2018 IEEE Conference on Decision and Control (CDC)
Miami Beach, FL, USA, Dec. 17-19, 2018

Global Stability and Asymptotic Gain Imply Input-to-State Stability for State-Dependent Switched Systems

Shenyu Liu Daniel Liberzon

Abstract— **In this paper we study several stability properties for state-dependent switched systems. We examine the gap between global asymptotic stability and uniform global asymptotic stability, and illustrate it with an example. Several regularity assumptions are proposed in order to obtain the equivalence between these two stability properties. Based on this equivalence, we are able to show that global stability and asymptotic gain imply input-to-state stability for state-dependent switched systems, which is the main result of the paper. The proof consists of a bypass via an auxiliary system which takes in a bounded disturbance, and showing that this system is uniformly globally asymptotically stable.**

I. Introduction

Input-to-State Stability (ISS), first introduced by Sontag in [1], turned out to be an important and widely used concept for characterizing a system's response to inputs. While ISS is normally defined in terms of the sum of an initial-state-dependent, time-decaying estimate and an input-dependent estimate, it also has many other characterizations, each with its own advantages. For example, ISS is equivalent to the validity of a dissipation inequality for an appropriately defined energy storage function; ISS is also equivalent to the Uniform Asymptotic Gain (UAG) property (see, e.g., [2]). Here we are interested in the close relation of ISS with the Global Stability (GS) property and the Asymptotic Gain (AG) property; these two properties combined were shown to be equivalent to ISS for single-mode, Lipschitz systems in [3].

In our prior work, we have designed state feedback controllers with quantized state measurements, via zoom-in/out techniques, for achieving disturbance attenuation. This controller design can be applied to single-mode linear systems with inputs [4], or to switched linear systems with inputs [5]. The closed-loop system was proven to be GS and AG with respect to the external disturbance, yet this does not immediately result in ISS as the closed-loop system is a switched system and so the theorem from [3] is not directly applicable. A strictly weaker version of ISS with parametrization was shown in [6], with significant extra effort.

Motivated by the above reasons, we want to study ISS for switched systems, in particular the implication from GS plus AG to ISS. It is observed that in quantized controller design, the zoom events and transitions of control law typically occur when the error exceeds certain bounds; in other words, the switch is triggered when the system state reaches certain regions in the state space. Accordingly, we choose to focus on state-dependent switched systems; see, e.g., [7]. (We note that event-triggered control systems [8] can also be captured in a similar modeling framework.) As a popular type of hybrid systems, state-dependent switched systems have attracted a lot of research recently (see, e.g., [9], [10] among many other works). Our main task in this paper is to formulate assumptions under which the implication from GS plus AG to ISS holds for state-dependent switched systems.

It is identified in this paper that the major gap between GS plus AG and ISS is the uniformity of convergence time. Briefly speaking, the lack of uniformity lies in the nature of state-dependent switched systems, namely, in the fact that solutions evolving from adjacent initial states may behave very differently because they are in different modes. As a result, while AG guarantees that all solutions will converge to the equilibrium, the time to converge to a small set is no longer continuous with respect to the initial states and hence a uniform upper bound on the convergence time may not exist; consequently the system may not be ISS. This gap can be filled by imposing suitable regularity conditions; for example, in the hybrid system framework of [11], the system solution space is closed, and it is concluded that global pre-asymptotic stability is equivalent to uniform global pre-asymptotic stability. It is also noted that GS plus AG is related to the nonuniform ISS defined in [12], which is shown to imply ISS if this nonuniform ISS still holds when either the dynamics of the system or switch guards/rules are perturbed.

Motivated by [13], we would like to impose transversality of solutions with respect to switch guards in our model. The idea of transversal solutions can be traced back to [14] in the 1980s. In [15] the transversality condition is also shown to be essential for trajectory sensitivity analysis. With this assumption of transversality, we can eventually draw the equivalence between GS plus AG and ISS.

This paper is divided into 8 sections. Section II introduces the necessary notation and stability-related definitions. Section III provides an ill-behaved example and proposes the assumptions needed for the problem to be well posed. Section IV states the main theorem. Sections V and VI contain the supporting lemmas. Section VII compares our model with another one from the literature and discusses some possible improvements. Finally, Section VIII concludes the paper.

S. Liu and D. Liberzon are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, U.S.A. This work was supported by the NSF grant CMMI-1662708 and the AFOSR grant FA9550-17-1-0236.

