Input-to-State Stability for Switched Systems with Unstable Subsystems: A Hybrid Lyapunov Construction

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Abstract— The input-to-state stability (ISS) of a nonlinear switched system is investigated in the scenario where there may exist some subsystems that are not input-to-state stable (non-ISS). We show that, providing the switching signal neither switches too frequently nor activates non-ISS subsystems for too long, a hybrid ISS Lyapunov function can be constructed to guarantee ISS of the switched system. With the constraints on the switching signal being modeled by a novel auxiliary timer, a hybrid system is defined so that the solutions to the two systems are correspondent. After the construction and verification of an ISS Lyapunov function, ISS of all complete solutions to the hybrid system, and therefore all solutions to the switched system, is conveniently proved.

I. INTRODUCTION

In this paper, we explore the stability property of nonlinear switched systems using hybrid system techniques. The study of switched systems has attracted a lot of attention in recent years (see, e.g., [1] and references therein). It is well-known that, a switched system does not necessarily inherit the stability properties of its subsystems. For example, in [1, Part II] it was shown that a switched system consisting of two asymptotically stable subsystems may not be stable. For linear systems, it was proved in [2] that such a switched system can achieve asymptotic stability providing the switching signal satisfies a certain dwell-time condition. This approach was then generalized to the nonlinear system context and to the concept of average dwell-time condition in [3]. In [4], a similar result was developed for a linear switched system with both stable and unstable subsystems by restricting the fraction of time in which the unstable subsystems are active.

When disturbances or controls are present, the notion of input-to-state stability (ISS) proposed by Sontag [5] has proved to be valuable in the stability analysis of nonlinear systems. In this context, an important result is that ISS is equivalent to the existence of an ISS Lyapunov function [6]. The dual concept of ISS, output-to-state stability (OSS), and their combination, input/output-to state stability (IOSS), were introduced in [7] and [8], respectively. The study of stability property inheritance in nonlinear switched systems was extended to the ISS context by Xie et al. [9] under dwell-time conditions, Vu et al. [10] under average dwell-time conditions, and to the IOSS context by Müller and Liberzon [11] under average dwell-time conditions as well. Furthermore, in [11] the IOSS property of a switched nonlinear system

was studied also for the general case where some of the subsystems are not input/output-to-state stable.

In this work, we consider the same general scenario as in [11]: in the switched system there may exist some subsystems that are not input-to-state stable (non-ISS). It is proved that, providing the switching signals neither switch too frequently (average dwell-time constraint) nor activate non-ISS subsystems for too long (time-ratio constraint), an ISS Lyapunov function can be constructed by introducing an auxiliary timer and adopting hybrid system techniques. In particular, a hybrid system is defined such that the solutions to the two systems are correspondent and the constraints on the switching signal are modeled by the auxiliary timer. For the hybrid system, an ISS Lyapunov function is then constructed to establish the ISS property for all complete solutions to the hybrid system, and therefore all solutions to the switched system. Although the result that a switched system with not necessarily ISS subsystems is ISS under certain average dwell-time condition and time-ratio condition has already been proved in [11], the Lyapunov-based formulation in this paper exhibits improvements: it not only generates an ISS Lyapunov function which can be used later in the study of interconnected systems [12], but provides means for robustness analysis as well. Also, our construction of the novel hybrid timer is of interest in its own right.

Hybrid systems are dynamic systems that possess both continuous-time and discrete-time features. In our analysis of hybrid systems, we adopt the modeling framework proposed by Goebel et al. [13], which proved to be general and natural from the viewpoint of Lyapunov stability theory. The concepts of ISS and ISS Lyapunov function were extended to hybrid systems in [14]. In the hybrid system context, a detailed study of constructing ISS Lyapunov functions under undesired flow or jump behaviors can be found in [15, Section IV]. Comparing to [15], our result on modifying the ISS Lyapunov function to guarantee its decrease along solutions is more general in the sense that it applies to the situation where the original ISS Lyapunov functions are increasing both at the jumps and during some of the flows. Based on the idea of restricting non-ISS subsystems' total activation time proportion proposed in [4] and [11], an aforementioned auxiliary timer is introduced in the construction of the hybrid system to manage the non-ISS flows.

This paper is structured as follows: In Section II, we introduce some preliminaries. Our main result—the sufficient condition that guarantees ISS of a nonlinear switched system with both ISS and non-ISS subsystems—is presented and clarified in Section III. A detailed proof, prefaced by an

introduction to hybrid systems, is provided in Section IV. In Section V, our approach is demonstrated in a simulation example. Section VI concludes the paper.

II. PRELIMINARIES

Consider a family of dynamic systems

$$\dot{x} = f_p(x, u), \qquad p \in \mathcal{P} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $\mathcal{P}$ is the *index set* (which can in principle be arbitrary). For all $p \in \mathcal{P}$, f_p is locally Lipschitz and $f_p(0, 0) = 0$. Given the family (1), a *switched system*

$$\dot{x} = f_\sigma(x, u) \qquad (2)$$

is generated by a *switching signal* $\sigma \colon \mathbb{R}_{\geq 0} \to \mathcal{P}$ which specifies the index of the active system at time t. The switching signal σ is assumed to be piecewise constant and right-continuous. Let ψ_k ($k \in \mathbb{Z}_{>0}$) denote the time when the k-th switch occurs and define $\Psi := \{\psi_k : k \in \mathbb{Z}_{>0}\}$ as the set of switching time instants, which is assumed to contain no accumulation points. (Thus the switched system (2) has at most one switch at any time instant and finitely many switches in any finite time interval.) A function u is an admissible input to the switched system (2) if it is measurable and locally essentially bounded.

