

Connections between Stability Conditions for Slowly Time-Varying and Switched Linear Systems

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Abstract—This paper establishes an explicit relationship between stability conditions for slowly time-varying linear systems and switched linear systems. The concept of total variation of a matrix-valued function is introduced to characterize the variation of the system matrix. Using this concept, a result generalizing existing stability conditions for slowly time-varying linear systems is derived. As a special case of this result, it is shown that a switched linear system is globally exponentially stable if the average dwell time of the switching signal is large enough, which qualitatively matches known results in the literature.

I. INTRODUCTION

Stability of slowly time-varying linear systems and switched linear systems have been extensively studied during the past decades. Earlier results on stability of slowly time-varying linear systems were derived by the *frozen-time* approach [1]–[5]. That is, if the system is stable for any frozen time and varies slowly enough, then the system is globally exponentially stable. There are in fact a couple of ways to characterize the rate of system variation. In the work of [1]–[3], it is shown that the system is globally exponentially stable if the time derivative of the system matrix is sufficiently small. In the work of [4], [5], global exponential stability can be established if either of the following conditions holds: (i) the system matrix is globally Lipschitz in time and the Lipschitz’s constant is sufficiently small; (ii) the time integral of the norm of the time derivative of the system matrix is bounded by some affine function of the length of the time interval and the slope of the affine function is sufficiently small.

The above earlier results in the literature [1]–[5] all impose assumptions on the stability of system matrix at each instant of time and the continuity of the system matrix, which are somewhat conservative. Recent works on stability of slowly time-varying linear systems have relaxed these assumptions [6]–[8]. In [6], Solo showed that the system is exponentially stable even if the eigenvalues of the system matrix do not always have negative real parts, as long as their time averages have negative real parts. The work by Zhang [7] showed that the system is globally exponentially stable if the system matrix is stable at a sequence of times and varies sufficiently slowly during the time interval between successive stable times. In [8], Jetto and Orsini considered a similar case where they replace the slowly time-varying assumption with

the assumption that in each time interval between successive stable times, the weighted cumulative time of unstable system matrix is smaller than the weighted cumulative time of the stable system matrix. Under this assumption, it was shown that the system is uniformly asymptotically stable. The results in [6]–[8] did not assume time continuity of the system matrix.

For a switched linear system, the results by Morse [9], as well as by Hespanha and Morse [10], stated that if each subsystem is stable and if the system switches sufficiently slowly, then the switched system is stable. The rate of switching is characterized by the *dwell-time*, or the *average dwell-time*, which describes the time, or average time, respectively, between two successive switches. The follow-up work by Zhai *et al.* [11] relaxed the assumption on the stability of all subsystems, by allowing switched linear systems with unstable subsystems. It was shown that the system is exponentially stable if the average dwell-time is sufficiently large and the ratio between the activation time of unstable subsystems and the activation time of stable subsystems is sufficiently small.

It is natural to view switched linear systems as a special class of slowly time-varying linear systems. Although there are some similarities, to the best of our knowledge there is no explicit relationship bridging the two sets of results. To be more specific, the stability conditions available in one set cannot be applied directly to the other. With this in mind, we study in this paper the gap between the two sets of results. Our aim is to build connections between stability results for slowly time-varying linear systems derived by the frozen-time approach and stability results for switched linear systems consisting of stable subsystems. Inspired by the concept of *total variation* of a real-valued function, we introduce the total variation of a matrix-valued function to characterize the variation of the system matrix. We apply the concept of total variation to derive a more general result for slowly time-varying linear systems where the system matrix could be piecewise differentiable with discontinuities at the non-differentiable points. We then apply the derived result to switched linear systems and show that a switched linear system is globally exponentially stable if the average dwell time of the switching signal is large enough, which qualitatively matches the results by Hespanha and Morse [10].

A. Preliminaries

We denote by $\|\cdot\|$ the Euclidean norm for a vector and induced norm for a matrix. We write A^T for the transpose of

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a matrix A . We use I to denote the $n \times n$ identity matrix. For any complex number x , $\text{Re}\{x\}$ denotes the real part of x . For an $n \times n$ matrix A , $\lambda_1(A), \dots, \lambda_n(A)$ denote the eigenvalues of A . A is Hurwitz if

$$\text{Re}\{\lambda_i(A)\} < 0, \quad \forall i = 1, 2, \dots, n$$

B. Organization

The remainder of this paper is organized as follows. In Section II, we first review some existing results for the stability of slowly time-varying linear systems, and then we introduce the concept of total variation, and extend the existing results to the case where the system matrix $A(t)$ is piecewise differentiable, using the concept of total variation. In Section III, we apply the extended result to derive stability conditions for switched linear systems and compare them with the results in the literature. Finally, in Section IV, we draw conclusions and discuss several future directions.

