Lyapunov-Based Small-Gain Theorems for Hybrid Systems

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***Abstract*—Constructions of strong and weak Lyapunov functions are presented for a feedback connection of two hybrid systems satisfying certain Lyapunov stability assumptions and a small-gain condition. The constructed strong Lyapunov functions can be used to conclude input-to-state stability (ISS) of hybrid systems with inputs and global asymptotic stability (GAS) of hybrid systems without inputs. In the absence of inputs, we also construct weak Lyapunov functions nondecreasing along solutions and develop a LaSalle-type theorem providing a set of sufficient conditions under which such functions can be used to conclude GAS. In some situations, we show how average dwell time (ADT) and reverse average dwell time (RADT) "clocks" can be used to construct Lyapunov functions that satisfy the assumptions of our main results. The utility of these results is demonstrated for the "natural" decomposition of a hybrid system as a feedback connection of its continuous and discrete dynamics, and in several design-oriented contexts: networked control systems, event-triggered control, and quantized feedback control.**

***Index Terms*—Hybrid system, input-to-state stability, Lyapunov function, small-gain theorem.**

I. Introduction

Stability theory for nonlinear systems benefits immensely from the consideration of interconnections of dissipative systems, which allows one to build the analysis of large systems from properties of smaller subsystems. In this context, passivity and small-gain theorems play a central role as they apply to a feedback connection of two systems which is canonical in control engineering and commonly arises in a range of other situations. These results are invaluable tools in the analysis and design of nonlinear systems for establishing stability and robustness properties of the feedback interconnection.

Small-gain theorems involving linear input-output gains are now regarded as classical and a good account of these techniques can be found in [10]. In the nonlinear context, it was realized in [30] that working with linear gains is too restrictive and a small-gain result for monotone stability was developed. Moreover, the notion of input-to-state stability (ISS) proposed by Sontag [37] turned out to be very natural for formulating general small-gain theorems with nonlinear gains, as first illustrated in [17] for continuous-time systems. Further results on small-gain theorems can be found in [8], [20], [27], [28] and references cited therein. Such results were shown to be extremely useful in design of general control systems and they have already become a part of standard texts on nonlinear control [15].

Lyapunov functions are central tools in this context as they not only serve as certificates of stability and simplify stability proofs, but also provide means to quantify robustness or redesign the controller to improve robustness of the feedback connection [21]. Small-gain theorems are particularly useful for construction of Lyapunov functions by using ISS Lyapunov functions of the subsystems in the feedback loop with an appropriate small-gain condition. This approach was first used for the special case of cascades of continuous-time systems [38] and discrete-time systems [33]. A Lyapunov-based small-gain theorem for general feedback connections was first reported for continuous-time systems [16] and then for discrete-time systems [22].

Hybrid systems combine features of continuous-time and discrete-time systems and, hence, are harder to analyze than their continuous and discrete counterparts. Viewing hybrid systems as feedback connections of smaller subsystems opens the door for the application of small-gain theorems to hybrid systems. For example, many hybrid systems can be regarded as feedback connections of their continuous and discrete dynamics. First trajectory-based small-gain theorems for classes of hybrid systems were reported in [32] and [19]. Lyapunov-based small-gain theorems for a class of hybrid systems were first presented in [24]. These theorems were shown to be useful in a range of applications, such as networked and quantized control systems [6], [32].

Recent progress in the area of hybrid control systems [11] has led to a new class of hybrid models that are proving to be very general and natural from the point of view of Lyapunov stability theory [5]. An appropriate extension of ISS Lyapunov functions for this class of hybrid systems was reported in [2]. A Lyapunov small-gain theorem that proposes a construction of strict Lyapunov functions via a small-gain argument and ISS Lyapunov functions of two subsystems modeled via the framework of [11] was reported in [35]; results in [35] are similar to [24] but they are more general and apply to a different class

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of models. Results in [35] were recently extended in [9] to construct strict Lyapunov functions for a network of hybrid systems. A very similar construction is given in [36] to provide a Lyapunov function for verifying input-output stability of hybrid systems. The more recent results in [26] use constructions similar to [35] to construct weak Lyapunov functions for hybrid systems that can be used to conclude stability via LaSalle-type theorems. We concentrate on Lyapunov small-gain theorems within the hybrid modeling framework proposed in [11] but the idea is more general and naturally applies to hybrid systems modeled differently too; see [24] and [29].

This paper is motivated by early work in [24] and [32] and builds directly on results in [26] and [35]; its goal is to unify and generalize the ideas and results from these preliminary conference papers. We first propose a construction of strict Lyapunov functions via small-gain theorems. As the assumptions needed are strong in general, we show how one can use average dwell time (ADT) and reverse average dwell time (RADT) conditions to modify Lyapunov functions so that they satisfy our assumptions. Then, we construct weak (nonstrictly decreasing) Lyapunov functions via small-gain arguments. A novel LaSalle-type theorem is presented which generalizes a result from [26]; this theorem can be used in conjunction with our Lyapunov constructions to infer global asymptotic stability of the hybrid system. Finally, we show how our results can be used to unify, generalize, derive new and interpret some known results in the literature. In particular, we demonstrate that quantized control systems [1] and event-triggered control [39] can be analyzed in a novel manner. We also consider a "natural" decomposition of the hybrid system as a feedback connection of its continuous and discrete parts. We show that results on networked control systems [34] can be interpreted within the proposed analysis framework (a different construction of Lyapunov functions for networked control systems that does not directly fit our framework was derived in [6]).

The paper is organized as follows. In Section II we present background and mathematical preliminaries. Section III contains the main results of the paper. Modification of Lyapunov functions via ADT and RADT conditions is discussed in Section IV. Our results are applied to several examples of hybrid systems in Section V. A summary concludes the paper.

II. Preliminaries

Our results are presented for locally Lipschitz Lyapunov functions for which we cannot use classical derivatives; we opt to use the so-called Clarke derivative which is a widely accepted generalization of the classical derivatives in non-smooth analysis [7]. It plays the same role for locally Lipschitz functions as the classical derivative does for continuously differentiable (C^1) functions. We note that our construction of Lyapunov functions for feedback systems is such that even if the Lyapunov functions for subsystems are C^1, the composite Lyapunov function is typically not C^1 since it is defined to be a maximum of two functions. The Clarke derivative is defined as follows: for a locally Lipschitz function $U:\mathbb{R}^n\to\mathbb{R}$ and a vector $v\in\mathbb{R}^n$, $U^\circ(x;v):=\limsup_{h\to 0^+,y\to x}(U(y+hv)-U(y))/h$. For a C^1 function $U(\cdot)$, the Clarke derivative $U^\circ(x;v)$ reduces to the standard directional derivative $\langle\nabla U(x),v\rangle$, where $\nabla U(\cdot)$ is the (classical) gradient. The following is a direct consequence of [7, Propositions 2.1.2 and 2.3.12].

Lemma II.1: Consider two functions $U_1:\mathbb{R}^n\to\mathbb{R}$ and $U_2:\mathbb{R}^n\to\mathbb{R}$ that have well-defined Clarke derivatives for all $x\in\mathbb{R}^n$ and $v\in\mathbb{R}^n$. Introduce three sets $A:=\{x:U_1(x)>U_2(x)\}$, $B:=\{x:U_1(x)<U_2(x)\}$, $\Gamma:=\{x:U_1(x)=U_2(x)\}$. Then, for any $v\in\mathbb{R}^n$, the function $U(x):=\max\{U_1(x),U_2(x)\}$ satisfies $U^\circ(x;v)=U_1^\circ(x;v)$ for all $x\in A$, $U^\circ(x;v)=U_2^\circ(x;v)$ for all $x\in B$, and $U^\circ(x;v)\le\max\{U_1^\circ(x;v),U_2^\circ(x;v)\}$ for all $x\in\Gamma$.

The following lemma was proved in [16] and it will be used in our main results.

Lemma II.2: Let[1] $\chi_1,\chi_2\in\mathcal{K}_\infty$ satisfy $\chi_1\circ\chi_2(r)<r$ for all $r>0$ ("small-gain condition"). Then, there exists a function $\rho\in\mathcal{K}_\infty$ that is C^1 on $(0,\infty)$ and satisfies $\chi_1(r)<\rho(r)<\chi_2^{-1}(r)$ and $\rho'(r)>0$ for all $r>0$.

Motivated by hybrid system models proposed in [12], we consider hybrid systems with inputs described by a combination of continuous flow and discrete jumps, of the form (see also [2])

$$\begin{aligned}\dot{x}\in F(x,w),&\quad (x,w)\in C\\ x^+\in G(x,w),&\quad (x,w)\in D\end{aligned}\tag{1}$$

where $x\in\mathbb{R}^n$, $w\in\mathbb{R}^m$, C and D are closed subsets of $\mathbb{R}^n\times\mathbb{R}^m$, and F and G are set-valued maps from $\mathbb{R}^n\times\mathbb{R}^m$ to $\mathbb{R}^n$. The solutions of the hybrid system are defined on so-called hybrid time domains. A set $E\subset\mathbb{R}_{\ge 0}\times\mathbb{Z}_{\ge 0}$ is called a *compact hybrid time domain* if $E=\cup_{j=0}^{J}([t_j,t_{j+1}],j)$ for some finite sequence of times $0=t_0\le t_1\le\cdots\le t_{J+1}$. E is a *hybrid time domain* if for all $(T,J)\in E$, $E\cap([0,T]\times\{0,1,\ldots,J\})$ is a compact hybrid time domain. A *hybrid signal* is a function defined on a hybrid time domain. A *hybrid input* is a hybrid signal $w:\mathrm{dom}w\to\mathbb{R}^m$ such that $w(\cdot,j)$ is Lebesgue measurable and locally essentially bounded for each j. A *hybrid arc* is a hybrid signal $x:\mathrm{dom}x\to\mathbb{R}^n$ such that $x(\cdot,j)$ is locally absolutely continuous for each j. A hybrid arc $x:\mathrm{dom}x\to\mathbb{R}^n$ and a hybrid input $w:\mathrm{dom}w\to\mathbb{R}^m$ are a solution pair to the hybrid model (1) if: $\mathrm{dom}x=\mathrm{dom}w$; for all $j\in\mathbb{Z}_{\ge 0}$ and almost all $t\in\mathbb{R}_{\ge 0}$ such that $(t,j)\in\mathrm{dom}x$ we have $(x(t,j),w(t,j))\in C$ and $\dot{x}(t,j)\in F(x(t,j),w(t,j))$; for $(t,j)\in\mathrm{dom}x$ such that $(t,j+1)\in\mathrm{dom}x$ we have $(x(t,j),w(t,j))\in D$ and $x(t,j+1)\in G(x(t,j),w(t,j))$. Here, $x(t,j)$ represents the state of the hybrid system after t time units and j jumps. Under suitable assumptions on the data (C,D,F,G) of the hybrid system (see, e.g., [12, Prop. 2.4] or [11, p. 44]) one can establish local existence of solutions, which may be non-unique; this basically boils down to checking that flow is possible from every $x\in C\setminus D$. While not directly needed for our Lyapunov function constructions, local existence of solutions will be assumed

[1] A continuous function $\gamma:\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is of *class* $\mathcal{K}$ if it is zero at zero and strictly increasing. It is of *class* $\mathcal{K}_\infty$ if it is unbounded; note that $\mathcal{K}_\infty$ functions are globally invertible. A function $\beta:\mathbb{R}_{\ge 0}\times\mathbb{R}_{\ge 0}\to\mathbb{R}_{\ge 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t\ge 0$ and $\beta(r,t)$ is decreasing to zero as $t\to\infty$ for each fixed $r\ge 0$.

whenever we talk about properties of system trajectories. A solution is called *complete* if its domain is unbounded.