Following Morse [2], we say that a switching signal σ satisfies the *dwell-time condition* if there exists a $\tau_d \in \mathbb{R}_{>0}$, called the *dwell-time*, such that for all consecutive switching time instants $\psi_k, \psi_{k+1} \in \Psi$,

$$\psi_{k+1} - \psi_k \geq \tau_d. \qquad (3)$$

A generalization of this concept was introduced by Hespanha and Morse [3]: a switching signal σ is said to satisfy the *average dwell-time condition* if there exists a $\tau_a \in \mathbb{R}_{>0}$, called the *average dwell-time*, and $N_0 \in \mathbb{Z}_{\geq 0}$ such that

$$N(t_2, t_1) \leq N_0 + \frac{t_2 - t_1}{\tau_a} \qquad \forall\, t_2 \geq t_1 \geq 0, \qquad (4)$$

where $N(t_2, t_1)$ denotes the number of switches in the time interval $(t_1, t_2]$. Note that the dwell-time condition can be interpreted as a special case of the average dwell-time condition with $N_0 = 1$ and $\tau_a = \tau_d$.

For two vectors x_1 and x_2, let $(x_1, x_2) := (x_1^\top, x_2^\top)^\top$. For a vector $x \in \mathbb{R}^n$, we use $|x|$ to denote its Euclidean norm. For a compact set $A \subset \mathbb{R}^n$, we use $|x|_A$ to denote the Euclidean distance from a vector x to A. For a function $u \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $\|u\|_t$ is used to denote its essential supremum (Euclidean) norm on the interval $[0, t]$.

A function $\alpha \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class $\mathcal{K}$* if it is continuous, strictly increasing and positive definite. It is of *class $\mathcal{K}_\infty$* if $\alpha \in \mathcal{K}$ and $\lim_{r \to \infty} \alpha(r) = \infty$. In particular, this implies that α is globally invertible. A function $\beta \colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class $\mathcal{KL}$* if $\beta(\cdot, t) \in \mathcal{K}$ for all fixed t, $\beta(r, \cdot)$ is decreasing and $\lim_{t \to \infty} \beta(r, t) = 0$ for all fixed r.

As introduced by Sontag [5], a dynamic system from family (1) is called *input-to-state stable* (ISS) if there exist functions $\gamma \in \mathcal{K}_\infty, \beta \in \mathcal{KL}$ such that for all initial states $x(0) \in \mathbb{R}^n$ and all inputs $u \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^m$,

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_t) \qquad \forall\, t \in \mathbb{R}_{\geq 0}. \qquad (5)$$

The definition of input-to-state stability (ISS) also applies to switched systems. Note that for an autonomous dynamic system (i.e. $u \equiv 0$), inequality (5) is equivalent to the notion of *global asymptotic stability* (GAS) [16, Proposition 2.5].

III. MAIN RESULT

Consider the switched system (2) with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $\sigma \in \mathcal{P}$, which may contain both ISS and non-ISS subsystems. Let $\mathcal{P}_s$ and $\mathcal{P}_u$ denote the subsets of $\mathcal{P}$ containing the indices of ISS and non-ISS subsystems, respectively. Then $(\mathcal{P}_s, \mathcal{P}_u)$ forms a partition of $\mathcal{P}$ (i.e., $\mathcal{P}_s \cup \mathcal{P}_u = \mathcal{P}$ and $\mathcal{P}_s \cap \mathcal{P}_u = \varnothing$). Following Müller and Liberzon [11], we define $T_s(t_2, t_1)$ as the total activation time of ISS subsystems (i.e., subsystems from $\mathcal{P}_s$) on the time interval $(t_1, t_2]$ and $T_u(t_2, t_1)$ that for non-ISS subsystems. Then $T_s(t_2, t_1) + T_u(t_2, t_1) = t_2 - t_1$.

We introduce three constraints in the following assumption; the first two are frequently used in the context of switched systems, while the last one (Time-Ratio Constraint) is somewhat less standard. The idea of restricting the fraction of time during which non-ISS subsystems are active in the third constraint is essentially introduced in [4] and [11].