II. STABILITY OF SLOWLY TIME-VARYING LINEAR SYSTEMS

Consider a real n -dimensional linear time-varying system described by

$$\dot{x}(t) = A(t)x(t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $A(t)$ is an $n \times n$ real matrix for each time t , which is called the system matrix. The system described by (1) is *globally exponentially stable* if there exist finite positive constants m and γ such that for any initial condition $x(0) \in \mathbb{R}^n$, the corresponding solution satisfies

$$\|x(t)\| \leq m\|x(0)\|e^{-\gamma t}, \quad \forall t \geq 0$$

In the special case when $A(t) = A$ is independent of time, it is well known that the system (1) is global exponentially stable if and only if A is Hurwitz. The conditions for global exponential stability of (1) have been widely studied [1]–[5]. Here we introduce two previous results for slowly time-varying cases.

Theorem 1: [1] The system (1) is globally exponentially stable if the following conditions are satisfied:

- 1) There exists a positive constant L such that for all t ,

$$\|A(t)\| \leq L$$

- 2) There exists a positive constant σ_s such that for all t ,

$$\text{Re}\left\{\lambda_i\left(A(t)\right)\right\} \leq -\sigma_s \quad \forall i = 1, 2, \dots, n$$

By the first two conditions, there exist positive constants c and λ_0 (which depend only on L and σ_s) such that for all t ,

$$\left\|e^{A(t)s}\right\| \leq ce^{-\lambda_0 s} \quad \text{for all } s \geq 0$$

- 3) For all t , $A(t)$ is differentiable and

$$\left\|\dot{A}(t)\right\| \leq \frac{4\lambda_0^2}{3c^4}$$

Here $\|\dot{A}(t)\|$ can be regarded as the rate at which the system changes over time. Hence, the result of Theorem 1 implies that a linear time-varying system (1) is globally exponentially stable if the system matrix is Hurwitz for each fixed time, uniformly bounded, and changes at a sufficiently small rate.

An improvement on the sufficient condition just described is obtained by replacing $\|\dot{A}(t)\|$ with the integral of $\|\dot{A}(t)\|$, as follows.

Theorem 2: [5] The system (1) is globally exponentially stable if the following conditions are satisfied:

- 1) Condition 1 in Theorem 1.
- 2) Condition 2 in Theorem 1.
- 3) $A(t)$ is differentiable and there exist scalars $\alpha > 0$ and $0 < \mu < \frac{\beta_1}{2\beta_2^2}$ such that for all $t, T \geq 0$,

$$\int_t^{t+T} \left\|\dot{A}(s)\right\| ds \leq \mu T + \alpha$$

$$\text{where } \beta_1 = \frac{1}{2L} \text{ and } \beta_2 = \frac{c^2}{2\lambda_0}.$$

The first two conditions are the same as those in Theorem 1 while the third one is in terms of the integral of $\|\dot{A}(t)\|$ on each interval $[t, t+T]$, which is required to be bounded by some affine function of the length of the time interval and the slope of the affine function is sufficiently small.

All the sufficient conditions above assume that $A(t)$ is differentiable for all t . In the sequel, we will relax this assumption and consider a more general case in which $A(t)$ is piecewise differentiable. Our approach will entail appealing to total variation of piecewise differentiable functions.

A. Total Variation

The total variation of a real valued function $f(t)$ on a closed interval $[a, b]$ is defined as follows [12]:

A *partition* P of a closed interval $[a, b]$ is a finite set of $n > 1$ points $t_i, i \in \{1, 2, \dots, n\}$, such that $a = t_1 < t_2 < \dots < t_n = b$. Let \mathbb{P} denote the set of all partitions of $[a, b]$. The *total variation* of a real-valued function f on the closed interval $[a, b]$, denoted by $V_a^b(f)$, is defined as

$$V_a^b(f) = \sup_{P \in \mathbb{P}} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

In the special case where f is piecewise differentiable on $[a, b]$ with discontinuities at the non-differentiable points, define

$$f(t^+) = \lim_{x \rightarrow t^+} f(x), \quad \text{for all } t \in [a, b]$$

$$f(t^-) = \lim_{x \rightarrow t^-} f(x), \quad \text{for all } t \in (a, b]$$

and then we have the following lemma.