We now define basic asymptotic properties of solutions that are of interest to us, and which we will be able to establish as eventual consequences of our Lyapunov function constructions. The first property is useful mainly for systems with no disturbances (or for when the disturbance is 0 or does not have any influence on the system dynamics); the second property characterizes the desired response to inputs. These stability properties are standard in the nonlinear systems literature; for hybrid system models considered here, they are discussed in [11] and [2], respectively. For simplicity, we limit ourselves here to global properties. Consider a compact set $\mathcal{A} \subset \mathbb{R}^n$, and let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$. A continuous function $\omega : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a *proper indicator for* $\mathcal{A}$ if $\omega(x) = 0$ if and only if $x \in \mathcal{A}$, and $\omega(x) \to \infty$ when $|x| \to \infty$. The hybrid system (1) is *globally pre-asymptotically stable (pre-GAS)* with respect to the set $\mathcal{A}$ if all its solutions satisfy[2]

$$\omega\left(x(t,j)\right) \leq \beta\left(\omega\left(x(0,0)\right), t+j\right) \quad \forall (t,j) \in \text{ dom } x \quad (2)$$

where ω is a proper indicator for $\mathcal{A}$ and β is a function of class $\mathcal{KL}$. When $\mathcal{A} = \{0\}$, we just say that the system is pre-GAS.

Remark II.1: We work with proper indicator functions for $\mathcal{A}$ since our results lead naturally to such stability properties. However, the bound with a proper indicator for $\mathcal{A}$ is (qualitatively[3]) equivalent to the more common set-stability bound in terms of the distance to $\mathcal{A}$:

$$|x(t,j)|_{\mathcal{A}} \leq \tilde{\beta}\left(|x(0,0)|_{\mathcal{A}}, t+j\right) \quad \forall (t,j) \in \text{ dom } x. \quad (3)$$

It is obvious that (3) implies (2) since $|\cdot|_{\mathcal{A}}$ is a proper indicator function for $\mathcal{A}$. The converse implication follows from the fact that for any proper indicator function ω for $\mathcal{A}$ there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that $\psi_1(|x|_{\mathcal{A}}) \leq \omega(x) \leq \psi_2(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$; this can be proved in the same manner as [21, Lemma 3.5]. Note also that from Theorem 14 of [11, p. 56] we have that (3) is equivalent to stability plus pre-attractivity for the compact set $\mathcal{A}$.

The hybrid system (1) is *pre-input-to-state stable (pre-ISS)* with respect to the input w and the set $\mathcal{A}$ if all its solutions satisfy

$$\omega\left(x(t,j)\right) \leq \max\left\{\beta\left(\omega\left(x(0,0)\right), t+j\right), \kappa\left(\|w\|_{(t,j)}\right)\right\} \quad (4)$$

for all $(t,j) \in \text{dom}x$, where ω is a proper indicator for $\mathcal{A}$ on $\mathbb{R}^n$, β is a function of class $\mathcal{KL}$, κ is a function of class $\mathcal{K}_\infty$ (the *ISS gain function*), and $\|w\|_{(t,j)}$ stands for the supremum norm of w up to the hybrid time (t,j) (modulo a set of measure zero not including jump times; see [2] for a precise definition). When $\mathcal{A} = \{0\}$, we just say that the system is pre-ISS.

We choose to work with the pre-GAS notion instead of the more standard GAS notion because it corresponds more directly to the existence of Lyapunov functions, as will be clear from the results given below. If a system is pre-GAS then all complete solutions converge to $\mathcal{A}$. Completeness is not part of the stability definition, and needs to be checked separately. As shown in [11, Theorem S3], for hybrid systems with local existence of solutions, establishing completeness of solutions amounts to ruling out the possibility of finite escape time (during flow) and of jumping out of $C \cup D$; the former can be done using well-known results on ODEs, and the latter is automatic when $C \cup D = \mathbb{R}^n \times \mathbb{R}^m$. Local existence of solutions, in turn, can be checked as we explained in the paragraph following (1). Similar comments apply to the pre-ISS notion. If all solutions are complete, then the prefix "pre-" is dropped; Section V will contain examples of such situations.

In this paper, we are concerned with situations where the hybrid system (1) is decomposed as

$$\begin{array}{lll} \dot{x}_1 \in F_1(x_1, x_2, w), & \dot{x}_2 \in F_2(x_1, x_2, w), & (x, w) \in C \\ x_1^+ \in G_1(x_1, x_2, w), & x_2^+ \in G_2(x_1, x_2, w), & (x, w) \in D \end{array} \quad (5)$$

where $x := (x_1, x_2)$ which is a shorthand notation we use for $(x_1^T, x_2^T)^T$, $x_i \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^m$, $F := (F_1, F_2)$, $G := (G_1, G_2)$ and $n := n_1 + n_2$ (i.e., $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$). We regard this system as a feedback connection of two hybrid subsystems with states x_1 and x_2. Decomposing the hybrid system (1) in this way is very natural and not restrictive; for example, we can always view it as a feedback connection of its continuous and discrete dynamics, which yields what we may call the "natural decomposition" (see Section V-A).

III. Main Technical Results

In this section, we present the main results of the paper, which specify how to construct a strong or weak Lyapunov function by using suitable ISS Lyapunov functions for subsystems in a feedback connection with an appropriate small-gain condition. We will see later how these results can be used to verify stability and ISS of various examples recently considered in the literature. We note that in order to use our results we would sometimes need to modify the given Lyapunov functions for subsystems to satisfy all assumptions needed in our main results. These constructions require the use of various "clocks" that restrict the set of solutions of the hybrid system; this is demonstrated in the next section.

A. Construction of Strong Lyapunov Functions

The following assumption is an appropriate generalization of assumptions typically used for continuous-time [16] and discrete-time [22] Lyapunov-based small-gain theorems (cf. Remark III.2 below).

Assumption III.1: For $i, j \in \{1, 2\}$, $i \neq j$ there exist locally Lipschitz functions $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ such that the following hold:

1) There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ and continuous proper[4] functions $h_i : \mathbb{R}^{n_i} \to \mathbb{R}^{\ell_i}$ for some ℓ_i such that for all $x_i \in \mathbb{R}^{n_i}$ we have $\psi_{i1}(|h_i(x_i)|) \leq V_i(x_i) \leq \psi_{i2}(|h_i(x_i)|)$.

[2]We work here with $\mathcal{KL}$ functions rather than $\mathcal{KLL}$ functions that are sometimes used for hybrid time domains, see for instance [4]; there is no loss of generality in working with $\mathcal{KL}$ functions, see proof of Lemma 6.1 in [4].

[3]In other words, the $\mathcal{KL}$ functions β in (2) and $\tilde{\beta}$ in (3) are different in general.

[4]By *proper* here we mean that $|h_i(x_i)| \to \infty$ when $|x_i| \to \infty$.

2) There exist functions $\chi_i, \gamma_i \in \mathcal{K}_\infty$ and positive definite functions $\alpha_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $(x, w) \in C$ we have

$$V_i(x_i) \geq \max\{\chi_i(V_j(x_j)), \gamma_i(|w|)\} \Rightarrow V_i^\circ(x_i; z_i) \leq -\alpha_i(V_i(x_i)) \quad \forall z_i \in F_i(x, w). \tag{6}$$

3) There exist positive definite functions $\lambda_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lambda_i(s) < s\ \forall s > 0$ such that for all $(x, w) \in D$ we have, with the same χ_i and γ_i as in item 2,

$$V_i(z_i) \leq \max\{\lambda_i(V_i(x_i)), \chi_i(V_j(x_j)), \gamma_i(|w|)\} \tag{7}$$

$\forall z_i \in G_i(x, w)$.

4) The following small-gain condition holds: $\chi_1 \circ \chi_2(s) < s\ \forall s > 0$.

The functions χ_i, γ_i in the Lyapunov-based conditions (6) and (7) play a role similar to that of the ISS gain function κ in the definition (4) of pre-ISS, and they can be used to arrive at the ISS gain of the overall system via the results presented below; with a slight abuse of terminology, we will refer to these functions also as "gain functions" or "gains." We note that we use the same functions χ_i, γ_i in (6) and in (7). We could work with different functions and at the end take the maximum to arrive at the overall ISS gain of the system; on the other hand, we can always take the maximum at the start. Moreover, note that we use different forms of ISS Lyapunov conditions on the sets C and D because this simplifies the proofs.

Defining the set

$$\mathcal{A} := \{(x_1, x_2) : h_1(x_1) = 0, \quad h_2(x_2) = 0\} \tag{8}$$

we can now state our first main result.

Theorem III.1: Consider the hybrid system (5). Suppose that Assumption III.1 holds. Let $\rho \in \mathcal{K}_\infty$ be generated via Lemma II.2 using χ_1, χ_2. Let

$$V(x) := \max\{V_1(x_1), \rho(V_2(x_2))\}. \tag{9}$$

Then, there exist functions $\psi_1, \psi_2, \gamma \in \mathcal{K}_\infty$ and positive definite functions $\alpha, \lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\lambda(s) < s \forall s > 0$, such that the following hold:

1) For all $x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ we have

$$\psi_1(|(h_1(x_1), h_2(x_2))|) \leq V(x) \leq \psi_2(|(h_1(x_1), h_2(x_2))|). \tag{10}$$

2) For all $(x, w) \in C$ with $x \notin \mathcal{A}$ we have

$$V(x) \geq \gamma(|w|) \Rightarrow V^\circ(x; z) \leq -\alpha(V(x)) \quad \forall z \in F(x, w). \tag{11}$$

3) For all $(x, w) \in D$ we have

$$V(z) \leq \max\{\lambda(V(x)), \gamma(|w|)\} \quad \forall z \in G(x, w). \tag{12}$$

Remark III.1: Note that the function V constructed in (9) is not guaranteed to be locally Lipschitz everywhere because the derivative of ρ may grow unbounded as its argument approaches 0. For this reason, we added the quantifier $x \notin \mathcal{A}$ in item 2 of the theorem to ensure the existence of the Clarke derivative. However, it is not difficult to check that the theorem remains valid if (9) is generalized to $V(x) := \hat{\rho}(\max\{\tilde{\rho}(V_1(x_1)), \rho \circ \tilde{\rho}(V_2(x_2))\})$ with arbitrary C^1 and $\mathcal{K}_\infty$ functions $\hat{\rho}$ and $\tilde{\rho}$.

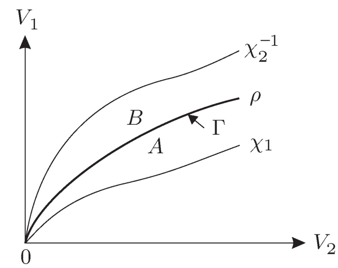

Fig. 1. Sets A (below the middle curve), B (above the middle curve), and Γ (the middle curve).

Using the extra freedom in choosing these functions, we can always arrange V to be locally Lipschitz everywhere. Then, the quantifier $x \notin \mathcal{A}$ in item 2 can also be dropped because V° satisfies (11) as long as it exists.