Assumption 1 The following three constraints are satisfied:
UNIFORM ISS LYAPUNOV-TYPE CONSTRAINT   There exists a family of positive definite $\mathcal{C}^1$ functions $V_p \colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}, p \in \mathcal{P}$ such that the following conditions hold:
1. $\exists\, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $p \in \mathcal{P}$,

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|). \qquad (6)$$

2. $\exists\, \phi \in \mathcal{K}_\infty, \lambda_s, \lambda_u \in \mathbb{R}_{>0}$ such that for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $p_s \in \mathcal{P}_s, p_u \in \mathcal{P}_u$,

$$|x| \geq \phi(|u|) \Rightarrow$$
$$\begin{cases} \dfrac{\partial V_{p_s}(x)}{\partial x} \cdot f_{p_s}(x, u) \leq -\lambda_s V_{p_s}(x), \\ \dfrac{\partial V_{p_u}(x)}{\partial x} \cdot f_{p_u}(x, u) \leq \lambda_u V_{p_u}(x). \end{cases} \qquad (7)$$

3. $\exists\, \mu \in \mathbb{R}_{\geq 1}$ such that for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$,

$$V_p(x) \leq \mu V_q(x). \qquad (8)$$

AVERAGE DWELL-TIME CONSTRAINT   The switching signal σ satisfies the average dwell-time condition (4) with constants $\tau_a \in \mathbb{R}_{>0}$ and $N_0 \in \mathbb{Z}_{\geq 0}$.
TIME-RATIO CONSTRAINT   There exists $\rho \in [0, 1)$ and $T_0 \in \mathbb{R}_{\geq 0}$ such that the total activation time of non-ISS subsystems satisfies

$$T_u(t_2, t_1) \leq T_0 + \rho(t_2 - t_1) \qquad \forall\, t_2 \geq t_1 \geq 0. \qquad (9)$$

In Assumption 1, the Uniform ISS Lyapunov-Type Constraint is a constraint on the subsystems' dynamics, while the Time-Ratio Constraint and the Average Dwell-Time Constraint are constraints on the switching signal.

Remark 1 The Uniform ISS Lyapunov-Type Constraint in Assumption 1 is "Lyapunov-type" in the sense that it constrains not only the ISS subsystems, but the non-ISS subsystems as well. The existence of functions V_{p_s} satisfying (7) for $p_s \in \mathcal{P}_s$ follows from the fact that these subsystems are ISS [17], while the existence of functions V_{p_u} satisfying (7) for $p_u \in \mathcal{P}_u$ is equivalent to the forward completeness property of non-ISS subsystems [18].

Remark 2 The Uniform ISS Lyapunov-Type Constraint in Assumption 1 is "uniform" since it is satisfied by ISS Lyapunov functions V_p for all subsystems, with fixed class $\mathcal{K}_\infty$ functions α_1, α_2, ϕ and constants $\lambda_s, \lambda_u, \mu$. This uniformity can be concluded automatically for some particular types of index sets. For example, (6) is guaranteed if $\mathcal{P}$ is finite and all subsystems are ISS [10, Remark 1]. Besides, for positive definite functions V_p, the existence of the uniform ratio bound μ in (8) is a sufficient condition for the existence of the uniform comparison functions α_1, α_2 in (6).

One of our main contributions is to give a Lyapunov-based proof of the following theorem, established earlier in [11] using trajectory-based arguments:

Theorem 1 ([11, Theorem 2]) *Consider the switched system (2). Suppose that Assumption 1 holds with*

$$\lambda_s > \frac{\ln(\mu)}{\tau_a} + \rho(\lambda_s + \lambda_u) =: \gamma. \tag{10}$$

Then the switched system is input-to-state stable.

Remark 3 Note that equation (10) can be rewritten as

$$(1 - \rho)\lambda_s - \rho\lambda_u - \frac{\ln(\mu)}{\tau_a} > 0,$$

which helps provide a clearer interpretation of this condition. Here $(1 - \rho)\lambda_s$ measures the average rate of exponential decay of the ISS Lyapunov functions due to the ISS subsystems, while $\rho\lambda_u$ and $\ln(\mu)/\tau_a$ measure their exponential growth due to the non-ISS subsystems and the switches, respectively. Thus this condition can be interpreted as saying that the ISS Lyapunov functions are decreasing on average.

IV. PROOF OF THE MAIN RESULT

A detailed proof of Theorem 1 is presented in this section. We start by introducing some preliminaries for hybrid systems in Subsection IV-A. In Subsection IV-B, a correspondent hybrid system is constructed for the switched system (2) under Assumption 1. An ISS Lyapunov function for the hybrid system is defined and verified in Subsection IV-C. Subsection IV-D concludes the proof by establishing the ISS property for all complete solutions to the hybrid system, and therefore all solutions to the switched system (2).

A. Preliminaries for Hybrid Systems

Following Goebel et al. [13, Chapter 2], a *hybrid system* with inputs can be modeled as

$$\begin{cases} \dot{z} \in F(z, u), & z \in C, \\ z^+ \in G(z, u), & z \in D, \end{cases} \tag{11}$$

where $z \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $C \subset \mathbb{R}^n$ is the *flow set*, $D \subset \mathbb{R}^n$ is the *jump set*, $F \colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the *flow map* and $G \colon \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is the *jump map*.[1] (In this model, if $z \in C$, then the state can flow at a velocity $\dot{z} \in F(z, u)$; if $z \in D$, then the state can jump to a point $z^+ \in G(z, u)$; if $z \in C \cap D$, then there are two possibilities: the state can either flow or jump.) $\mathcal{H} = (C, F, D, G)$ is called the *data* of the hybrid system. The solutions to the hybrid system are defined on the so-called hybrid time domain. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is a *compact hybrid time domain* if