978-1-5386-1394-8/18/$31.00 ©2018 IEEE

II. PRELIMINARIES

A. Basic definitions and notations

Our state-dependent switched system deploys a model from [13], which has a similar setup as the state-dependent switched system model in [7] and the references therein. Let $I = \{1, 2, \cdots, l\}$ be the set of modes of the system and for each $i \in I$, define functions

$$f_i(x, u) : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n.$$

These are the dynamics for each mode and we require $f_i(x, u)$ to be locally Lipschitz in both x and u for all $i \in I$. Here $\mathcal{U} \subseteq \mathbb{R}^m$ is the input value set. We then define $\mathcal{M}_\mathcal{U}$ as the set of all locally essentially bounded functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{U}$. Let $S_i \subseteq \mathbb{R}^n$ be the admissible regions of the state x in mode i. S_i's are not necessarily disjoint, meaning the system can have same state while in different modes. Define the total admissible hybrid state space to be $\mathcal{S} = \cup_{i \in I}(S_i, \{i\})$. Define the switch guards $E_{i,j} \subset \mathbb{R}^n$ so that a switch from mode i to j occurs when $x \in E_{i,j}$ and $\sigma = i$. By convention $E_{i,i} = \emptyset$ and $E_{i,j}$ can be empty for lots of other indices j, meaning that the switch from mode i to j will never happen. Here are some regularity assumptions on the switch guards:

A1:

$$E_{i,j} \subseteq \text{int} S_j \quad \forall i, j \in I, \quad \text{and} \tag{1}$$

$$\cup_{j \in I} E_{i,j} = \partial S_i \quad \forall i \in I. \tag{2}$$

A2: each $E_{i,j}$ is closed and

$$E_{i,j} \cap E_{i,k} = \emptyset \quad \forall j \neq k,\ i, j, k \in I. \tag{3}$$

Here (1) ensures that the solution is still in the admissible hybrid state space after each switch. Equation (2) ensures the occurrence of a switch when the state is at the boundary of an admissible region, and (3) guarantees that when a switch is about to occur, the mode-to-be is unique.

The dynamics of a forward complete, state-dependent switched system (Σ) is defined as follows:

$$\begin{cases} \dot{x} = f_\sigma(x, u) & \text{if } x \in \text{int} S_\sigma \\ x^+ = x & \text{if } x \in \partial S_\sigma \end{cases} \tag{4}$$

$$\begin{cases} \sigma^+ = \sigma & \text{if } x \in \text{int} S_\sigma \\ \sigma^+ = j & \text{if } x \in E_{\sigma,j} \end{cases} \tag{5}$$

with initial condition $(x_0, \sigma_0) \in \mathcal{S}$. We denote the state and mode of the solution at time t as $x(t, x_0, \sigma_0, u), \sigma(t, x_0, \sigma_0, u)$ respectively. When $(x_0, \sigma_0) \in \mathcal{S}$ is given and $u \in \mathcal{M}_\mathcal{U}$ is fixed, we can simplify the two notations to be $x(t), \sigma(t)$, respectively. Sometimes we will simply call $x(t, x_0, \sigma_0, u)$ the solution of system (Σ) while ignoring the current modes the system is in. Because of Assumption A1, we see that $(x(t), \sigma(t)) \in \mathcal{S}$ for all $t \geq 0, u \in \mathcal{M}_\mathcal{U}$. In addition, (5) is well defined when $x \in \partial S_\sigma$ because (3) in Assumption A2 tells us that the mode-to-be is unique.

For two sets $A, B \subset \mathbb{R}^n$ define the metric

$$d(A, B) := \inf_{x \in A, y \in B} |x - y|.$$

It naturally reduces to the case when one of them is only a single point $x \in \mathbb{R}^n$ and we abuse the same notation

$$d(A, x) := \inf_{y \in A} |x - y|.$$

B. Auxiliary system

Let $r > 0$ and let ρ be a class $\mathcal{K}^\infty$ function. Define

$$f_i^\rho(x, d) := f_i(x, \rho(|x|)d), \quad i \in I \tag{6}$$

Define the auxiliary system (Σ^ρ) for (Σ) as follows:

$$\begin{cases} x = f_\sigma^\rho(x, d) & \text{if } x \in \text{int} S_\sigma \\ x^+ = x & \text{if } x \in \partial S_\sigma \end{cases} \tag{7}$$

$$\begin{cases} \sigma^+ = \sigma & \text{if } x \in \text{int} S_\sigma \\ \sigma^+ = j & \text{if } x \in E_{\sigma,j} \end{cases} \tag{8}$$

with initial condition $(x_0, \sigma_0) \in \mathcal{S}$ and disturbance $d \in \mathcal{M}_\mathcal{D}$, $\mathcal{D} = \{d \in \mathbb{R}^m : |d| \leq 1\}$. Similarly we denote the state and mode of this auxiliary system (Σ^ρ) by $x^\rho(t, x_0, \sigma_0, d), \sigma^\rho(t, x_0, \sigma_0, d)$ respectively. Notice that by this definition, $x^\rho(t, x_0, \sigma_0, d) = x(t, x_0, \sigma_0, \rho(|x|)d)$, $\sigma^\rho(t, x_0, \sigma_0, d) = \sigma(t, x_0, \sigma_0, \rho(|x|)d)$ for all $t \geq 0, (x_0, \sigma_0) \in \mathcal{S}, d \in \mathcal{M}_\mathcal{D}$. The construction of an auxiliary system is a common technique practiced in the literature (see, e.g., [3],[16]) and we also would like to mimic those techniques in this paper. The relation between (Σ) and (Σ^ρ) will be discussed in Section V.

C. Stability definitions

First of all, $f_i(0, 0) = 0$ for all $i \in I$ such that $0 \in S_i$ imply $x(t, 0, \sigma_0, 0) \equiv 0 \quad \forall t \geq 0, (0, \sigma_0) \in \mathcal{S}$. In this case we say 0 is an equilibrium to the system (Σ).