Proof: Since ρ is generated using χ_1, χ_2 via Lemma II.2, we have

$$\chi_1(r) < \rho(r) \text{ and } \chi_2(r) < \rho^{-1}(r) \quad \forall r > 0. \tag{13}$$

Denote $q(r) := \rho'(r)$. The proof of item 1 is straightforward and it is omitted. We now establish item 2. Let $\gamma(s) := \max\{\rho \circ \gamma_2(s), \gamma_1(s)\}$ and $\alpha(s) := \min\{q \circ \rho^{-1}(s) \cdot \alpha_2 \circ \rho^{-1}(s), \alpha_1(s)\}$. Suppose that $V(x) \geq \gamma(|w|)$. Now we introduce three subsets of $\mathbb{R}^n \times \mathbb{R}^m$ (shown in Fig. 1) and investigate $V^\circ(x, z)$, $z \in F(x, w)$ on each of them intersected with C. Define $A := \{(x_1, x_2, w) : V_1(x_1) < \rho(V_2(x_2))\}$, $B := \{(x_1, x_2, w) : V_1(x_1) > \rho(V_2(x_2))\}$, and $\Gamma := \{(x_1, x_2, w) : V_1(x_1) = \rho(V_2(x_2))\}$. Consider first $(x, w) \in A \cap C$. In this case $V(x) = \rho(V_2(x_2))$ and we have that $V_1(x_1) < \rho(V_2(x_2))$ which implies $V_2(x_2) > \chi_2(V_1(x_1))$ using (13). Hence, (6) applies with $(i, j) = (2, 1)$ and so, whenever $V(x) \geq \rho \circ \gamma_2(|w|)$ and $z \in F(x, w)$, we have $z_2 \in F_2(x, w)$ and $V^\circ(x; z) = q(V_2(x_2)) V_2^\circ(x_2; z_2) \leq -q(V_2(x_2))\alpha_2(V_2(x_2)) = -q \circ \rho^{-1}(V(x)) \cdot \alpha_2 \circ \rho^{-1}(V(x))$. Now, consider $(x, w) \in B \cap C$. Since $V_1(x_1) > \rho(V_2(x_2))$, we have using (13) that $V_1(x_1) > \chi_1(V_2(x_2))$ and $V(x) = V_1(x_1)$. Hence, (6) applies with $(i, j) = (1, 2)$ and, whenever $V(x) \geq \gamma_1(|w|)$ and $z \in F(x, w)$, we have $z_1 \in F_1(x, w)$ and $V^\circ(x; z) = V_1^\circ(x_1; z_1) \leq -\alpha_1(V(x))$. Finally, consider $(x, w) \in \Gamma \cap C$. Then, using the definition of V and Lemma II.1 and noting that the previous inequalities remain valid on the closure of A and B, we have for $V(x) \geq \max\{\rho \circ \gamma_2(|w|), \gamma_1(|w|)\}$ and $z \in F(x, w)$ that $V^\circ(x; z) \leq \max\{V_1^\circ(x_1; z_1), q(V_2(x_2)) \cdot V_2^\circ(x_2; z_2)\} \leq -\min\{\alpha_1(V(x)), q \circ \rho^{-1}(V(x)) \cdot \alpha_2 \circ \rho^{-1}(V(x))\} = -\alpha(V(x))$ when $x \notin \mathcal{A}$. Hence, (11) holds. We now show that item 3 holds. Let $\lambda(s) := \max\{\lambda_1(s), \chi_1 \circ \rho^{-1}(s), \rho \circ \lambda_2 \circ \rho^{-1}(s), \rho \circ \chi_2(s)\}$ and $\gamma(s) := \max\{\gamma_1(s), \rho \circ \gamma_2(s)\}$. Note that $\lambda(s) < s$ for all $s > 0$. Indeed, $\lambda_1(s) < s$ and $\lambda_2(s) < s$ for all $s > 0$ by assumption. The latter implies that $\rho \circ \lambda_2 \circ \rho^{-1}(s) < s$ for all $s > 0$. By construction of ρ (see (13) and Fig. 1) we have that $\chi_1 \circ \rho^{-1}(s) < s$ and $\rho \circ \chi_2(s) < s$ for all $s > 0$, which shows that $\lambda(s) < s$ for all $s > 0$. Using the definition of V in (9) and (7), we can write for all $(x, w) \in D$, $z \in G(x, w)$ that $V(z) = \max\{V_1(z_1), \rho(V_2(z_2))\} \leq \max\{\lambda_1(V_1(x_1)), \chi_1(V_2(x_2)), \gamma_1(|w|), \rho \circ \lambda_2(V_2(x_2)), \rho \circ \chi_2(V_1(x_1)), \rho \circ \gamma_2(|w|)\} = \max\{\lambda_1(V_1(x_1)), \chi_1 \circ \rho^{-1} \circ \rho(V_2(x_2)), \gamma_1(|w|), \rho \circ \lambda_2 \circ \rho^{-1} \circ$

$\rho(V_2(x_2)),\ \ \rho\circ\chi_2(V_1(x_1)),\ \ \rho\circ\gamma_2(|w|)\} \le \max\{\lambda_1(V(x)),$ $\chi_1\circ\rho^{-1}(V(x)),\ \gamma_1(|w|),\ \rho\circ\lambda_2\circ\rho^{-1}(V(x)),\ \rho\circ\chi_2(V(x)),$ $\rho\circ\gamma_2(|w|)\} \le \max\{\lambda(V(x)),\gamma(|w|)\}$. Hence, (12) holds. □

The calculations used to prove [2, Proposition 2.7] (see also [3]) can be used to show that, under items 1–3 of Assumption III.1, the i-th subsystem ($i=1,2$) is pre-ISS with respect to the input (x_j,w), $j\neq i$ and the set $\mathcal{A}_i := \{x_i : h_i(x_i)=0\}$. Specifically, $\omega_i(x_i) := |h_i(x_i)|$ is a proper indicator for $\mathcal{A}_i$ and we have an ISS estimate $|h_i(x_i(t,k))| \le \max\{\beta_i(|h_i(x_i(0,0))|,t+k),\kappa_i(\|(x_j,w)\|_{(t,k)})\}\quad \forall (t,k)\in \mathrm{dom}x_i$. Similarly, the conclusions of Theorem III.1 (with the observation of Remark III.1) guarantee that the overall hybrid system is pre-ISS with respect to the input w and the set $\mathcal{A}$ defined in (8). Note that $\mathcal{A}$ is compact because $\omega(x) := |(h_1(x_1),h_2(x_2))|$ is a proper indicator for $\mathcal{A}$ which is continuous and radially unbounded. An ISS estimate takes the form

$$\left|\begin{pmatrix} h_1\left(x_1(t,j)\right) \\ h_2\left(x_2(t,j)\right)\end{pmatrix}\right| \le \max\left\{\beta\left(\left|\begin{pmatrix} h_1\left(x_1(0,0)\right) \\ h_2\left(x_2(0,0)\right)\end{pmatrix}\right|, t+j\right),\right.$$
$$\left.\kappa\left(\|w\|_{(t,j)}\right)\right\} \quad \forall (t,j)\in \ \mathrm{dom}\ x.$$

Hence, we can state the following corollary of [2, Proposition 2.7] and our Theorem III.1.

Corollary III.2: If the hybrid system (5) fulfills Assumption III.1, then it is pre-ISS with respect to the input w and the set $\mathcal{A}$ defined in (8).

Remark III.2: The formula (9) is the same as the one used for the special cases of purely continuous-time systems ($D=\emptyset$) in [16] and purely discrete-time systems ($C=\emptyset$) in [22], and the above proof (which also appeared in [35]) is essentially a streamlined combination of the arguments from those references. However, our formulation differs from those previous works in several aspects. In particular, our condition (10) is more general than those in [16], [22] since we consider ISS with respect to sets, whereas in the cited references only ISS with respect to the origin (i.e., the case $h_i(x_i)=x_i$) is considered. While this generalization is easily achieved if we revisit results in [16], [22], it is very useful in the context of hybrid systems in situations when additional "clock" variables are introduced to constrain the hybrid time domain with the aim of ensuring that all conditions of Assumption III.1 hold. The use of clock variables will be illustrated in the next section. Another difference with [16] in the treatment of continuous dynamics is the use of the Clarke derivative, which makes the analysis of Γ in the proof of Theorem III.1 more elegant.

B. Construction of Weak Lyapunov Functions

In this subsection, we consider a version of the hybrid system (5) without disturbances

$$\begin{array}{lll} \dot{x}_1 \in F_1(x_1,x_2), & \dot{x}_2 \in F_2(x_1,x_2), & x\in C \\ x_1^+ \in G_1(x_1,x_2), & x_2^+ \in G_2(x_1,x_2), & x\in D \end{array} \tag{14}$$

where $x_i\in\mathbb{R}^{n_i}$ and all other notation is the same as in the previous subsection but applied to the above system without disturbances. We need the following assumption.

Assumption III.2: For $i=1,2$ there exist locally Lipschitz functions $V_i:\mathbb{R}^{n_i}\to\mathbb{R}_{\ge 0}$ such that:

1) Item 1 of Assumption III.1 holds.
2) There exist functions $\chi_i\in\mathcal{K}_\infty$, a positive definite function $\alpha_1:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$, and a function $R:\mathbb{R}^{n_2}\to\mathbb{R}_{\ge0}$ such that for all $x\in C$, we have

$$\begin{aligned} &V_1(x_1)\ge\chi_1\left(V_2(x_2)\right)\\ &\qquad\Rightarrow V_1^\circ(x_1;z_1)\le-\alpha_1\left(V_1(x_1)\right)\quad\forall z_1\in F_1(x),\end{aligned} \tag{15}$$

$$\begin{aligned} &V_2(x_2)\ge\chi_2\left(V_1(x_1)\right)\\ &\qquad\Rightarrow V_2^\circ(x_2;z_2)\le-R(x_2)\quad\forall z_2\in F_2(x).\end{aligned} \tag{16}$$

3) There exists a positive definite function $\lambda_2:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ with $\lambda_2(s)<s\ \forall s>0$ and a function $Y:\mathbb{R}^{n_1}\to\mathbb{R}_{\ge0}$ such that for all $x\in D$ we have, with the same χ_i as in item 2,

$$V_1(z_1)\le\max\left\{V_1(x_1)-Y(x_1),\chi_1\left(V_2(x_2)\right)\right\}\quad\forall z_1\in G_1(x), \tag{17}$$

$$V_2(z_2)\le\max\left\{\lambda_2\left(V_2(x_2)\right),\chi_2\left(V_1(x_1)\right)\right\}\quad\forall z_2\in G_2(x). \tag{18}$$

4) Item 4 of Assumption III.1 (the small-gain condition) holds.

Note that the individual ISS Lyapunov functions V_1, V_2 in Assumption III.2 are "weak" in the sense that they are allowed to decrease nonstrictly along the continuous dynamics for one subsystem and the discrete dynamics for the other subsystem, respectively; thus the subsystems are not required to be ISS. The next result asserts the existence of a weak Lyapunov function nondecreasing along trajectories of the overall hybrid system, suitable for an application of a Barbashin-Krasovskii-LaSalle-type theorem as we show afterwards.

Theorem III.3: Consider the hybrid system (14). Suppose that Assumption III.2 holds. Let $\rho\in\mathcal{K}_\infty$ be generated via Lemma II.2 using χ_1,χ_2. Let V be defined via (9). Then, there exist $\psi_1,\psi_2\in\mathcal{K}_\infty$, a positive definite function $\sigma:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ with $\sigma(s)<s\ \forall s>0$, and a positive semi-definite function $S:\mathbb{R}_{\ge0}\to\mathbb{R}_{\ge0}$ such that the following hold:

1) Item 1 of Theorem III.1 holds.
2) $V^\circ(x;z)\le\max\{-\alpha_1(V(x)),-S(x_2)\}\quad\forall x\in C\setminus\mathcal{A}$, $\forall z\in F(x)$.
3) $V(z)\le\max\{V(x)-Y(x_1),\sigma(V(x))\}\ \forall x\in D,\ \forall z\in G(x)$.

Proof: Denote $q(r):=\rho'(r)$ and let V be defined as in (9). The proof of item 1 is straightforward and it is omitted. We now establish item 2. Let $S(\cdot):=q(V_2(\cdot))\cdot R(\cdot)$ and $\sigma(\cdot):=\max\{\chi_1\circ\rho^{-1}(\cdot),\rho\circ\lambda_2\circ\rho^{-1}(\cdot),\rho\circ\chi_2(\cdot)\}$. By construction of ρ (see (13) and Fig. 1) we easily see that $\sigma(s)<s$ for all $s>0$. Similarly to the proof of Theorem III.1, we introduce three subsets of $\mathbb{R}^n$ and investigate the behavior of V on each one intersected with C. Define $A:=\{(x_1,x_2):V_1(x_1)<\rho(V_2(x_2))\}$, $B:=\{(x_1,x_2):V_1(x_1)>\rho(V_2(x_2))\}$, and $\Gamma:=\{(x_1,x_2):V_1(x_1)=\rho(V_2(x_2))\}$. Consider first $x\in A\cap C$.

