

Lemma 5 *Suppose that $V(x_0)$ satisfies* (12) *and $V(x_0) \ge R$ for some $R > M_1\rho$ such that the interval $[\frac{1}{\xi a}\ln\frac{1}{\gamma}, T^+_\rho(R)]$ is nonempty, and that for some T in this interval we have $W_{\rho,\gamma,R}(T) \le 0$. Then $x(t) \in D$ for $t \in [0, T]$ and* (14) *holds for the same T.*

PROOF. Use Lemma 2 to pick a point $\bar{x}_0 \in B_\rho(x_0)\cap G_a$, and let $\bar{x}(\cdot)$ be the solution of (1) with $\bar{x}(0) = \bar{x}_0$. Since $V(x_0) \ge R$ and $|\bar{x}_0 - x_0| \le \rho$, by Lemma 1 we have $V(\bar{x}_0) \ge R - M_1\rho$. Comparing (13) with (16) we see that $T^+_\rho(R) \le T^+_{\bar{x}_0}$ and hence by Lemma 4 we have $\bar{x}(t) \in G_{\xi a}$ for all $t \in [0, T^+_\rho(R)]$. As long as $t \le T^+_\rho(R)$ and $x(t) \in D$, this implies

$$\begin{aligned}
V(x(t)) &\le V(\bar{x}(t)) + M_1|x(t) - \bar{x}(t)| \\
&\le e^{-\xi at}V(\bar{x}_0) + M_1|x_0 - \bar{x}_0|e^{L_1t} \\
&\le e^{-\xi at}\big(V(x_0) + M_1\rho\big) + M_1\rho e^{L_1t}
\end{aligned}$$

where the first inequality follows from Lemma 1, the second inequality relies on Lemmas 3 and 4, and to arrive at the last inequality we used Lemma 1 again and the fact that $|x_0 - \bar{x}_0| \le \rho$. Now, let T be as in the statement of the

[3]We note that the reasoning employed here has some resemblance to arguments showing finiteness of entropy for Lipschitz systems, such as the proof of [7, Theorem 3.3].

lemma, and note that $e^{-\xi a T} - \gamma \leq 0$ because $T \geq \frac{1}{\xi a} \ln \frac{1}{\gamma}$. Thus we have

$$V(x(T)) - \gamma V(x_0) \leq M_1 \rho (e^{L_1 T} + e^{-\xi a T}) + V(x_0)(e^{-\xi a T} - \gamma) \leq W_{\rho,\gamma,R}(T) \leq 0$$

as claimed. The proof will be complete when we show that $x(t) \in D$ for $t \in [0, T]$. For this, consider the function $t \mapsto e^{-\xi a t}\big(V(x_0) + M_1\rho\big) + M_1 \rho e^{L_1 t}$ which as we know provides an upper bound for $V(x(t))$ as long as $x(t) \in D$. This function takes the value $V(x_0) + 2M_1\rho$ at $t = 0$, which is strictly smaller than c by (12), and its value at T does not exceed $\gamma V(x_0)$ which is also smaller than c. Moreover, it is easy to check that this function is convex (its second derivative is positive). It follows that this function stays strictly below c on the interval $[0, T]$, hence $x(t)$ cannot escape D during this time interval. □

Let $T_{\rho,\gamma}(R)$ be the time (if one exists) when the function $W_{\rho,\gamma,R}$ changes sign from positive to negative, i.e., the time instant with the properties

$$W_{\rho,\gamma,R}(T_{\rho,\gamma}(R))) = 0, \qquad W'_{\rho,\gamma,R}(T_{\rho,\gamma}(R))) \leq 0$$

By convexity of $W_{\rho,\gamma,R}$ (which is easily seen from the fact that its second derivative is positive), if $T_{\rho,\gamma}(R)$ exists then it is unique. It also must satisfy $T_{\rho,\gamma}(R) \geq \frac{1}{\xi a} \ln \frac{1}{\gamma}$ because for $t < \frac{1}{\xi a} \ln \frac{1}{\gamma}$ we have $e^{-\xi a t} - \gamma > 0$ hence $W_{\rho,\gamma,R}(t) > 0$ by (15).