Consider the case when $\mathcal{U} = \mathbb{R}^m$; that is, when the control is unconstrained. We say the system (Σ) has *global stability (GS)* property if bounded initial states and controls produce uniformly bounded trajectories and, in addition, small initial states and controls produce uniformly small trajectories:

$$\exists \sigma, \gamma \in \mathcal{K}_\infty \text{ s.t. } \forall (x_0, \sigma_0) \in \mathcal{S}, \forall u \in \mathcal{M}_\mathcal{U},$$
$$\sup_{t \geq 0} |x(t, x_0, \sigma_0, u)| \leq \max\{\sigma(|x_0|), \gamma(\|u\|_\infty)\}.$$

The system (Σ) has *asymptotic gain (AG)* property if every trajectory must ultimately stay not far from the origin, depending on the magnitude of the input:

$$\exists \gamma \in \mathcal{K}_\infty \text{ s.t. } \forall (x_0, \sigma_0) \in \mathcal{S}, \forall u \in \mathcal{M}_\mathcal{U},$$
$$\limsup_{t \to \infty} |x(t, x_0, \sigma_0, u)| \leq \gamma(\|u\|_\infty).$$

The system (Σ) is *input-to-state stable* (ISS) if

$$\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty \text{ s.t. } \forall (x_0, \sigma_0) \in \mathcal{S}, \forall u \in \mathcal{M}_\mathcal{U},$$
$$|x(t, x_0, \sigma_0, u)| \leq \beta(|x_0|, t) + \gamma(\|u\|_{[0,t]}).$$

The next few stability definitions will only be used on the auxiliary system (Σ^ρ) whose input value set $\mathcal{D}$ is the unit ball. Nevertheless, we state the definitions for the general state-dependent switched systems (Σ) when $\mathcal{U}$ is bounded.

We say a system (Σ) is *globally asymptotically stable* (GAS) if the system is *stable* in the sense that

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall (x_0, \sigma_0) \in \mathcal{S} \text{ with } |x_0| \leq \delta, \\ \sup_{t \geq 0, u \in \mathcal{M}_{\mathcal{U}}} |x(t, x_0, \sigma_0, u)| \leq \epsilon$$

and is *attractive* in the sense that

$$\forall (x_0, \sigma_0) \in \mathcal{S}, u \in \mathcal{M}_{\mathcal{U}}, \quad \lim_{t \to 0} x(t, x_0, \sigma_0, u) = 0.$$

Further, the system (Σ) is said to be *uniformly globally asymptotically stable* (UGAS) if the system is stable and is *uniformly attractive* in the sense that

$$\forall \epsilon > 0, \kappa > 0, \exists T \geq 0 \text{ s.t. } \forall (x_0, \sigma_0) \in \mathcal{S} \text{ with } |x_0| \leq \kappa, \\ \sup_{t \geq T, u \in \mathcal{M}_{\mathcal{U}}} |x(t, x_0, \sigma_0, u)| \leq \epsilon.$$

Notice that the uniformity in UGAS refers to the existence of uniform time T for attractivity. In the special case when $\mathcal{U} = \{0\}$, the system becomes autonomous and stability, attractivity, uniform attractivity, GAS and UGAS reduce to the classical definitions (see, e.g., [17]) for autonomous systems.

III. Motivation

Before studying the state-dependent switched system, we would like to review some ideas behind the elegant proof in [3] of the equivalence between GS plus AG and ISS for single-mode Lipschitz systems. Figure 1 shows a proof flow of the main result in that paper:

$$\begin{array}{cccccccc}
\Sigma: & \text{GS} & + & \text{AG} & & \xleftarrow{\text{(f)}} & & \text{ISS} \\
 & \downarrow\text{(a)} & & \downarrow\text{(b)} & & & & \uparrow\text{(c)} \\
\Sigma^{\rho}: & \text{stability} & + & \text{attractivity} & \xrightarrow{\text{(d)}} & \text{GAS} & \xrightarrow{\text{(e)}} & \text{UGAS}
\end{array}$$

Fig. 1. Proof flow of AG+GS=ISS

In their proof, while the implication (f) in Figure 1 is trivial, the proof of the other direction is done by a detour via arguments on the auxiliary system (Σ^{ρ}). Firstly (a) and (b) are proven by straightforward comparison function manipulation; (c) can be either concluded directly by invoking the converse Lyapunov theorem from [16], or again proven via comparison functions. In addition, (d) is the definition of GAS. The essential step is (e), which heavily depends on the property of continuous dependence of solutions on initial conditions induced by a Lipshitz vector field and an approximation of the limit of a sequence of infinite time horizon solutions with arbitrarily small error. Thanks to this key result of (e), there is no necessity to mention uniform convergence time for systems with Lipschitz vector fields whenever we are dealing with stability of systems and convergence of solutions. For example, ISS (with uniform convergence time implicitly embedded in the class $\mathcal{KL}$ function β) applied to an autonomous system yields the so-called 0-GAS property, which in fact should be more precisely referred as 0-UGAS. However, this equivalence between GAS and UGAS cannot be simply transferred to state-dependent switched systems, as illustrated by the following counterexample.

