Chapter 7
Observer Design for Switched Linear Systems with State Jumps

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Abstract An observer design for switched linear systems with state resets is proposed based on the geometric conditions for large-time observability from our recent work. Without assuming the observability of individual subsystems, the basic idea is to combine the maximal information available from each mode to obtain a good estimate of the state after a certain time interval (over which the switched system is observable) has passed. We first study systems where state reset maps at switching instants are invertible, in which case it is possible to collect all the observable and unobservable information separately at one time instant. One can then annihilate the unobservable component of all the modes and obtain an estimate of the state by introducing an error correction map at that time instant. However, for the systems with non-invertible jump maps, this approach needs to be modified and a recursion-based error correction scheme is proposed. In both approaches, the criterion for choosing the output injection matrices is given, which leads to the asymptotic recovery of the system state.

7.1 Introduction

State estimation in dynamical systems is one of the classical control-theoretic problems that relates to constructing estimates of the state of the system using the measurements of the inputs and outputs. This chapter studies the problem of state

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estimation for a class of hybrid systems characterized by linear continuous-time dynamics, switching vector fields, and state jumps, which are described as

$$\dot{x}(t) = A_q x(t) + B_q u(t), \qquad t \in [t_{q-1}, t_q), \tag{7.1a}$$

$$x(t_q) = G_q x(t_q^-) + H_q v_q, \tag{7.1b}$$

$$y(t) = C_q x(t) + D_q u(t), \qquad t \in [t_{q-1}, t_q) \tag{7.1c}$$

where $x : [t_0, \infty) \to \mathbb{R}^{\mathbf{n}}$ is the state, $y : [t_0, \infty) \to \mathbb{R}^{\mathbf{y}}$ is the output, $v_i \in \mathbb{R}^{\mathbf{v}}$ and $u : [t_0, \infty) \to \mathbb{R}^{\mathbf{u}}$ are the inputs, and u is a measurable function. The index $q \in \mathbb{N}$ determines the active subsystem over the interval $[t_{q-1}, t_q)$ and the system trajectories are right-continuous. It is assumed that there are a finite number of switching times in any finite time interval, thus we rule out the Zeno phenomenon in our problem formulation. The switching mode $q \in \mathbb{N}$ and the switching times $\{t_q\}$ may be governed by a supervisory logic controller, or determined internally depending on the system state, or considered as an external input. In any case, it is assumed in this paper that the active subsystem and the switching times $\{t_q\}$ as well are known. For estimation of the active subsystem, one may be referred to, e.g., [4, 6, 8, 18, 25, 26].

Over the past decade and a half, the structural properties of switched systems have been investigated by many researchers and observability along with observer construction has been one of them. For switched systems, observability can be studied from various perspectives. If we allow for the use of the differential operator in the observer, then it may be desirable to determine the state of the system instantaneously from the measured output. This in turn requires each subsystem to be observable; however, the problem becomes nontrivial when the switching signal is treated as a discrete state and simultaneous recovery of the discrete and continuous state is required for observability. Some results on this problem are published in [2, 5, 25].

On the other hand, with the knowledge of switching signal, even though the subsystems at individual modes are not observable, it is possible to recover the initial state $x(t_0)$ when the output is observed over an interval $[t_0, T)$ that involves multiple switching instants. This phenomenon is of particular interest for switched systems as the notion of instantaneous observability and observability over an interval[1] coincide for linear time invariant systems. This variant of the observability in switched systems has been studied most notably by [4, 9, 10, 17, 27], and we refer the reader to Chap. 8 for more references on different notions of observability. The observer design has also received some attention in the literature [1, 3, 13], where authors have assumed that each mode in the system is in fact observable, hence admitting a state observer, and have treated the switching as a source of perturbation effect. This approach immediately incurs the need of a common Lyapunov function for the switched error dynamics, or a fixed amount of dwell-time between switching instants, because it is intrinsically a stability problem of the error dynamics.

[1] See Definition 7.1 for precise meaning.

The approach adopted for observer design in our work is based on the notion of observability over an interval, and is conceptually similar to the work of [4]. This relaxed notion of observability for switched systems does not require observability of individual subsystems in the classical sense. As a result, one *cannot* simply take the Luenberger observer for individual subsystem and work out the stability of the error dynamics using slow switching, or common Lyapunov function approach. Since the amount of observable information coming from different modes may vary, the interesting aspect of this approach is to design an algorithm that recovers the maximal possible information available from each mode and combines this information in an appropriate manner that results in an asymptotically converging state estimate. This is the fundamental idea behind our recent papers on observer design for switched linear systems [19, 20], switched nonlinear systems [14, 15], and systems with switched linear differential-algebraic equations [22].

In this chapter, we address the problem of observer design in the context of switched systems with linear ordinary differential equations, and the technical content is primarily based on our papers [19–21]. We focus only on the linear case because this relatively simpler class of systems brings out our design methodology in the most transparent manner. The construction of the observer is based on the necessary and sufficient conditions of *forward observability* (see Definition 7.1), or what is also called determinability in [17], and final-state observability in [16]. The detailed treatment of this notion of observability is considered in Chap. 8 of this book, and here in Sect. 7.2, for the sake of completeness, we will only recall the definition and the related formulae that set-up the ground work for observer construction.

Section 7.3 then considers the construction of the observers. The key idea is to combine the partial information available from each mode and collect them at one instant of time to get the estimate of the state at that time. We show that under mild assumptions, such estimates converges to the actual state of the plant. More emphasis will be given to the case when the individual modes of the system (7.1) are not observable (in the classical sense of linear time-invariant systems theory) since it is obvious that the system becomes immediately observable when the system is switched to the observable mode. The distinct feature of our observer design is that we do not inject error in the continuous dynamics of the proposed observer, but rather apply the error correction at discrete switching instants. This way the state estimation error may grow in between two consecutive switches, but the error correction terms are designed in a manner such that the error eventually converges to zero. One can already see that, in contrast to slow switching, our approach would work only if the switching is persistent and the estimation gets better if the frequency of switching is high since we can apply error corrections more often in that case. We basically treat two different cases: in the first case [19, 20], we assume that all the jump maps G_q are invertible, because it is relatively easier to do computations for this case. The second case [21] allows for jump maps G_q to be non-invertible, but the calculations are more involved in this case. As a result the error correction term in the later case is computed using a recursion-based algorithm instead of a direct formula.

In the end, we give some concluding remarks on how the ideas presented in this paper have been applied to more general classes of switched systems, and where the proposed observer has been applied in practice.

Notation: For a matrix A, $\mathscr{R}(A)$ denotes the column space (range space) of A. The sum of two subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$ is defined as $\mathscr{V}_1 + \mathscr{V}_2 := \{v_1 + v_2 : v_1 \in \mathscr{V}_1, v_2 \in \mathscr{V}_2\}$. For a possibly non-invertible matrix A, the pre-image of a subspace $\mathscr{V}$ under A is given by $A^{-1}\mathscr{V} = \{x : Ax \in \mathscr{V}\}$. Let $\ker A := A^{-1}\{0\}$; then it is seen that $A^{-1} \ker C = \ker(CA)$ for a matrix C. For convenience of notation, let $A^{-\top}\mathscr{V} := (A^\top)^{-1}\mathscr{V}$ where $A^\top$ is the transpose of A, and it is understood that $A_2^{-1}A_1^{-1}\mathscr{V} = A_2^{-1}(A_1^{-1}\mathscr{V})$. Also, we denote the products of matrices A_i as $\prod_{i=j}^{k} A_i := A_j A_{j+1} \dots A_k$ when $j < k$, and $\prod_{i=j}^{k} A_i := A_j A_{j-1} \dots A_k$ when $j > k$. The notation $\mathrm{col}(\mathrm{A}_1, \dots, \mathrm{A}_\mathrm{k})$ means the vertical stack of matrices $A_1, \dots, A_k$, that is, $[A_1^\top, \dots, A_k^\top]^\top$.

7.2 Preliminaries: Observability Notion

As mentioned earlier, our observer design is based on geometric conditions for forward observability. The related observability notions are treated in detail in this chapter. For the sake of completeness, here we recall the definition, and the related formulae that will be used in the design of observer. The notion of observability that we consider is formulated as follows:

Definition 7.1 (*Forward observability*) The system (7.1) is said to be *forward observable* if, and only if, for every pair of solutions $(u^1, v^1, y^1, x^1), (u^2, v^2, y^2, x^2)$, there exists $T > t_0$, such that the following implication holds:

$$(u^1, v^1, y^1) = (u^2, v^2, y^2) \quad \Rightarrow \quad x^1_{[T,\infty)} = x^2_{[T,\infty)}$$

where $x^i_{[T,\infty)}$, $i = 1, 2$, denotes the restriction of x^i over the interval $[T, \infty)$.

Since the value of the state at time T, $x(T)$, and the inputs (u, v) uniquely determine x on $[T, \infty)$ through the Eq. (7.1), forward observability is achieved if and only if $x(T)$, for some $T > t_0$, is uniquely determined by the inputs and the output. In case, all the jump maps are invertible, one can, in theory, recover $x(t_0)$ from the knowledge of $x(T)$ and in that case forward observability implies global (in time) observability. However, if the state reset maps G_q are not invertible, then the two notions may not coincide. We refer the reader to Chap. 8 for more details and examples related to this issue.