Lemma 1: Suppose that a real-value function f satisfies the following conditions:

- 1) $f(a^+)$ and $f(b^-)$ exist and are bounded, and $f(a) = f(a^+)$.
- 2) f has a finite number of discontinuities on (a, b) , denoted by $\{d_1, d_2, \dots, d_m\}$, such that $a < d_1 < d_2 < \dots < d_m < b$.
- 3) $f(d_i^-)$ and $f(d_i^+)$ exist, and $f(d_i) = f(d_i^+)$ for all $i \in \{1, 2, \dots, m\}$.

- 4) f is differentiable and \dot{f} is integrable on (a, d_1) , (d_m, b) , and (d_i, d_{i+1}) for all $i \in \{1, 2, \dots, m-1\}$.

Then, the total variation of f on $[a, b]$ is

$$\begin{aligned} V_a^b(f) &= \int_a^{d_1} |\dot{f}(t)| dt + \int_{d_m}^b |\dot{f}(t)| dt \\ &+ \sum_{i=1}^{m-1} \int_{d_i}^{d_{i+1}} |\dot{f}(t)| dt + |f(b) - f(b^-)| \quad (2) \\ &+ \sum_{i=1}^m |f(d_i^+) - f(d_i^-)| \end{aligned}$$

The proof of this lemma is omitted due to space limitations and will appear in an expanded version of this paper.

Inspired by the expression of total variation of a real-valued piecewise differentiable function, described by (2), we define the *total variation* of a piecewise differentiable matrix-valued function. First define

$$\begin{aligned} A(t^+) &= \lim_{x \rightarrow t^+} A(x), \quad \text{for all } t \in [a, b) \\ A(t^-) &= \lim_{x \rightarrow t^-} A(x), \quad \text{for all } t \in (a, b] \end{aligned}$$

We then introduce a number of assumptions.

Assumption 1:

- 1) $A(a^+)$ and $A(b^-)$ exist, and $A(a) = A(a^+)$.
- 2) A has a finite number of discontinuities on (a, b) , denoted by $\{d_1, d_2, \dots, d_m\}$, such that $a < d_1 < d_2 < \dots < d_m < b$.
- 3) $A(d_i^-)$ and $A(d_i^+)$ exist, and $A(d_i) = A(d_i^+)$ for all $i \in \{1, 2, \dots, m\}$.
- 4) A is differentiable and $\|\dot{A}\|$ is integrable on (a, d_1) , (d_m, b) , and (d_i, d_{i+1}) for all $i \in \{1, 2, \dots, m-1\}$.

Definition 1: Suppose that a matrix-valued function $A(t)$ satisfies the regularity conditions of Assumption 1. Then, the total variation of A on $[a, b]$, denoted by $\int_a^b \|dA\|$, is defined as follows,

$$\begin{aligned} \int_a^b \|dA\| &:= \int_a^{d_1} \|\dot{A}(t)\| dt + \int_{d_m}^b \|\dot{A}(t)\| dt \\ &+ \sum_{i=1}^{m-1} \int_{d_i}^{d_{i+1}} \|\dot{A}(t)\| dt + \|A(b) - A(b^-)\| \\ &+ \sum_{i=1}^m \|A(d_i) - A(d_i^-)\| \end{aligned}$$

Remark 1: We write $\int_a^b \|dA\|$ for the total variation of a matrix-valued function, instead of $V_a^b(A)$, to make our main result more comparable with the previous results, i.e., Theorems 1 and 2.

B. Main Result

With the concept of total variation introduced in the previous subsection, we are able to extend the existing results in the literature (i.e., Theorem 1 and Theorem 2) to a more general case, as stated in the following theorem.

Theorem 3: The system (1) is globally exponentially stable if the following conditions are satisfied:

- 1) Condition 1 in Theorem 1.
- 2) Condition 2 in Theorem 1.
- 3) For any closed interval $[a, b]$, $A(t)$ satisfies Assumption 1 in Section II-A.
- 4) There exist scalars $\alpha > 0$ and $0 < \mu < \frac{\beta_1}{2\beta_2^3}$ such that for all $t, T \geq 0$,

$$\int_t^{t+T} \|dA\| \leq \mu T + \alpha$$

where $\beta_1 = \frac{1}{2L}$, $\beta_2 = \frac{c^2}{2\lambda_0}$, and $\int_t^{t+T} \|dA\|$ was defined in Definition 1.

Remark 2: When $A(t)$ is differentiable on $[a, b]$, Theorem 3 collapses to Theorem 2.

To prove the theorem, we need the following lemmas.

