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Controllability with a finite data-rate of switched linear systems

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Abstract— **In this work, we argue that the usual notion of controllability is unfit for systems that operate with finite data-rate constraints. We deal with this issue by defining a new concept of controllability with finite data-rate. Then, we specialize our discussion to the case of switched linear systems. We state a necessary condition and a sufficient condition for our new controllability notion to hold. Next, we take advantage of the switched linear system's structure to present a simple sufficient condition for controllability with finite data-rate that only involves the controllable subspace of the individual modes and some mild assumptions about the switching signal that guarantee that our sufficient condition holds. We also present another sufficient condition for systems that activate some controllable mode often enough. In particular, we illustrate the power of this result by deriving relations between the sampling time and the Average Dwell-Time (ADT) of the switching signal that guarantee that the switched system is controllable with finite data-rate. Finally, we discuss the gap between the necessary and the sufficient conditions and show that the sufficient condition is not necessary.**

I. INTRODUCTION

Computers and electronic circuits appear everywhere in modern control systems practice. The digital nature of modern controllers and sensors forces our control system not only to work with discretized time but also with quantized measurements. Moreover, since these digital controllers only have a finite number of possible outputs for any given clock cycle, they must operate with a finite data-rate. This latter fact restricts what control problems we can solve with such controllers. Historically, the discovery of new fundamental limitations in control systems guided us to new ideas that helped develop new controller design techniques [14]. For instance, in [8], Kalman introduced the concept of controllability to explain what plant properties hinder our ability to design controllers that solve specific problems. In that same work, he showed how to construct a controller for a controllable plant that sends the system's state to zero as fast as possible, extending the work [2].

In light of this discussion, we ask a natural question: what new constraints arise from the fact that our controller must operate with a finite data-rate? The so-called data-rate theorems [12], which provide the minimum data-rates for stabilizing plants, give part of the answer. Indeed, the control over communication networks community devoted much of its attention to studying such theorems [17], [11], [9] since communication channels restrict the data-rate of the control laws used. Nonetheless, these theorems are not the only restrictions to finite data-rate control. In this article, we argue that, in general, a finite data-rate controller cannot make the system's state norm decay at a rate that is faster than exponentially, in a sense we define formally later. This fact shows us that the usual concept of controllability, as defined in [8], is unfit for studying the problem of making the state go to the origin as quickly as possible when data-rate constraints are present. Thus, this motivates us to introduce a new controllability notion suited to this case. We do so with the help of concepts from [3]. In that article, the author introduced a concept of stabilization with a finite data-rate, which, loosely speaking, is the ability to drive the state of a system to zero with a prescribed exponential rate of decay. In our work, we strengthen that notion to allow for arbitrary exponential rates of decay. This latter concept is compatible with the idea of being able to drive the state norm to zero with the fastest rate of decay, as we discuss later.

The general problem of controllability with finite data-rate for linear time-varying (LTV) systems is more difficult to analyze than its classical counterpart. However, it is possible to get simple and interesting results for the class of switched linear systems [10]. For this case, we provide a necessary condition and a sufficient condition for our system to be controllable with a finite data-rate. The sufficient condition we present, however, might be hard to check. To address this issue, we exploit the switched system's structure to help us derive simple conditions that we can verify by imposing mild assumptions on the modes and on the switching signal. One of these conditions has a geometric nature, involving the controllable subspace of the modes and an easily checked assumption on the switching signal. Another condition requires our system to switch to a controllable mode frequently enough, in a sense specified later. When we assume that all the modes are controllable and our system satisfies an average dwell-time condition, this latter result allows us to derive a simple inequality involving the average dwell-time, the sampling frequency, and the chatter bound that guarantees controllability with a finite data-rate. Finally, we discuss the gap between our necessary condition and the sufficient one. Closing this gap is the topic of future research.

We take this opportunity to connect this work with [18], where we discussed controllability with a finite data-rate for linear time-varying (LTV) systems. In that document, we also presented a necessary condition and a sufficient condition for a general LTV system to be controllable with a finite data-rate. The necessary condition we state here is the same as

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the one from [18]. On the other hand, the sufficient condition we present here is different and more directly checked than the one we stated in [18]. In fact, our new condition is a consequence of the one we discussed in [18], as we prove in the present paper. Nonetheless, by asking more of our system, i.e., by imposing the switched linear structure, we get several elementary conditions for controllability with finite data-rate that are, in some sense, more realistically verified. Thus, we can see the current paper as a continuation of that one where we impose more structure to our system to get readily checkable sufficient conditions for the class of switched linear systems.