7.2.1 Characterization of Forward Observability

Roughly speaking, the switched system (7.1) is forward observable in the sense of Definition 7.1 if there exists $m \in \mathbb{N}$ such that $x(t_m)$ could be determined from the knowledge of external signals (u, v, y) measured over the interval $[t_0, t_{m+1})$. Because $x(t_{m+1}^-) = e^{A_{m+1}(t_{m+1}-t_m)}x(t_m)$, uncertainty in the knowledge of $x(t_m)$ and $x(t_{m+1}^-)$ is the same, so that, recovering $x(t_m)$ is equivalent to recovering $x(t_{m+1}^-)$. We now proceed toward quantifying the unknown information about the state using the measurements of (u, v, y) over a certain interval. Since Definition 7.1 does not require individual subsystems to be observable, the basic idea in formulating the geometric conditions that quantify the unknown information is to characterize how much information could be extracted from each subsystem about the state by measuring the output over a certain interval. To do so, it is seen that system (7.1) is an LTI system between two consecutive switching times, so that its unobservable subspace on the interval $[t_{q-1}, t_q)$ is simply given by the largest A_q-invariant subspace contained in $\ker C_q$, i.e., $\ker O_q$ where

$$O_q := \mathrm{col}(C_q, C_q A_q, \ldots, C_q A_q^{n-1}).$$

For system (7.1), let $\mathcal{N}_q^m$ be the subspace such that $x(t_m^-)$ is determined modulo $\mathcal{N}_q^m$ using the knowledge of external signals (u, v, y) over the interval $[t_{q-1}, t_m)$. We call $\mathcal{N}_q^m$ the *unobservable subspace for* $[t_{q-1}, t_m)$ and compute it recursively as follows, for $q \geq 1$:

$$\begin{aligned} \mathcal{N}_q^q &:= \ker O_q \\ \mathcal{N}_q^k &:= \ker O_k \cap G_{k-1} e^{A_{k-1}\tau_{k-1}} \mathcal{N}_q^{k-1}, \quad q+1 \leq k \leq m, \end{aligned} \tag{7.2}$$

where $\tau_k := t_k - t_{k-1}$.

An alternative dual characterization of forward observability is possible by inspecting whether the complete state information is available. This is achieved in terms of the *observable subspace* $\mathcal{Q}_q^m$, defined in this chapter as the orthogonal complement of $\mathcal{N}_q^m$. It is noted that a recursive expression for $\mathcal{Q}_q^m$ is given by

$$\begin{aligned} \mathcal{Q}_q^q &= \mathcal{R}(O_q^\top) \\ \mathcal{Q}_q^k &= G_{k-1}^{-\top} e^{-A_{k-1}^\top \tau_{k-1}} \mathcal{Q}_q^{k-1} + \mathcal{R}(O_k^\top), \quad q+1 \leq k \leq m. \end{aligned} \tag{7.3}$$

We characterize the observability of system (7.1) using these subspaces in the following result, which is essentially a restatement of [20, Theorem 2]:

Theorem 7.2 (Forward observability characterization) *Consider the switched system (7.1) with $(u, v) \equiv 0$. Then, $\mathcal{N}_q^m$ for some $m \geq q \geq 1$ characterizes the unobservable space in the following sense:*

$$y_{[t_{q-1}, t_m)} \equiv 0 \quad \Leftrightarrow \quad x(t_{m-1}) \in \mathcal{N}_q^m.$$

In particular, if there exists $m \geq q$ such that $\mathcal{N}_q^m = \{0\}$, or equivalently $\mathcal{Q}_q^m = \mathbb{R}^{\mathbf{n}}$, then the state $x(t_{m-1})$ (and hence the complete future trajectory) can be determined from the knowledge of (u, v, y) on the interval $[t_{q-1}, t_m)$.

We are often interested in deriving a direct formula for $\mathscr{Q}_q^m$ instead of the recursive one given in (7.2). For that, let us consider the matrix

$$\Phi_j^k = G_{k-1} e^{A_{k-1}\tau_{k-1}} \ldots G_j e^{A_j \tau_j}, \quad k > j$$

which defines the flow of system (7.1) with zero inputs from t_{j-1} to t_{k-1}, and assume that, for $k \geq q+2$, $i = 1, 2, \ldots, k-q-1$, $q \in \mathbb{N}$, the following condition holds[2]:

$$\Phi_{k-i}^k (\ker O_{k-i} \cap \Phi_{k-i-1}^{k-i} \mathscr{N}_q^{k-i-1}) = \Phi_{k-i}^k \ker O_{k-i} \cap \Phi_{k-i-1}^k \mathscr{N}_q^{k-i-1}. \quad (7.4)$$

It is readily checked that, if (7.4) holds, then the sequential definition (7.2) leads to another equivalent expression for $\mathscr{N}_q^m$, $m \geq q \geq 1$, given by:

$$\begin{aligned} \mathscr{N}_q^m &= \bigcap_{j=m,\ldots,q} \Phi_j^m \ker O_j \\ &= \ker O_m \cap G_{m-1} \ker(O_{m-1}) \cap \left(\bigcap_{i=q}^{m-2} \prod_{l=m-1}^{i+1} G_l e^{A_l \tau_l} G_i \ker O_i \right), \end{aligned} \quad (7.5)$$

where Φ_k^k denotes the identity matrix, and we used the fact that $e^{A_j \tau_j} \ker G_j = \ker G_j$. Condition (7.4) indeed holds when each of the matrix G_q, $q \in \mathbb{N}$, is invertible because in that case the mapping Φ_j^k, for all $j, k \in \mathbb{N}$, $k > j$, is invertible.

Similarly, when (7.4) holds, $\mathscr{Q}_q^m$ in (7.3) is equivalently expressed as:

$$\begin{aligned} \mathscr{Q}_q^m &= (\mathscr{N}_q^m)^\perp \\ &= \sum_{i=q}^{m-2} \prod_{l=m-1}^{i+1} G_l^{-\top} e^{-A_l^\top \tau_l} G_i^{-\top} \mathscr{R}(O_i^\top) + G_{m-1}^{-\top} \mathscr{R}(O_{m-1}^\top) + \mathscr{R}(O_m^\top). \end{aligned} \quad (7.6)$$

7.3 Observer Design

Using the geometric conditions for forward observability stated in the previous section, we now proceed to design an observer. Our proposed observer is given by:

$$\dot{\hat{x}}(t) = A_q \hat{x}(t) + B_q u(t), \qquad t \in [t_{q-1}, t_q), \quad (7.7a)$$

$$\hat{x}(t_q) = G_q(\hat{x}(t_q^-) - \xi_q) + H_q v_q, \quad (7.7b)$$

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[2] Note that, $A(\mathscr{V}_1 \cap \mathscr{V}_1) \subset A\mathscr{V}_1 \cap A\mathscr{V}_2$, and the equality does not hold in general. The necessary and sufficient condition for equality to hold is that $(\mathscr{V}_1 + \mathscr{V}_2) \cap \ker A = \mathscr{V}_1 \cap \ker A + \mathscr{V}_2 \cap \ker A$, which is the case when A is invertible. For systems with non-invertible jump maps, the flow matrix Φ_i^j is not necessarily invertible and (7.4) does not hold in general.

with an arbitrary initial condition $\hat{x}(t_0) \in \mathbb{R}^n$ and the expression for ξ_q will be computed in the sequel. It is seen that the observer consists of a system copy and unlike classical methods where the continuous dynamics of the estimate are driven by an error injection term, the observer (7.7) updates the state estimate only at discrete switching instants by an error correction vector ξ_q. If for some $q \in \mathbb{N}$, ξ_q equals the state estimation error $\hat{x}(t_q^-) - x(t_q^-)$, then the Eq. (7.7) gives $\hat{x}(t_q) = x(t_q)$, and from then onward we can recover the exact value of the trajectory x. However, in practice, where we do not use the derivatives of the output, it is not easy to recover the exact value of the state estimation error. Thus, our goal is to compute ξ_q, for each $q \in \mathbb{N}$, such that it *approximates* the value of state estimation error at time t_q^-, which will result in $\hat{x}(t)$ converging to $x(t)$ as t increases.

With this motivation, we introduce the state estimation error $\tilde{x} := \hat{x} - x$, and the error dynamics are given by

$$\dot{\tilde{x}}(t) = A_q \tilde{x}(t), \qquad t \in [t_{q-1}, t_q), \tag{7.8a}$$

$$\tilde{x}(t_q) = G_q(\tilde{x}(t_q^-) - \xi_q). \tag{7.8b}$$

The corresponding output error is defined as

$$\tilde{y}(t) := C_q \hat{x}(t) + D_q u(t) - y(t) = C_q \tilde{x}(t), \quad t \in [t_{q-1}, t_q).$$

The basic idea in computing ξ_q is to

- Firstly, identify the observable components of the individual subsystems that can be estimated using classical state-estimation techniques. For subsystem $q \in \mathbb{N}$, let $z_q \in \mathscr{R}(O_q^\top)$ denote the vector of such observable component.
- Secondly, derive an equation for $\tilde{x}(t_q^-)$ of the form[3]

$$\tilde{x}(t_q^-) = \Xi_q(z_q, z_{q-1}, \ldots, z_{q-\mathtt{m}^*}, \xi_{q-1}, \ldots, \xi_{q-\mathtt{m}^*}) \tag{7.9}$$

  for some $\mathtt{m}^* \in \mathbb{N}$.
- Finally, let

$$\xi_q = \Xi_q(\hat{z}_q, \hat{z}_{q-1}, \ldots, \hat{z}_{q-\mathtt{m}^*}, \xi_{q-1}, \ldots, \xi_{q-\mathtt{m}^*}). \tag{7.10}$$

We will develop calculations for each of the aforementioned steps in detail and arrive at a formal statement on error convergence that results from the observer. To do that, we need to introduce some assumptions that allow us to follow this proposed line of thought.