Remark 5 In view of the properties of $W_{\rho,\gamma,R}$ just noted above, a necessary condition for the existence of $T_{\rho,\gamma}(R)$ is that $W'_{\rho,\gamma,R}(t)$ be negative at $t = \frac{1}{\xi a} \ln \frac{1}{\gamma}$. We can calculate that this derivative is $M_1 \rho L_1 e^{\frac{L_1}{\xi a} \ln \frac{1}{\gamma}} - (R + M_1\rho)\xi a \gamma$, and after simplification we get a more conservative but simpler necessary condition

$$R > M_1\big(1 + \tfrac{L_1}{a}\big)\rho = \frac{M_1 L_1 + a M_1}{a}\rho \tag{17}$$

Lemma 6 *For R large enough, $T_{\rho,\gamma}(R)$ exists and satisfies $T_{\rho,\gamma}(R) \leq T_\rho^+(R)$.*

PROOF. Fix some $\bar{t} > \frac{1}{\xi a} \ln \frac{1}{\gamma}$. For this $\bar{t}$ we have $e^{-\xi a \bar{t}} - \gamma < 0$, hence $W_{\rho,\gamma,R}(\bar{t}) < 0$ for R large enough. This means that for large R the time $T_{\rho,\gamma}(R)$ exists and is smaller than $\bar{t}$. Next, consider a pair of values $R' > R$ for which the corresponding times $T_{\rho,\gamma}(R')$ and $T_{\rho,\gamma}(R)$ exist. Since $e^{-\xi a t} - \gamma$ is negative at both of these times, it is clear that $W_{\rho,\gamma,R'}(T_{\rho,\gamma}(R)) < 0$, hence $T_{\rho,\gamma}(R') < T_{\rho,\gamma}(R)$. In other words, $T_{\rho,\gamma}$ is monotonically decreasing in R. On the other hand, $T_\rho^+(R)$ is monotonically increasing to ∞ as $R \to \infty$ in view of (16), and the lemma follows. □

We let $\mathcal{S}_{\rho,\gamma}$ denote the set of values of R with the properties stated in Lemma 6:

$$\mathcal{S}_{\rho,\gamma} := \{R > 0 : T_{\rho,\gamma}(R) \text{ exists and } T_{\rho,\gamma}(R) \leq T_\rho^+(R)\}$$

This set is closed (as its complement is open due to continuous dependence of $W_{\rho,\gamma,R}$ and T_ρ^+ on their arguments) and thus admits a minimum

$$\bar{R}_{\rho,\gamma} := \min \mathcal{S}_{\rho,\gamma}$$

It actually follows from the proof of Lemma 6 that the set $\mathcal{S}_{\rho,\gamma}$ has very simple structure: $\mathcal{S}_{\rho,\gamma} = [\bar{R}_{\rho,\gamma}, \infty)$. The minimum, $\bar{R}_{\rho,\gamma}$, corresponds to either $W'_{\rho,\gamma,R}(T_{\rho,\gamma}(\bar{R}_{\rho,\gamma}))) = 0$ or $T_{\rho,\gamma}(\bar{R}_{\rho,\gamma}) = T_\rho^+(\bar{R}_{\rho,\gamma})$. We must have $\bar{R}_{\rho,\gamma} > M_1\rho$ because $T_\rho^+(M_1\rho) = 0$ and $T_\rho^+(R)$ is not defined for $R < M_1\rho$. Note that this bound is already subsumed by (17).

Lemma 5 guarantees that when $V(x_0)$ satisfies (12) and $V(x_0) \geq \bar{R}_{\rho,\gamma}$, the solution $x(\cdot)$ remains in D on $[0, T_{\rho,\gamma}(\bar{R}_{\rho,\gamma})]$ and satisfies

$$V(x(T_{\rho,\gamma}(\bar{R}_{\rho,\gamma})) \leq \gamma V(x_0)$$

At $t = T_{\rho,\gamma}(\bar{R}_{\rho,\gamma})$, we can reset the time to 0 and repeat the above analysis. This means that $V(x(t))$ decreases at least by the factor of γ at the end of each time interval of length $T_{\rho,\gamma}(\bar{R}_{\rho,\gamma})$, as long as $V(x(t)) \geq \bar{R}_{\rho,\gamma}$. Therefore, $V(x(t))$ will reach the value $\bar{R}_{\rho,\gamma}$ in finite time. From that time onwards, we know from the proof of Lemma 5 that $V(x(t))$ will remain bounded from above by $\bar{R}_{\rho,\gamma} + 2M_1\rho$.