The structure of this document is as follows: First, in Section II, we present the model we want to study and describe why we need a new controllability notion. Still in Section II, we define controllability with finite data-rate. Next, in Section III, we state a necessary condition and a sufficient condition for controllability with finite data-rate. Then, by constraining the switching signal and the sampling times, we derive a sufficient condition for controllability using the controllable subspaces of the modes. After that, in Section IV, we present another sufficient condition that requires our system to activate some controllable mode often enough. In this section, we present our condition that involves average dwell-time and the sampling frequency. We finish this section with a discussion on the gap between the necessary condition and the sufficient one. In Section V, we conclude and present future research directions.

Notation: We denote by $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) the set of the positive (nonnegative) integers. We denote by $\mathbb{R}$ the set of real numbers. We denote by $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) the set of positive (nonnegative) real numbers. Given $m \in \mathbb{Z}_{>0}$, we define the set $[m] := \{1, \ldots, m\}$. Given a set S, we denote by $\#S$ its cardinality. Let $d_x \in \mathbb{Z}_{>0}$ and $d_u \in \mathbb{Z}_{>0}$, we denote by $\mathcal{M}^{d_x \times d_u}$ the set of $d_x \times d_u$ real matrices. Let $d_x \in \mathbb{Z}_{>0}$, then we denote by I_{d_x} the $d_x \times d_x$ identity matrix. We denote the transpose of a matrix $A \in \mathcal{M}^{d_x \times d_u}$ by $A' \in \mathcal{M}^{d_u \times d_x}$. Given a pair of matrices (A, B) with $A \in \mathcal{M}^{d_x \times d_x}$ and $B \in \mathcal{M}^{d_x \times d_u}$, we denote by $\langle A|B \rangle$ their controllable subspace. Given $A \in \mathcal{M}^{d_x \times d_x}$ and $B \in \mathcal{M}^{d_x \times d_x}$ two symmetric positive semi-definite matrices, we write that $A \geq B$ ($A > B$) if $A - B$ is positive semidefinite (definite). If A is a $d_x \times d_x$ real matrix and $|\cdot|$ is a vector norm[1] in $\mathbb{R}^{d_x}$, we denote by $\|A\| := \max\{|Ax| : |x| = 1, x \in \mathbb{R}^{d_x}\}$ the norm induced by that vector norm. For a set $S \subset \mathbb{R}^{d_x}$, we define its maximum distance from the origin as $\operatorname{dist}(S) := \sup\{|x| : x \in S\}$. We denote by $\log(a)$ the natural logarithm of $a \in \mathbb{R}_{>0}$. We denote by $L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^{d_u})$ the set of all Lebesgue integrable (see, e.g., Chapter 2 of [4]) locally essentially bounded functions from $[t_0, \infty)$ to $\mathbb{R}^{d_u}$ where $t_0 \in \mathbb{R}_{\geq 0}$ and $d_u \in \mathbb{Z}_{>0}$. Finally, given a function $u : I \subset \mathbb{R} \to \mathbb{R}^{d_u}$ and a set $J \subset I$, we denote by $u_{|J}(\cdot)$ the restriction of the function $u(\cdot)$ to the subset J.

[1]If not stated otherwise, we assume that $|\cdot|$ is the Euclidean norm.

II. MODEL AND MOTIVATION

A. The Model

In this work, we study the controllability with finite data-rate of switched linear systems, i.e., systems described by equation:

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) \tag{1}$$

where the current time is $t \in [t_0, \infty)$, the initial time is $t_0 \in \mathbb{R}_{\geq 0}$, the initial state is $x(t_0) = x_0$ and it belongs to a compact set with nonempty interior $K \subset \mathbb{R}^{d_x}$, $m \in \mathbb{Z}_{>0}$ is the number of modes, $\sigma : [t_0, \infty) \to [m]$ is the switching signal, $u : [t_0, \infty) \to \mathbb{R}^{d_u}$ is the control function, and $A_p \in \mathcal{M}^{d_x \times d_x}$ and $B_p \in \mathcal{M}^{d_x \times d_u}$ are the matrices of each mode $p \in [m]$. We also assume that $u(\cdot) \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^{d_u})$ and that $\sigma(\cdot)$ is a càdlàg function[2]. We denote by[3] t_n the n-th discontinuity point of $\sigma(\cdot)$ and we call such points the *switching times*. Finally, we define by $\Phi_\sigma(t, \tau)$ for $t \in \mathbb{R}$ and $\tau \in \mathbb{R}$ the *state-transition matrix* associated with the autonomous part of system (1), i.e., $\Phi_\sigma(t, \tau)$ is the unique solution to the differential equation $\dot{\Phi}_\sigma(t, \tau) = A_{\sigma(t)} \Phi_\sigma(t, \tau)$ with $\Phi_\sigma(\tau, \tau) = I_{d_x}$.