Lemma 2: [13] Suppose that $A(t)$ satisfies the first two conditions in Theorem 3. For each fixed t , let $P(t)$ be the symmetric positive definite solution of the Lyapunov equation $P(t)A(t) + A^T(t)P(t) = -I$ and consider the candidate Lyapunov function $V(t, x) = x^T P(t)x$. Then,

$$\begin{aligned} \beta_1 &\leq \|P(t)\| \leq \beta_2 \quad \forall t \geq 0 \\ \beta_1 \|x\|^2 &\leq V(t, x) \leq \beta_2 \|x\|^2 \quad \forall x \in \mathbb{R}^n \quad \forall t \geq 0 \end{aligned}$$

where $\beta_1 = \frac{1}{2L}$, and $\beta_2 = \frac{c^2}{2\lambda_0}$

Lemma 3: Consider two matrices A_1, A_2 satisfying the first two conditions in Theorem 3, namely

$$\begin{aligned} \|A_1\|, \|A_2\| &\leq L \\ \operatorname{Re} \left\{ \lambda_i(A_1), \lambda_i(A_2) \right\} &\leq -\sigma_s \quad \forall i = 1, 2, \dots, n \\ \|e^{A_1 s}\|, \|e^{A_2 s}\| &\leq ce^{-\lambda_0 s} \quad \text{for all } s \geq 0 \end{aligned}$$

Let P_1, P_2 be respectively the solutions of the Lyapunov equations $P_1 A_1 + A_1^T P_1 = -I$ and $P_2 A_2 + A_2^T P_2 = -I$. Then,

$$\|P_1 - P_2\| \leq 2\beta_2^2 \|A_1 - A_2\|$$

with $\beta_2 = \frac{c^2}{2\lambda_0}$ as defined above in Theorem 3.

Proof: The proof of this lemma is a discrete-time version of the proof of Theorem 3.4.11 in [5], where an upper bound of $\|\dot{P}(t)\|$ is derived in terms of $\|\dot{A}(t)\|$. Since P_1 and P_2 are solutions of the Lyapunov equations, we have

$$\begin{aligned} (P_1 - P_2)A_2 + A_2^T(P_1 - P_2) \\ = P_1(A_2 - A_1) + (A_2^T - A_1^T)P_1 := -Q \end{aligned} \quad (3)$$

where Q is a symmetric matrix. Since A_2 is Hurwitz, by [14], the Lyapunov equation (3) has a unique solution, which is

$$P_1 - P_2 = \int_0^\infty e^{A_2^T s} Q e^{A_2 s} ds$$

Hence,

$$\begin{aligned} \|P_1 - P_2\| &\leq \|Q\| \int_0^\infty \|e^{A_2^T s}\| \cdot \|e^{A_2 s}\| ds \\ &\leq \|Q\| \int_0^\infty c^2 e^{-2\lambda_0 s} ds \\ &= \frac{c^2}{2\lambda_0} \cdot \|Q\| = \beta_2 \|Q\| \end{aligned} \quad (4)$$

By Lemma 2 and the definition of Q in (3),

$$\begin{aligned}\|Q\| &= \|-P_1(A_2 - A_1) - (A_2^T - A_1^T)P_1\| \\ &\leq \|P_1(A_1 - A_2)\| + \|(A_1^T - A_2^T)P_1\| \\ &= 2\|P_1\| \cdot \|A_1 - A_2\| \leq 2\beta_2\|A_1 - A_2\|\end{aligned}\quad (5)$$

Combining inequalities (4) and (5), we conclude the proof of Lemma 3. \blacksquare

Lemma 4: Suppose that $A(t)$ satisfies the first three conditions in Theorem 3. Let $P(t)$ be the symmetric positive definite solution of the Lyapunov equation $P(t)A(t) + A^T(t)P(t) = -I$, $V(t, x) = x^T P(t)x$ be the candidate Lyapunov function, and $V(t) = V(t, x(t))$ be the candidate Lyapunov function evaluated along system solution $x(t)$. For any $t_b > 0$, introduce

$$V(t_b^+) := \lim_{t \rightarrow t_b^+} V(t), \quad V(t_b^-) := \lim_{t \rightarrow t_b^-} V(t)$$

Then, $V(t_b^+)$ and $V(t_b^-)$ exist and satisfy

$$V(t_b^+) = V(t_b) \leq e^{2\beta_2^2\beta_1^{-1}\|A(t_b) - A(t_b^-)\|} V(t_b^-) \quad (6)$$

where $\beta_1 = \frac{1}{2L}$ and $\beta_2 = \frac{c^2}{2\lambda_0}$.