A. Counterexample

Consider a 2-dimensional, 2-mode system with

$$S_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 1\}, \; S_2 = \mathbb{R}^2, \\ E_{1,2} = \partial S_1 = \{(1, x_2) : x_2 \in \mathbb{R}\}.$$

The subsystem dynamics of each mode is given by:

$$f_1(x) = (\sqrt{x_1^2 + x_2^2} - 1) \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \; f_2(x) = -\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

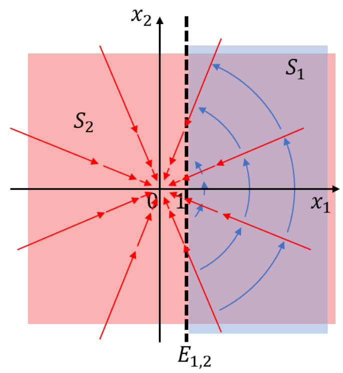

Fig. 2. A 2-dimensional example which is GAS but not UGAS

The mode regions and corresponding vector fields are shown in Fig 2. It is not hard to see that in Mode 1, the system solution is rotating counter-clockwise around the origin with angular velocity $|x| - 1$. Since S_1 is only the right half-plane with respect to the line $x_1 = 1$, the rotation velocity is always positive in intS_1 and the solution will eventually hit the boundary and switch to Mode 2. In Mode 2, the solution converges to the origin exponentially fast. Therefore, this system is stable and attractive, so it is GAS. Nevertheless, consider a solution with initial condition $x_0 = (r, 0), \sigma_0 = 1$ where $r > 1$ but very close to 1. It needs to rotate an angle of $\arccos(\frac{1}{r})$ before it hits $E_{1,2}$, hence it has to stay in mode 1 for a time $\frac{\arccos(\frac{1}{r})}{r-1}$, which tends to infinity when $r \to 1^+$. Thus the convergence time is not uniformly bounded; the system is not uniformly attractive. Therefore, this system is not UGAS.

B. Additional assumptions

For simplicity, the assumptions in this subsection are expressed in terms of f_i^{ρ}, which can be translated to assumptions in terms of f_i via (6). It is observed that in the previous example, the ill behavior of solutions arises in the neighborhood of state $(1, 0)$ in S_1, on which $f_1(x)$ becomes parallel to the boundary $x_1 = 1$ and hence the time needed for a switch to occur approaches infinity. Therefore, we need a suitable transversality assumption imposed on the system:

A3: There exist functions $g_i \in C^1(\mathbb{R}^n)$ such that each admissible region S_i can be defined by g_i:

$$S_i = \{x \in \mathbb{R}^n : g_i(x) \geq 0\}, \quad i \in I.$$

In addition,

$$f_i^{\rho}(x, d) \cdot \nabla g_i(x) < 0 \quad \forall d \in \mathcal{D}, x \in \partial S_i, i \in I. \quad (9)$$

By this definition, the boundaries of regions of system modes are $\partial S_i = \{x \in S_i : g_i(x) = 0\}$. For any $K \subset \mathcal{S}$ (in most cases the mode element in K is a singleton), the reachable set of the solutions of (Σ^ρ) over the time interval $[0,T]$ starting from K is denoted to be $\mathcal{R}^T(K)$. In other words,

$$\mathcal{R}^T(K) := \{x^\rho(t, x_0, \sigma_0, d) : t \in [0,T], \\ (x_0, \sigma_0) \in K, d \in \mathcal{M}_\mathcal{D}\}$$

To make the analysis easier, we also impose the two following assumptions here:

A4: For any $T \geq 0$ and compact set $K \subset \mathcal{S}$, there exists $c > 0$ such that $\mathcal{R}^T(K) \subseteq B_c$.

A5: The sets $F_i(x) := \{f_i^\rho(x,d) : d \in \mathcal{D}\}$ are convex for all $x \in \mathbb{R}^n, i \in I$.

Assumption A4 means the reachable space over a compact set of initial conditions and finite time horizon is bounded. While this assumption is always true for single-mode, Lipschitz systems (see [16]), it is not clear for state-dependent switched systems. Nevertheless, if we are working on a compact state space, or $|f_i|$ are globally bounded, or some more knowledge of the system directly tells that every solution is bounded, then A4 would be true. We postpone the discussion of A5 to Lemma 6 where it is used.

IV. Main results

With the assumptions proposed in the previous section, we can prove the following theorem regarding GAS and UGAS in this paper:

Theorem 1 *Let a state-dependent switched system (Σ^ρ) be defined via* (7), (8). *Under assumptions A1–A5, (Σ^ρ) is GAS if and only if it is UGAS.*

Theorem 1 also leads to the main result of our work:

Theorem 2 *Let a state-dependent switched system (Σ) be defined via* (4), (5) *and assume it is GS and AG. There exists $\rho \in \mathcal{K}_\infty$ such that if assumptions A1–A5 are satisfied with f_i^ρ defined via* (6)*, then (Σ) is ISS.*

Referring to Figure 1 and following the same proof flow, we will first prove some simple arrows in the figure, that is, (a) by Lemma 1, (b) by Lemma 2, and (c) by Lemma 3, respectively. We will then prove Theorem 1, which also leads to the arrow (e) in the figure. As that proof is the most critical component of this paper, it will be contributed by the entire Section VI, consisting of several lemmas. Now notice that the arrow (d) is simply the definition of GAS and (f) is still trivial in this case, subsequently we can conclude Theorem 2.