[3] With slight abuse of notation, the vectors z_j in (7.9) will be replaced by $z_j(t_j^-)$, so that the notation z_j will be used to denote a function.

The identification of observable components in the first step could be achieved easily by Kalman-like decomposition without imposing any constraints on system structure. For the second step, however, where we want to write $\tilde{x}(t_q^-)$, for each $q \in \mathbb{N}$, in terms of the observable components of the currently active mode and some past modes, we need the following assumption on the switching signal and system dynamics:

Assumption 7.3 The switched system (7.1) is persistently forward observable in the sense that there exists an $\mathtt{m}^* \in \mathbb{N}$ such that

$$\dim \mathscr{Q}^q_{q-\mathtt{m}^*} = \mathtt{n}, \qquad \forall\, q \geq \mathtt{m}^* + 1. \tag{7.11}$$

The integer $\mathtt{m}^*$ in Assumption 7.3 is interpreted as the minimal number of switches required to gain forward observability.

For the third step, it is seen that if $\hat{z}_q$ closely approximates z_q, and Ξ_q is globally Lipschitz (in our calculations, it will be linear), then it follows from (7.8) that the norm of the state estimation error at switching instants $\tilde{x}(t_q)$ becomes small. Since there is no error correction between the switching instants, it is important to update the estimate repeatedly for asymptotic convergence and also make sure that the error does not get arbitrarily large between the two switching instants. This motivates us to introduce the following assumptions for our observer design:

Assumption 7.4 The switching is persistent in the sense that a switch occurs at least once in any time interval of length D; that is,

$$t_q - t_{q-1} < D, \qquad \forall\, q \in \mathbb{N}. \tag{7.12}$$

Assumption 7.5 The induced matrix norms $\|A_q\|$ and $\|G_q\|$ are uniformly bounded for all $q \in \mathbb{N}$.

Note that Assumption 7.5 holds when A_q, G_q belong to a set of finite elements. Assumption 7.4 is in contrast to the conditions proposed for observer designs in [1] in the sense that we require the switching to be sufficiently fast and not too slow. This is not surprising because the works like [1] assume the observability of individual modes, so that the resulting error dynamics are stable for each subsystem, and a result on stability of switched systems with slow switching could be invoked [7] to show error convergence. In our work, however, since the individual subsystems are not assumed to observable, so that the resulting error dynamics for a particular mode are not necessarily stable, we need to switch fast enough between the unstable (or, partly stable) switched systems to obtain error convergence.

In the remainder of this section, the above thought process is formalized by following the steps outlined earlier to compute the correction vector ξ_q. The identification of observable components for individual subsystems is carried out in Sect. 7.3.1. For the computation of the map Ξ_q, we will discuss two cases separately: in the first case, we assume that all the jump maps G_q, $q \in \mathbb{N}$, are invertible and in the second case, we allow for non-invertible jump maps. For each of these cases, we show that the resulting state estimation error converges to zero (Theorems 7.6 and 7.7, respectively) if the observable components are estimated accurately enough. Followed by the theorem statements, we will arrive at a specific criteria for obtaining the estimates by carrying out the error analysis for each case. Our computations are summed up in Algorithm 7.1 (for invertible jump maps) and Algorithm 7.2 (for non-invertible jump maps).

7.3.1 Observability Decomposition of Error Dynamics

As a first step in computing ξ_q, $q \in \mathbb{N}$, we want to write $\tilde{x}$ in terms of observable components of individual subsystems. To do that, we first find a coordinate change for each mode, similar to the Kalman decomposition. For each $q \in \mathbb{N}$, choose a matrix Z_q such that its columns are an orthonormal basis of $\mathscr{R}(O_q^\top)$, so that $\mathscr{R}(Z_q) = \mathscr{R}(O_q^\top)$. Similarly, choose a matrix W_q such that its columns are an orthonormal basis of $\ker O_q$. From the construction, there are matrices $S_q \in \mathbb{R}^{r_q \times r_q}$ and $R_q \in \mathbb{R}^{\mathtt{y} \times r_q}$, where $r_q = \operatorname{rank} O_q$, such that $Z_q^\top A_q = S_q Z_q^\top$ and $C_q = R_q Z_q^\top$, and that the pair (S_q, R_q) is observable. Let $z_q := Z_q^\top \tilde{x} \in \mathbb{R}^{r_q}$ and $w_q := W_q^\top \tilde{x} \in \mathbb{R}^{\mathtt{n}-r_q}$. So, for the interval $[t_{q-1}, t_q)$, we obtain,

$$\dot{z}_q = Z_q^\top A_q \tilde{x} = S_q z_q, \quad \tilde{y} = C_q \tilde{x} = R_q z_q, \tag{7.13a}$$

$$z_q(t_{q-1}) = Z_q^\top \tilde{x}(t_{q-1}). \tag{7.13b}$$

Since z_q is observable over the interval $[t_{q-1}, t_q)$, a standard Luenberger observer is designed as

$$\dot{\hat{z}}_q = S_q \hat{z}_q + L_q(\tilde{y} - R_q \hat{z}_q), \quad t \in [t_{q-1}, t_q), \tag{7.14a}$$

$$\hat{z}_q(t_{q-1}) = 0, \tag{7.14b}$$

whose role is to estimate $z_q(t_q^-)$ at the end of the interval. Note that we have fixed the initial condition of the estimator to be zero for each interval. A consequence of introducing the observable and unobservable components is that the vector $\tilde{x}(t_q^-)$ can be written as,

$$\tilde{x}(t_q^-) = \begin{bmatrix} Z_q^\top \\ W_q^\top \end{bmatrix}^{-1} \begin{bmatrix} z_q(t_q^-) \\ w_q(t_q^-) \end{bmatrix} = Z_q z_q(t_q^-) + W_q w_q(t_q^-), \tag{7.15}$$

where $w_q(t_q^-)$ on the right-hand side remains unknown. Our objective now is to write $\tilde{x}(t_q^-)$ only in terms of known or recoverable quantities, that is, only as a function of the vectors $z_j(t_j^-)$, $j = q, q-1, \ldots, q-\mathtt{m}^*$, and ξ_k, $k = q-1, \ldots, q-\mathtt{m}^*$, for $\mathtt{m}^*$ given in Assumption 7.3. The calculations for arriving at such a formula for the general case are given in Sect. 7.3.3, but for the case when all jumps are invertible, one can derive a simpler formula. Because of simplicity, and to give an intuition about the calculations leading up to the computable expression for ξ_q, we choose to treat the case with invertible jumps first in the following section.

7.3.2 Error Correction with Invertible State Reset Maps

The goal of this subsection is to derive an expression of the form (7.9) when the jump maps G_q are invertible. For that, we first define the state-flow matrix Ψ_p^q, $p, q \in \mathbb{N}$, $p < q$, as

$$\Psi_p^q := e^{A_q \tau_q} G_q e^{A_{q-1}\tau_{q-1}} G_{q-1}, \ldots, e^{A_{p+1}\tau_{p+1}} G_p,$$

which transports $\tilde{x}(t_p^-)$ to $\tilde{x}(t_q^-)$ along (7.8) by

$$\tilde{x}(t_q^-) = \Psi_p^q \tilde{x}(t_p^-) - \sum_{k=p}^{q-1} \Psi_k^q \xi_k, \tag{7.16}$$

where for convenience, we let Ψ_q^q to be the identity matrix. For $q > \mathbf{m}^*$, we now have the following series of equivalent expressions for $\tilde{x}(t_q^-)$:

$$\begin{aligned}
\tilde{x}(t_q^-) &= Z_q z_q(t_q^-) + W_q w_q(t_q^-) \\
&= \Psi_{q-1}^q Z_{q-1} z_{q-1}(t_{q-1}^-) + \Psi_{q-1}^q W_{q-1} w_{q-1}(t_{q-1}^-) - \Psi_{q-1}^q \xi_{q-1} \\
&= \Psi_{q-2}^q Z_{q-2} z_{q-2}(t_{q-2}^-) + \Psi_{q-2}^q W_{q-2} w_{q-2}(t_{q-2}^-) - \sum_{k=1}^{2} \Psi_{q-k}^q \xi_{q-k} \\
&\ \ \vdots \\
&= \Psi_{q-\mathbf{m}^*}^q Z_{q-\mathbf{m}^*} z_{q-\mathbf{m}^*}(t_{q-\mathbf{m}^*}^-) + \Psi_{q-\mathbf{m}^*}^q W_{q-\mathbf{m}^*} w_{q-\mathbf{m}^*}(t_{q-\mathbf{m}^*}^-) - \sum_{k=q-\mathbf{m}^*}^{q-1} \Psi_k^q \xi_k.
\end{aligned} \tag{7.17}$$

In these equations, the vector $\tilde{x}(t_q^-)$ is not only expressed in terms of the observable and unobservable components of mode $q \in \mathbb{N}$, but also those of previously active $q - \mathbf{m}^*$ modes. In other words, for each $q - \mathbf{m}^* \le k \le q$, the term $\Psi_k^q Z_k z_k(t_k^-)$ transports the observable information of the kth mode from the interval $[t_{k-1}, t_k)$ to the time instant t_q^-. However, in each equation, there is an added unknown term $w_k(t_k^-)$. Thus, in order to obtain an explicit expression for $\tilde{x}(t_q^-)$ in terms of z_k, $k = q, q-1, \ldots, q - \mathbf{m}^*$, we must

- first, eliminate $w_k(t_k^-)$ from each equation in (7.17), and

- secondly, make sure that the resulting set of equations is not under-determined with $\tilde{x}(t_q^-)$ as the unknown.