A *control law* is a set $\mathscr{U}(K)$ of functions $u(x, \cdot)$ indexed by initial conditions $x \in K \subset \mathbb{R}^{d_x}$, i.e., each initial state $x \in K$ corresponds to a unique control $u(x, \cdot) \in \mathscr{U}(K)$. Denote by $\mathscr{U}_T(K) := \{v_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0, T], \mathbb{R}^{d_u}) : v(\cdot) \in \mathscr{U}(K)\}$ the set of restrictions of functions from our control law. We define the *data-rate of the control law* $\mathscr{U}(K)$ as $b(\mathscr{U}(K)) := \limsup_{T \to \infty} \frac{1}{T} \log(\#\mathscr{U}_T(K))$ and we say that the control law $\mathscr{U}(K)$ *operates with a finite data-rate* if it satisfies $b(\mathscr{U}(K)) < \infty$.

B. The Need for a New Controllability Notion

In this subsection, we explain why the usual notion of controllability of LTV systems is not suitable when we consider control systems that operate with finite data-rate. To do that, we start by recalling the usual controllability notion (see, e.g., Chapter 9 of [13]) for LTV systems.

Definition 2.1: We say that system (1) is *controllable in the usual sense* on $[t_0, T]$, where $T \geq t_0$, if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^{d_x}$ there exists a function $u : [t_0, T] \to \mathbb{R}^{d_u}$ such that $x(T) = 0$.

To see why this notion is unfit when we work with a finite data-rate, we consider the following simple Example 2.1, which also appeared in a slightly different form in [18].

Example 2.1: Let $\dot{x}(t) = u(t)$ where $t_0 \in \mathbb{R}_{\geq 0}$, $t \in \mathbb{R}_{>t_0}$, $x_0 \in K \subset \mathbb{R}$ with K compact with a nonempty interior, and $u(t) \in \mathbb{R}$. We can easily solve this equation to get that $x(T) = x_0 + \int_{t_0}^T u(\tau) d\tau$. Note that, if $u(t) \in \mathbb{R}^{d_u}$, this system is controllable in the usual sense on the interval $[t_0, T]$. If we impose that this control function comes from a control law that operates with finite data-rate, we have that the set of possible controls $u_{[t_0,T]}(\cdot)$ on any interval of time $t \in [t_0, T]$ has a finite cardinality. Therefore, the integral

[2]A function that is right-continuous and has a left limit everywhere.

[3]We consider t_0 the 0-th discontinuity point of $\sigma(\cdot)$ to keep the notation simple.

of $u(\cdot)$ on $[0,T]$ attains at most finitely many values, but x_0 belongs to the set K, which has infinitely many points. Hence, it is not possible to make $x(T)=0$ for an arbitrary initial condition in K.

The goal of the previous example is to make the straightforward observation that we cannot have $x(T)=0$ for an arbitrary initial condition in K, which supports the claim that the usual controllability notion is unfit for the case where we have a finite data-rate. Thus, we must define a new notion of controllability in this setting. One way of doing so, is to think of controllability as the property of being able to drive the state with an arbitrarily fast rate to the origin. This was Kalman's original idea when he introduced the concept of controllability [8]. The following Proposition 2.2 shows that, in general, the fastest mode of decay for the norm of the state of system (1) using finite data-rate is exponentially fast. Indeed, a stronger claim is true for a much larger class of systems.

Proposition 2.2: Let the set of possible initial states $K \subset \mathbb{R}^{d_x}$ have a nonempty interior, let $m \in \mathbb{Z}_{>0}$ be the number of modes, and let $t_0 \in \mathbb{R}_{\geq 0}$ be the initial time. Consider the switched nonlinear time-varying dynamics given by

$$\dot{x}(t) = f(t, \sigma(t), u(x_0, t), x(t)), \tag{2}$$

where $x(t_0) = x_0 \in K$ is a initial state, $u(x_0, \cdot) \in \mathscr{U}(K)$ is the control function that corresponds to the initial state x_0, $\mathscr{U}(K)$ is a control law that operates with a finite data-rate, $\sigma : [t_0, \infty) \to [m]$ is a càdlàg switching signal, and $f : \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x}$. Also, define $\mathcal{R}_u := \overline{\{u(x,t) \in \mathbb{R}^{d_u} : u(\cdot,\cdot) \in \mathcal{U}(K), (x,t) \in P\}}$, where[4] $P := \{(x,t) \in K \times [t_0, \infty) : |u(x,t)| < \infty\}$. We assume that:

- Equation (2) has a unique forward-complete[5] Caratheodory solution for each initial state $x_0 \in K$ and the initial time t_0. We denote by[6] $\xi(t, t_0, x_0)$ the Caratheodory solution of (2) at time t when the initial time is t_0 and the initial state is x_0.
- There exists a compact set $B_x \subset \mathbb{R}^{d_x}$ such that[7] $\{|\xi(t, t_0, x_0)| : x_0 \in K, t \in [t_0, \infty)\} \subset B_x$.
- $f(\cdot,\cdot,\cdot,\cdot)$ is continuously differentiable in its fourth argument. Define the Jacobian of $f(\cdot,\cdot,\cdot,\cdot)$ in its fourth argument as $f_x : \mathbb{R}_{\geq 0} \times [m] \times \mathbb{R}^{d_u} \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_x}$ where $(f_x(\cdot,\cdot,\cdot,\cdot))_{(i,j)} := \frac{\partial f_i}{\partial x_j}(\cdot,\cdot,\cdot,\cdot)$ for each pair $(i,j) \in [d_x]^2$. We assume that $f_x(\cdot,\cdot,\cdot,\cdot)$ is a continuous function. Further, the quantity $\underline{a} := \operatorname{ess\,sup}\{\|f_x(p_1,p_2,p_3,p_4)\| : p_1 \in [t_0,\infty), p_2 \in [m], p_3 \in \mathcal{R}_u, p_4 \in B_x\}$ is finite.

Denote by $\operatorname{dist}(t, t_0, K) := \sup\{|\xi(t, t_0, x_0)| : x_0 \in K\}$ the maximum distance from a point in the reachable set of (2) at time $t \in [t_0, \infty)$ and the origin of $\mathbb{R}^{d_x}$ when the initial condition belongs to K. Then, we have that

$$\liminf_{t \to \infty} \frac{\log(\operatorname{dist}(t, t_0, K))}{t} > -\infty.$$

In particular, if $f(t, \sigma(t), u(x_0,t), x(t)) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(x_0, t)$, $\mathcal{R}_u$ is a bounded subset of $\mathbb{R}^{d_u}$, and the second bullet above is true, then this result holds.

Thus, it seems natural to relax the usual controllability notion by asking the norm of the state to converge to zero with an arbitrary exponential rate of decay instead of asking the state to equal zero in finite time. To formally state our controllability notion, we use the following Definition 2.2, which is an adaptation from the definitions given in [3] about stabilization with finite data-rate. To improve readability, we name some sets and properties that were not named in [3].

Definition 2.2: We say that system (1) satisfies the *exponential decay condition* with rate $\alpha \in \mathbb{R}_{\geq 0}$, with $M \in \mathbb{R}_{>0}$, and $\epsilon \in \mathbb{R}_{>0}$ if for each $x_0 \in K \subset \mathbb{R}^{d_x}$ there exists $u(\cdot) \in L^\infty_{\text{loc}}([t_0,\infty), \mathbb{R}^{d_u})$ such that the corresponding solution satisfies

$$|x(t)| \leq \big(M|x_0| + \epsilon\big) e^{-\alpha(t - t_0)} \tag{3}$$

for all $t \in \mathbb{R}_{\geq t_0}$. For given $\alpha \in \mathbb{R}_{\geq 0}$, $M \in \mathbb{R}_{>0}$, $\epsilon \in \mathbb{R}_{>0}$, and $K \subset \mathbb{R}^{d_x}$ as above, we call a set[8] $\mathscr{R}(\epsilon, M, K, \alpha) \subset L^\infty_{\text{loc}}([t_0,\infty), \mathbb{R}^{d_u})$ a *stabilizing control set* of system (1) if for every $x_0 \in K$, there exists a control function $u(\cdot) \in \mathscr{R}(\epsilon, M, K, \alpha)$ such that (3) holds for the corresponding solution. Furthermore, we denote by

$$\mathscr{R}_T(\epsilon, M, K, \alpha) := \{u_{|[t_0,T]}(\cdot) \in L^\infty_{\text{loc}}([t_0,T], \mathbb{R}^{d_u}) : \quad u(\cdot) \in \mathscr{R}(\epsilon, M, K, \alpha)\} \tag{4}$$

a *set of restrictions of stabilizing controls*, where $T > t_0$ is arbitrary. We define the *data-rate* associated with system (1) in the following manner. First, given a stabilizing control set $\mathscr{R}(\epsilon, M, K, \alpha)$, we define the *data-rate of the stabilizing control set* $\mathscr{R}(\epsilon, M, K, \alpha)$ as[9]

$$b(\mathscr{R}(\epsilon, M, K, \alpha)) := \limsup_{T \to \infty} \frac{1}{T} \log(\#\mathscr{R}_T(\epsilon, M, K, \alpha)).$$

Next, we define the *data-rate of system* (1) as[10]

$$b(M, \alpha) := \lim_{\epsilon \to 0} \big(\inf\{b(\mathscr{R}(\epsilon, M, K, \alpha)) : \mathscr{R}(\epsilon, M, K, \alpha) \text{ is a stabilizing control set of (1)}\}\big). \tag{5}$$

Finally, we say that system (1) can be *stabilized with finite data-rate* with $M \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}_{>0}$ if $b(M, \alpha) < \infty$.

The reader might wonder if we can remove the ϵ term from inside inequality (3) and still get a reasonable notion of stabilizability with finite data-rate. The answer is negative,

[4] Note that $(K \times [0,\infty)) \setminus P$ has measure zero since $u(\cdot,\cdot)$ is locally essentially bounded.