Here $V(x)=\rho(V_2(x_2))$ and so (16) applies by virtue of (13), hence for all $z\in F(x)$ we have $V^\circ(x;z)=q(V_2(x_2))\cdot V_2^\circ(x_2;z_2)\le -q(V_2(x_2))R(x_2)=-S(x_2)$. Next, consider $x\in B\cap C$ so that $V(x)=V_1(x_1)$. By (15), for all $z\in F(x)$ we have

$$V^\circ(x;z)=V_1^\circ(x_1;z_2)\le -\alpha_1(V_1(x_1))=-\alpha_1\left(V(x)\right). \quad (19)$$

Finally, consider $x\in(\Gamma\cap C)\setminus\mathcal{A}$. Using Lemma II.1 and noting that Γ is contained in the closure of both A and B, we can use the same inequalities as above to obtain for all $z\in F(x)$ that $V^\circ(x;z)\le\max\{V_1^\circ(x_1;z_1),q(V_2(x_2))\cdot V_2^\circ(x_2;z_2)\}\le\max\{-\alpha_1(V(x)),-q(V_2(x_2))R(x_2)\}=\max\{-\alpha_1(V(x)),-S(x_2)\}$. Therefore, item 2 holds.

We now establish item 3. Using the definition of V in (9) and item 3 of Assumption III.2, we can write for all $x\in D$ and $z\in G(x)$ that $V(z)=\max\{V_1(z_1),\ \rho(V_2(z_2))\}\le\max\{V_1(x_1)-Y(x_1),\ \ \chi_1(V_2(x_2)),\ \ \rho\circ\lambda_2(V_2(x_2)),\ \ \rho\circ\chi_2(V_1(x_1))\}=\max\{V_1(x_1)-Y(x_1),\ \ \chi_1\circ\rho^{-1}\circ\rho(V_2(x_2)),\ \rho\circ\lambda_2\circ\rho^{-1}\circ\rho(V_2(x_2)),\ \rho\circ\chi_2(V_1(x_1))\}\le\max\{V(x)-Y(x_1),\ \chi_1\circ\rho^{-1}(V(x)),\ \ \rho\circ\lambda_2\circ\rho^{-1}(V(x)),\ \ \rho\circ\chi_2(V(x))\}\le\max\{V(x)-Y(x_1),\sigma(V(x))\}$, and item 3 is verified. □

Remark III.3: The same comments as in Remark III.1 apply here concerning the exclusion of points $x\in\mathcal{A}$ from item 2 in Theorem III.3. Moreover, we can sometimes draw stronger conclusions if either $R(\cdot)$ or $Y(\cdot)$ or both are positive definite functions rather than merely nonnegative. For instance, assuming that $R(x_2)=\alpha_2(V_2(x_2))$ where $\alpha_2(\cdot)$ is positive definite, we can replace item 2 in Theorem III.3 by the following item: *2') For all $x\in C$ we have $V^\circ(x;z)\le-\tilde{\alpha}(V(x))\ \forall z\in F(x)$, where $\tilde{\alpha}$ is a positive definite function.* A similar modification can be made if $Y(\cdot)$ is positive definite or if both $R(\cdot)$ and $Y(\cdot)$ are positive definite; in the latter case, we recover Theorem III.1 (for no w). On the other hand, if we were to allow α_1 and/or (id$-\lambda_2$) to be just nonnegative instead of positive definite, then Theorem III.3 would remain valid but would no longer be useful for us later because Proposition III.5 will not be possible to derive under such weaker assumptions.

We can translate the properties of the weak Lyapunov function V established in Theorem III.3 into a stability property of the system trajectories by using Theorem 23 of [11], which is a version of the Barbashin-Krasovskii-LaSalle theorem for hybrid systems. That result and the conclusion of Theorem III.3 imply that the system (14) is pre-GAS with respect to the compact set $\mathcal{A}$ defined by (8) if V does not stay constant and positive along any complete solution. All complete solutions of the hybrid system can be classified into the following three types: (i) (eventually) continuous solutions, i.e., solutions which (possibly after jumping finitely many times) only flow; (ii) (eventually) discrete solutions, i.e., solutions which (possibly after flowing for some finite time) only jump; and (iii) solutions that continue to have both flow and jumps for arbitrarily large times. Ruling out the possibility of V staying constant and positive along each of the above solution types is not convenient to do directly. The next result gives more constructive sufficient conditions that are easier to check. We state it as a general principle independent of the feedback interconnection structure of the hybrid system and of the specific function V being used. Consider the hybrid system (1) without disturbances:

$$\begin{aligned}\dot{x}&\in F(x),\quad x\in C\\ x^+&\in G(x),\quad x\in D\end{aligned}\quad (20)$$

Assumption III.3: Let $V:\mathbb{R}^n\to\mathbb{R}_{\ge0}$ be a locally Lipschitz function. Let $h:\mathbb{R}^n\to\mathbb{R}^\ell$ be continuous and proper, and define the compact set $\mathcal{A}:=\{x\in\mathbb{R}^n: h(x)=0\}\subset\mathbb{R}^n$. Suppose that V satisfies the following:

1) For all $x\in\mathbb{R}^n$ we have $\psi_1(|h(x)|)\le V(x)\le\psi_2(|h(x)|)$ for some $\psi_1,\psi_2\in\mathcal{K}_\infty$.
2) For all $x\in C$ we have $V^\circ(x;z)\le 0\ \forall z\in F(x)$.
3) For all $x\in D$ we have $V(z)\le V(x)\ \forall z\in G(x)$.

Theorem III.4: Consider the hybrid system (20). Suppose that Assumption III.3 holds, and that:

1) There are no complete, purely continuous solutions that keep V equal to a nonzero constant.
2) There are no complete, purely discrete solutions that keep V equal to a nonzero constant.
3) For each point $\xi\in(C\cap D)\setminus\mathcal{A}$ one of the following holds:
   a) $V^\circ(\xi;z)<0\ \forall z\in F(\xi)$.
   b) $V(z)<V(\xi)\ \forall z\in G(\xi)$.
   c) $R_c^{<0}(\xi)\cap L_V(V(\xi))=\emptyset$, where $L_V(c):=\{x: V(x)=c\}$ and $R_c^{<0}(\xi)$ denotes the reachable set from ξ in strictly negative time (i.e., backward time) for the continuous system $\dot{x}\in F(x), x\in C$.
   d) For all η in $R_d(\xi)\cap L_V(V(\xi))\cap C$ we have $V^\circ(\eta;z)<0\ \ \forall z\in F(\eta)$, where $R_d(\xi)$ denotes the reachable set from ξ in forward time for the discrete system $x^+\in G(x), x\in D$.

Then, the system is pre-GAS with respect to $\mathcal{A}$.

Proof: In light of [11, Theorem 23], we just need to prove that there are no complete solutions that keep $V(x(t,j))$ equal to a nonzero constant. We have assumed there are no such complete solutions with time domain $[0,\infty)\times\{0\}$ (purely continuous) or $\{0\}\times\{0,1,2,\ldots\}$ (purely discrete). Next, we observe that there exists a solution with an eventually continuous domain that keeps V equal to a nonzero constant if and only if there exists a purely continuous solution that keeps V equal to the same constant; however, this situation is ruled out by item 1 of the theorem. Similarly, there exists a solution with an eventually discrete domain that keeps V equal to a nonzero constant if and only if there exists a purely discrete solution that keeps V equal to the same constant, but this situation is ruled out by item 2 of the theorem.

In view of the classification of all complete solutions into three types given before Theorem III.4, it remains to analyze solutions of the third type, i.e., solutions that continue to have both flow and jumps. For every such solution, there exist nonnegative integers $j<k$ and three real numbers $t_j<t_{j+1}<t_{k+1}$ such that (t_j,j), (t_{j+1},j), (t_{j+1},k), and (t_{k+1},k) belong to the domain; in other words, we have "flow followed by at least one jump followed by flow." Let $\xi:=x(t_{j+1},j)$. By construction, $\xi\in(C\cap D)\setminus\mathcal{A}$ and so one of the four items in condition 3 of the theorem holds. If item 3(c) of the theorem applies to ξ, then it is impossible for V to remain constant

between (t_j, j) and (t_{j+1}, j) (i.e., during the first flow interval) and thus we have $V(x(t_{j+1}, j)) < V(x(t_j, j))$. If item 3(a) of the theorem applies to ξ, then V must be decreasing while flowing just before the jump, and we reach the same conclusion. If item 3(b) applies to ξ, then V decreases at the jump, i.e., $V(t_{j+1}, k) < V(t_{j+1}, j)$. Finally, suppose that item 3(d) of the theorem applies to ξ. Then, even if V did not decrease through all of the jumps, it still decreases during the flow just after the last jump. In other words, item 3(d) implies that $V(t_{k+1}, k) < V(t_{j+1}, k)$. This analysis shows that V cannot equal a nonzero constant along any complete solution with "flow followed by at least one jump followed by flow," as needed. □

Remark III.4: In practice it is not always necessary to invoke all the items in condition 3 of Theorem III.4. Indeed, we will soon see that when using the weak Lyapunov function V given by (9) in the context of Theorem III.3, each point $\xi \in (C \cap D) \setminus \mathcal{A}$ satisfies one of the items 3(a–c) so item 3(d) is not needed. We will also give an alternative construction of a weak Lyapunov function for which each $\xi \in (C \cap D) \setminus \mathcal{A}$ satisfies either 3(a) or 3(b). It is clear from the proof of Theorem III.4 that when item 3(d) is not needed, the second flow interval in "flow followed by at least one jump followed by flow" is not used; instead, we only need to know that our solution contains (t_j, j), (t_{j+1}, j), and $(t_{j+1}, j+1)$ in its domain, i.e., that it has "flow followed by a jump," and the proof shows that V cannot stay constant and positive along any such solution. It is also clear from the above proof that if a point $\xi \in (C \cap D) \setminus \mathcal{A}$ satisfies 3(a) then it also satisfies 3(c); however, we stated item 3(a) separately because it may be simpler to apply it first.

We now return to the more specific setting of Theorem III.3 and demonstrate how, combining it with Theorem III.4, we can establish the desired (pre-)GAS property of our hybrid system.

Proposition III.5: Consider the hybrid system (14). Let the hypotheses of Theorem III.3 hold, let V be defined via (9), and define the set $\mathcal{A}$ by (8). Then, condition 3 of Theorem III.4 holds.

Proof: We continue to use the notation and calculations of the proof of Theorem III.3. First, note that V decreases strictly on $B \cap C$ (during flow) and on $A \cap D$ (during jumps), i.e.,

$$V^{\circ}(x; z) < 0 \quad \forall x \in B \cap C,\ \forall z \in F(x) \tag{21}$$

$$V(z) < V(x) \quad \forall x \in A \cap D,\ \forall z \in G(x). \tag{22}$$

The first of these properties is an immediate consequence of (19). To see why the second one is true, consider $x \in A \cap D$ so that $V(x) = \rho(V_2(x_2)) > V_1(x_1)$. For $z \in G(x)$ we have $V(z) = \max\{V_1(z_1), \rho(V_2(z_2))\}$. By (17) and (13), $V_1(z_1) \le \max\{V_1(x_1) - Y(x_1), \chi_1(V_2(x_2))\} < \rho(V_2(x_2)) = V(x)$. On the other hand, by (18) and (13) again, $\rho(V_2(z_2)) \le \max\{\rho \circ \lambda_2(V_2(x_2)),\ \rho \circ \chi_2(V_1(x_1))\} \le \max\{\rho \circ \lambda_2 \circ \rho^{-1}(V(x)),\ \rho \circ \chi_2(V(x))\} < V(x)$. Hence, (22) is established.