$$E = \bigcup_{k=0}^{K} ([\theta_k, \theta_{k+1}], k) \tag{12}$$

for some finite sequence of times $0 = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{K+1}$. E is a *hybrid time domain* if for all $(T, K) \in E$, $E \cap ([0, T] \times \{0, 1, \dots, K\})$ is a compact hybrid time domain. A *hybrid arc* is a function $z \colon \operatorname{dom} z \to \mathbb{R}^n$ defined on a hybrid time domain such that for each fixed $k \in \mathbb{Z}_{\geq 0}$, $z(\cdot, k)$ is locally absolutely continuous on $\{t : (t, k) \in \operatorname{dom} z\} =: \Theta_k^z$. A hybrid arc is *complete* if its domain is unbounded. A *hybrid input* is a function $u \colon \operatorname{dom} u \to \mathbb{R}^m$ defined on a hybrid time domain such that for each fixed $k \in \mathbb{Z}_{\geq 0}$, $u(\cdot, k)$ is Lebesgue measurable and locally essentially bounded on $\{t : (t, k) \in \operatorname{dom} u\} = \Theta_k^u$. A hybrid arc $z \colon \operatorname{dom} z \to \mathbb{R}^n$ is a solution to a hybrid system $\mathcal{H} = (C, F, D, G)$ with hybrid input $u \colon \operatorname{dom} u \to \mathbb{R}^m$ if the following conditions hold:

1. $\operatorname{dom} z = \operatorname{dom} u$.
2. $z(t, k) \in C$ and $\dot{z}(t, k) \in F(z(t, k), u(t, k))$ for all $k \in \mathbb{Z}_{\geq 0}$ and almost all $t \in \Theta_k^z$.[2]
3. $z(t, k) \in D$ and $z(t, k+1) \in G(z(t, k), u(t, k))$ for all $(t, k) \in \operatorname{dom} z$ such that $(t, k+1) \in \operatorname{dom} z$.

With proper assumptions on the data $\mathcal{H}$, one can establish the local existence of solutions to the hybrid system, which may not be necessarily unique; cf. [13, Proposition 2.10].

Following Cai and Teel [14], for a function defined on a hybrid time domain $z \colon \operatorname{dom} z \to \mathbb{R}^n$, the essential supremum (Euclidean) norm up to hybrid time (t, k) is denoted by $\|z\|_{(t,k)}$ and defined as

$$\|z\|_{(t,k)} :=$$

$$\max \left\{ \operatorname*{ess\,sup}_{\substack{(s,l) \in \operatorname{dom} z \setminus J(z) \\ s \leq t, l \leq k}} |z(s, l)|, \sup_{\substack{(s,l) \in J(z), \\ s \leq t, l \leq k}} |z(s, l)| \right\},$$

where $J(z)$ is the set of all $(s, l) \in \operatorname{dom} z$ such that $(s, l+1) \in \operatorname{dom} z$. (Note that the set of measure 0 that can be ignored when computing this essential supremum norm cannot include any jump time instants.)

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[1]We use "$\rightrightarrows$" to denote a set-valued mapping.
[2]Here $z(t, k)$ represents the state of the system at time t and after k jumps.

B. A Correspondent Hybrid System

In this subsection, we construct a hybrid system whose state consists of variables representing the state of the switched system, the switching signal and an auxiliary timer τ. The dynamics of the timer is specifically designed to not only incorporate the effect of the Average Dwell-Time Constraint and Time-Ratio Constraint in Assumption 1, but also enable us to construct an ISS Lyapunov function for the hybrid system in Subsection IV-C.

Consider the hybrid system with state $z = (\widetilde{x}, \widetilde{\sigma}, \tau) \in \mathbb{R}^n \times \mathcal{P} \times [0, \Gamma] =: \mathcal{Z}$ and input $\widetilde{u} \in \mathbb{R}^m$ defined as follows:

$$\begin{cases} \dot{z} \in F(z, \widetilde{u}), & z \in C, \\ z^+ \in G(z), & z \in D, \end{cases} \tag{13}$$

where

$$F(z, \widetilde{u}) := \begin{cases} \begin{bmatrix} \{f_{\widetilde{\sigma}}(\widetilde{x}, \widetilde{u})\} \\ \{0\} \\ [0, \gamma] \end{bmatrix}, & \text{if } \widetilde{\sigma} \in \mathcal{P}_s, \\ \begin{bmatrix} \{f_{\widetilde{\sigma}}(\widetilde{x}, \widetilde{u})\} \\ \{0\} \\ \{\gamma - (\lambda_s + \lambda_u)\} \end{bmatrix}, & \text{if } \widetilde{\sigma} \in \mathcal{P}_u, \end{cases} \tag{14}$$

$C := \mathbb{R}^n \times \mathcal{P} \times [0, \Gamma],$
$G(z) := \{\widetilde{x}\} \times (\mathcal{P} \backslash \{\widetilde{\sigma}\}) \times \{\tau - \ln(\mu)\},$
$D := \mathbb{R}^n \times \mathcal{P} \times [\ln(\mu), \Gamma].$

with

$$\Gamma := N_0 \ln(\mu) + T_0(\lambda_s + \lambda_u) \tag{15}$$

and the constant γ is defined in (10). We will show that the following proposition holds:

Proposition 1 *Consider a solution x to the switched system (2) with an input u and a switching signal σ. Suppose that Assumption 1 is satisfied. Then there exists a complete solution $z = (\widetilde{x}, \widetilde{\sigma}, \tau)$ to the hybrid system (13) with a hybrid input $\widetilde{u}$ such that*

$$\begin{cases} \widetilde{u}(t, k) = u(t) \\ \widetilde{x}(t, k) = x(t) \end{cases} \forall\, (t, k) \in \operatorname{dom} z. \tag{16}$$ [3]

Proof: Suppose x is a solution to the switched system (2) with input u and switching signal σ. We construct a hybrid arc z and a hybrid input $\tilde{u}$ in a recursive manner. Define $\Psi := \{\psi_k : k \in \mathbb{Z}_{>0}\}$ as the set of the switching time instants of σ and let $\psi_0 = 0$. For all $T \in \mathbb{R}_{\geq 0}$, define the number of switches on $[0, T]$ as $K_T := \max\{\bar{k} \in \mathbb{Z}_{\geq 0} : \psi_k \leq T\}$ and let

$$E_T := \left(\bigcup_{k=0}^{K_T - 1} ([\psi_k, \psi_{k+1}], k) \right) \cup ([\psi_{K_T}, T], K_T). \tag{17}$$

Then E_T is a compact hybrid time domain. Consider the hybrid input $\widetilde{u}$ and the hybrid arc $z = (\widetilde{x}, \widetilde{\sigma}, \tau)$ defined so that for all $T \in \mathbb{R}_{\geq 0}$, the following conditions hold:

- $\operatorname{dom} z \cap ([0, T] \times \{0, 1, \ldots, K_T\}) = E_T$, $\operatorname{dom} \widetilde{u} = \operatorname{dom} z$;

- For all $(t, k) \in E_T$, $\widetilde{u}(t, k) = u(t)$, $\widetilde{x}(t, k) = x(t)$ and $\widetilde{\sigma}(t, k) = \sigma(\psi_k)$;
- For all $(t, k) \in E_T$,

$$\tau(t, k) = \begin{cases} \Gamma, & \text{if } k = 0, \\ \min\{\Gamma, \bar{\tau}_s(t, k)\}, & \text{if } k > 0, \sigma(\psi_k) \in \mathcal{P}_s, \\ \bar{\tau}_u(t, k), & \text{if } k > 0, \sigma(\psi_k) \in \mathcal{P}_u, \end{cases} \tag{18}$$

where

$$\begin{aligned} \bar{\tau}_s(t, k) &:= \tau(\psi_k, k - 1) - \ln(\mu) + \gamma(t - \psi_k), \\ \bar{\tau}_u(t, k) &:= \bar{\tau}_s(t, k) - (\lambda_s + \lambda_u)(t - \psi_k). \end{aligned} \tag{19}$$

We will show that, if Assumption 1 is satisfied, the hybrid arc z is a complete solution to the hybrid system (13) with hybrid input $\widetilde{u}$.

Indeed, by construction, z and $\widetilde{u}$ are defined on the same hybrid time domain and satisfy the dynamics of the hybrid system (13). Then it remains to prove that z is complete and for all $(t, k) \in \operatorname{dom} z$, $z(t, k) \in C \cup D = \mathcal{Z}$, which amount to showing the following three properties:

1. By the Uniform ISS Lyapunov-Type Constraint in Assumption 1, the solution x to the switched system (2) is forward complete and is thus defined for all $t \in \mathbb{R}_{\geq 0}$. Therefore $\operatorname{dom} z$ is unbounded in the t-direction and $\widetilde{x}(t, k) \in \mathbb{R}^n$ for all $(t, k) \in \operatorname{dom} z$.
2. Since $\sigma : \mathbb{R}_{\geq 0} \to \mathcal{P}$, $\widetilde{\sigma}(t, k) \in \mathcal{P}$ for all $(t, k) \in \operatorname{dom} z$.
3. From (18) and (19), it is clear that $\tau(t, k) \leq \Gamma$ for all $(t, k) \in \operatorname{dom} z$. On the other hand, for any $(t, k) \in \operatorname{dom} z$, let $(t_0, k_0) := \arg\max_{(s,l) \in \operatorname{dom} z} \{s + l \leq t + k : \tau(s, l) = \Gamma\}$. (Such (t_0, k_0) always exists since $\tau(0, 0) = \Gamma$.) Then according to the Time-Ratio Constraint and the Average Dwell-Time Constraint in Assumption 1 and the definitions of γ in (10) and Γ in (15), we have

$$\begin{aligned} \tau(t, k) &= \tau(t_0, k_0) - N(t, t_0) \ln(\mu) \\ &\quad + T_s(t, t_0)\gamma + T_u(t, t_0)(\gamma - (\lambda_s + \lambda_u)) \\ &\geq \Gamma - (N_0 + (t - t_0)/\tau_a) \ln(\mu) \\ &\quad + (t - t_0)\gamma - (T_0 + \rho(t - t_0))(\lambda_s + \lambda_u) \\ &= 0. \end{aligned}$$