Proof: By condition 3 in Theorem 3 (Assumption 1), $A(t_b^+)$ and $A(t_b^-)$ exist, and $A(t_b^+) = A(t_b)$. Moreover, $\|A\|$, $\text{Re}\{\lambda_i(A)\}$ and e^{As} are continuous functions of A , therefore $A(t_b^+)$ and $A(t_b^-)$ satisfy the first two conditions in Theorem 3. Let $P(t_b^+)$ and $P(t_b^-)$ be the solutions of Lyapunov equations $P(t_b^+)A(t_b^+) + A^T(t_b^+)P(t_b^+) = -I$, and $P(t_b^-)A(t_b^-) + A^T(t_b^-)P(t_b^-) = -I$. By the definitions of $A(t_b^+)$ and $A(t_b^-)$, given any $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$\|A(t_b^+) - A(t)\| \leq \frac{\epsilon}{2\beta_2^2}, \quad \forall t \in (t_b, t_b + \delta_1)$$

$$\|A(t_b^-) - A(t)\| \leq \frac{\epsilon}{2\beta_2^2}, \quad \forall t \in (t_b - \delta_2, t_b)$$

Since $A(t)$ satisfies the first two conditions in Theorem 3 for all $t \geq 0$, by Lemma 3,

$$\|P(t_b^+) - P(t)\| \leq \epsilon, \quad \forall t \in (t_b, t_b + \delta_1)$$

$$\|P(t_b^-) - P(t)\| \leq \epsilon, \quad \forall t \in (t_b - \delta_2, t_b)$$

which implies $P(t_b^+) = \lim_{t \rightarrow t_b^+} P(t)$ and $P(t_b^-) = \lim_{t \rightarrow t_b^-} P(t)$. Furthermore, since $A(t_b) = A(t_b^+)$ is Hurwitz, the Lyapunov equations $P(t_b)A(t_b) + A^T(t_b)P(t_b) = -I$ and $P(t_b^+)A(t_b^+) + A^T(t_b^+)P(t_b^+) = -I$ have unique solutions and $P(t_b^+) = P(t_b)$.

$A(t)$ is piecewise continuous, hence $x(t)$ is always continuous regardless of the initial condition. Then,

$$x(t_b^+) := \lim_{t \rightarrow t_b^+} x(t) = x(t_b) = \lim_{t \rightarrow t_b^-} x(t) =: x(t_b^-)$$

Consider $V(t_b^+)$ and $V(t_b^-)$,

$$V(t_b^+) = \lim_{t \rightarrow t_b^+} V(t) = \lim_{t \rightarrow t_b^+} x^T(t)P(t)x(t)$$

$$V(t_b^-) = \lim_{t \rightarrow t_b^-} V(t) = \lim_{t \rightarrow t_b^-} x^T(t)P(t)x(t)$$

We have shown that $x(t_b^+)$, $x(t_b^-)$, $P(t_b^+)$, and $P(t_b^-)$ exist. Hence, $V(t_b^+)$ and $V(t_b^-)$ exist, and

$$V(t_b^+) = x^T(t_b^+)P(t_b^+)x(t_b^+) = x^T(t_b)P(t_b^+)x(t_b)$$

$$V(t_b^-) = x^T(t_b^-)P(t_b^-)x(t_b^-) = x^T(t_b)P(t_b^-)x(t_b)$$

Furthermore,

$$V(t_b^+) = x^T(t_b)P(t_b^+)x^T(t_b) = x^T(t_b)P(t_b)x(t_b) = V(t_b)$$

Now we can prove the inequality (6). Consider the function

$$g(x) = e^{x-1} - x$$

where $x \in \mathbb{R}$. It is straightforward to check that $g(x) \geq 0$, $\forall x \in \mathbb{R}$. Then, $x \leq e^{x-1}$, $\forall x \in \mathbb{R}$. Therefore,

$$\frac{V(t_b)}{V(t_b^-)} \leq e^{\frac{V(t_b)}{V(t_b^-)} - 1} = e^{\frac{V(t_b) - V(t_b^-)}{V(t_b^-)}} \quad (7)$$