Remark 6 Note that the choice of ξ affects both T_ρ^+ (explicitly) and $T_{\rho,\gamma}$ (implicity through $W_{\rho,\gamma,R}$). An interesting question is which value of ξ is optimal in the sense of giving the smallest $\bar{R}_{\rho,\gamma}$.

C. The limit $\gamma \to 1$ and behavior for small ε

In the above analysis, $\gamma \in (0, 1)$ was a design parameter, and we will now let it approach 1 from below. We extend the definitions of the function $W_{\rho,\gamma,R}$, its first zero-crossing time $T_{\rho,\gamma}$, the set $\mathcal{S}_{\rho,\gamma}$, and its minimum $\bar{R}_{\rho,\gamma}$ in the obvious way to include the value $\gamma = 1$.

Lemma 7 *$\bar{R}_{\rho,\gamma}$ is continuously decreasing in γ, and $\bar{R}_{\rho,1} = \lim_{\gamma \nearrow 1} \bar{R}_{\rho,\gamma}$.*

PROOF. From (15) it is clear that $W_{\rho,\gamma,R}$ decreases when γ increases. Thus if $R \in \mathcal{S}_{\rho,\gamma}$ and $\gamma' > \gamma$, then $W_{\rho,\gamma',R}(T_{\rho,\gamma}(R)) < 0$. This means that R is in the interior of $\mathcal{S}_{\rho,\gamma'}$, which gives the decreasing property. Due to the continuous dependence of $W_{\rho,\gamma,R}$ on R and γ, it is not hard to see that $\bar{R}_{\rho,\gamma}$ is continuous in γ and that for every $\delta > 0$ we have $\bar{R}_{\rho,1} + \delta \in \mathcal{S}_{\rho,\gamma}$ for some $\gamma < 1$. Together with the decreasing property this gives the last claim. □

We can now examine the conclusions of Section IV-B in the limit as $\gamma \to 1$. Suppose that $V(x_0)$ satisfies (12) and $V(x_0) > \bar{R}_{\rho,1}$. Then by Lemma 7, we have $V(x_0) \geq \bar{R}_{\rho,\gamma}$ for all γ sufficiently close to 1. For each such γ, we know that $V(x(t))$ will reach the value $\bar{R}_{\rho,\gamma}$ in finite time, and from that time onwards $V(x(t))$ will remain bounded from above by $\bar{R}_{\rho,\gamma} + 2M_1\rho$. This corresponds to the claims of the theorem if we let $\varepsilon \mapsto \bar{R}(\varepsilon)$ be any function strictly larger than $\bar{R}_{\rho(\varepsilon),1}$ for $\varepsilon > 0$.

It remains to show that the conditions (12) and $V(x_0) > \bar{R}_{\rho,1}$ can be simultaneously satisfied if ε is small enough, and that we can choose $\bar{R}(\cdot)$ to be 0 at 0. Since ρ was defined by (11), $\varepsilon \to 0$ if and only if $\rho \to 0$.

Lemma 8 *$\bar{R}_{\rho,\gamma}$ is continuously increasing in ρ, and $\lim_{\rho\searrow 0}\bar{R}_{\rho,1}=0$.*

PROOF. The first claim is proved similarly to the first claim of Lemma 7. To prove the second claim, we note from (15) with $\gamma=1$ that as $\rho\to 0$, with $R>0$ arbitrary and fixed, $T_{\rho,1}(R)$ becomes well defined and approaches 0. Indeed, if we fix an arbitrary $\bar{t}>0$ and define two positive numbers $\delta_1 := M_1(e^{L_1\bar{t}}+e^{-\xi a\bar{t}})$ and $\delta_2 := -e^{-\xi a\bar{t}}+1$, then $W_{\rho,\gamma,R}(\bar{t})=\delta_1\rho-\delta_2 R$ hence $T_{\rho,1}(R)$ exits and is smaller than $\bar{t}$ for $R>(\delta_1/\delta_2)\rho\to 0$ as $\rho\to 0$. At the same time, $T^+_\rho(R)$ approaches $\frac{(1-\xi)aR}{(M_1L_1+L_0M_2+\xi aM_1)L_0}$ in view of (16). Thus, every $R>0$ belongs to $\mathcal{S}_{\rho,1}$ for ρ small enough, which gives the result. □