V. Connection between (Σ) and (Σ^ρ)

Without loss of generality we can assume the two γ functions in the definition of GS and AG are identical and smooth. Define

$$\rho(s) := \gamma^{-1}(\frac{s}{2}). \tag{10}$$

Since $\gamma \in \mathcal{K}_\infty$, $\rho(s)$ is also a class $\mathcal{K}_\infty$ function and $\gamma \circ \rho(s) = \frac{s}{2}$. Use this ρ and define the corresponding auxiliary system, we can prove several relations between (Σ) and (Σ^ρ):

Lemma 1 *If (Σ) is GS, then its auxiliary system (Σ^ρ) is stable, where ρ is defined via* (10).

Lemma 2 *If (Σ) is AG, then its auxiliary system (Σ^ρ) is attractive, where ρ is defined via* (10).

Lemma 3 *Assume that the system (Σ) is GS. Then it is also ISS if and only if its auxiliary system (Σ^ρ) is UGAS, where ρ is defined via* (10).

The proofs of the above three lemmas can be found in the full version of this paper [18]. It is worth pointing out that since converse Lyapunov theorem may not hold for state-dependent switched systems, the existence of a Lyapunov function V can not be assumed when proving Lemma 3; like the other two lemmas, Lemma 3 is then proven via comparison functions.

VI. GAS to UGAS

The special properties of state-dependent switched systems are not required for the proofs for the lemmas in Section V; they will only appear when we show the implication from GAS to UGAS. The proofs for the next two lemmas are omitted here due to space constraint but again they can be found in [18]. We first conclude an important result from the transversality assumption A3, which suggests that whenever a solution is very close to a switch guard, it is guaranteed to hit the switch guard within a time that is proportional to the distance the current state is away from the guard.

Lemma 4 *When A3 is true, for any $T > 0$ and any compact set $K \subseteq \mathcal{S}$, there exists $r > 0, \mu > 0$ such that if $|x^\rho(s, x_0, \sigma_0, d) - y| \leq r$ for some $s \leq T, (x_0, \sigma_0) \in K, d \in \mathcal{M}_\mathcal{D}$ and $y \in \partial S_{\sigma(s)}$, then $x^\rho(s+\Delta, x_0, \sigma_0, d) \in \partial S_{\sigma(s)}$ for some $\Delta \leq \mu|x^\rho(s, x_0, \sigma_0, d) - y|$.*

With the help of Lemma 4 and the other assumptions in the theorem statement, we can now show that adjacent solutions of the state-dependent switched system switch at similar time. To be more precise, let $K \subseteq \mathcal{S}$ be a compact set and pick a convergent sequence of initial conditions$(x_0^k, \sigma_0) \in K$. Denote $x^k(t) := x^\rho(t, x_0^k, \sigma_0, d^k)$, $\sigma^k(t) := \sigma^\rho(t, x_0^k, \sigma_0, d^k)$ where $d^k \in \mathcal{M}_\mathcal{D}$. Suppose that $x^k(t) \to \theta(t) \in \mathbb{R}^n$ for all $t \geq 0$ point-wise. Clearly we should have $\theta(0) = \lim_{t\to\infty} x_0^k$. Because each $x^k(t)$ is continuous with $|\dot{x}^k|$ locally uniformly bounded, $x^k(t)$ are locally equicontinuous so the limit $\theta(t)$ is continuous. Keep in mind that θ may not be a solution so "switches" on θ are not defined. Alternatively, we can recursively define

$$t_0 = 0,\ t_j = \min\{t \geq t_{j-1} : \theta(t) \in \partial S_{\sigma_{j-1}}\}, \tag{11}$$

with σ_j defined such that $\theta(t_j) \in E_{\sigma_{j-1},\sigma_j}$ for $j \geq 1$. Similar switching time means:

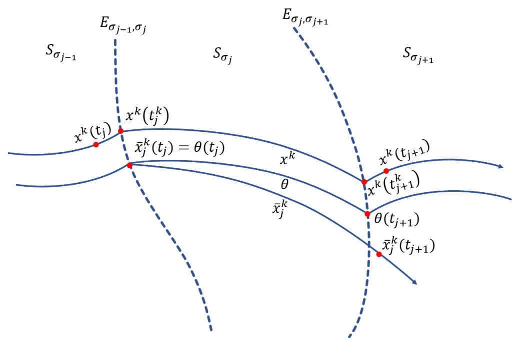

Fig. 3. An illustration of x^k, θ and $\bar{x}_j^k$

Lemma 5 *For any $T > 0$, there exists a $\bar{k}$ such that for each $j \geq 1$ and $t_j < T$ as defined via* (11)*, there will be a sequence of time t_j^k when all the solutions $x^k(t)$ with $k \geq \bar{k}$ will switch, in the sense that $\sigma^k(t_j^k) = \sigma_{j-1}$, $x^k(t_j^k) \in E_{\sigma_{j-1},\sigma_j}$. In addition, $\lim_{k\to\infty} t_j^k = t_j$ and $\lim_{k\to\infty} x^k(t_j^k) = \theta(t_j)$.*