Now, consider a point $\xi \in (C \cap D) \setminus \mathcal{A}$. If $\xi \in B$ then $\xi \in B \cap C$ and by (21) we have that item 3(a) holds. If $\xi \in A$ then $\xi \in A \cap D$ and by (22) we have that item 3(b) holds. The only remaining case to consider is when $\xi \in \Gamma \setminus \mathcal{A}$. We claim that in this case item 3(c) holds. Seeking a contradiction, suppose there exists a point $\eta \in R_c^{<0}(\xi) \cap L_V(V(\xi))$. This means that there is a solution $x : [-s, 0] \to C$ of the differential inclusion $\dot{x} \in F(x)$ such that $x(-s) = \eta$ and $x(0) = \xi$ and along which $V(x(\cdot))$ remains constant. We cannot have $x(t) \in B \cap C$ for any $t \in [-s, 0)$, because then (21) applied with $x = x(t)$ would force $V(x(\cdot))$ to decrease during flow. Hence, $x(t) \in A \cup \Gamma\ \forall t \in [-s, 0)$. We thus have $V(x(t)) = \rho(V_2(x(t))$ on this interval, and $V_2(x(\cdot))$ must remain constant during flow. As for $V_1(x(\cdot))$, by (15) and (13) it decreases during flow when $x(\cdot)$ is in A sufficiently close to Γ or in Γ itself (here we are using the fact that $\xi \notin \mathcal{A}$ and so the value of V along $x(\cdot)$ is not 0). However, considering the trajectory $x(\cdot)$ in the (V_1, V_2)-plane, we see that it is impossible for it to reach $\xi \in \Gamma$ while remaining in $A \cup \Gamma$ if V_1 decreases and V_2 stays constant. The resulting contradiction proves the claim.[5] □

We have the following direct corollary of Theorem III.3, Theorem III.4, and Proposition III.5.

Corollary III.6: Consider the hybrid system (14). Let the hypotheses of Theorem III.3 hold, let V be defined via (9), and let the set $\mathcal{A}$ be defined by (8). If V does not stay constant and positive along any complete solution that is either purely continuous or purely discrete, then the system is pre-GAS with respect to $\mathcal{A}$.

Corollary III.6 tells us that only purely continuous and purely discrete solutions require further analysis. In practice, these classes of solutions are not very rich and we expect to be able to rule out either the existence of such solutions or the possibility of V staying constant and positive along them. We will see examples of such reasoning in Section V, where it will go through thanks to additional structure relating the flow and jump sets to the gain functions. What we mean by this is that in the general setting of Lyapunov-based ISS small-gain theorems considered so far, the flow and jump sets C and D are completely separate from the gain functions χ_1 and χ_2, while in the design examples treated in Sections V-C and V-D there is a close relation between them. For this reason, in the context of these examples we can reach stronger conclusions than what the general results of this section can provide. We will also be able to show there that all solutions are complete and hence the system is GAS and not just pre-GAS.

We end this section with an interesting alternative construction of a weak Lyapunov function V which, while somewhat more complicated, allows us to reach the same conclusions with a simpler proof (in contrast with the proof of Proposition III.5, there is no set Γ on which additional analysis is needed).

Proposition III.7: Let the hypotheses of Theorem III.3 hold, and let the set $\mathcal{A}$ be defined by (8). Let $\rho_1, \rho_2 \in \mathcal{K}_\infty$ be both generated via Lemma II.2 using χ_1, χ_2 such that for all $r > 0$,

$$\chi_1(r) < \rho_1(r) < \rho_2(r) \text{ and } \chi_2(r) < \rho_2^{-1}(r) < \rho_1^{-1}(r). \tag{23}$$

Let W_1 and W_2 be defined via (9) using ρ_1 and ρ_2, respectively, and let

$$V(x) := W_1(x) + W_2(x) = \max\{V_1(x_1), \rho_1(V_2(x_2))\} + \max\{V_1(x_1), \rho_2(V_2(x_2))\}. \tag{24}$$

[5] Alternatively, we could finish the proof by showing that for $\xi \in \Gamma \setminus \mathcal{A}$ item 3(d) holds. Indeed, if $\eta \in R_d(\xi) \cap L_V(V(\xi)) \cap C$ then along the jump(s) that take x from ξ to η we must have that V_2 becomes strictly less than $V_2(\xi)$ by virtue of (18) while V_1 remains constant, but this implies that $\eta \in B \cap C$ and hence (21) establishes item 3(d).

Then, there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, positive definite functions $\sigma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ with $\sigma_i(s) < s$ $\forall s > 0$, and positive semi-definite functions $S_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$ such that:

1) Item 1 of Theorem III.1 holds.
2) $V^\circ(x; z) \leq \max\{-\alpha_1(W_1(x)), -S_1(x_2)\} + \max\{-\alpha_1(W_2(x)), -S_2(x_2)\}$ $\forall x \in C, \forall z \in F(x)$.
3) $V(z) \leq \max\{W_1(x) - Y_1(x_1), \sigma_1(W_1(x))\} + \max\{W_2(x) - Y_2(x_1), \sigma_2(W_2(x))\}$ $\forall x \in D, \forall z \in G(x)$.
4) For each $\xi \in (C \cap D) \setminus \mathcal{A}$ we have that either item 3(a) or item 3(b) of Theorem III.4 holds.

Proof (sketch): The proofs of items 1–3 are straightforward and we omit them. To prove item 4, let A_i, B_i for $i = 1, 2$ be the sets defined as $A_i := \{x : V_1(x_1) < \rho_i(V_2(x_2))\}$ and $B_i := \{x : V_1(x_1) > \rho_i(V_2(x_2))\}$, respectively. Using the calculations from the proof of Proposition III.5 which led us to (21) and (22), it is not difficult to check that V decreases strictly on $B_1 \cap C$ during flow and it also decreases strictly on $A_2 \cap D$ during jumps. Since the sets A_2 and B_1 overlap and cover $C \cup D$, we have that either item 3(a) or 3(b) of Theorem III.4 holds. Indeed, $\xi \in (C \cap D) \setminus \mathcal{A}$ implies either $\xi \in B_1 \cap C$, in which case item 3(a) holds, or $\xi \in A_2 \cap D$, in which case item 3(b) holds. □

Remark III.5: We note that (23) can always be achieved via Lemma II.2. Indeed, we can first pick $\rho_1(\cdot)$ that satisfies $\chi_1(r) < \rho_1(r) < \chi_2^{-1}(r)$ via Lemma II.2 and then apply Lemma II.2 again to construct $\rho_2(\cdot)$ to satisfy $\rho_1(r) < \rho_2(r) < \chi_2^{-1}(r)$ for all $r > 0$.

It follows that Corollary III.6 remains valid if V is defined via (24) in place of (9).

IV. Average Dwell Time and Reverse Average Dwell Time Conditions

In general, we cannot expect a hybrid system of interest to satisfy the assumptions of Theorem III.1. For instance, the Lyapunov function may not decrease either along flow or along jumps. Sometimes it is possible to use the construction in Theorem III.3 together with a LaSalle theorem for hybrid systems, as explained above, to conclude asymptotic stability of the hybrid system. When this is not possible, we can try to modify the hybrid system by augmenting it with a clock that restricts the set of all trajectories in such a way that conditions of Theorem III.1 are satisfied. Such constructions also require a modification of the Lyapunov function and we present two such cases next. We note that these results are of interest in their own right and their various versions have been used, e.g., in [13], [34]. We have not seen, however, the general constructions that we present here; preliminary results can be found in our earlier conference papers [24], [35].

Consider the following system:

$$\dot{x} \in F(x, z, w), \quad (x, z, w) \in \widetilde{C} \tag{25}$$

$$x^+ \in G(x, z, w), \quad (x, z, w) \in \widetilde{D} \tag{26}$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^k$. Here x may correspond to either x_1 or x_2 for subsystems in a feedback connection considered in the previous section, and z is the state of the other subsystem. There are two interesting cases which we consider next. The first one is when the Lyapunov function decreases strictly along flows but does not decrease (or potentially increases) along jumps. The second case covers the situation when the Lyapunov function decreases strictly along jumps but does not decrease (or potentially increases) along flows; this situation arises in networked control systems considered in [34] and we revisit it in the next section.

Remark IV.1: The Lyapunov conditions that we use in this section are restrictive because of the exponential decay and/or growth assumptions on the Lyapunov function. These stronger conditions allow us to state global results with suitable dwell time clocks and they are appropriate for our illustrative examples. Such conditions can be relaxed to include non-exponential decay and/or growth of the Lyapunov function but then the conclusions would be semi-global practical in the dwell time parameters; see for instance Remark 3 in [24].

A. Decreasing Flows, Non-Decreasing Jumps

Consider the system (25), (26). The starting point in our analysis is the following assumption, where we suppose that we have found a Lyapunov function $W(\cdot)$ for one of the subsystems which satisfies an appropriate decrease condition along flow (25) but potentially increases along jumps (26). We show how to augment the system with an average dwell time (ADT) clock that restricts the set of all trajectories of the original system so that an appropriate Lyapunov function can be constructed from $W(\cdot)$ that satisfies suitable decrease conditions along both flows and jumps of the augmented system. In particular, the constructed Lyapunov function satisfies all assumptions needed in Theorem III.1. We can think of $U(\cdot)$ in the assumption as the Lyapunov function for the other subsystem.

Assumption IV.1: There exist class $\mathcal{K}_\infty$ functions $\tilde{\psi}_1, \tilde{\psi}_2$, nondecreasing functions[6] $\tilde{\chi}_c, \tilde{\chi}_d, \tilde{\gamma}_c, \tilde{\gamma}_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a continuous proper function $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^\ell$ for some ℓ, a locally Lipschitz function $U : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$, a locally Lipschitz function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and constants $c > 0$, $d \geq 0$ such that:

1) $\tilde{\psi}_1(|\tilde{h}(x)|) \leq W(x) \leq \tilde{\psi}_2(|\tilde{h}(x)|)$ $\forall x \in \mathbb{R}^n$.
2) $W(x) \geq \max\{\tilde{\chi}_c(U(z)), \tilde{\gamma}_c(|w|)\} \Rightarrow W^\circ(x; y) \leq -cW(x)$ $\forall (x, z, w) \in \widetilde{C}, \forall y \in F(x, z, w)$.
3) $W(y) \leq \max\{e^d W(x), \tilde{\chi}_d(U(z)), \tilde{\gamma}_d(|w|)\}$ $\forall (x, z, w) \in \widetilde{D}, \forall y \in G(x, z, w)$.

We embed the original hybrid system (25), (26) into a bigger system augmented with the following ADT clock, for some $\delta > 0$, $N_0 \geq 1$:

$$\dot{\tau} \in [0, \delta], \quad \tau \in [0, N_0]; \quad \tau^+ = \tau - 1, \quad \tau \in [1, N_0]. \tag{27}$$

This models exactly the ADT constraint [14]

$$j - i \leq \delta(t - s) + N_0 \tag{28}$$

where $1/\delta$ is the ADT. This is to say that a hybrid time domain satisfies (28) if and only if it is the domain of some solution

[6] Note that we allow these functions to be identically equal to zero.

to the hybrid system (27); see the appendix of [5] for a proof of this fact (see also [31] for a similar construction). A more familiar special case is just the dwell time (DT) condition which is obtained when $N_0 = 1$. In this case, consecutive jumps cannot be separated by less than $1/\delta$ units of time.