Recall that $A(t_b^-)$ satisfies the first two conditions in Theorem 3; therefore, by Lemma 2,

$$0 < \frac{1}{V(t_b^-)} \leq \frac{1}{\beta_1\|x(t_b)\|^2} \quad (8)$$

Moreover,

$$\begin{aligned}V(t_b) - V(t_b^-) &= x^T(t_b)\left(P(t_b) - P(t_b^-)\right)x(t_b) \\ &\leq \|P(t_b) - P(t_b^-)\| \cdot \|x(t_b)\|^2 \\ &\leq 2\beta_2^2\|A(t_b) - A(t_b^-)\| \cdot \|x(t_b)\|^2\end{aligned}\quad (9)$$

where the last inequality is due to Lemma 3. Combining equations (7), (8), (9), we have

$$\begin{aligned}\frac{V(t_b)}{V(t_b^-)} &\leq e^{\frac{2\beta_2^2\|A(t_b) - A(t_b^-)\|\|x(t_b)\|^2}{\beta_1\|x(t_b)\|^2}} \\ &= e^{2\beta_2^2\beta_1^{-1}\|A(t_b) - A(t_b^-)\|}\end{aligned}$$

which implies the inequality (6). \blacksquare

We are now in a position to prove Theorem 3.

Proof: (Proof of Theorem 3) Let $P(t)$ be the symmetric positive definite solution of the Lyapunov equation $P(t)A(t) + A^T(t)P(t) = -I$, $V(t, x) = x^T P(t)x$ be the candidate Lyapunov function, and $V(t) = V(t, x(t))$ be the candidate Lyapunov function evaluated along system solution $x(t)$. By condition 3 in Theorem 3 (Assumption 1), within any time interval $[a, b]$, there are finitely many discontinuities of $A(t)$, denoted by $\{d_1, d_2, \dots, d_m\}$, where $a < d_1 < d_2 < \dots < d_m < b$. Consider any sub-interval of $[a, b]$ among $[a, d_1]$, $[d_1, d_2]$, \dots , and $[d_m, b]$, call it $[t_1, t_2]$, then $A(t)$ has no discontinuity on (t_1, t_2) . By the proof of Theorem 3.4.11 in [5], we have

$$V(t_2^-) \leq e^{-\int_{t_1}^{t_2} \beta_2^{-1} dt + 2\beta_2^2\beta_1^{-1}(\int_{t_1}^{t_2} \|\dot{A}(t)\| dt)} V(t_1^+)$$

Then by Lemma 4,

$$\begin{aligned}V(t_2) &\leq e^{-\beta_2^{-1}(t_2 - t_1)} \\ &\cdot e^{2\beta_2^2\beta_1^{-1}(\int_{t_1}^{t_2} \|\dot{A}(t)\| dt + \|A(t_2) - A(t_2^-)\|)} V(t_1)\end{aligned}\quad (10)$$

Apply inequality (10) to sub-intervals $[a, d_1], [d_1, d_2], \dots$, and $[d_m, b]$,

$$\begin{aligned} V(d_1) &\leq e^{-\beta_2^{-1}(d_1-a)} \\ &\quad \cdot e^{2\beta_2^2\beta_1^{-1}\left(\int_a^{d_1}\|\dot{A}(t)\|dt+\|A(d_1)-A(d_1^-)\|\right)}V(a) \\ V(d_2) &\leq e^{-\beta_2^{-1}(d_2-d_1)} \\ &\quad \cdot e^{2\beta_2^2\beta_1^{-1}\left(\int_{d_1}^{d_2}\|\dot{A}(t)\|dt+\|A(d_2)-A(d_2^-)\|\right)}V(d_1) \\ &\quad \vdots \\ V(b) &\leq e^{-\beta_2^{-1}(b-d_m)} \\ &\quad \cdot e^{2\beta_2^2\beta_1^{-1}\left(\int_{d_m}^b\|\dot{A}(t)\|dt+\|A(b)-A(b^-)\|\right)}V(d_m) \end{aligned}$$

Combining all the inequalities above and recall Definition 1 of the total variation of $A(t)$ on $[a, b]$, we have

$$V(b) \leq e^{-\beta_2^{-1}(b-a)+2\beta_2^2\beta_1^{-1}\left(\int_a^b\|dA\|\right)}V(a)$$

By condition 4 in Theorem 3,

$$\begin{aligned} V(b) &\leq e^{-\beta_2^{-1}(b-a)+2\beta_2^2\beta_1^{-1}(\mu(b-a)+\alpha)}V(a) \\ &= e^{2\beta_2^2\beta_1^{-1}\alpha} \cdot e^{-(\beta_2^{-1}-2\beta_2^2\beta_1^{-1}\mu)(b-a)}V(a) \end{aligned}$$

where $e^{2\beta_2^2\beta_1^{-1}\alpha} > 0$, and $\beta_2^{-1} - 2\beta_2^2\beta_1^{-1}\mu > 0$. Hence, the Lyapunov function decays exponentially along system solutions, which implies the global exponential stability of the system. ■

III. STABILITY OF SWITCHED LINEAR SYSTEMS

In this section, we will apply the generalized stability conditions for slowly time-varying linear systems and derive stability conditions for switched linear systems, which bridges the gap between the two sets of results.