For any set $\Omega \subseteq \mathbb{R}^n$, define function $\tau_\Omega : C^0(\mathbb{R}^n_{\geq 0} \to \mathbb{R}^n) \to \mathbb{R}^n_{\geq 0}$:

$$\tau_\Omega(x) := \inf_{t \geq 0}\{t : x(t) \in \Omega\}.$$

This is the hitting time of a solution to the set Ω. To be complete, we say $\tau_\Omega(x) = \infty$ if $x(t) \notin \Omega$ for all $t \geq 0$. The last lemma we are going to state suggests that there is always a convergent sequence of solutions of this state-dependent switched system, and indeed the limit of this sequence of solutions is again a solution to this system:

Lemma 6 *Let $K \subseteq \mathcal{S}$ be a compact subset and $\Omega \subseteq \mathbb{R}^n$ be an open subset. If*

$$\sup_{(x_0,\sigma_0)\in K, d\in\mathcal{M}_\mathcal{D}} \tau_\Omega(x^\rho(t, x_0, \sigma_0, d)) = \infty,$$

then there exists $(x^*, \sigma^*) \in K, v \in \mathcal{M}_\mathcal{D}$ such that

$$\tau_\Omega(x^\rho(t, x^*, \sigma^*, v)) = \infty.$$

Proof: The proof consists of two parts. The first part is to show that under the hypothesis, there exists a curve that never intersects Ω. The proof of the first part is almost identical to the proof of Lemma III.2 in [3]. Interested readers are referred to that paper for a detailed construction of a limit curve θ so that $\theta(t) \notin \Omega$ for all $t \geq 0$.

The second part is to show that θ indeed is a solution to the system (Σ^ρ). Without loss of generality assume the second element in K is a singleton; hence we must have $\sigma^* = \sigma_0$. Define the sequence of t_j on $\theta(t)$ as in (11). By A1 we know that in fact $t_j < t_{j+1}$ for all j. We define a solution $\bar{x}_j^k(t)$ over the time interval $[t_j, t_{j+1}]$ by the dynamics

$$\dot{\bar{x}}_j^k(t) = f^\rho_{\sigma_j}(\bar{x}_j^k, d^k)$$

with initial condition $\bar{x}_j^k(t_j) = \theta(t_j)$. Figure 3 is an illustration of the relation between x^k, θ and $\bar{x}_j^k$. In order to prove that θ is a solution, we first show that $\bar{x}_j^k \to \theta$ uniformly over $[t_j, t_{j+1}]$ for all $j \geq 0$ and $t_{j+1} \leq T$. Pick any arbitrary $\delta_1, \delta_2 > 0$. By Lemma 5, there exists $k_1 \in \mathbb{N}$ such that as long as $k \geq k_1$, $|t_j^k - t_j| \leq \delta_1$, $|t_{j+1}^k - t_{j+1}| \leq \delta_1$ and $x^k(t_j^k) \in E_{\sigma_{j-1},\sigma_j}, x^k(t_{j+1}^k) \in E_{\sigma_j,\sigma_{j+1}}$. In addition because $x^k(t)$ converges to $\theta(t)$ uniformly over $[t_j, t_{j+1}] \subseteq [0, T]$, we should have $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$, $|x^k(t) - \theta(t)| \leq \delta_2$ for all $t \in [t_j, t_{j+1}]$. Now if $t_j^k \leq t_j$, $|x^k(t_j) - \bar{x}_j^k(t_j)| = |x_k(t_j) - \theta(t_j)| \leq \delta_2$. Otherwise, $\sigma^k(t) = \sigma_{j-1}$ for all $t \in [t_j, t_j^k]$, meaning there is no switch on the solution x^k over this time interval so $|x^k(t) - x^k(t_j)| \leq M|t - t_j|$ where M is the local upper bound for $|f_i^\rho|$. Thus we have

$$\begin{aligned}|x^k(t) - \bar{x}_j^k(t)| \leq &|x^k(t) - x^k(t_j)| + |x^k(t_j) - \bar{x}_j^k(t_j)| \\ &+ |\bar{x}_j^k(t_j) - \bar{x}_j^k(t)| \\ \leq &2M(t_j^k - t_j) + \delta_2 \leq 2M\delta_1 + \delta_2\end{aligned}$$