Lemma 8 confirms that the constraints (12) and $V(x_0)>\bar{R}_{\rho,1}$ are indeed consistent for ρ small enough, hence for ε small enough, and that it is possible to have $\bar{R}(\varepsilon)\to 0$ as $\varepsilon\to 0$. For $\bar{\varepsilon}$ in the theorem statement, we can take any positive number satisfying $\bar{R}(\bar{\varepsilon})<c-2M_1\rho(\bar{\varepsilon})$. This completes the proof of the theorem.

V. Numerical example

The purpose of this section is to illustrate how Theorem 1 can be applied in practice on a specific example. Consider the system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}-\lambda & \mu\\ -\mu & -\lambda\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{18}$$

We can take D to be a ball $B_d(0)$, and for V we can take $\alpha(|x|)$ for some function α (here $\alpha(d)=c$). We easily calculate that $L_0=d\sqrt{\lambda^2+\mu^2}$, $L_1=\sqrt{\lambda^2+\mu^2}$. For concreteness, here we choose the candidate Lyapunov function $V(x)=|x|^2$, i.e., $\alpha(r)=r^2$. Then we have $M_1=2d$, $M_2=2$, and $a=-2\lambda$.

Next, we must select the following parameters: $\rho>0$ such that the right-hand side of (12) is positive; $\xi\in(0,1)$ whose selection is in principle arbitrary but can be tuned to give the best results (in the sense of Remark 6); γ which for practical purposes we can take equal to 1; and R which initially can be any number satisfying (17) and no larger than $c-2M_1\rho$. (We should decrease ρ if necessary so that this range of values for R is nonempty).

We can now calculate $T^+_\rho(R)$ from the formula (16). The next step is to plot the bounding function $W_{\rho,\gamma,R}$ given by (15) for $t\in[0,T^+_\rho(R)]$ and check for a zero crossing on this interval. If a zero crossing does not occur, we have to increase R until a zero crossing is observed. We let $\bar{R}$ denote the smallest value of R for which this happens (this value can be computed to a desired accuracy by a simple bisection procedure). Finally, we check that

$$\bar{R}+2M_1\rho\le c. \tag{19}$$

If this inequality does not hold, then ρ must be decreased. Lemma 8 guarantees that if we keep decreasing ρ to 0, eventually (19) will be verified. Once we find the value $\bar{\rho}$ for which (19) holds with equality (again, this can be approximately computed by a bisection procedure), the constant $\bar{\varepsilon}$ in Theorem 1 can be any positive number smaller than $\pi\bar{\rho}^2$, and ρ should be taken smaller than $\bar{\rho}$. We then conclude that every system whose right-hand side differs from that of (18) on some set of volume less than $\pi\rho^2$ (or on some set not containing a ball of radius ρ, according to Remark 1)—*and whose constants L_0 and L_1 are not larger than those for the original system*—has the properties listed in Theorem 1 with the corresponding value $\bar{R}$ found by the above procedure. (Here $\bar{R}$ depends on our choice of ρ, which in turn is related to ε via (11).)

For our numerical study, we took $\lambda=\mu=d=c=1$ and $\xi=0.5$. For $\rho=0.01$, we have $c-2M_1\rho=0.96$ and the above procedure terminates with $\bar{R}\approx 0.7$ and $\bar{R}+2M_1\rho\approx 0.74$, which gives a meaningful ultimate bound on solutions. For $\rho=0.015$, we obtain $\bar{R}\approx 0.86$ and $\bar{R}+2M_1\rho\approx 0.92$. The upper bound on feasible values of ρ in this case is $\bar{\rho}\approx 0.018$. Regarding the choice of ξ, values between 0.4 and 0.5 seem to give the best results for this example.