A. Application of Generalized Stability Condition for Slowly Time-Varying Linear Systems

We first apply the generalized stability conditions for slowly time-varying linear systems (Theorem 3) to switched linear systems. Suppose we are given a family of linear systems

$$\dot{x} = A_p x \quad p \in \mathcal{P}$$

where $x \in \mathbb{R}^n$ is the system state, p is the index of the linear system A_p in the family, and \mathcal{P} is the index set. Consider a switched linear system

$$\dot{x} = A_{\sigma(t)}x(t) \quad (11)$$

For each fixed t , $A_{\sigma(t)}$ is an $n \times n$ real matrix, and $A_{\sigma(t)} \in \{A_p, p \in \mathcal{P}\}$, which is the set of subsystems. The function $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is called the switching signal. We assume that σ is a piecewise constant function, which has a finite number of discontinuities on every bounded time interval. It is also assumed that σ is continuous from the right everywhere. Denote by $N_\sigma(t, t+T)$ the number of switches (number of discontinuities of σ) on the time interval $(t, t+T]$. If there exist two positive constants τ_a and N_0 such that

$$N_\sigma(t, t+T) \leq N_0 + \frac{T}{\tau_a} \quad \forall t, T \geq 0, \quad (12)$$

then the switching signal $\sigma(t)$ is said to have the *average dwell time* τ_a . We have the following set of stability conditions for switched linear systems.

Corollary 1: The system (11) is globally exponentially stable if:

- 1) Conditions 1 and 2 in Theorem 1 hold for all subsystems $A_p, p \in \mathcal{P}$.
- 2) The switching signal $\sigma(t)$ has average dwell time τ_a and

$$\tau_a > \frac{2\beta_2^3}{\beta_1^2} \quad (13)$$

where $\beta_1 = \frac{1}{2L}$ and $\beta_2 = \frac{e^2}{2\lambda_0}$.

The proof of this corollary is omitted due to space limitations and will appear in an expanded version of this paper. However, one should see that due to the piecewise constant property of switching signal, the total variation of system matrix is only induced by switching from A_p to A_q . By triangle inequality, $\|A_p - A_q\|$ is uniformly bounded over all $p, q \in \mathcal{P}$. Therefore, the total variation of the system matrix over a long time interval can be made small enough by restricting the system to switch slowly enough (or the average dwell time to be large enough), resulting in stability by Theorem 3.

B. Comparison with the Existing Stability Condition for Switched Systems

Given a family of systems

$$\dot{x} = f_p(x), \quad p \in \mathcal{P}$$

consider a general switched system

$$\dot{x} = f_{\sigma(t)}(x) \quad (14)$$

where $x \in \mathbb{R}^n$ is the system state. For each fixed t , $f_{\sigma(t)}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $f_{\sigma(t)} \in \{f_p, p \in \mathcal{P}\}$, which is the set of subsystems. Similarly, \mathcal{P} is the index set, and $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is the switching signal. The definition of average dwell time is the same as above. By [10], we have the following theorem.

Theorem 4: [10] The general switched system (14) is globally exponentially stable if:

- 1) There exist continuously differentiable (\mathcal{C}^1) functions $V_p(x) : \mathbb{R}^n \rightarrow \mathbb{R}, p \in \mathcal{P}$ and two class \mathcal{K}_∞ functions α_1 and α_2 such that

$$\alpha_1(\|x\|) \leq V_p(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathbb{R}^n, \quad \forall p \in \mathcal{P}$$

- 2) There exists a positive constant λ_0 such that,

$$\frac{\partial V_p}{\partial x} f_p(x) \leq -2\lambda_0 V_p(x) \quad \forall x \in \mathbb{R}^n, \quad \forall p \in \mathcal{P}$$

- 3) There exists a positive constant μ such that,

$$V_p(x) \leq \mu V_q(x) \quad \forall x \in \mathbb{R}^n, \quad \forall p, q \in \mathcal{P}$$

- 4) The switching signal $\sigma(t)$ has average dwell time τ_a and

$$\tau_a > \frac{\log \mu}{2\lambda_0}$$

We now apply Theorem 4 to the switched linear system (11) to derive stability conditions for switched linear systems. Consider the switched linear system (11) satisfying the first condition in Theorem 1. For each subsystem A_p , define $V_p(x) := x^T P_p x$ (which is \mathcal{C}^1), where P_p is the solution to the Lyapunov equation $P_p A_p + A_p^T P_p = -I$. Then, by Lemma 2,

$$\beta_1 \|x\|^2 \leq V_p(x) \leq \beta_2 \|x\|^2, \quad \forall x \in \mathbb{R}^n, \quad \forall p \in \mathcal{P}$$