[5] This means that the solution is defined for all $t \in [t_0, \infty)$. See, e.g., Section 1.5 from [5] for sufficient conditions on $f(\cdot,\cdot,\cdot,\cdot)$ for this assumption to hold.

[6] Note that the control is defined by the initial state.

[7] Informally, we are asking the control law to keep the state bounded uniformly over all possible initial states.

[8] We note that this set can be infinite in general.

[9] The corresponding quantity in [3] uses the limit inferior instead of limit superior. Because of that, if the quantity given in [3] is infinite, ours is also infinite.

[10] Note that $b(M, \alpha)$ also depends on the set of initial conditions K. We drop that dependence to make the notation simpler.

and is proved in Proposition 2.2 of [3] where the author showed that, for any pair $(\alpha, M) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, LTI systems with poles with a nonnegative real part cannot satisfy (3) with $\epsilon = 0$ and have $b(M, \alpha) < \infty$. Also, we take this opportunity to note that the limit on the right-hand side of equation (5) exists. That happens because the infimum on the right-hand side of that equality is a monotonically decreasing function of ϵ. Consequently, that limit can be replaced by the supremum over $\epsilon \in \mathbb{R}_{>0}$. We also note that $\mathscr{R}(\epsilon, M, K, \alpha)$ is a control law[11] that operates with the data-rate $b(\mathscr{R}(\epsilon, M, K, \alpha))$. Now, we are ready to define controllability with a finite data-rate.

Definition 2.3: We say that system (1) is *controllable with finite data-rate* if for every $\alpha \in \mathbb{R}_{>0}$, there exists $M \in \mathbb{R}_{\geq 0}$ such that system (1) can be stabilized with finite data-rate $b(M, \alpha) < \infty$.

In light of our discussion, Definition 2.3 captures the property of the norm of the state converging to zero with the fastest rate possible in our setting. We believe that it is a natural candidate for extending the concept of controllability to switched linear systems with finite data-rate. It is important to remark that the previous definition is new and it differs from the definition of stabilization with a finite data-rate, originally given in [3], in the sense that it captures the possibility of stabilization with an arbitrary convergence rate $\alpha \in \mathbb{R}_{\geq 0}$, while in [3] α was taken to be a fixed parameter.

III. CHARACTERIZING CONTROLLABILITY WITH A FINITE DATA-RATE

In this section, we characterize controllability with finite data-rate of switched linear systems using classical controllability notions. We recall the concept of complete controllability to state our necessary condition. Next, we use the well-known uniform complete controllability notion to present our sufficient condition. Finally, we provide a geometric characterization for uniform complete controllability of switched linear systems using the controllable subspaces of each mode and the switching signal.

A. The Necessary Condition

We start this subsection by recalling the notion of complete controllability, first stated in [8] and [7].

Definition 3.1: We say that system (1) is *completely controllable* if, for each $\bar{t} \in [t_0, \infty)$, there exists $t_1 \in (\bar{t}, \infty)$ such that (1) is controllable in the usual sense[12] on the time interval $[\bar{t}, t_1]$.

We have two remarks about this definition. First, some authors, such as [15] in Chapter 4, use the term "complete controllability" to refer to usual controllability on a given time interval. The difference is that Definition 3.1 requires system (1) to be controllable over infinitely many intervals, while the definition given in [15] requires the system to be controllable on a single time interval. Second, recall that the *controllability Gramian* of system (1) is given by $W(t, s) := \int_s^t \Phi_\sigma(t, \tau) B_{\sigma(\tau)} B'_{\sigma(\tau)} \Phi'_\sigma(t, \tau) d\tau$ for any $t \in \mathbb{R}_{>0}$ and $s \in \mathbb{R}_{>0}$. Then, it is a well-known fact (see, e.g., [8]) that complete controlabillity is equivalent to the statement: for every $\bar{t} \in \mathbb{R}_{\geq 0}$ there exists some $t_1 > \bar{t}$ such that $W(t_1, \bar{t})$ is invertible. This result gives us an operational way to check if a system is completely controllable. Now, we are ready to state our necessary condition in Theorem 3.1.

[11]See Subsection II-A

[12]See Definiton 2.1.

Theorem 3.1: System (1) is controllable with finite data-rate only if it is completely controllable.

This statement is interesting because it gives a simple condition that guarantees that, if not satisfied, we can rule out the possibility of our system being controllable with finite data-rate. This theorem appears in a slightly different form, stated for a more general class of LTV systems, in [18]. We also refer to [18] for an example of a system that does not satisfy the assumption of Theorem 3.1.

B. The Sufficient Condition

To state the sufficient condition, we must first recall a classical controllability notion for LTV systems.