Combining the hybrid system (25), (26) with the clock (27), we arrive at the following hybrid system with state (x, τ) and flow and jump sets $C := \widetilde{C} \times [0, N_0]$ and $D := \widetilde{D} \times [1, N_0]$:

$$\begin{array}{lll} \dot{x} \in F(x,z,w), & \dot{\tau} \in [0,\delta], & (x,z,\tau,w) \in C \\ x^+ \in G(x,z,w), & \tau^+ = \tau - 1, & (x,z,\tau,w) \in D. \end{array} \tag{29}$$

The clock restricts the set of trajectories to only those that satisfy the ADT constraint (28) and we can state the following result.

Proposition IV.1: Consider the hybrid system (29). Suppose that Assumption IV.1 holds with $c/\delta > d$. Let $V(x,\tau) := e^{L\tau}W(x)$, where $L \in (d, c/\delta)$. Then:

1) For all $(x,\tau) \in \mathbb{R}^n \times [0, N_0]$ we have $\psi_1(|h(x,\tau)|) \leq V(x,\tau) \leq \psi_2(|h(x,\tau)|)$, where $\psi_1 := \tilde{\psi}_1$, $\psi_2 := e^{LN_0}\tilde{\psi}_2$, and $h(x,\tau) := \tilde{h}(x)$.
2) For all $(x,z,w,\tau) \in C$ we have $V(x,\tau) \geq \max\{\chi_c(U(z)), \gamma_c(|w|)\} \Rightarrow V^\circ((x,\tau);\ (y_1,y_2)) \leq -\alpha(V(x,\tau))\ \forall y_1 \in F(x,z,w),\ \forall y_2 \in [0,\delta]$, where $\chi_c := e^{LN_0}\tilde{\chi}_c$, $\gamma_c := e^{LN_0}\tilde{\gamma}_c$, and $\alpha(r) := (c - L\delta)r$.
3) For all $(x,z,w,\tau) \in D$ we have $V(y,\tau-1) \leq \max\{\lambda(V(x,\tau)), \chi_d(U(z)), \gamma_d(|w|)\}\ \forall y \in G(x,z,w)$, where $\lambda(r) := e^{-L+d}r$, $\chi_d := e^{L(N_0-1)}\tilde{\chi}_d$, and $\gamma_d := e^{L(N_0-1)}\tilde{\gamma}_d$.

Proof: First, seeing that item 1 holds is straightforward. To prove item 2, note that $V(x,\tau) \geq e^{LN_0}\max\{\tilde{\chi}_c(U(z)), \tilde{\gamma}_c(|w|)\}$ implies $W(x) \geq \max\{\tilde{\chi}_c(U(z)), \tilde{\gamma}_c(|w|)\}$ from which it follows by item 2 of Assumption IV.1 that $W^\circ(x;y) \leq -cW(x)\ \forall y \in F(x,z,w)$, from which we have that $V^\circ((x,\tau);(y_1,y_2)) = L\dot{\tau}e^{L\tau}W(x) + e^{L\tau}W^\circ(x;y_1) \leq L\delta e^{L\tau}W(x) - ce^{L\tau}W(x) = -(c-L\delta)V(x,\tau) \forall y_1 \in F(x,z,w)$, $\forall y_2 \in [0,\delta]$. To prove item 3, we use item 3 of Assumption IV.1 to write $V(y,\tau-1) = e^{L(\tau-1)}W(g(x,z,w)) \leq e^{L(\tau-1)} \max\{e^d W(x),\ \tilde{\chi}_d(U(z)),\ \tilde{\gamma}_d(|w|)\} = \max\{e^{-L+d}V(x,\tau), e^{L(N_0-1)}\tilde{\chi}_d(U(z)), e^{L(N_0-1)}\tilde{\gamma}_d(|w|)\}\ \forall y \in G(x,z,w)$. □

Remark IV.2: We see that the h function does not involve the clock variables, hence we need it even if we start with $\tilde{h}(x) = x$. The same is true for the result in the next subsection.

Remark IV.3: We made a distinction between the functions $\tilde{\chi}_c$ and $\tilde{\chi}_d$ in Assumption IV.1 as well as functions $\tilde{\gamma}_c$ and $\tilde{\gamma}_d$ although no such distinction was made in conditions used in Theorem III.1. The reason is that by making this distinction, we can obtain less conservative gains using the construction in Proposition IV.1. We do the same in Assumption IV.2 in the next subsection.

Next, we state two time-domain implications of Proposition IV.1. The first corollary provides a conclusion for trajectories of the augmented system with clock (29).

Corollary IV.2: Let all conditions of Proposition IV.1 hold. Then, there exist $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}_\infty$ such that all solutions of the system (29) satisfy

$$\begin{aligned} &\left|\tilde{h}\left(x(t,j)\right)\right| \\ &\quad \leq \max\left\{\beta\left(\left|\tilde{h}\left(x(0,0)\right)\right|, t+j\right), \kappa\left(\|(U(z),w)\|_{(t,j)}\right)\right\} \end{aligned} \tag{30}$$

for all $(t,j) \in \operatorname{dom} x$.

Corollary IV.2 implies that the system (29) is pre-ISS with respect to the input $(U(z), w)$ and the compact set $\mathcal{A} := \{(x,\tau) : \tilde{h}(x) = 0, \tau \in [0, N_0]\}$. The second corollary translates this result into a property of a subset of solutions of the original system without clocks (25), (26).

Corollary IV.3: Let all conditions of Proposition IV.1 hold. Then, (30) holds for all solutions of the system (25), (26) that satisfy the ADT constraint (28).

Corollary IV.3 implies that all solutions of the original system (25), (26) for which the ADT hybrid time domain constraint (28) holds satisfy a pre-ISS bound with respect to the input $(U(z), w)$ and the compact set $\tilde{\mathcal{A}} := \{x : \tilde{h}(x) = 0\}$ (see also Remark II.1).

B. Decreasing Jumps, Non-Decreasing Flows

In this subsection we cover the situation where we have found a Lyapunov function $W(\cdot)$ for one of the subsystems which is not decreasing (or potentially increasing) along the flow (25) but decreases along jumps (26). We show how to augment the system with a reverse average dwell time (RADT) clock that restricts the set of all trajectories of the original system. We construct an appropriate Lyapunov function from $W(\cdot)$ and show that it satisfies decrease conditions along both flows and jumps of the augmented system. In particular, the constructed Lyapunov function satisfies all assumptions needed in Theorem III.1.

Assumption IV.2: There exist class $\mathcal{K}_\infty$ functions $\tilde{\psi}_1, \tilde{\psi}_2$, nondecreasing functions[7] $\tilde{\chi}_c, \tilde{\chi}_d, \tilde{\gamma}_c, \tilde{\gamma}_d$, a continuous proper function $\tilde{h} : \mathbb{R}^n \to \mathbb{R}^\ell$ for some ℓ, a locally Lipschitz function $U : \mathbb{R}^k \to \mathbb{R}_{\geq 0}$, a locally Lipschitz function $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and constants $c \geq 0$, $d > 0$ such that:

1) $\tilde{\psi}_1(|\tilde{h}(x)|) \leq W(x) \leq \tilde{\psi}_2(|\tilde{h}(x)|)\ \forall x \in \mathbb{R}^n$.
2) $W(x) \geq \max\{\tilde{\chi}_c(U(z)), \tilde{\gamma}_c(|w|)\} \Rightarrow W^\circ(x;y) \leq cW(x)\ \forall (x,z,w) \in \widetilde{C}, \forall y \in F(x,z,w)$.
3) $W(y) \leq \max\{e^{-d}W(x), \tilde{\chi}_d(U(z)), \tilde{\gamma}_d(|w|)\}$ $\forall (x,z,w) \in \widetilde{D}, \forall y \in G(x,z,w)$.

We embed the original hybrid system (25), (26) into a bigger system augmented with the following RADT clock, for some $\delta > 0$, $N_0 \geq 1$:

$$\dot{\tau} = 1, \quad \tau \in [0, N_0\delta]; \quad \tau^+ = \max\{0, \tau - \delta\}, \quad \tau \in [0, N_0\delta]. \tag{31}$$

This models exactly the RADT constraint [13]

$$t - s \leq \delta(j - i) + N_0\delta \tag{32}$$

[7]We allow these functions to be identically zero.

or, equivalently, $j - i \geq (t-s)/\delta - N_0$, where δ is the reverse ADT. This is to say that a hybrid time domain satisfies (32) if and only if it is the domain of some solution to the hybrid system (31); see the Appendix of [5] for a proof of this fact. A more familiar special case is the reverse DT when $N_0 = 1$, which enforces jumps at least every δ units of time.

Remark IV.4: It would be more consistent with the previous case (but equivalent in terms of hybrid time domains this generates) to write $\tau^+ \in [\max\{0, \tau - \delta\}, \tau]$ in (31). However, we want to work with the simplest clock that gives the stated equivalence.

Combining the system (25), (26) with the clock (31), we arrive at the following hybrid system with state (x, τ) and flow and jump sets $C := \widetilde{C} \times [0, N_0\delta]$ and $D := \widetilde{D} \times [0, N_0\delta]$:

$$\begin{aligned} \dot{x} &\in F(x,z,w), & \dot{\tau} &= 1, & (x,z,\tau,w) &\in C \\ x^+ &\in G(x,z,w), & \tau^+ &= \max\{0, \tau-\delta\}, & (x,z,\tau,w) &\in D \end{aligned} \tag{33}$$

We can state the following result for this augmented system.

Proposition IV.4: Consider the hybrid system (33). Suppose that Assumption IV.2 holds with $d > \delta c$. Let $V(x,\tau) := e^{-L\tau} W(x)$, where $L \in (c, d/\delta)$. Then:

1) For all $(x,\tau) \in \mathbb{R}^n \times [0, N_0]$ we have $\psi_1(|h(x,\tau)|) \leq V(x,\tau) \leq \psi_2(|h(x,\tau)|)$, where $\psi_1 := e^{-LN_0\delta}\tilde{\psi}_1$, $\psi_2 = \tilde{\psi}_2$, and $h(x,\tau) := \tilde{h}(x)$.
2) For all $(x,z,w,\tau) \in C$ we have $V(x,\tau) \geq \max\{\chi_c(U(z)), \gamma_c(|w|)\} \Rightarrow V^\circ((x,\tau);(y,1)) \leq -\alpha(V(x,\tau))\ \forall y \in F(x,z,w)$, where $\chi_c := \tilde{\chi}_c$, $\gamma_c := \tilde{\gamma}_c$, and $\alpha(r) := (L-c)r$.
3) For all $(x,z,w,\tau) \in D$ we have $V(y, \max\{0, \tau-\delta\}) \leq \max\{\lambda(V(x,\tau)), \chi_d(U(z)),\ \gamma_d(|w|)\}\ \forall y \in G(x,z,w)$, where $\lambda(r) := e^{L\delta - d}r$, $\chi_d := \tilde{\chi}_d$, and $\gamma_d := \tilde{\gamma}_d$.

Proof: The proof is similar to that of Proposition IV.1. It is easy to see that item 1 holds. To show item 2, note that $V(x,\tau) \geq \max\{\tilde{\chi}_c(U(z)), \tilde{\gamma}_c(|w|)\}$ implies $W(x) \geq \max\{\tilde{\chi}_c(U(z)), \tilde{\gamma}_c(|w|)\}$ from which it follows that $W^\circ(x;y) \leq cW(x)\ \forall y \in F(x,z,w)$, hence $V^\circ((x,\tau);(y,1)) = -L\dot{\tau}V(x,\tau) + e^{-L\tau}W^\circ(x;y) \leq -LV(x,\tau) + cV(x,\tau) = -(L-c)V(x,\tau)\ \ \forall y \in F(x,z,w)$. As for item 3, $V(y, \max\{0,\tau-\delta\}) = e^{-L\max\{0,\tau-\delta\}}W(y) \leq e^{-L\max\{0,\tau-\delta\}} \max\{e^{-d}W(x),\ \tilde{\chi}_d(U(z)),\ \tilde{\gamma}_d(|w|)\} \leq \max\{e^{-L\max\{0,\tau-\delta\}-d} e^{L\tau}V(x,\tau), \tilde{\chi}_d(U(z)), \tilde{\gamma}_d(|w|)\} \leq \max\{e^{L\delta-d}V(x,\tau), \tilde{\chi}_d(U(z)),\ \tilde{\gamma}_d(|w|)\}\ \ \forall y \in G(x,z,w)$, where we used the identity $\tau - \max\{0,\tau-\delta\} \leq \delta$. □

Similarly to the previous subsection, we state two time-domain implications of Proposition IV.4. The first one provides a conclusion for trajectories of the augmented system (33).