To achieve the first objective, we introduce the matrices Θ_k^q whose columns form the basis of the subspace $\mathscr{R}(\Psi_k^q W_k)^\perp$; that is,

$$\mathscr{R}(\Theta_k^q) = \mathscr{R}(\Psi_k^q W_k)^\perp, \quad k = q - \mathbf{m}^*, \dots, q.$$

Then, for each equality in (7.17), we obtain the relation

$$\Theta_k^{q\top} \tilde{x}(t_q^-) = \Theta_k^{q\top} \left(\Psi_k^q Z_k z_k(t_k^-) - \sum_{j=k}^{q-1} \Psi_j^q \xi_j \right), \quad k = q - \mathbf{m}^*, \dots, q. \tag{7.18}$$

It now follows that if the matrix

$$\Theta_q^\top := \mathrm{col}(\Theta_q^{q\top}, \Theta_{q-1}^{q\top}, \dots, \Theta_{q-\mathbf{m}^*}^{q\top})$$

has full column rank equal to $\mathbf{n}$, then the set of equations (7.18) can be solved for $\tilde{x}(t_q^-)$. It can be shown that the matrix Θ_q, $q > \mathbf{m}^*$, has rank $\mathbf{n}$ if, and only if, Assumption 7.3 holds. Indeed, $\Theta_q^\top$ has full column rank $\mathbf{n}$ if, and only if, $\ker(\Theta_q^\top) = \{0\}$, or equivalently,

$$\mathscr{R}(\Theta_q^q) + \mathscr{R}(\Theta_{q-1}^q) + \cdots + \mathscr{R}(\Theta_{q-\mathbf{m}^*}^q) = \mathbb{R}^{\mathbf{n}}.$$

Using the fact that $\mathscr{R}(W_k)^\perp = (\ker O_k)^\perp = \mathscr{R}(O_k^\top)$, $e^{-A_k^\top \tau_k} \mathscr{R}(O_k^\top) = \mathscr{R}(O_k^\top)$, and the expression (7.6), it follows under Assumption 7.3 that

$$\begin{aligned}
&\mathscr{R}(W_q)^\perp + \mathscr{R}(\Psi_{q-1}^q W_{q-1})^\perp + \cdots + \mathscr{R}(\Psi_{q-\mathbf{m}^*}^q W_{q-\mathbf{m}^*})^\perp \\
&= e^{-A_q^\top \tau_q} \left(\mathscr{R}(O_q^\top) + G_{q-1}^{-\top} \mathscr{R}(O_{q-1}^\top) + \sum_{i=q-\mathbf{m}^*}^{q-2} \prod_{l=q-1}^{i+1} G_l^{-\top} e^{-A_l^\top \tau_l} G_i^{-\top} \mathscr{R}(O_i^\top) \right) \\
&= e^{-A_q^\top \tau_q} \mathscr{Q}_{q-\mathbf{m}^*}^q = \mathbb{R}^{\mathbf{n}},
\end{aligned} \tag{7.19}$$

where we recall from Sect. 7.2.1 that the second to last equality only holds when the jump maps G_q are invertible. Thus, the matrix $\Theta_q^\top$ is left-invertible, so that $(\Theta_q^\top)^\dagger = (\Theta_q \Theta_q^\top)^{-1} \Theta_q$, where $\dagger$ denotes the left-pseudo-inverse. Let us now introduce the matrix

$$\Omega_{q-\mathbf{m}^*}^q := \begin{bmatrix} \Theta_q^{q\top} \Psi_q^q Z_q z_q(t_q^-) \\ \vdots \\ \Theta_{q-\mathbf{m}^*}^{q\top} \left(\Psi_{q-\mathbf{m}^*}^q Z_{q-\mathbf{m}^*} z_{q-\mathbf{m}^*}(t_{q-\mathbf{m}^*}^-) - \sum_{l=q-\mathbf{m}^*}^{q-1} \Psi_l^q \xi_l \right) \end{bmatrix},$$

so that the arguments of the matrix $\Omega^q_{q-\mathtt{m}^*}$ are $z_q(t_q^-), z_{q-1}(t_{q-1}^-), \ldots, z_{q-\mathtt{m}^*}(t_{q-\mathtt{m}^*}^-)$ and $\xi_{q-1}, \ldots, \xi_{q-\mathtt{m}^*}$. It then follows that

$$\begin{aligned}\tilde{x}(t_q^-) &= (\Theta_q^\top)^\dagger \Omega^q_{q-\mathtt{m}^*}(z_q(t_q^-), \ldots, z_{q-\mathtt{m}^*}(t_{q-\mathtt{m}^*}^-), \xi_{q-1}, \ldots, \xi_{q-\mathtt{m}^*}) \\ &=: \Xi_q(z_q(t_q^-), \ldots, z_{q-\mathtt{m}^*}(t_{q-\mathtt{m}^*}^-), \xi_{q-1}, \ldots, \xi_{q-\mathtt{m}^*}). \end{aligned} \tag{7.20}$$

Once again, it is seen from (7.20) that, if we can estimate $z_k(t_k^-)$, $k = q - \mathtt{m}^*, \ldots, q$, without error, then by (7.20) the plant state $x(t_q^-)$ is exactly recovered because $x(t_q^-) = \hat{x}(t_q^-) - \tilde{x}(t_q^-)$, and both entities on the right side of the equation are known. However, since this is not the case, we set ξ_q to be an estimate of $\tilde{x}(t_q^-)$ through

$$\boxed{\xi_q = \Xi_q(\hat{z}_q(t_q^-), \hat{z}_{q-1}(t_{q-1}^-), \ldots, \hat{z}_{q-\mathtt{m}^*}(t_{q-\mathtt{m}^*}^-), \xi_{q-1}, \ldots, \xi_{q-\mathtt{m}^*}), \quad q > \mathtt{m}^*} \tag{7.21}$$

and for $1 \leq q \leq \mathtt{m}^*$, we let $\xi_q = 0$. The following theorem now states that if the estimates used in (7.21) are good enough, then the resulting estimate converges to the actual state asymptotically.

Theorem 7.6 *Consider the observer proposed in* (7.7) *under Assumptions 7.3–7.5 and also assume that the jump maps G_q, $q \in \mathbb{N}$, are invertible. If the error correction vector ξ_q is computed using* (7.21) *in which $\hat{z}_j$, $j = q, \ldots, q - \mathtt{m}^*$ are obtained from the Luenberger observers* (7.14)*, then the output injection gains L_j in* (7.14) *can be chosen such that*

$$\lim_{t \to \infty} |\hat{x}(t) - x(t)| = 0.$$

To complete the design procedure, we need to choose the gain matrices $L_q, q \in \mathbb{N}$. This is done in Sect. 7.3.2.1, where we analyze the state estimation error resulting from injecting the expression for ξ_q from (7.21) in Eq. (7.8).

7.3.2.1 Error Analysis and Gain Criterion

The gain matrices L_q, $q \in \mathbb{N}$, are basically chosen such that $\tilde{x}(t_q)$ converges to zero as q increases, because it follows from (7.8) and Assumptions 7.4 and 7.5 that the estimation error $\tilde{x}(t)$ for the interval $[t_q, t_{q+1})$ is bounded by

$$|\tilde{x}(t)| = |e^{A_{q+1}t}\tilde{x}(t_q)| \leq e^{a(t-t_q)}|\tilde{x}(t_q)|$$

with constant a such that $\|A_q\| \leq a$, and thus,

$$|\tilde{x}(t)| \leq e^{aD}|\tilde{x}(t_q)|.$$

Therefore, if $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$, then we achieve that

$$\lim_{t \to \infty} |\tilde{x}(t)| = 0. \tag{7.22}$$

Since the operator $\Omega^q_{q-\mathbf{m}^*}$ is linear in its arguments, it is noted that,

$$\tilde{x}(t_q) = G_q(\tilde{x}(t_q^-) - \xi_q) \tag{7.23a}$$

$$= G_q\Big(\Xi_q(z_q(t_q^-), \dots, z_{q-\mathbf{m}^*}(t^-_{q-\mathbf{m}^*}), \xi_{q-1}, \dots, \xi_{q-\mathbf{m}^*}) - \Xi_q(\hat{z}_q(t_q^-), \dots, \hat{z}_{q-\mathbf{m}^*}(t^-_{q-\mathbf{m}^*}), \xi_{q-1}, \dots, \xi^-_{q-\mathbf{m}^*})\Big) \tag{7.23b}$$

$$= -G_q(\Theta_q^\top)^\dagger \Omega^q_{q-\mathbf{m}^*}(\tilde{z}_q(t_q^-), \dots, \tilde{z}_{q-\mathbf{m}^*}(t^-_{q-\mathbf{m}^*}), 0, \dots, 0) \tag{7.23c}$$

where $\tilde{z}_j := \hat{z}_j - z_j$, $j = q, \dots, q - \mathbf{m}^*$. It follows from (7.13) and (7.14) that

$$\tilde{z}_j(t_{j-1}) = \hat{z}_j(t_{j-1}) - z_j(t_{j-1}) = 0 - Z_j^\top \tilde{x}(t_{j-1}),$$

and

$$\tilde{z}_j(t_j^-) = e^{(S_j - L_j R_j)\tau_j}\tilde{z}_j(t_{j-1}) = -e^{(S_j - L_j R_j)\tau_j} Z_j^\top \tilde{x}(t_{j-1}).$$

Plugging this expression in (7.23), and introducing the matrices M_j^q, for $j = q, q-1, \dots, q-\mathbf{m}^*$, as follows:

$$[M_q^q, M_{q-1}^q, \dots, M^q_{q-\mathbf{m}^*}] := G_q(\Theta_q^\top)^\dagger \times \operatorname{blockdiag}\left(\Theta_q^{q^\top}\Psi_q^q, \Theta_{q-1}^{q^\top}\Psi_{q-1}^q, \dots, \Theta_{q-\mathbf{m}^*}^{q^\top}\Psi^q_{q-\mathbf{m}^*}\right), \tag{7.24}$$

we obtain

$$\tilde{x}(t_q) = \sum_{j=q-\mathbf{m}^*}^{q} M_j^q Z_j e^{(S_j - L_j R_j)\tau_j} Z_j^\top \tilde{x}(t_{j-1}), \tag{7.25}$$

where we note that the argument of every $\mathbf{n} \times \mathbf{n}$ matrix M_j^q, $j = q - \mathbf{m}^*, \cdots, q$, is the vector $(\tau_q, \dots, \tau_{q-\mathbf{m}^*+1})$ because of the matrices Ψ_j^q in the definition.