Definition 3.2: We say that system (1) is *uniformly completely controllable* (UCC) if there exist $T \in \mathbb{R}_{>0}$ and some $\underline{w} \in \mathbb{R}_{>0}$ such that the controllability Gramian satisfies $\underline{w} I_{d_x} \leq W(t + T, t)$ for all $t \in \mathbb{R}_{>0}$, where the inequality here denotes the partial order relation on symmetric positive definite matrices.

We remark that this concept was introduced by Kalman in works [8] and [7] using different conditions from the one we stated. It was [1] who proved that, if $A_{\sigma(\cdot)}$ and $B_{\sigma(\cdot)}$ are uniformly bounded for all times, then the condition we present in Definition 3.2 is equivalent to UCC. Now, we are ready to state our sufficient condition:

Theorem 3.2: System (1) is controllable with finite data-rate if it is UCC.

This result is a consequence of Theorem 3.1 from [18]. In that work, we came up with a different sufficient condition for general LTV systems to be controllable with a finite data-rate. It happens that being UCC is a stronger condition than the one state in that paper. Thus, an LTV UCC system is controllable with finite data-rate.

The previous result applies to any LTV system[13], and it requires us to prove that our system is UCC, which might be difficult in general. However, assuming that our system is given by the switched linear dynamics (1), we can prove results that involve the controllable subspaces of the modes and some properties of the switching signal.

C. Geometric Characterization of UCC for switched systems

Up to this point, we did not mention sampling. The reason is because all the previous results do not need an explicit sampling strategy to hold. In this subsection, however, the relationship between the sampling times and the switching times will be instrumental. Thus, we start it by defining the sampling times and some related concepts. Then, we present

[13]Any system with $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, where the function $u(\cdot)$ is integrable and locally essentially bounded an the matrix functions $A(\cdot)$ and $B(\cdot)$ are locally integrable and bounded.

a controllability notion that allows us to give a geometric criteria for a switched linear system to be controllable with finite data-rate.

We define the sequence of *sampling times* $(\tau_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [t_0, \infty)$ by

$$\tau_n := t_0 + nT_p, \tag{6}$$

where $T_p \in \mathbb{R}_{>0}$ is the *sampling period*. Next, recall that $(t_n)_{n\in\mathbb{Z}\geq 0}$ is the sequence of switching times. Note that the sequences $(\tau_n)_{n\in\mathbb{Z}_{>0}}$ and $(t_n)_{n\in\mathbb{Z}_{>0}}$ are not, in principle, related. When $(\tau_n)_{n\in\mathbb{Z}_{\geq 0}} \subset (t_n)_{n\in\mathbb{Z}_{\geq 0}}$, we say that the switching happens *synchronously* with the sampling. At this point, it is convenient to introduce some notation. Let $\mathcal{S} := \{n \in \mathbb{Z}_{\geq 0} : \sigma(\tau_n) \neq \sigma(t) \text{ for some } t \in [\tau_n, \tau_{n+1})\}$, i.e., $n \in \mathcal{S}$ if a switching occurs in the interior of the time interval $[\tau_n, \tau_{n+1})$. Note that $\mathcal{S} = \emptyset$ only if the switchings happen synchronously with the samplings, or if there are no switchings.

Now, to state our condition for uniform complete controllability, we must introduce a new controllability definition. We briefly recall that $\langle A|B\rangle$ denotes the controllable subspace of the pair (A, B).

Definition 3.3: Let $\ell \in \mathbb{Z}_{>0}$ be a discrete time-horizon and let $\mathcal{S} = \emptyset$. For each $k \in \mathbb{Z}_{\geq 0}$, let $n = n(k) := \lfloor \frac{k}{\ell} \rfloor$. Define $\mathcal{V}_k := \Phi_\sigma^{-1}(\tau_k, \tau_{\ell n})\langle A_{\sigma(\tau_k)}|B_{\sigma(\tau_k)}\rangle$. We say that system (1) is $\ell-$*uniformly completely controllable* if

$$\sum_{j=\ell n(k)}^{\ell(n(k)+1)-1} \mathcal{V}_j = \mathbb{R}^{d_x} \tag{7}$$

for each[14] $k \in \mathbb{Z}_{\geq 0}$.

To help the reader better understand the idea behind Definition 3.3, we first discuss its relationship with classical controllability notions. Notice that equation (7) is the same as the condition for complete controllability on the interval $[\tau_{\ell n}, \tau_{\ell(n+1)}]$ given in Chapter 4 of [15] for some fixed $n \in \mathbb{Z}_{\geq 0}$[15]. In fact, more is true. Since equation (7) holds for each $n \in \mathbb{Z}_{\geq 0}$, a stronger controllability property must hold. The following lemma shows that Definition 3.3 and UCC are equivalent when the switchings are synchronous. Therefore, the existence of $\ell \in \mathbb{Z}_{\geq 0}$ such that our system is ℓ-uniformly completely controllable is sufficient for our system to be controllable with finite data-rate.