Corollary IV.5: Let all conditions of Proposition IV.4 hold. Then, there exist $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}_\infty$ such that all solutions of the system (33) satisfy

$$\left|\tilde{h}\left(x(t,j)\right)\right| \leq \max\left\{\beta\left(\left|\tilde{h}\left(x(0,0)\right)\right|, t+j\right), \kappa\left(\|\left(U(z),w\right)\|_{(t,j)}\right)\right\} \tag{34}$$

for all $(t,j) \in \operatorname{dom} x$.

Corollary IV.5 implies that the system (33) is pre-ISS with respect to the input $(U(z), w)$ and the compact set $\mathcal{A} := \{(x,\tau) : \tilde{h}(x) = 0, \tau \in [0, N_0]\}$. The second corollary translates this result into a property of a subset of solutions of the original system without clocks (25), (26).

Corollary IV.6: Let all conditions of Proposition IV.4 hold. Then, (34) holds for all solutions of the system (25), (26) that satisfy the RADT constraint (32).

Corollary IV.6 implies that all solutions of the original system (25), (26) for which the RADT hybrid time domain constraint (32) holds satisfy a pre-ISS bound with respect to the input $(U(z), w)$ and the compact set $\tilde{\mathcal{A}} := \{x : \tilde{h}(x) = 0\}$.

C. Interconnection

We present here a generic case of an interconnection of two hybrid systems that possibly need ADT and RADT clocks to satisfy Assumption III.1. As a result of using clocks, the gains of the two subsystems and the small-gain condition need to be modified. The result that we present here is abstract but it is general and it covers all our examples, as illustrated in Section V. Consider the following system:

$$\begin{aligned} &\dot{x}_1 \in F_1(x,w), \dot{\tau}_1 \in H_1(\tau_1), & &\dot{x}_2 \in F_2(x,w), \\ &\dot{\tau}_2 \in H_2(\tau_2), & &(x_1,x_2,\tau_1,\tau_2,w) \in C \\ &x_1^+ \in G_1(x,w), \tau_1^+ \in L_1(\tau_1), & &x_2^+ \in G_2(x,w), \\ &\tau_2^+ \in L_2(\tau_2), & &(x_1,x_2,\tau_1,\tau_2,w) \in D \end{aligned} \tag{35}$$

where $x_i \in \mathbb{R}^{n_i}$ and $\tau_i \in \mathbb{R}^{p_i}$, $i = 1, 2$. We can think of x_1, x_2 as the states of subsystems that we are interested in and of τ_1, τ_2 as ADT and/or RADT clock variables that are needed to satisfy the conditions of Assumption III.1. When a clock is not needed for that subsystem, we will set $p_i = 0$ and $a_i = 0$ in what follows.

Assumption IV.3: For $i, j \in \{1,2\}$, $i \neq j$ there exist C^1 functions $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ and $V_i : \mathbb{R}^{n_i + p_i} \to \mathbb{R}_{\geq 0}$ such that the following hold:

1) For all $(x_1, x_2, \tau_1, \tau_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ we have $W_i(x_i) \leq e^{a_i} V_i(x_i, \tau_i)$, $i = 1, 2$ for some $a_1, a_2 \geq 0$.
2) For some functions $\chi_{c,i}, \gamma_{c,i} \in \mathcal{K}_\infty$ and positive definite functions α_i we have

$$\begin{aligned} &V_i(x_i,\tau_i) \geq \max\left\{\chi_{c,1}\left(W_j(x_j)\right), \gamma_{c,1}\left(|w|\right)\right\} \Rightarrow \\ &V_i^\circ\left((x_i,\tau_i);(y_1,y_2)\right) \leq -\alpha_i\left(V_i(x_i,\tau_i)\right) \\ &\forall (x_1,x_2,\tau_1,\tau_2,w) \in C, \forall y_1 \in F_i(x,w), \forall y_2 \in H_i(\tau_i). \end{aligned}$$

For some functions $\chi_{d,i}, \gamma_{d,i} \in \mathcal{K}_\infty$ and positive definite functions λ_i with $\lambda_i(s) < s\ \ \forall s > 0$ we have $V_i(y_1, y_2) \leq \max\{\lambda_i(V_i(x_i,\tau_i)),\ \chi_{1,d}(W_j(x_j)),\ \gamma_{d,1}(|w|)\}\ \ \forall (x_i, x_j, \tau_i, \tau_j, w) \in D,\ \ \forall y_1 \in G_i(x,w),\ \forall y_2 \in L_i(\tau_i)$.
3) With $\chi_1(r) := \max\{\chi_{c,1}(e^{a_2}r), \chi_{d,1}(e^{a_2}r)\}$ and $\chi_2(r) := \max\{\chi_{c,2}(e^{a_1}r), \chi_{d,2}(e^{a_1}r)\}$, the following small-gain condition holds: $\chi_1 \circ \chi_2(s) < s\ \forall s > 0$.

We note that the conditions in Assumption IV.3 match the situation in the previous two subsections where we can think of $W_i(\cdot)$ as the original functions and $V_i(\cdot,\cdot)$ as the modified Lyapunov functions that satisfy Assumption III.1; we did not state item 1 of Assumption III.1 as it is automatically satisfied under the Lyapunov function transformations that we use. Then, the following result is easily shown (we omit the proof):

Proposition IV.7: Consider the hybrid system (35). Suppose that Assumption IV.3 holds. Then, items 2, 3 and 4 from Assumption III.1 hold with χ_1, χ_2 as defined in item 4 of Assumption IV.3, $\gamma_1 := \max\{\gamma_{c,1}, \gamma_{d,1}\}$, and $\gamma_2 := \max\{\gamma_{c,2}, \gamma_{d,2}\}$.

V. Applications of Main Results

In this section we present several examples to which we apply our main results. The first one considers a "natural" decomposition of hybrid systems into its flow part and jump part. In this example we show how both constructions in Theorems III.1 and III.3 can be used under certain conditions; the approach based on Theorem III.1 is interesting since we need to use both ADT and RADT clocks with arbitrarily short ADT and arbitrarily long RADT. In our second example, we revisit networked control systems considered in [34]. In this case, we need to use simpler DT and reverse DT clocks which must be adjusted appropriately to achieve stability. In the third example, we revisit the problem of event-triggered sampling considered in [39]. We provide an alternative model and stability proof to [39] that uses our Theorem III.3. In our last example, we consider a class of linear systems with quantized control. This example provides another application of Theorem III.3 and an alternative analysis method to those used in [1], [23].

A. Natural Decomposition

The following "natural decomposition" of the hybrid system (without disturbances)

$$\begin{array}{lll} \dot{x}_1 \in F(x_1, x_2), & \dot{x}_2 = 0, & (x_1, x_2) \in \widetilde{C} \\ x_1^+ = x_1, & x_2^+ \in G(x_1, x_2), & (x_1, x_2) \in \widetilde{D} \end{array} \quad (36)$$

is often of interest, where we can call x_1 and x_2 the continuous and discrete state variables, respectively. Note that x_1 does not change during the jumps and x_2 does not change during the flow. This class of systems is useful for illustrating both Theorems III.1 and III.3. We first use directly Theorem III.3 to construct a weak Lyapunov function which can be used under appropriate conditions to conclude asymptotic stability via Theorem III.4. Then, we augment the system with ADT and RADT clocks and use Propositions IV.1, IV.4 and IV.7 to show that Theorem III.1 can be used to construct a strong Lyapunov function under appropriate conditions.

The following proposition is a direct consequence of Theorem III.3.

Proposition V.1: Suppose that there exist C^1 functions V_1 and V_2 and functions $\psi_{ij}, h_i, \chi_i, \alpha_1, \lambda_2$ such that (15), (18) and items 1 and 4 of Assumption III.2 hold with $\widetilde{C}, \widetilde{D}$ in place of C, D (all functions are from the same classes as in Assumption III.2). Let V be defined via (9). Then, there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that the item 1 of Theorem III.3 holds and we have $V^\circ((x_1, x_2); (z_1, 0)) \leq 0\ \forall (x_1, x_2) \in \widetilde{C}$, $\forall z_1 \in F(x_1, x_2)$ and $V(x_1, z_2) \leq V(x_1, x_2)\ \forall (x_1, x_2) \in \widetilde{D}$, $\forall z_2 \in G(x_1, x_2)$.

Proof: It is immediate that for any C^1 positive definite V_2 we have $V_2^\circ(x_2; 0) = \langle \nabla V_2, 0 \rangle = 0\ \forall (x_1, x_2) \in \widetilde{C}$ and, hence, (16) holds with $R(\cdot) \equiv 0$. Similarly, for any positive definite V_1 we have $V_1(x_1^+) = V_1(x_1)\ \forall (x_1, x_2) \in \widetilde{D}$ and, hence, (17) holds with $Y(\cdot) \equiv 0$. Hence, all conditions of Assumption III.2 hold and the conclusion follows from Theorem III.3. □

The next corollary shows how we can use this weak Lyapunov function construction with our LaSalle theorem (Theorem III.4) to conclude stability of the system (36); we specialize our conditions to investigate pre-GAS of the origin $\{x = (x_1, x_2) = (0, 0)\}$ for simplicity.

Corollary V.2: Suppose that all conditions of Proposition V.1 hold with $h_i(x_i) = x_i$ for $i = 1, 2$ and let V be as in Proposition V.1. If V does not stay constant and positive along any complete solution that is either purely continuous or purely discrete, then (36) is pre-GAS.

Now we augment the system (36) with ADT and RADT clocks:

$$\begin{array}{l} \dot{\tau}_1 \in [0, \delta_1], \tau_1 \in [0, N_1];\ \tau_1^+ = \tau_1 - 1, \tau_1 \in [1, N_1] \\ \dot{\tau}_2 = 1, \tau_2 \in [0, N_2\delta_2];\ \tau_2^+ = \max\{0, \tau_2 - \delta_2\}, \tau_2 \in [0, N_2\delta_2] \end{array} \quad (37)$$

Defining $C := \widetilde{C} \times [0, N_1] \times [0, N_2\delta_2]$ and $D = \widetilde{D} \times [1, N_1] \times [0, N_2\delta_2]$, we consider the system (36), (37) as a feedback connection of (x_1, τ_1) and (x_2, τ_2) subsystems.

Assumption V.1: The following hold:

1) Assumption IV.1 holds for x_1-subsystem when x_2 is regarded as its input, with the functions $W, U, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\chi}_c, \tilde{\chi}_d, \tilde{h}, \tilde{\gamma}_c, \tilde{\gamma}_d$ replaced respectively by $W_1, W_2, \tilde{\psi}_{11}, \tilde{\psi}_{21}, \tilde{\chi}_{c1}, \tilde{\chi}_{d1} \equiv 0, \tilde{h}_1, \tilde{\gamma}_{c1} \equiv 0, \tilde{\gamma}_{d1} \equiv 0$ and with $d = 0$ and some $c = c_1 > 0$.
2) Assumption IV.2 holds for x_2-subsystem when x_1 is regarded as its input, with the functions $W, U, \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\chi}_c, \tilde{\chi}_d, \tilde{h}, \tilde{\gamma}_c, \tilde{\gamma}_d$ replaced respectively by $W_2, W_1, \tilde{\psi}_{12}, \tilde{\psi}_{22}, \tilde{\chi}_{c2} \equiv 0, \tilde{\chi}_{d2}, \tilde{h}_2, \tilde{\gamma}_{c2} \equiv 0, \tilde{\gamma}_{d2} \equiv 0$ and with $c = 0$ and some $d = d_2 > 0$.
3) There exist numbers $\delta_1, \delta_2 > 0$, $L_1 \in (0, c_1/\delta_1)$, $L_2 \in (0, d_2/\delta_2)$, and $N_1, N_2 \geq 1$ such that we have the small-gain condition $\chi_1 \circ \chi_2(s) < s\ \forall s > 0$, where $\chi_1(s) := e^{L_1 N_1} \tilde{\chi}_{c1}(e^{L_2 N_2 \delta_2} s)$, $\chi_2(s) := \tilde{\chi}_{d2}(s)$.