In order to bound the norm of $\tilde{x}(t_q)$, one can always find the constants $\alpha_j, \gamma_j > 0$ such that $\|Z_j e^{(S_j - L_j R_j)\tau_j} Z_j^\top\| \le \alpha_j e^{-\gamma_j \tau_j}$. With constants $\lambda_j^q > 0$ denoting the induced norm of M_j^q, we get

$$|\tilde{x}(t_q)| \le \sum_{j=q-\mathbf{m}^*}^{q} \lambda_j^q \alpha_j e^{-\gamma_j \tau_j} |\tilde{x}(t_{j-1})|. \tag{7.26}$$

If for each $q > \mathbf{m}^*$, and $j = q - \mathbf{m}^*, \dots, q$, the gains L_j are chosen such that

$$\lambda_j^q \alpha_j e^{-\gamma_j \tau_j} \le c < \frac{1}{\mathbf{m}^* + 1}$$

(such a choice is always feasible [12, Lemma 1]), then

$$|\tilde{x}(t_q)| < c \sum_{j=q-\mathtt{m}^*}^{q} |\tilde{x}(t_{j-1})|.$$

One can now use [19, Lemma 1] to conclude that $\lim_{q\to\infty} |\tilde{x}(t_q)| = 0$.

Algorithm 7.1 Implementation of hybrid observer for invertible jump maps

Require: u, v, y

Ensure: Run (7.7) for $t \in [t_0, t_{\mathtt{m}^*+1})$ with some $\hat{x}(t_0)$

1: **for all** $q \geq \mathtt{m}^* + 1$ **do**

2: **for** $j = q - \mathtt{m}^*$ **to** q **do**

3: Compute the injection gain L_j such that

$$\|M_j^q Z_j e^{(S_j - L_j R_j)\tau_j} Z_j^\top\| \leq c < \frac{1}{\mathtt{m}^* + 1}. \tag{7.27}$$

4: Obtain $\hat{z}_j(t_j^-)$ by running the individual observer (7.14) for the j-th mode.

5: **end for**

6: Compute ξ_q from (7.21) , to implement (7.7).

7: Compute $\hat{x}(t_q)$ using (7.7) and run (7.7) over the interval $[t_q, t_{q+1})$.

8: **end for**

7.3.3 Error Correction for Non-invertible State Reset Maps

The formula for ξ_q computed in Sect. 7.3.2 is only valid for the case of invertible jump maps. To derive a more general formula, which is also valid for the case of non-invertible jump maps, we basically follow the same procedure but the details are slightly more involved.

For $p, q \in \mathbb{N}$ with $p < q$, let Q_p^q and N_p^q be matrices such that their columns are an orthonormal basis of $e^{-A_q^\top \tau_q} \mathcal{Q}_p^q$ and $e^{A_q \tau_q} \mathcal{N}_p^q$, respectively. The corresponding projections of $\tilde{x}(t_q^-)$ onto these subspaces are defined by letting $\chi_p^q := Q_p^{q\top} \tilde{x}(t_q^-)$ and $\nu_p^q := N_p^{q\top} \tilde{x}(t_q^-)$. Thus, it is seen that in addition to (7.15), another way of expressing $\tilde{x}(t_q^-)$ is:

$$\tilde{x}(t_q^-) = \begin{bmatrix} Q_p^{q\top} \\ N_p^{q\top} \end{bmatrix}^{-1} \begin{bmatrix} \chi_p^q \\ \nu_p^q \end{bmatrix} = Q_p^q \chi_p^q + N_p^q \nu_p^q. \tag{7.28}$$

The definition of χ_p^q implies that it contains the information of the error $\tilde{x}(t_q^-)$, which we are able to extract from the output on the interval $[t_{p-1}, t_q)$ as given by the observability space $\mathcal{Q}_p^q$. For $q > \mathtt{m}^*$, the forward observability assumption ensures that $\chi^q_{q-\mathtt{m}^*}$ contains all information of $\tilde{x}(t_q^-)$; in fact $Q^q_{q-\mathtt{m}^*}$ is then an invertible matrix, and hence the equation $\chi^q_{q-\mathtt{m}^*} = Q^{q\,\top}_{q-\mathtt{m}^*}\tilde{x}(t_q^-)$ is uniquely solvable for $\tilde{x}(t_q^-)$.

Thus, once again we are interested in representing $\tilde{x}(t_q^-)$ only in terms of the known vectors χ^q_j, and eliminate its dependency over the terms involving ν^q_j, $j = q, q-1, \ldots, q-\mathtt{m}^*$. For that, we once again introduce the matrix Θ^q_p whose columns form the basis of the subspace $\mathscr{R}(e^{A_{q+1}\tau_{q+1}}G_q N^q_p)^\perp$; that is,

$$\Theta^{q\,\top}_p e^{A_{q+1}\tau_{q+1}} G_q N^q_p = 0. \tag{7.29}$$

Compared to the case treated earlier, the key difference is that we do not transport the observable components of the individual subsystems to one time instant through the state-transition matrix. Instead, we gather all the observable information for $\tilde{x}(t_{q-1}^-)$ over the interval $[t_{p-1}, t_{q-1})$ into the vector χ^{q-1}_p, $p < q$, and combine it with the local observability information $z_q(t_q^-)$ for $\tilde{x}(t_q^-)$ obtained on the interval $[t_{q-1}, t_q)$ in order to recover more information for $\tilde{x}(t_q^-)$, represented by χ^q_p. For that, the following relationship between $\tilde{x}(t_q^-)$ and χ^{q-1}_p, $p < q$, is crucial:

$$\begin{aligned}\tilde{x}(t_q^-) &= e^{A_q\tau_q} G_{q-1}(\tilde{x}(t_{q-1}^-) - \xi_{q-1}) \\ &= e^{A_q\tau_q} G_{q-1}\left(Q^{q-1}_p \chi^{q-1}_p + N^{q-1}_p \nu^{q-1}_p - \xi_{q-1}\right).\end{aligned} \tag{7.30}$$

Combining this with (7.15) we obtain

$$\begin{pmatrix} Z_q^\top \\ \Theta^{q-1\,\top}_p \end{pmatrix} \tilde{x}(t_q^-) = \begin{pmatrix} z_q(t_q^-) \\ \Theta^{q-1\,\top}_p\left(e^{A_q\tau_q} G_{q-1}\left(Q^{q-1}_p \chi^{q-1}_p - \xi_{q-1}\right)\right) \end{pmatrix},$$

and hence we have more information about $\tilde{x}(t_q^-)$ by combining $z_q(t_q^-)$ and χ^{q-1}_p accordingly. Now, consider a full column rank matrix U^q_p such that

$$[Z_q, \Theta^{q-1}_p] U^q_p = Q^q_p.$$

Such a matrix always exists because from the definition of $\mathscr{Q}^q_p$ and Z_q, we have

$$\mathscr{R}(Q^q_p) = \mathscr{R}([Z_q, \Theta^{q-1}_p]).$$

From $\chi^q_p = Q^{q\,\top}_p \tilde{x}(t_q^-)$, it now follows that

$$\chi^q_p = U^{q\,\top}_p \begin{bmatrix} Z_q^\top \\ \Theta^{q-1\,\top}_p \end{bmatrix} \tilde{x}(t_q^-)$$

$$= U_p^{q\top} \begin{pmatrix} z_q(t_q^-) \\ \Theta_p^{q-1\top} \left(e^{A_q \tau_q} G_{q-1} \left(Q_p^{q-1} \chi_p^{q-1} - \xi_{q-1} \right) \right) \end{pmatrix} \tag{7.31}$$

$$= U_p^{q\top} \left[\begin{array}{c|c} Z_q^\top & 0 \\ \hline 0 & \Theta_p^{q-1\top} e^{A_q \tau_q} G_{q-1} \end{array} \right] \begin{pmatrix} Z_q z_q(t_q^-) \\ Q_p^{q-1} \chi_p^{q-1} - \xi_{q-1} \end{pmatrix} \tag{7.32}$$

$$\triangleq J_p^q Z_q z_q + K_p^q \left(Q_p^{q-1} \chi_p^{q-1} - \xi_{q-1} \right). \tag{7.33}$$

Note that (7.32) expresses the vector χ_p^q recursively in terms of χ_p^{q-1}. Recall that $\mathscr{Q}_p^p = \mathcal{R}(O_p^\top)^\perp = \mathscr{R}(Z_p)$, hence we can assume $Q_p^p = Z_p$ and we have the "initial value" for the recursion (7.32) given by $\chi_p^p = z_p$.