Thus $\tau(t, k) \geq 0$ for all $(t, k) \in \operatorname{dom} z$. [4],[5]

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C. A Hybrid ISS Lyapunov Function

Let a function $V : \mathcal{Z} \to \mathbb{R}_{\geq 0}$ be defined as

$$V(z) := V_{\widetilde{\sigma}}(\widetilde{x}) \exp(\tau), \tag{20}$$

where functions $V_p, p \in \mathcal{P}$ are the ISS Lyapunov functions in the Uniform ISS Lyapunov-Type Constraint in Assumption 1.

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[3]From the proof of Proposition 1, it will be clear that there exists a complete solution z such that in addition to (16), we also have $\widetilde{\sigma}(t, k) = \sigma(t)$ for all $(t, k) \in \operatorname{dom} z$. But only (16) is required in the proof.

[4]This property is equivalent to the fact that $\tau \geq \ln(\mu)$ whenever a jump occurs, since otherwise $\tau^+ < 0$.

[5]By Goebel et al. [13, Proposition 2.10], for a hybrid system with local existence of solutions, a solution is complete if it has no finite escape time and does not jump out of the union of the jump set and the closure of the flow set. Unfortunately, we cannot apply this result since in the hybrid system (13), the local existence of solutions is not satisfied everywhere. In particular, at $z = (\widetilde{x}, \widetilde{\sigma}, 0)$ where $\widetilde{\sigma} \in \mathcal{P}_u$, the condition (VC) in [13, Proposition 2.10] does not hold. However, the hybrid arcs we constructed will not arrive at such points.

For all $z = (\widetilde{x}, \widetilde{\sigma}, \tau) \in \mathcal{Z}$, since $V_{\widetilde{\sigma}}(\widetilde{x})$ is $\mathcal{C}^1$ with respect to $\widetilde{x}$, $V(z)$ is continuously differentiable with respect to $\widetilde{x}$ and τ. We will show that the following uniform ISS Lyapunov conditions are satisfied:

Proposition 2 *V satisfies the following conditions:*

1. $\exists\ \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ *such that for all $z \in \mathcal{Z}$,*

$$\underline{\alpha}(|z|_{\mathcal{A}}) \leq V(z) \leq \overline{\alpha}(|z|_{\mathcal{A}}), \qquad (21)$$

where

$$\mathcal{A} := 0^n \times \mathcal{P} \times [0, \Gamma]. \qquad (22)$$

2. $\exists\ \lambda \in \mathbb{R}_{>0}$ *such that for all $z \in C$, all $\widetilde{u} \in \mathbb{R}^m$ and all $v \in F(z, \widetilde{u})$,*

$$|z|_{\mathcal{A}} \geq \phi(|\widetilde{u}|) \ \Rightarrow\ \frac{\partial V(z)}{\partial z} \cdot v \leq -\lambda V(z). \qquad (23)$$

3. For all $z \in D$ and all $z^+ \in G(z)$,

$$V(z^+) \leq V(z). \qquad (24)$$

Proof: Based on the Uniform ISS Lyapunov-Type Constraint in Assumption 1, we have

1. Let $\underline{\alpha}(r) := \alpha_1(r)$, $\overline{\alpha}(r) := \alpha_2(r)\exp(\Gamma)$, then (21) is satisfied according to (6).
2. Let $\lambda := \lambda_s - \gamma$, then $\lambda > 0$ by (10). For all $z \in C$, all $\widetilde{u} \in \mathbb{R}^m$ and all $v \in F(z, \widetilde{u})$, since $V(z)$ is continuously differentiable with respect to $\widetilde{x}$ and τ, and $\dot{\widetilde{\sigma}} = 0$, the inner product in (23) is well-defined. According to (7), $|z|_{\mathcal{A}} \geq \phi(|\widetilde{u}|)$ implies the following:

   i. If $\widetilde{\sigma} \in \mathcal{P}_s$,
   $$\frac{\partial V(z)}{\partial z} \cdot v \leq \frac{\partial V(\widetilde{x}, \widetilde{\sigma}, \tau)}{\partial \widetilde{x}} \cdot f_{\widetilde{\sigma}}(\widetilde{x}, \widetilde{u}) + \frac{\partial V(\widetilde{x}, \widetilde{\sigma}, \tau)}{\partial \tau} \cdot \gamma$$
   $$\leq -\lambda_s V_{\widetilde{\sigma}}(\widetilde{x})\exp(\tau) + V_{\widetilde{\sigma}}(\widetilde{x})\exp(\tau)\gamma$$
   $$= -\lambda V(z).$$

   ii. If $\widetilde{\sigma} \in \mathcal{P}_u$,
   $$\frac{\partial V(z)}{\partial z} \cdot v = \frac{\partial V(\widetilde{x}, \widetilde{\sigma}, \tau)}{\partial \widetilde{x}} \cdot f_{\widetilde{\sigma}}(\widetilde{x}, \widetilde{u})$$
   $$+ \frac{\partial V(\widetilde{x}, \widetilde{\sigma}, \tau)}{\partial \tau} \cdot (\gamma - (\lambda_s + \lambda_u))$$
   $$\leq (\lambda_u + \gamma - (\lambda_s + \lambda_u))V_{\widetilde{\sigma}}(\widetilde{x})\exp(\tau)$$
   $$= -\lambda V(z).$$