Lemma 3.3: Let $\mathcal{S} = \emptyset$. Then, there exists some $\ell \in \mathbb{Z}_{>0}$ such that system (1) is $\ell-$uniformly completely controllable if, and only if, system (1) is UCC.

The following example should help us illustrate how we can apply Lemma 3.3 to show a nontrivial result.

Example 3.4: Let $\ell \in \mathbb{Z}_{>0}$, let $m = 2$, and let $t_0 = 0$. Let $\{e_1, e_2\} \subset \mathbb{R}^2$ be the canonical basis. Assume that, for each $n \in \mathbb{Z}_{\geq 0}$, there exists at least one integer $k_i(n)$ such that $\ell n \leq k_i(n) < \ell(n+1)$ and that $\sigma(k_i(n)) = i$ for $i \in [2]$. Also, let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$, $B_1 = e_1$, and $B_2 = e_2$. Note that each individual mode is unstabilizable. A simple calculation shows that $\langle A_i|B_i\rangle = \text{span}\{e_i\}$ for $i \in [2]$. Also, since the matrix A_i is diagonal for each $i \in [2]$, we have that $\Phi_\sigma(t, s)$ is diagonal for each $t \in \mathbb{R}_{\geq 0}$ and $s \in [t, \infty)$. This latter fact implies that $\Phi_\sigma^{-1}(t, s)\langle A_i|B_i\rangle = \langle A_i|B_i\rangle = \text{span}\{e_i\}$ for each $i \in [2]$, all $t \in \mathbb{R}_{\geq 0}$, and all $s \in [t, \infty)$. In particular, for each $n \in \mathbb{Z}_{\geq 0}$ and each $i \in [2]$, we have that $\mathcal{V}_{k_i(n)} = \text{span}\{e_i\}$. Thus, we conclude that $\sum_{j=\ell n}^{\ell(n+1)-1} \mathcal{V}_j \supset \mathcal{V}_{k_1(n)} + \mathcal{V}_{k_2(n)} = \mathbb{R}^2$, which implies that our system is ℓ-uniformly completely controllable. Thus, by Lemma 3.3, our system is controllable with a finite data-rate.

The previous example used the fact that the switchings are synchronous to conclude that the switched system is controllable with a finite data-rate, even though the modes are unstabilizable. In the next section, we deal with switching signals that might not be synchronous.

IV. THEORETICAL APPLICATIONS AND THE GAP BETWEEN CONDITIONS

A. Average Dwell-Time and Sampling

At this point, the reader might wonder if there are simple conditions that ensure that the conditions from Theorem 3.2 hold when we do not require the switchings to be synchronous. The next proposition answers this questions affirmatively.

Proposition 4.1: Let $\ell \in \mathbb{Z}_{\geq 0}$. If, for each index $n \in \mathbb{Z}_{\geq 0}$, there exists some index $k(n) \in \mathbb{Z}_{\geq 0}$ such that $\ell n \leq k(n) < \ell(n+1)$, that $k(n) \notin \mathcal{S}$, and that $\langle A_{\sigma(\tau_{k(n)})}|B_{\sigma(\tau_{k(n)})}\rangle = \mathbb{R}^d$, then system (1) is UCC.

Informally, the last proposition is saying the following: if each interval of the form $[\tau_{\ell n}, \tau_{\ell(n+1)})$, where $n \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 0}$ is given, has a sampling subinterval without a switching in its interior and a controllable mode is active on that subinterval, then the conditions of Theorem 3.2 hold. This latter condition is verified, for instance, when we have a "safe" mode, which we visit at least once in each time interval $[\tau_{\ell n}, \tau_{\ell(n+1)})$, i.e., we vist the controllable mode "frequently enough".

Interestingly, Proposition 4.1 has an immediate corollary of practical interest. First, we recall the definition of average dwell-time.

Definition 4.1 (Average Dwell-Time [6]): We say that system (1) satisfies an *average dwell-time condition* [6] if there exist a *chatter bound* $N_0 \in \mathbb{Z}_{\geq 0}$ and an *average dwell-time* $\tau_D \in \mathbb{R}_{>0}$ such that the number of switches $N_\sigma(t, \tau)$ on any time interval of the form $[\tau, t) \subset [t_0, \infty)$ satisfies $N_\sigma(t, \tau) \leq N_0 + \frac{t-\tau}{\tau_D}$.

The next result gives us a simple relation between the sampling period, the chatter bound, and the dwell-time of our switching signal that ensures that system (1) is controllable with finite data-rate.

Corollary 4.1: Assume that system (1) satisfies the ADT condition with average dwell-time $\tau_D \in \mathbb{R}_{>0}$ and chatter bound $N_0 \in \mathbb{Z}_{\geq 0}$. Further, assume that system (1) modes' are controllable. If $\frac{\tau_D}{N_0+2} \geq T_p$, then the system is controllable with finite data-rate.