If $z_{q-\mathbf{m}^*}, \ldots, z_q$ were known, then we would be able to compute the error $\tilde{x}(t_q^-)$ exactly and would pick $\xi_q = \tilde{x}(t_q^-)$. Since this is not the case, we work with the estimates $\hat{z}_{q-\mathbf{m}^*}, \ldots, \hat{z}_q$ to compute ξ_q.

In summary, having introduced the matrices Z_q as in (7.15), Q_p^q as in (7.28), and Θ_p^q as in (7.29), for $q \in \mathbb{N}$, we let

$$\boxed{\xi_q = \begin{cases} 0, & 1 \le q \le \mathbf{m}^* \\ Q_{q-\mathbf{m}^*}^q \hat{\chi}_{q-\mathbf{m}^*}^q, & q \ge \mathbf{m}^* + 1 \end{cases}} \tag{7.34}$$

where $\hat{\chi}_{q-\mathbf{m}^*}^{q-k}$, for $k = \mathbf{m}^* - 1, \ldots, 0$, is computed recursively as follows:

$$\boxed{\begin{aligned} \hat{\chi}_{q-\mathbf{m}^*}^{q-\mathbf{m}^*} &= \hat{z}_{q-\mathbf{m}^*} \\ \hat{\chi}_{q-\mathbf{m}^*}^{q-k} &= J_{q-\mathbf{m}^*}^{q-k} Z_{q-k} \hat{z}_{q-k} + K_{q-\mathbf{m}^*}^{q-k} \left(Q_{q-\mathbf{m}^*}^{q-k-1} \hat{\chi}_{q-\mathbf{m}^*}^{q-k-1} - \xi_{q-k-1} \right), \end{aligned}} \tag{7.35}$$

and

$$[J_{q-\mathbf{m}^*}^{q-k}, K_{q-\mathbf{m}^*}^{q-k}] := U_{q-\mathbf{m}^*}^{q-k\ \top} \begin{bmatrix} Z_{q-k}^\top & 0 \\ 0 & \Theta_{q-\mathbf{m}^*}^{q-k-1\top} e^{A_{q-k} \tau_{q-k}} G_{q-k-1} \end{bmatrix}. \tag{7.36}$$

Using the formula for ξ_q in (7.34), we can again state a result very similar to Theorem 7.6, but this time we do not place any constraints on the jump maps.

Theorem 7.7 *Consider observer (7.7) under Assumptions 7.3–7.5. If the error correction vector ξ_q is computed using (7.34) in which $\hat{\chi}_{q-\mathbf{m}^*}^q$ is computed in a recursive manner from Eqs. (7.35) and (7.36), then the output injection gains L_j in (7.14) can be chosen such that*

$$\lim_{t \to \infty} |\hat{x}(t) - x(t)| = 0.$$

It just remains to show how well $\hat{\chi}^q_{q-\mathbf{m}^*}$ should approximate $\chi^q_{q-\mathbf{m}^*}$ by appropriate choice of gains L_j, $j \in \mathbb{N}$. Once again we motivate the gain criterion by analyzing the error.

7.3.3.1 Error Analysis and Gain Criterion

As in Sect. 7.3.2.1, we want to derive a gain criterion for L_j, $j \in \mathbb{N}$, such that $\lim_{q\to\infty} |\tilde{x}(t_q)| = 0$. It is noted that, for $q > \mathbf{m}^*$:

$$\begin{aligned} \tilde{x}(t_q) &= G_q(\tilde{x}(t_q^-) - \xi_q) = G_q Q^q_{q-\mathbf{m}^*}(\chi^q_{q-\mathbf{m}^*} - \hat{\chi}^q_{q-\mathbf{m}^*}) \\ &= -G_q Q^q_{q-\mathbf{m}^*}\tilde{\chi}^q_{q-\mathbf{m}^*}, \end{aligned} \tag{7.37}$$

where $\tilde{\chi}^q_{q-\mathbf{m}^*} := \hat{\chi}^q_{q-\mathbf{m}^*} - \chi^q_{q-\mathbf{m}^*}$. In the sequel, we will derive an expression for $\tilde{\chi}^q_{q-\mathbf{m}^*}$ for a fixed $q > \mathbf{m}^*$ and plug it in (7.37) to show that $|\tilde{x}(t_q)|$ converges to zero as q increases for appropriate choice of the matrices L_j, $j \in \mathbb{N}$.

Toward this end, we first compute the difference $\tilde{z}_q := \hat{z}_q - z_q$, for $q \in \mathbb{N}$ as follows:

$$\tilde{z}_q(t_q^-) = \hat{z}_q(t_q^-) - z_q(t_q^-) = e^{(S_q - L_q R_q)\tau_q}\tilde{z}_q(t_{q-1}) = -\Lambda_q Z_q{}^\top \tilde{x}(t_{q-1}),$$

where we define $\Lambda_q := e^{(S_q - L_q R_q)\tau_q}$. As a first step in arriving at the expression for $\tilde{\chi}^q_{q-\mathbf{m}^*}$, we observe that $\tilde{\chi}^{q-\mathbf{m}^*}_{q-\mathbf{m}^*} = \tilde{z}_{q-\mathbf{m}^*}(t^-_{q-\mathbf{m}^*})$ and we compute $\tilde{\chi}^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*}$ as follows:

$$\begin{aligned} \tilde{\chi}^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*} &= \hat{\chi}^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*} - \chi^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*} \\ &= J^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*} Z_{q-\mathbf{m}^*+1}\tilde{z}_{q-\mathbf{m}^*+1}(t^-_{q-\mathbf{m}^*+1}) + K^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*} Z_{q-\mathbf{m}^*}\tilde{z}_{q-\mathbf{m}^*}(t^-_{q-\mathbf{m}^*}) \\ &= -\sum_{i=0}^{1}\big(V^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*,q-\mathbf{m}^*+i} Z_{q-\mathbf{m}^*+i}\Lambda_{q-\mathbf{m}^*+i} Z^\top_{q-\mathbf{m}^*+i}\tilde{x}(t_{q-\mathbf{m}^*+i-1})\big), \end{aligned}$$

where

$$V^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*,q-\mathbf{m}^*} := K^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*} \tag{7.38a}$$

$$V^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*,q-\mathbf{m}^*+1} := J^{q-\mathbf{m}^*+1}_{q-\mathbf{m}^*}. \tag{7.38b}$$

Finally, by introducing the matrices, $q > \mathbf{m}^*$, $k = \mathbf{m}^* - 2, \dots, 0$ and $i = 0, \dots, \mathbf{m}^* - k - 1$

$$V^{q-k}_{q-\mathbf{m}^*,q-k} := J^{q-k}_{q-\mathbf{m}^*} \tag{7.38c}$$

$$V^{q-k}_{q-\mathbf{m}^*,q-\mathbf{m}^*+i} := K^{q-k}_{q-\mathbf{m}^*} Q^{q-k-1}_{q-\mathbf{m}^*} V^{q-k-1}_{q-\mathbf{m}^*,q-\mathbf{m}^*+i} \tag{7.38d}$$

the expression for $\tilde{\chi}_{q-\mathtt{m}^*}^{q-k}$, $k = \mathtt{m}^* - 1, \ldots, 0$, is derived recursively:

$$
\begin{aligned}
\tilde{\chi}_{q-\mathtt{m}^*}^{q-k} &= \hat{\chi}_{q-\mathtt{m}^*}^{q-k} - \chi_{q-\mathtt{m}^*}^{q-k} \\
&= J_{q-\mathtt{m}^*}^{q-k} Z_{q-k} \tilde{z}_{q-k}(t_{q-k}^-) + K_{q-\mathtt{m}^*}^{q-k} Q_{q-\mathtt{m}^*}^{q-k-1} \tilde{\chi}_{q-\mathtt{m}^*}^{q-k-1} \\
&= -\sum_{i=0}^{\mathtt{m}^*-k} V_{q-\mathtt{m}^*, q-\mathtt{m}^*+i}^{q-k} Z_{q-\mathtt{m}^*+i} \Lambda_{q-\mathtt{m}^*+i} Z_{q-\mathtt{m}^*+i}^\top \tilde{x}(t_{q-\mathtt{m}^*+i-1}).
\end{aligned}
$$

Plugging this expression for $\tilde{\chi}_{q-\mathtt{m}^*}^{q}$ in (7.37), we now obtain

$$
\tilde{x}(t_q) = G_q Q_{q-\mathtt{m}^*}^q \sum_{i=q-\mathtt{m}^*}^{q} V_{q-\mathtt{m}^*,i}^q Z_i \Lambda_i Z_i^\top \tilde{x}(t_{i-1}). \tag{7.39}
$$

If, for each $k = 0, \ldots, \mathtt{m}^*$, and $q > \mathtt{m}^*$, the output injection matrices L_{q-k} are chosen to minimize the norm of Λ_{q-k} such that

$$
\| G_q Q_{q-\mathtt{m}^*}^q V_{q-\mathtt{m}^*,q-k}^q Z_{q-k} \Lambda_{q-k} Z_{q-k}^\top \| \le c < \frac{1}{\mathtt{m}^* + 1}, \tag{7.40}
$$

then it follows that

$$
|\tilde{x}(t_q)| \le c \sum_{i=q-\mathtt{m}^*}^{q} |\tilde{x}(t_{i-1})|.
$$

We can again invoke Lemma 1 from [19] to obtain $\lim_{q\to\infty} |\tilde{x}(t_q)| = 0$, which proves the desired result.