VI. Challenge and discussion

Theorem 1 allows us to establish convergence even if the decrease condition $\dot{V}\le -aV$ is not known to hold everywhere in the region of interest. A question that remains open is whether this result actually permits the existence of points outside the region of convergence at which $\dot{V}$ is positive, i.e., points $x\in D$ such that $x\notin G_0$ and $V(x)>\bar{R}$ (we omit the ε argument, treating ε as fixed). Suppose that such an x exists. By Remark 1, the ball of radius ρ around x cannot be completely contained in Ω, and thus contains a point $z\in G_a$. Then by Remark 4 with $\xi=0$, the ball of radius $\Delta x := \frac{aV(z)}{M_1L_1+L_0M_2}$ around z is completely contained in G_0. To avoid contradiction, we must have $\rho>\Delta x$. Remembering that $V(x)>\bar{R}$ and $|z-x|\le\rho$ and using Lemma 1, we arrive at the necessary condition $\rho>\frac{a(\bar{R}-M_1\rho)}{M_1L_1+L_0M_2}$ which is equivalent to $\bar{R}<\frac{M_1L_1+L_0M_2+aM_1}{a}\rho$. Earlier, we derived the necessary condition (17), and the good news is that the two conditions are consistent. In other words, so far there is no obvious reason why we cannot in principle have points of instability within the context of Theorem 1. The challenge is to construct an example where such points of instability are actually present.

Let us start again with the system (18) from Section V. Pick a number $\hat{R}>\alpha^{-1}(\bar{R})$ so that $V(x)>\bar{R}$ when $|x|=\hat{R}$, i.e., $\hat{R}$ is (slightly) larger than the convergence radius $\bar{R}$ provided by Theorem 1. Suppose we want to modify the right-hand side of (18) as follows: inside the ball of radius ρ around the point $\begin{pmatrix}\hat{R}\\ 0\end{pmatrix}$, perturb the vector field so that at the point $\begin{pmatrix}\hat{R}\\ 0\end{pmatrix}$ itself it becomes vertical (and thus critically stable with respect to our Lyapunov function). Outside this ball, the vector field should remain as is.

Consider the pair of points $x_{\text{rad},\pm} := \begin{pmatrix}\hat{R}\pm\rho\\ 0\end{pmatrix}$ on the boundary of this ball. At these points the velocity vector is (and will remain)

$$\dot{x}=\begin{pmatrix}-\lambda & \mu\\ -\mu & -\lambda\end{pmatrix}\begin{pmatrix}\hat{R}\pm\rho\\ 0\end{pmatrix}=-\begin{pmatrix}\lambda\\ \mu\end{pmatrix}(\hat{R}\pm\rho)$$

At the point $x = \begin{pmatrix} \hat{R} \\ 0 \end{pmatrix}$ the velocity vector originally is $\dot{x} = \begin{pmatrix} -\lambda\hat{R} \\ -\mu\hat{R} \end{pmatrix}$ but we want to change it to $\dot{x} = \begin{pmatrix} 0 \\ -\mu\hat{R} \end{pmatrix}$. For the modified vector field, the new value of L_1 will have to satisfy

$$L_1 \geq \frac{1}{\rho}\left|\begin{pmatrix} -\lambda(\hat{R}\pm\rho) \\ -\mu(\hat{R}\pm\rho) \end{pmatrix} - \begin{pmatrix} 0 \\ -\mu\hat{R} \end{pmatrix}\right| = \sqrt{\lambda^2\left(\tfrac{\hat{R}}{\rho}\pm 1\right)^2 + \mu^2}$$

Choosing the "+" sign gives a larger lower bound on L_1.