So we have $|x^k(t) - \bar{x}_j^k(t)| \leq 2M\delta_1 + \delta_2$ for all $t \in [t_j, \max\{t_j, t_j^k\}]$. Now for $t \in [\max\{t_j, t_j^k\}, \min\{t_{j+1}, t_{j+1}^k\}]$, we see that $\sigma^k(t) = \sigma_j$, that is, x^k follows dynamics $\dot{x}^k = f^\rho_{\sigma_j}(x^k, u^k)$, which is the same as of $\bar{x}_j^k$. Hence we can apply Grönwall's lemma, $|x^k(t) - \bar{x}_j^k(t)| \leq |x^k(t_j^k) - \bar{x}_j^k(t_j^k)|e^{L_1(t - t_j^k)} \leq (2M\delta_1 + \delta_2)e^{L_1(t_{j+1} - t_j)}$. In the case $t_{j+1}^k \geq t_{j+1}$, that is exactly the upper bound for the separation over whole time interval $[t_j, t_{j+1}]$. Otherwise, for $t \in [t_{j+1}^k, t_{j+1}]$,

$$\begin{aligned}|x^k(t) - \bar{x}_j^k(t)| \leq &|x^k(t) - x^k(t_j)| + |x^k(t_{j+1}) - \bar{x}_j^k(t_{j+1})| \\ &+ |\bar{x}_j^k(t_{j+1}) - \bar{x}_j^k(t)| \\ \leq &2M(t_j^k - t_j) + (2M\delta_1 + \delta_2)e^{L_1(t_{j+1} - t_j)} \\ \leq &2M\delta_1 + (2M\delta_1 + \delta_2)e^{L_1(t_{j+1} - t_j)}\end{aligned}$$

Comparing it with the earlier bounds, we see that the inequality above is in fact true for all $t \in [t_j, t_{j+1}]$. Using triangle inequality again, we have

$$\begin{aligned}|\bar{x}_j^k(t) - \theta(t)| &\leq |\bar{x}_j^k(t) - x^k(t)| + |x^k(t) - \theta(t)| \\ &\leq 2M\delta_1 + (2M\delta_1 + \delta_2)e^{L_1(t_{j+1} - t_j)} + \delta_2 \\ &= (2M\delta_1 + \delta_2)\left(1 + e^{L_1(t_{j+1} - t_j)}\right)\end{aligned}$$

For all $k \geq \max\{k_1, k_2\}, t \in [t_j, t_{j+1}]$. As δ_1, δ_2 are taken arbitrarily so the separation can be made arbitrarily small, we conclude that $\bar{x}_j^k(t)$ converges to $\theta(t)$ uniformly over $[t_j, t_{j+1}]$. Thus by Filippov's Theorem [19] and using the assumption A5 that f_i^ρ are convex, there exists a control $v_j \in \mathcal{M}_\mathcal{D}$ that $\dot{\theta} = f^\rho_{\sigma_j}(\theta, v_j)$ over $[t_j, t_{j+1}]$. By defining $x^* = \theta(0)$ and $v \in \mathcal{M}_\mathcal{D}$ by $v(t) := v_j(t) \quad \forall t \in [t_j, t_{j+1})$, we finally have $x(t, x^*, \sigma_0, v) = \theta(t)$ and hence $\tau(x^*, \sigma_0, \Omega, v) = \infty$. ∎

Proof of Theorem 1: Let $\kappa, \epsilon > 0$ be arbitrary. The system (Σ) being GAS means it is stable and attractive. Let $\delta > 0$ be given by stability so that $(\xi, i) \in \mathcal{S}$ with $|\xi| \leq \delta$ implies $|x^\rho(t, \xi, i, d)| \leq \epsilon$ for all $t \geq 0, d \in \mathcal{M}_\mathcal{D}$. Let $\Omega = \{x \in \mathbb{R}^n : |x| < \delta\}$ and $K = \{(\xi, i) \in \mathcal{S} : |\xi| \leq \kappa\}$. On the other hand, attractivity implies that $\tau(\xi, i, \Omega, d) < \infty$ for all $(\xi, i) \in \mathcal{S}, d \in \mathcal{M}_\mathcal{D}$. Hence by the contrapositive argument of Lemma 6 we conclude that $\sup_{(\xi,i)\in K, d\in\mathcal{M}_\mathcal{D}} \tau(\xi,i,\Omega,u) < \infty$. In other words, there exists $T := T(\kappa,\delta)$ such that $x(\tau,\xi,i,u) \in \bar{\Omega}$ for some $\tau \le T$ and all $(\xi,i) \in K, d \in \mathcal{M}_\mathcal{D}$. Because the system is time-invariant, with the aforementioned stability we conclude that $\limsup_{t\ge T, u\in\mathcal{M}_\mathcal{U}} |x(t,\xi,i,d)| \le \epsilon$ for all $(\xi,i) \in \mathcal{S}$ with $|\xi| \le \kappa$. Because T only depends on κ and δ, which further depends on ϵ, the system (Σ) is uniformly attractive in addition to being stable, and hence it is UGAS. ■

VII. Discussion and future work

We would like to discuss another possible approach to showing the equivalence between GAS and UGAS via some results given in [11]. For a hybrid system $\mathcal{H}$ defined via

$$\begin{cases} \dot{x} \in F(x) & \text{if } x \in C \\ x^+ \in G(x) & \text{if } x \in D \end{cases},$$