Algorithm 7.2 Hybrid observer for systems with non-invertible jump maps

---

Require: u, v, y
Ensure: Run (7.7) for $t \in [t_0, t_{\mathtt{m}^*+1})$ with some $\hat{x}(t_0)$

1: **for all** $q \ge \mathtt{m}^* + 1$ **do**
2:     **for** $j = q - \mathtt{m}^*$ **to** q **do**
3:         Compute the injection gain L_j such that (7.40) holds.
4:         Obtain $\hat{z}_j(t_j^-)$ by running the individual observer (7.14) for the j-th mode.
5:     **end for**
6:     Compute ξ_q from (7.34), (7.35), and (7.36), to implement (7.7).
7:     Compute $\hat{x}(t_q)$ using (7.7) and run (7.7) over the interval $[t_q, t_{q+1})$.
8: **end for**

7.4 Illustrative Examples

We will now apply our results for two academic examples. The first one considers the case without any state resets, and in the second case, we consider switching dynamics with non-invertible state reset maps.

7.4.1 Invertible State Reset Maps

Consider a switched system given by:

$$A_{2k-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_{2k} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad k \geq 1$$
$$C_{2k-1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_{2k} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad k \geq 1.$$

with $G_k = I$, $H_k = 0$, $B_k = 0$, and $D_k = 0$ for each $k \geq 1$.

We assume that each mode is activated for τ seconds and $\tau \neq \kappa\pi$ for any $\kappa \in \mathbb{N}$. For simplicity, let us call $[(2k-2)\tau, (2k-1)\tau)$, $k \in \mathbb{N}$, an *odd* interval, and the mode active on the odd intervals as the *odd* mode. Similarly, the intervals $[(2k-1)\tau, 2k\tau)$, $k \in \mathbb{N}$ are called *even* intervals, and the mode active on these intervals is called the *even* mode. We also use the notation $q_{\mathrm{o}}, k_{\mathrm{o}}$ for odd positive integers and $q_{\mathrm{e}}, k_{\mathrm{e}}$ for even positive integers. It can be verified that the system is forward observable over a time interval that involves the mode sequence *odd* → *even* → *odd*. Hence, we pick $\mathtt{m}^* = 3$ so that Assumption 7.3 holds. With an arbitrary initial condition $\hat{x}(0)$, the observer to be implemented is:

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A_{2k-1}\hat{x}(t) \\ \hat{y}(t) &= C_{2k-1}\hat{x}(t) \end{aligned}\right\}, \quad t \in [(2k-2)\tau, (2k-1)\tau), \tag{7.41a}$$

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A_{2k}\hat{x}(t) \\ \hat{y}(t) &= C_{2k}\hat{x}(t) \end{aligned}\right\}, \quad t \in [(2k-1)\tau, 2k\tau), \tag{7.41b}$$

$$\hat{x}(t_k) = \hat{x}(t_k^-) - \xi_k, \quad k \in \mathbb{N}. \tag{7.41c}$$

In order to determine the value of ξ_k, we start off with the estimators for the observable part of each subsystem, denoted by z_q in (7.13). Note that the *odd* mode has a one-dimensional unobservable subspace, whereas for *even* mode, the unobservable subspace is $\mathbb{R}^2$. Let $z_{q_{\mathrm{o}}}$ represent the partial information obtained from the *odd* mode, and $z_{q_{\mathrm{e}}}$ be a null vector as no information is gathered from the *even* mode. So the one-dimensional partial observer in (7.14) is implemented only for odd intervals. Also, for the *odd* mode, we obtain:

$$O_{q_{\mathrm{o}}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; W_{q_{\mathrm{o}}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; Z_{q_{\mathrm{o}}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that $S_{q_o} = 0$ and $R_{q_o} = 1$, which yields the observer in (7.14) as

$$\dot{\hat{z}}_{q_o} = -l_{q_o} \hat{z}_{q_o} + l_{q_o} \tilde{y}, \qquad t \in [(q_o - 1)\tau, q_o \tau),$$

with the initial condition $\hat{z}_{q_o}((q_o - 1)\tau) = 0$, and $\tilde{y}$ being the difference between the measured output and the estimated output of (7.41). The gain l_{q_o} will be chosen later (see (7.42)). For the *even* mode, we get $W_{q_e} = I_{2\times 2}$, and $O_{q_e} = 0_{2\times 2}$, so that Z_{q_e}, S_{q_e}, and R_{q_e} are null-matrices.

The next step is to use the value of $\hat{z}_{q_o}(t_{q_o}^-)$ to compute ξ_k, $k \in \mathbb{N}$, using the procedure outlined in Sect. 7.3.2. The matrices appearing in the computation of ξ_k are given as follows. For every $q_e > 3$:

$$\begin{aligned}
\Psi_{q_e-3}^{q_e} &= \begin{bmatrix} \cos 2\tau & \sin 2\tau \\ -\sin 2\tau & \cos 2\tau \end{bmatrix} \Rightarrow \Theta_{q_e-3}^{q_e} = \begin{bmatrix} \cos 2\tau \\ -\sin 2\tau \end{bmatrix}, \\
\Psi_{q_e-2}^{q_e} &= \begin{bmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{bmatrix} \Rightarrow \Theta_{q_e-2}^{q_e} = \text{null}, \\
\Psi_{q_e-1}^{q_e} &= \begin{bmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{bmatrix} \Rightarrow \Theta_{q_e-1}^{q_e} = \begin{bmatrix} \cos \tau \\ -\sin \tau \end{bmatrix}, \\
\Psi_{q_e}^{q_e} &= I_{2\times 2} \Rightarrow \Theta_{q_e}^{q_e} = \text{null}
\end{aligned}$$

where, as a convention, we have taken $\Theta_j^{q_e}$ as a null matrix whenever $\mathscr{R}(\Psi_j^{q_e} W_j)^\perp = \{0\}$. Using the matrices $\Theta_j^{q_e}$, $j = q_e - 3, \ldots, q_e$, we obtain for every $q_e > 3$:

$$\Theta_{q_e} = \begin{bmatrix} \Theta_{q_e-1}^{q_e} & \Theta_{q_e-3}^{q_e} \end{bmatrix} = \begin{bmatrix} \cos \tau & \cos 2\tau \\ -\sin \tau & -\sin 2\tau \end{bmatrix}.$$

Thus, for every $q_e > 3$, the error correction term ξ_{q_e} can be computed by the formula:

$$\xi_{q_e} = \Theta_{q_e}^{-\top} \begin{bmatrix} \hat{z}_{q_e-1}(t_{q_e-1}^-) - \xi_{q_e-1}(1) \\ \hat{z}_{q_e-3}(t_{q_e-3}^-) - \xi_{q_e-3}(1) - [\cos \tau \;\; -\sin \tau](\xi_{q_e-2} + \xi_{q_e-1}) \end{bmatrix},$$

where we use the notation $\xi_q(j)$ to denote the jth component of the vector ξ_q. Next, for every $q_o > 3$, we repeat the same calculations and obtain

$$\Theta_{q_o} = \begin{bmatrix} \Theta_{q_o}^{q_o} & \Theta_{q_o-2}^{q_o} \end{bmatrix} = \begin{bmatrix} 1 & \cos \tau \\ 0 & -\sin \tau \end{bmatrix}$$

which further gives

$$\xi_{q_o} = \Theta_{q_o}^{-\top} \begin{bmatrix} \hat{z}_{q_o}(t_{q_o}^-) \\ \hat{z}_{q_o-2}(t_{q_o-2}^-) - \xi_{q_o-2}(1) - [\cos \tau \;\; -\sin \tau]\, \xi_{q_o-1} \end{bmatrix}.$$

To compute the gain l_{q_o}, we note that $M^{q_e}_{q_e}$, $M^{q_e}_{q_e-2}$ are null matrices, and

$$M^{q_e}_{q_e-1} = \begin{bmatrix} \frac{\sin 2\tau}{\sin \tau} & 0 \\ \frac{\cos 2\tau}{\sin \tau} & 0 \end{bmatrix} \quad \text{and} \quad M^{q_e}_{q_e-3} = \begin{bmatrix} -1 & 0 \\ -\frac{\cos \tau}{\sin \tau} & 0 \end{bmatrix}.$$

Also, for $q_o > 3$, $M^{q_o}_{q_o-1}$ and $M^{q_o}_{q_o-3}$ are null matrices, and

$$M^{q_o}_{q_o} = \begin{bmatrix} 1 & 0 \\ \frac{\cos \tau}{\sin \tau} & 0 \end{bmatrix} \quad \text{and} \quad M^{q_o}_{q_o-2} = \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sin \tau} & 0 \end{bmatrix}.$$

By taking l_{q_o} equal to l for each q_o, and computing the induced 2-norm of the matrix, it is seen that, $\max_{q-3 \le j \le q, j:\text{odd}, q>3} \|M^q_j Z^j e^{(S_j - lR_j)\tau} Z^{j\top}\| = e^{-l\tau}/|\sin \tau|$. So, the lower bound for the gain l, is obtained as follows:

$$\frac{e^{-l\tau}}{|\sin \tau|} < \frac{1}{\mathtt{m}^* + 1} = \frac{1}{4} \quad \Rightarrow \quad l > \frac{1}{\tau} \ln \frac{4}{|\sin \tau|}. \tag{7.42}$$

It can be seen that the singularity occurs when τ is an integer multiple of π. Moreover, if τ approaches this singularity, then the gain required for convergence gets arbitrarily large. This shows that even though the condition $\sin \tau \neq 0$ guarantees observability, it may cause some difficulty in practice if $\sin \tau \approx 0$. This also explains why the knowledge of the switching signal is required in general to compute the observer gains.