3. For all $z \in D$ and all $z^+ \in G(z)$, according to (8),
   $$V(z^+) = V_{\widetilde{\sigma}^+}(\widetilde{x}^+)\exp(\tau^+)$$
   $$\leq \mu V_{\widetilde{\sigma}}(\widetilde{x})\exp(\tau - \ln(\mu)) = V(z).$$
   $\blacksquare$

Remark 4 From the proof of Proposition 2, we see that the auxiliary timer τ is designed so that it compensates the increases in the concatenation of the ISS Lyapunov-type function of the active subsystem, $V_\sigma(x)$. Similar techniques can be found in [1] for switched systems, [19] for impulsive systems and [15] for hybrid systems. Our timer is more general in the sense that it is able to handle the undesired

increases of $V_\sigma(x)$ both at the switches and when a non-ISS subsystem is active. In the latter case, our construction provides more decay in the auxiliary timer τ_i for compensation, as we can see in the definition of the flow map F_i (14).

D. ISS of the Switched System

In this subsection, we will show that the hybrid ISS Lyapunov function V defined in (20) can be conveniently used to prove the ISS property of the switched system.

Let $\alpha \in \mathcal{K}_\infty, \beta \in \mathcal{KL}$ be defined as
$$\begin{cases} \alpha(r) := \underline{\alpha}^{-1}(\overline{\alpha}(\phi(r))), \\ \beta(r, t) := \underline{\alpha}^{-1}(\overline{\alpha}(r)\exp(-\lambda t)), \end{cases}$$
then we have the following proposition:

Proposition 3 *Suppose z is a complete solution to the hybrid system (13) with hybrid input $\widetilde{u}$. Then,*

$$|z(t,k)|_{\mathcal{A}} \leq \beta(|z(0,0)|_{\mathcal{A}}, t) + \alpha(\|\widetilde{u}\|_{(t,k)}), \qquad (25)$$

for all $(t,k) \in \mathrm{dom}\, z$, where the set $\mathcal{A}$ is defined in (22).

Proof: Let z be a complete solution to the hybrid system (13) with a hybrid input $\widetilde{u}$. According to (23) and (24), for all $(t,k), (t_0, k_0) \in \mathrm{dom}\, z$ such that $t + k \geq t_0 + k_0$, if $|z(s,l)|_{\mathcal{A}} \geq \phi(\|\widetilde{u}\|_{(s,l)})$ for all $(s,l) \in \mathrm{dom}\, z \cap ([t_0, t] \times \{k_0, k_0 + 1, \ldots, k\})$, then

$$V(z(t,k)) \leq V(z(t_0, k_0))\exp(-\lambda(t - t_0)). \qquad (26)$$

From here, our proof follows similar arguments to the proof of [14, Proposition 2.7]. Consider the following cases:

1. $|z(t,k)|_{\mathcal{A}} \leq \phi(\|\widetilde{u}\|_{(t,k)})$. According to (21),
   $$|z(t,k)|_{\mathcal{A}} \leq \phi(\|\widetilde{u}\|_{(t,k)})$$
   $$\leq \underline{\alpha}^{-1}(\overline{\alpha}(\phi(\|\widetilde{u}\|_{(t,k)})))$$
   $$\leq \beta(|z(0,0)|_{\mathcal{A}}, t) + \alpha(\|\widetilde{u}\|_{(t,k)}).$$

2. For all $(s,l) \in \mathrm{dom}\, z \cap ([0,t] \times \{0, 1, \ldots, k\})$, $|z(s,l)|_{\mathcal{A}} > \phi(\|\widetilde{u}\|_{(s,l)})$. By (26), $V(z(t,k)) \leq V(z(0,0))\exp(-\lambda t)$. According to (21),
   $$|z(t,k)|_{\mathcal{A}} \leq \underline{\alpha}^{-1}(\overline{\alpha}(|z(0,0)|_{\mathcal{A}})\exp(-\lambda t))$$
   $$\leq \beta(|z(0,0)|_{\mathcal{A}}, t) + \alpha(\|\widetilde{u}\|_{(t,k)}).$$

3. $|z(t_0, k_0)|_{\mathcal{A}} = \phi(\|\widetilde{u}\|_{(t_0, k_0)})$ for some $(t_0, k_0) \in \mathrm{dom}\, z \cap ([0,t] \times \{0, 1, \ldots, k\})$ and for all $(s,l) \in \mathrm{dom}\, z \cap ([t_0, t] \times \{k_0, k_0 + 1, \ldots, k\}) \backslash (t_0, k_0)$, $|z(s,l)|_{\mathcal{A}} > \phi(\|\widetilde{u}\|_{(s,l)})$. By (26), $V(z(t,k)) \leq V(z(t_0, k_0))\exp(-\lambda(t - t_0))$. According to (21),
   $$|z(t,k)|_{\mathcal{A}} \leq \underline{\alpha}^{-1}(\overline{\alpha}(|z(t_0, k_0)|_{\mathcal{A}}))$$
   $$= \alpha(\|\widetilde{u}\|_{(t_0, k_0)})$$
   $$\leq \beta(|z(0,0)|_{\mathcal{A}}, t) + \alpha(\|\widetilde{u}\|_{(t,k)}).$$
   $\blacksquare$