Now consider another pair of points, $x_{\text{ang},\pm} := \hat{R}\begin{pmatrix} \cos\alpha \\ \pm\sin\alpha \end{pmatrix}$, where α is a (small) positive angle. For a value of α close to $\rho/\hat{R}$, these two points also lie on the boundary of the ball of radius ρ centered at $\begin{pmatrix} \hat{R} \\ 0 \end{pmatrix}$. At these points the velocity vector is then unchanged and equals $\hat{R}\begin{pmatrix} -\lambda\cos\alpha \pm \mu\sin\alpha \\ -\mu\cos\alpha \mp \lambda\sin\alpha \end{pmatrix}$. This gives us another lower bound

$$\begin{aligned} L_1 &\geq \frac{\hat{R}}{\rho}\left|\begin{pmatrix} -\lambda\cos\alpha \pm \mu\sin\alpha \\ -\mu\cos\alpha \mp \lambda\sin\alpha \end{pmatrix} - \begin{pmatrix} 0 \\ -\mu \end{pmatrix}\right| \\ &= \frac{\hat{R}}{\rho}\sqrt{\lambda^2 + 2\mu\big(\mu(1-\cos\alpha) \mp \lambda\sin\alpha\big)} \end{aligned}$$

Here choosing the point $x_{\text{ang},-}$ gives the "+" sign in the formula and hence a larger lower bound. This bound is monotonically increasing in μ, while the bound corresponding to $x_{\text{ang},+}$ has a minimum for a positive μ. (This can also be seen from geometric considerations.)

We see that L_1 does get larger as a result of the modification. (On the other hand, L_0 does not change because we are not increasing the velocity.) Thinking naively, we may hope that we can calculate $\bar{R}$ for a given ρ and a slightly enlarged L_1, and then show that the above modification increases L_1 by a small enough factor so that the same $\bar{R}$ is still valid after the modification. Unfortunately, this approach does not appear to be feasible, for the following reason. Suppose that our result applies to the vector field modified as explained above and indicates convergence to the ball $B_{\alpha^{-1}(\bar{R})}(0)$. Let us now treat μ as a parameter and make it approach 0. In the limit as $\mu \to 0$, the point $x = \begin{pmatrix} \hat{R} \\ 0 \end{pmatrix}$ becomes an equilibrium of the modified vector field. Neither a (which depends only on λ) nor L_0 change as we take this limit, and the above calculations show that L_1 does not increase. Thus the convergence claim must remain valid for $\mu = 0$, but of course it cannot be valid in the presence of an equilibrium outside the convergence ball.

We note that the above example is characterized by the following aspects (revealed by taking the limit $\mu \to 0$): starting with the assumption that the (modified) vector field is tangent to a level set of V at some point, by canceling the "angular" component of the vector field (the component tangential to the level sets of V) while preserving the radial component, we were able to create another vector field which has the same (or at least not larger) constants a, L_0, L_1 and has an equilibrium at that point. This led us to a contradiction. Therefore, to construct an example where our theorem allows points of instability, we need to find an example where this procedure cannot be carried out because canceling the angular component would increase L_1.

In view of our analysis of the points $x_{\text{ang},\pm}$ in the above example, it appears promising to consider a candidate Lyapunov function whose c-level set has an "inflection point" and to make the vector field f tangent to this level set at that point; see Figure 1. The following values appear to be

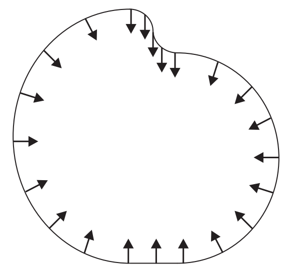

Fig. 1. A possible level set of V and a vector field f on it.

consistent with this situation: $c = a = L_0 = L_1 = M_1 = 1$, $M_2 = \sqrt{2}/\rho$. However, due to the fact that M_2 is large when ρ is small, these values are not feasible for Theorem 1. More effort is needed to realize this idea.

VII. Conclusions

We presented a result (Theorem 1) which establishes convergence of system trajectories from a given set to a smaller set, based on an "almost Lyapunov" function which is known to decrease along solutions on the complement of a set of small enough volume. The proof was based on trading off convergence speed of trajectories along which the Lyapunov function decreases against the expansion rate of the distance between nearby trajectories of the system.

One limitation of Theorem 1 is that it is designed to apply to *every* system with given values of the constants c, a, L_0, L_1, M_1, M_2. Taking into account specific structure of the system dynamics (e.g., rotation rate in the plane) may pave the way to a less conservative result.

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