Therefore, we have $\alpha_1(\|x\|) = \beta_1 \|x\|^2$, $\alpha_2(\|x\|) = \beta_2 \|x\|^2$. Furthermore,

$$\frac{\partial V_p}{\partial x} f_p(x) = -\|x\|^2 \leq -\frac{1}{\beta_2} V_p(x), \quad \forall x \in \mathbb{R}^n \quad \forall p \in \mathcal{P}$$

which implies $2\lambda_0 = \frac{1}{\beta_2}$. Moreover,

$$V_p(x) \leq \beta_2 \|x\|^2 \leq \frac{\beta_2}{\beta_1} V_q(x), \quad \forall x \in \mathbb{R}^n \quad \forall p, q \in \mathcal{P}$$

Hence, $\mu = \frac{\beta_2}{\beta_1}$. Applying Theorem 4, we conclude that the switched linear system is stable if

$$\tau_a > \frac{\log \mu}{2\lambda_0} = \beta_2 \log \frac{\beta_2}{\beta_1} \quad (15)$$

Comparing (13) and (15),

$$\frac{2\beta_2^3}{\beta_1^2} = 2 \cdot \beta_2 \cdot \frac{\beta_2}{\beta_1} \cdot \frac{\beta_2}{\beta_1} > 1 \cdot \beta_2 \cdot \log \frac{\beta_2}{\beta_1} \cdot 1$$

Therefore, the stability condition in terms of average dwell time for switched linear systems, which is derived from generalized stability condition for slowly time-varying linear systems, matches the existing result qualitatively but not quantitatively.

Remark 3: The comparison above does not imply the conservativeness of generalized stability conditions for slowly time-varying linear systems. By Theorem 3, a switched linear system is stable if the total variation of system matrix over a long time interval is small enough. Small variation of system matrix can be achieved in two ways: 1) The variation caused by each switch is large while the system switches slowly enough. 2) The system switches fast while the variation caused by each switch is small enough. The comparison above is under the first scenario. However, under the second scenario, the switching signal might not even have an average dwell time. Then we cannot apply Theorem 4, but can apply Theorem 3 to establish the stability result, as illustrated in the following example.

Example: Let $L > 0$ and $\sigma_s > 0$ defined in Theorem 1 be given. Then β_1 , β_2 , and μ in Theorem 3 are well defined. Suppose there exists a family of linear systems with system matrices $\{A_i | i \in \mathbb{N}\}$ satisfying conditions 1 and 2 in Theorem 1. Furthermore,

$$\|A_i - A_0\| < \frac{\mu}{2i} \quad \forall i \in \mathbb{N}_+ \quad (16)$$

The assumptions above can be satisfied by (1) choose A_0 satisfying conditions 1 and 2 in Theorem 1 with strict inequality. (2) Let $A_i = A_0 + \epsilon_i \cdot I$ and choose ϵ_i small

enough. Then A_i satisfies conditions 1 and 2 in Theorem 1 and (16).

Consider the switching signal $\sigma(t)$ such that during time interval $[i-1, i]$, $i \in \mathbb{N}_+$, $A_{\sigma(t)}$ switches $2 \cdot i$ times, between A_0 and A_i , uniformly over $[i-1, i]$. Then the time between two successive switches is $\frac{1}{2i}$. By assumption (16), condition 4 in Theorem 3 is satisfied with $\alpha = 2L$. Hence by Theorem 3, the switched linear system is stable. On the other hand, since the number of switches during time interval $[i-1, i]$ is $2i$, there exists no τ_a and N_0 satisfying (12), which means the switching signal does not have average dwell time and thus, we cannot apply Theorem 4 to draw any conclusions on the stability of the switched linear system.

IV. CONCLUSION AND FUTURE WORK

In this paper, we have derived a set of generalized stability conditions for slowly time-varying linear systems and applied it to derive a set of stability conditions for switched linear systems. By doing this we have unified stability conditions for slowly time-varying linear systems and stability conditions for switched linear systems.

Several issues remain open for future research. First, there is the need to build relationships between stability conditions for slowly time-varying linear systems and switched linear systems in the cases when $A(t)$ may be unstable for some time t [6], [11]. Second, there is the need to establish relationships between stability conditions for nonlinear time-varying systems and switched nonlinear systems. Third, one has to quantitatively improve the result in Theorem 1 so as to better match the result in Theorem 4.

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