Combining Propositions 1 and Proposition 3 gives that, for each solution x to the switched system (2) with an input u and a switching signal σ, if Assumption 1 holds, then

$$|x(t)| \leq \beta(|x(0)|, t) + \alpha(\|u\|_t) \qquad \forall\, t \in \mathbb{R}_{\geq 0}, \qquad (27)$$

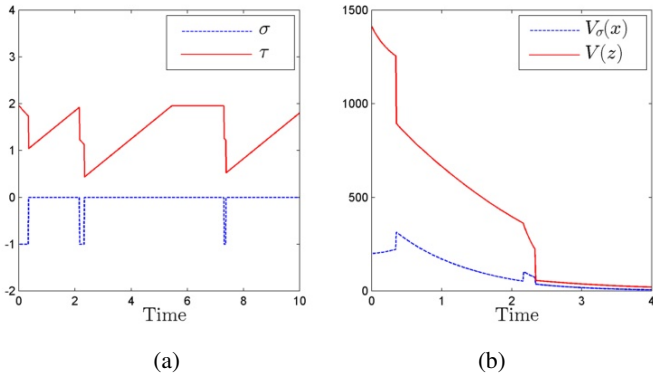that is, the switched system (2) is input-to-state stable. This completes the proof of Theorem 1.

(a)   (b)

Fig. 1: Simulation example. (a) Switching signal σ and auxiliary timer τ; (b) Concatenation of ISS Lyapunov-type functions $V_\sigma(x)$ and hybrid ISS Lyapunov function $V(z)$.

V. SIMULATION EXAMPLE

In this subsection, we demonstrate our approach by a simulation example. Consider the following system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - x_2 + u,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the state and $u \in \mathbb{R}$ is the input (disturbance). Suppose that due to a fault, sometimes the dynamics of x_2 will become

$$\dot{x}_2 = -x_2 + u.$$

Then the system becomes a switched system with the index set $\mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_u$, where $\mathcal{P}_s = \{0\}$ and $\mathcal{P}_u = \{-1\}$.[6] Define $V_0(x) := x_1^2 + x_1 x_2 + x_2^2$ and $V_{-1}(x) := x_1^2 + x_2^2$, it is easy to verify that the Uniform ISS Lyapunov-Type Constraint in Assumption 1 is satisfied with $\alpha_1(x) := x^2$, $\alpha_1(x) := 3x^2/2$, $\phi(x) := 4\sqrt{5}x$, $\lambda_s = 1/2$, $\lambda_u = \sqrt{2} - 1 + \sqrt{5}/10$ and $\mu = 2$. In the simulation, we generated a random switching signal satisfying the Average Dwell-Time Constraint and Time-Ratio Constraint in Assumption 1 with $N_0 = 2$, $\tau_a = 2$, $T_0 = 0.5$ and $\rho = 1/8$. Then condition (10) in Theorem 1 is satisfied. To focus on demonstrating the modification of the ISS Lyapunov functions, we used a random input signal u with small amplitude so that the condition $|x| \geq \phi(|u|)$ in (7) is always satisfied.

The simulation result is shown in Fig. 1. In particular, from Fig. 1(b), we see that if we concatenate the function V_i ($i \in \mathcal{P}$) of the active subsystem directly, the result, $V_\sigma(x)$, will have "spikes" at switches and may be increasing if the non-ISS subsystem is active ($\sigma = -1$); on the other hand, the hybrid ISS Lyapunov function $V(z)$ is always decreasing along the solution, which is consistent with Proposition 2.

Fig. 1(a) plots the switching signal σ we used and the behavior of the auxiliary timer τ. We see that τ decreases when the non-ISS subsystem is active, jumps down by a constant value $\ln(\mu)$ (≈ 0.693) when a switch occurs, increases when the ISS subsystem is active and saturates at $\Gamma = N_0 \ln(\mu) + T_0(\lambda_s + \lambda_u)$ (≈ 1.955).

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[6]The indices 0 and -1 are simply chosen to separate the plots of the switching signal σ and the timer τ in Fig. 1(a).

VI. CONCLUSIONS

We have studied the ISS property of a nonlinear switched system in a general scenario where there may exist some subsystems that are not input-to-state stable. A sufficient condition that guarantees the input-to-state stability of the switched system has been established via hybrid system techniques and the construction of an appropriate hybrid ISS Lyapunov function. An auxiliary timer was designed to handle the effect of undesirable switches and non-ISS subsystems. An application of this approach can be found in [12], where two such ISS Lyapunov functions are used to establish stability of an interconnected switched system.

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