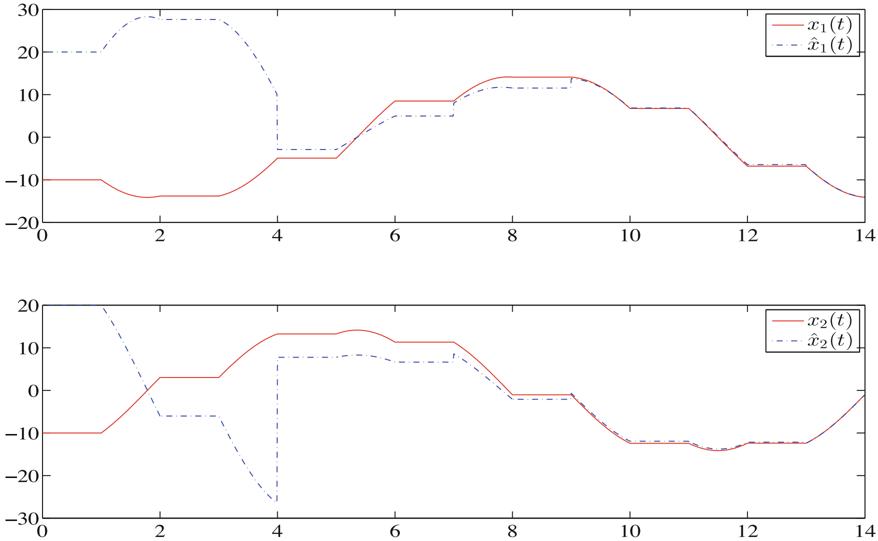

Fig. 7.1 State estimation error in x_1 and x_2

The results of simulations with $\tau = 1$, and $l = 2$, are illustrated in Fig. 7.1. The error initially evolves according to the (marginally stable) system dynamics as no correction is applied till t_4. When the error correction is applied, there is a jump in the state estimation error, which highlights the hybrid nature of the proposed observer.

7.4.2 Non-invertible State Reset Maps

We next consider an academic example of a third order ($\mathtt{n} = 3$) switched system with three modes where $A_q, B_q, H_q, D_q, q \in \mathbb{N}$, are zero matrices of appropriate dimensions. The output measurements are given by:

$$C_{3k-2} = [1\ 0\ 0], \quad C_{3k-1} = [0\ 1\ 0], \quad C_{3k} = [0\ 0\ 1], \quad k \geq 1,$$

and the state reset maps are:

$$G_{3k-2} = G_{3k} = I_{3\times3}, \quad G_{3k-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad k \geq 1.$$

For this system, it can be checked that the Assumption 7.3 indeed holds, that is, $\dim \mathcal{Q}^q_{q-\mathtt{m}^*} = 3$, for each $q > 2$, where we take $\mathtt{m}^* = 2$. The observer (7.7) is now implemented to obtain the state estimate in which we let $\xi_1 = \xi_2 = 0$.

For $q \geq 3$, the following expressions are obtained for the vector ξ_q using the calculations in Sect. 7.3.3:

$$\xi_{3k} = \begin{pmatrix} \hat{z}_{3k-2} + \hat{z}_{3k-1} \\ \hat{z}_{3k-2} + \hat{z}_{3k-1} \\ \hat{z}_{3k} \end{pmatrix} - \begin{pmatrix} \xi_{3k-2}(1) + \xi_{3k-1}(1) + \xi_{3k-1}(2) \\ \xi_{3k-2}(1) + \xi_{3k-1}(1) + \xi_{3k-1}(2) \\ 0 \end{pmatrix}, \quad k \geq 1,$$

$$\xi_{3k+1} = \begin{pmatrix} \hat{z}_{3k+1} \\ 0 \\ \hat{z}_{3k} \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}}\xi_{3k}(1) - \frac{1}{\sqrt{2}}\xi_{3k}(2) \\ \frac{1}{\sqrt{2}}\xi_{3k}(2) - \frac{1}{\sqrt{2}}\xi_{3k}(1) \\ \xi_{3k}(3) \end{pmatrix}, \quad k \geq 1,$$

$$\xi_{3k+2} = \begin{pmatrix} \hat{z}_{3k+1} \\ \hat{z}_{3k+2} \\ \hat{z}_{3k} \end{pmatrix} - \begin{pmatrix} \xi_{3k+1}(1) \\ 0 \\ \xi_{3k+1}(3) + \xi_{3k}(3) \end{pmatrix}, \quad k \geq 1,$$

where the short-hand $\hat{z}_q$ is used to denote $\hat{z}_q(t_q^-)$, which for each $q \in \mathbb{N}$ is obtained from the following equation:

$$\dot{\hat{z}}_q(t) = -l_q \tilde{y}(t), \quad t \in [t_{q-1}, t_q), \qquad \hat{z}(t_{q-1}) = 0.$$

For simplicity, if we let $l_q = l$, and $\tau_q = \tau$ for some $l, \tau > 0$ and each $q \in \mathbb{N}$, then the condition (7.40) boils down to:

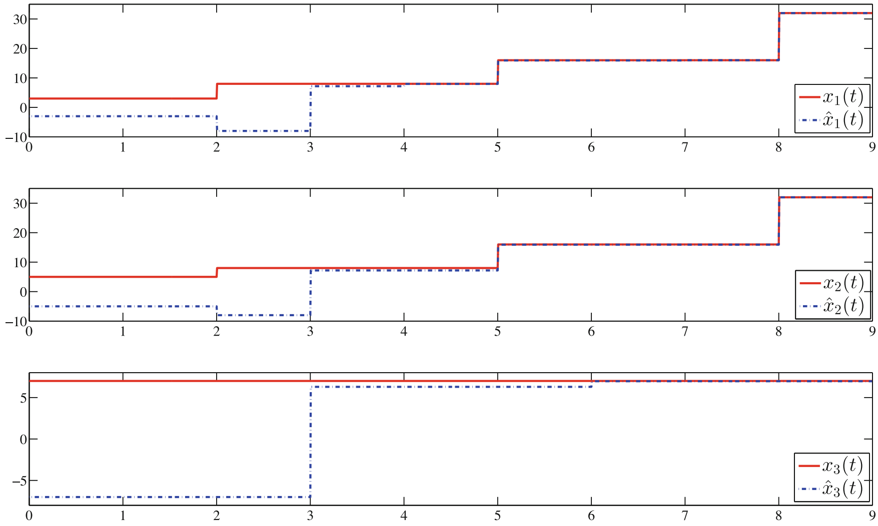

Fig. 7.2 The plot shows the state estimates $\hat{x}_i$, $i = 1, 2, 3$ (*dashed lines* in *blue*) converging to the actual states of the plant x_i, $i = 1, 2, 3$ (*solid lines* in *red*)

$$\sqrt{2} \cdot e^{-l\tau} < \frac{1}{3} \quad \Leftrightarrow \quad l > \frac{\log 3\sqrt{2}}{\tau}.$$

For $\tau = 1$, the simulation results are shown in Fig. 7.2. The plot shows the continuous and discrete nature of the error dynamics where the estimate does not improve between the two switching instants and only when the correction ξ_q is applied, the estimate gets closer to the actual state value.

7.5 Conclusions

This chapter has addressed the problems of observer design for switched linear systems with state reset maps based on a notion of observability, which does not necessarily require the observability of individual subsystems in the classical sense. The proposed state estimators apply error correction only at discrete switching instants and are inherently hybrid in nature. The examples considered in this chapter are purely academic, but it is not difficult to encounter practical systems where such techniques could be applied. First and foremost application that comes to mind are the electrical circuits: Multicellular converters could be modeled as switched systems where each mode is not observable. Our observer design has been used to study diagnostic problems in such systems [24]. Another instance of the utility of our observer design in a power converter has been reported in [11].

Although, we only consider the linear systems with ordinary differential equations in this chapter, the ideas presented in this chapter have been applied to a more general class of systems. The first of these extensions has been studied for the case when the dynamics of individual subsystems are represented by differential-algebraic equations [22]. Such systems have more structure because the solution only evolves in the consistency space determined by the algebraic constraints of individual modes. The state jumps in these systems are also determined by the algebraic constraints, and moreover the solutions of such systems may contain derivatives of jumps for which we adopt the distributional framework proposed in [23].

In another related work, we have used similar ideas to study the problem of observer design in switched nonlinear systems [14, 15]. The major difficulty in dealing with nonlinear systems is that one cannot explicitly solve the system equations to transport the observable information from one time instant to another and neither this map is expected to be linear. Thus, we have to introduce some additional assumptions on the dynamics of individual subsystems that allow for previously recovered information (or part of it) to flow through the unobservable manifold of the following subsystems without being perturbed by the unknown variables. This approach leads to a sufficient condition for forward observability, and the observer design based on this approach has somewhat different structure than the one proposed in this chapter, as one would expect it to be the case when making transition from linear to nonlinear systems.

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