[14]Note that for each $l \in \mathbb{Z}_{\geq 0}$ there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $l = \lfloor \frac{k}{\ell} \rfloor$. Thus, $n(k)$ is a surjective function.

[15]We also notice that there exists an analogous characterization for the concept of complete observability, given in [16].

B. The Gap Between Conditions

We see that the difference between the assumptions of Theorem 3.1 and 3.2 is just uniformity. It seems natural to ask if the sufficient condition is actually necessary. The answer is negative as Example 4.2 shows. Before we formally state that example, we take this opportunity to recall some concepts and results presented in [18]. We recall that system (1) is called *persistently completely controllable* if there exists an increasing sequence of times $(s_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [t_0, \infty)$ such that $s_0 = t_0$, that $\lim_{n\to\infty} s_n = \infty$, that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} < \infty$, and that $W(s_{n+1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. We also recall that system (1) satisfies the *exponential energy-growth condition* if there exist constants $N \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}_{\geq 0}$ such that $\|W^{-1}(s_{n+1}, s_n)\| \leq Ne^{\theta s_{n+1}}$ for each $n \in \mathbb{Z}_{\geq 0}$. The latter condition is related to the minimum control energy needed to drive the state $x(s_n)$ at time s_n to zero at time s_{n+1} for each $n \in \mathbb{Z}_{\geq 0}$. We refer to [18] for a discussion on this latter point. Now, Theorem 3.1 from [18] says that if an LTV system is persistently completely controllable and satisfies the exponential energy-growth condition, then it is controllable with finite data-rate. We use this result in our next example to show that UCC is not a necessary condition.

Example 4.2: Let $t_0 = 2$ and consider the equation $\dot{x}(t) = b_{\sigma(t)}u(t)$ with $b_{\sigma(t)} = 1$, when $t \in \cup_{n\geq 1}[2^n, 2^n + 1)$, and $b_{\sigma(t)} = 0$, otherwise. We claim that this system is controllable with a finite data-rate but it is not UCC. We start by choosing a sequence $(s_n)_{n\in\mathbb{Z}_{\geq 0}} \subset [2, \infty)$ such that $s_n = 2^{n+1}$ for $n \in \mathbb{Z}_{\geq 0}$. Naturally, $(s_n)_{n\in\mathbb{Z}_{\geq 0}}$ is an increasing sequence that grows to infinity. Also, we have that $\limsup_{n\to\infty} \frac{s_{n+1}}{s_n} = 2$. Further, for each $n \in \mathbb{Z}_{\geq 0}$, on the interval $[2^{n+1}, 2^{n+2})$, we have that $b_{\sigma(t)} = 1$ only on the time subinterval $[2^{n+1}, 2^{n+1} + 1)$ and $b_{\sigma(t)} = 0$ for the remainder of the total interval. Therefore, we get that $W(s_{n+1}, s_n) = \int_{s_n}^{s_{n+1}} b^2_{\sigma(\tau)} d\tau = \int_{2^{n+1}}^{2^{n+1}+1} 1 d\tau = 1$ for each $n \in \mathbb{Z}_{\geq 0}$, i.e., $W(s_{n+1}, s_n)$ is invertible for each $n \in \mathbb{Z}_{\geq 0}$. Finally, we can easily see that $|W^{-1}(s_{n+1}, s_n)| = 1$ for every $n \in \mathbb{Z}_{\geq 0}$, which implies that our system satisfies the exponential energy-growth condition with $N = 1$ and $\theta = 0$. Thus, our system satisfies all the conditions for Theorem 3.1 from [18] to hold. We therefore conclude that this system is controllable with a finite data-rate. Nonetheless, this system is not UCC. To see that, note that for every $T \in \mathbb{R}_{>0}$ there exists some $n \in \mathbb{Z}_{\geq 0}$ so that $W(s_n + 1 + T, s_n + 1) = 0$. Indeed, this follows from the fact that $b_{\sigma(t)} = 0$ for all $t \in [s_n + 1, s_n + 1 + T)$ if $T < 2^{n+1} - 1$ since $s_n + 1 + T < 2^{n+2}$. This proves the claim.

V. CONCLUSIONS

In this work, we discussed why we need a new controllability notion for systems that operate with finite data-rate. Then, we presented a necessary condition and a sufficient condition for switched linear systems to be controllable with finite data-rate. Next, we took advantage of the switched linear system's structure to get simpler sufficient conditions. The first condition, stated in Lemma 3.3, uses the controllable subspaces of the modes and a mild assumption on the switching signal to establish controllability with finite data-rate. The second one, stated in Proposition 4.1, required us to activate some controllable mode frequently enough. In particular, when all the modes are controllable, this latter condition boils down to a simple inequality for the sampling frequency that guarantees that a system that satisfies an ADT condition is controllable with finite data-rate.

In future works, we want to study similar conditions for nonlinear systems. Also, want to understand and close the gap between the necessary and the sufficient conditions presented.

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