Theorem 7.12 in this reference says that as long as C, D are closed, G, F are outer semicontinuous, locally bounded and $F(x)$ is convex for all $x \in C$, then GAS is equivalent to UGAS. In order to deploy this theorem, we need to combine state x and mode σ as the hybrid state as well as define $C := \cap_{i\in I}(\bar{S}_i, \{i\}), D := \cap_{i\in I}(\partial S_i, \{i\})$. Notice that by this transformation, although C and D are both closed, there are possible overlaps between them. As a result, when a solution reaches ∂S_i, $\mathcal{H}$ either allows the solution to keep flowing continuously inside ∂S_i without switch, or a switch occurs and the mode jumps. In other words, $\mathcal{H}$ is different from (Σ^ρ) as it allows non-unique solutions. Nevertheless, under the transversality assumption, the first situation cannot happen; thus indeed the system $\mathcal{H}$ has same solutions as (Σ^ρ). Additionally, using this approach we see that Lipschitzness on f_i's can be replaced by outer semicontinuity and Assumption 4 becomes redundant. Nevertheless, it is worth pointing out that our approach used in this paper is from scratch and does not require the framework of hybrid systems from [11], the transformation of models from state-dependent switched system to hybrid system is not apparent, and analysis in hybrid time domain requires some extra work. Besides, Lemma 4, Lemma 5 and their proofs also reveal some robustness related properties on the state-dependent switched system with the presence of transversality.

In addition, we have required in A5 that the vector fields f_i's be convex. As discussed earlier, this assumption is needed to show that the limit of sequence of solutions is also a solution. In fact, we can relax this; as long as we can approximate the limit of a sequence of solutions by a solution on the infinite time horizon within an arbitrary uniform ε-tube, we can still show the existence of a uniform convergence time. This is closely related to the Filippov-Wazewski Relaxation Theorem, and [20] gives an infinite time horizon version. This relaxation in the context of state-dependent switched systems will be another possible research direction.

At last, as partially revealed in the proof of Lemma 6, it is observed that A1-A4 also imply that our system has the property of continuous dependence of solutions on initial conditions. This continuous dependence is considered to be an overkill for showing the equivalence between GAS and UGAS, and therefore we would like to develop another set of assumptions which does not restrict the system studied to have continuous dependence on initial conditions yet still allows us to show the implication from GS and AG to ISS.

VIII. Conclusion

In this paper we have first proposed several additional assumptions for a GAS state-dependent switched system to be UGAS. Based on this equivalence we then proved that if a state-dependent switched system has the GS and AG properties, then it is ISS as well.

References

[1] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, Apr 1989.

[2] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems and Control Letters*, vol. 24, no. 5, pp. 351 – 359, 1995.

[3] ——, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, Sep 1996.

[4] D. Liberzon and D. Nesic, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Transactions on Automatic Control*, vol. 52, no. 5, pp. 767–781, May 2007.

[5] G. Yang and D. Liberzon, "Feedback stabilization of a switched linear system with unknown disturbances under data-rate constraints," *IEEE Transactions on Automatic Control*, p. to apear, 2018.

[6] Y. Sharon and D. Liberzon, "Input to state stabilizing controller for systems with coarse quantization," *IEEE Transactions on Automatic Control*, vol. 57, no. 4, pp. 830–844, April 2012.

[7] D. Liberzon, *Switching in Systems and Control*. Boston, MA: Birkhäuser, 2003.

[8] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, Dec 2012, pp. 3270–3285.

[9] Z. Wu, M. Cui, P. Shi, and H. R. Karimi, "Stability of stochastic nonlinear systems with state-dependent switching," *IEEE Transactions on Automatic Control*, vol. 58, no. 8, pp. 1904–1918, Aug 2013.

[10] L. I. Allerhand and U. Shaked, "Robust state-dependent switching of linear systems with dwell time," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 994–1001, April 2013.

[11] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[12] C. Cai and A. Teel, "Robust input-to-state stability for hybrid systems," *SIAM Journal on Control and Optimization*, vol. 51, no. 2, pp. 1651–1678, 2013.

[13] M. Broucke and A. Arapostathis, "Continuous selections of trajectories of hybrid systems," *Systems and Control Letters*, vol. 47, no. 2, pp. 149 – 157, 2002.

[14] L. Tavernini, "Differential automata and their discrete simulators," *Non-Linear Anal.*, vol. 11, no. 6, pp. 665–683, Jun. 1987.

[15] I. A. Hiskens and M. A. Pai, "Trajectory sensitivity analysis of hybrid systems," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 47, no. 2, pp. 204–220, Feb 2000.

[16] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse lyapunov theorem for robust stability," *SIAM Journal on Control and Optimization*, vol. 34, no. 1, pp. 124–160, 1996.

[17] H. Khalil, *Nonlinear Systems, 3rd ed.* Prentice Hall, 2002.

[18] S. Liu and D. Liberzon, "Global stability and asymptotic gain imply input-to-state stability for state-dependent switched systems." [Online]. Available: http://liberzon.csl.illinois.edu/research/SDSS.pdf

[19] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Springer, Dordrecht, 1988.

[20] B. Ingalls, E. D. Sontag, and Y. Wang, "An infinite-time relaxation theorem for differential inclusions," in *proceedings of the American Mathmatical Society*, 2003.