Remark V.1: Items 1 and 2 in Assumption V.1 can always be satisfied with $\tilde{\chi}_{d1} \equiv 0$, $d = 0$ and $\tilde{\chi}_{c2} \equiv 0$, $c = 0$, respectively, since x_1 does not change during jumps and x_2 does not change during flow. Also note that for linearly bounded gains $\tilde{\chi}_{c1}, \tilde{\chi}_{d2}$ satisfying $\tilde{\chi}_{c1} \circ \tilde{\chi}_{d2}(s) < s\ \forall s > 0$, the small-gain condition in item 3 can always be satisfied by using *arbitrary* $\delta_1, \delta_2, N_1, N_2$ and then choosing L_1, L_2 sufficiently small. This means that the ADT can be arbitrarily small and the RADT can be arbitrarily large.

Proposition V.3: Suppose that Assumption V.1 holds. Then, all conditions of Assumption III.1 hold for the system (36), (37) with[8] $V_1(x_1, \tau_1) := e^{L_1 \tau_1} W_1(x_1)$ and $V_2(x_2, \tau_2) :=$

[8]This holds modulo a change of notation: the vector (x_i, τ_i) here plays the role of x_i in Assumption III.1, for $i = 1, 2$.

$e^{-L_2\tau_2}W_2(x_2)$ and, hence, the conclusion of Theorem III.1 holds.

Proof: Since Assumptions IV.1 and IV.2 hold for subsystems x_1 and x_2 respectively (items 1 and 2 of Assumption V.1), we have from conclusion 1 in Propositions IV.1 and IV.4 that item 1 of Assumption III.1 holds. Using Propositions IV.1 and IV.4 and our small-gain condition in item 3 of Assumption V.1 we conclude that Assumption IV.3 holds with $a_1 = 0$ and $a_2 = L_2N_2\delta_2$. Hence, from Proposition IV.7 we conclude that items 2, 3 and 4 of Assumption III.1 hold. □

The next two results consider the special case when we are interested in stability of the origin $\{x = (x_1, x_2) = (0, 0)\}$ and they are direct consequences of Proposition V.3 and Corollary III.2.

Corollary V.4: Suppose that Assumption V.1 holds with $\tilde{h}_i(x_1) = x_i$, $i = 1, 2$. Then, the system (36), (37) is pre-GAS with respect to the set $\mathcal{A} := \{(x_1, x_2) = (0, 0), \tau_1 \in [0, N_1], \tau_2 \in [0, \delta_2 N_2]\}$.

Corollary V.5: Suppose that Assumption V.1 holds with $\tilde{h}_i(x_1) = x_i$, $i = 1, 2$. Then, all trajectories of the system (36) whose hybrid time domains satisfy (28) with $(N_0, \delta) = (N_1, \delta_1)$ and (32) with $(N_0, \delta) = (N_2, \delta_2)$ satisfy a pre-GAS stability bound (2) with respect to the set $\tilde{\mathcal{A}} = \{(x_1, x_2) = (0, 0)\}$.

Remark V.2: Note a subtle difference between Corollaries V.2 and V.5. Both results are stated for the proper indicator function $\omega(x) = |(x_1, x_2)|$. Corollary V.2 concludes a bound of the form (2) for all trajectories if the system (36) does not have purely continuous or purely discrete complete trajectories that keep the constructed V equal to a non-zero constant. On the other hand, Corollary V.5 proves a bound of the form (2) for those trajectories of the system (36) that satisfy the indicated ADT and RADT conditions. The second conclusion is weaker but there are no trajectories to check separately.

B. Networked Control Systems

Motivated by results in [6], [34] we consider a class of networked control systems that contain the following equations:

$$\begin{aligned} &\dot{x} = \tilde{f}_1(x, e, w), \quad \dot{e} = \tilde{f}_2(x, e, w), \quad \dot{s} = 0 \\ &x^+ = x, \quad e^+ = h(s, e), \quad s^+ = s + 1. \end{aligned} \tag{38}$$

The above system can be obtained by following an emulation-like procedure and the variable x represents the combined states of the plant and the controller, whereas the variable e represents an error that captures the mismatch between the networked and actual values of the inputs and outputs that are sent over the network. The variable s can be thought of as the variable that counts the number of transmissions. It was shown in [34] that the jump equation for e is solely described by the network protocol. To model transmission times, we use two clocks which are special cases of the earlier ADT and reverse ADT clocks. Namely, we consider a combination of DT and reverse DT clocks, as follows:

$$\begin{aligned} &\dot{\tau}_1 \in [0, 1/\underline{\varepsilon}], && \dot{\tau}_2 = 1, && (\tau_1, \tau_2) \in C \\ &\tau_1^+ = \tau_1 - 1, && \tau_2^+ = \max\{0, \tau_2 - \bar{\varepsilon}\}, && (\tau_1, \tau_2) \in D \end{aligned} \tag{39}$$

where $C := [0, 1] \times [0, \bar{\varepsilon}]$ and $D := \{1\} \times [0, \bar{\varepsilon}]$ and we assume $\underline{\varepsilon} < \bar{\varepsilon}$. This gives exactly the hybrid time domains satisfying $j - i \le (t - s)/\underline{\varepsilon} + 1$ and $j - i \ge (t - s)/\bar{\varepsilon} - 1$. This implies that the transmission times $t_k, k \in \mathbb{N}$ satisfy

$$\underline{\varepsilon} \le t_{k+1} - t_k \le \bar{\varepsilon}, \tag{40}$$

which is a condition used in [34] and references cited therein. The overall hybrid system consists of the earlier differential equations (38) and these clocks. We assume the following (cf. [34]):

Assumption V.2: There exist C^1 functions W_1, W_2 such that:

1) There exist $\gamma_1, c_1 > 0$, $\mathcal{K}_\infty$ functions $\overline{\psi}_{i1}, i = 1, 2$ and $\overline{\gamma}$ such that for all x, e, s, w we have $\overline{\psi}_{11}(|x|) \le W_1(x) \le \overline{\psi}_{21}(|x|)$ and

$$\begin{aligned} W_1(x) &\ge \max\{\gamma_1 W_2(s, e), \overline{\gamma}(|w|)\} \\ &\Rightarrow \left\langle \nabla W_1(x), \tilde{f}_1(x, e, w) \right\rangle \le -c_1 W_1(x). \end{aligned} \tag{41}$$

2) There exist $\gamma_2, d_2 > 0$, $\mathcal{K}_\infty$ functions $\overline{\psi}_{i2}, i = 1, 2$ and $\overline{\gamma}$ such that for all x, e, s, w we have $\overline{\psi}_{12}(|e|) \le W_2(s, e) \le \overline{\psi}_{22}(|e|)$,

$$W_2(s + 1, h(s, e)) \le e^{-d_2} W_2(s, e), \tag{42}$$

and $W_2(s, e) \ge \max\{\gamma_2 W_1(x), \bar{\gamma}(|w|)\} \Rightarrow \langle \nabla W_2(e), \tilde{f}_2(x, e, w)\rangle \le c_2 W_2(s, e)$.

3) The following condition holds: $\bar{\varepsilon} < \min\{(1/L_2)\ln(e^{-L_1}/(\gamma_1\gamma_2)), d_2/c_2\}$, where $L_1 \in (0, \underline{\varepsilon} c_1)$ and $L_2 \in (c_2, d_2/\bar{\varepsilon})$.

The condition (42) characterizes the so-called UGES protocols that were introduced in [34]. We only consider ISS with linear gain in (41) in order to state an explicit condition on $\bar{\varepsilon}$.

Remark V.3: Note that we can take L_2 to be as close as we want to (but larger than) c_2, and L_1 can be taken as close to 0 as we want.

Proposition V.6: Suppose that Assumption V.2 holds for the system (38). Then, all conditions of Assumption III.1 hold for the system (38), (39) with[9] $V_1(x, \tau_1) := e^{L_1\tau_1}W_1(x)$ and $V_2(s, e, \tau_2) := e^{-L_2\tau_2}W_2(s, e)$ and, hence, the conclusion of Theorem III.1 holds.

Proof: Since Assumption V.2 holds for subsystems x and (s, e) respectively, we have from conclusion 1 in Propositions IV.1 and IV.4 that item 1 of Assumption III.1 holds. Using Propositions IV.1 and IV.4 and item 3 of Assumption V.2 we conclude that Assumption IV.3 holds with $a_1 = 0$ and $a_2 = L_2\bar{\varepsilon}$. Hence, from Proposition IV.7 we conclude that items 2, 3, and 4 of Assumption III.1 hold. □

A direct consequence of Proposition V.6 is ISS of the system (38), (39). In this case, we can show ISS (and not only pre-ISS) since all solutions can be shown to be complete. Indeed, $C \cup D = \mathbb{R}^n$ and all solutions (x, e) are bounded for all essentially bounded inputs. We have:

[9] The vectors (x, τ_1) and (s, e, τ_2) here play the roles of x_1 and x_2, respectively, in Assumption III.1.

Corollary V.7: Suppose that Assumption V.2 holds for the system (38). Then the system (38), (39) is ISS with respect to the input w and the set $\mathcal{A} := \{(x, e, s, \tau_1, \tau_2) : x = 0, e = 0, s \in \mathbb{R}, \tau_1 \in [0, \underline{\varepsilon}], \tau_2 \in [0, \bar{\varepsilon}]\}$.

The above result uses a different (more conservative) Lyapunov construction than [6]; however, the result is simpler and fits directly our framework so it is appropriate for illustration purposes. The following result provides a time-domain conclusion for the trajectories of the original system:

Corollary V.8: Suppose that Assumption V.2 holds for the system (38). Then all solutions of the system (38) for which (40) holds satisfy an ISS bound with respect to the input w and the set $\tilde{\mathcal{A}} = \{(x, e, s) : x = 0, e = 0, s \in \mathbb{R}\}$.

C. Emulation With Event-Triggered Sampling

In this section, we revisit results in [39]. Consider a continuous-time plant $\dot{x} = f(x, u)$ for which a state feedback controller $u = k(x)$ was designed to globally asymptotically stabilize the closed-loop system. Suppose that we want to implement the controller in a sampled-data fashion so that we take samples of $x(\cdot)$ at times $t_k, k \in \mathbb{N}$ and let $u(t) = k(x(t_k)), t \in [t_k, t_{k+1})$. The sampling times t_k will be designed in an event-driven fashion. To this end, introduce an auxiliary variable $e(t) := x(t_k) - x(t)$ and assume that there exist C^1 functions $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ and $\psi_{ij}, \chi_1, \alpha_1 \in \mathcal{K}_\infty, i, j \in \{1, 2\}$ such that for all x and e we have

$$\psi_{11}(|x|) \le V_1(x) \le \psi_{21}(|x|),\ \psi_{12}(|e|) \le V_2(e) \le \psi_{22}(|e|) \tag{43}$$

and

$$V_1(x) \ge \chi_1(V_2(e)) \Rightarrow \langle \nabla V_1, f(x, k(x+e)) \rangle \le -\alpha_1(V_1(x)). \tag{44}$$

Let $\chi_2 \in \mathcal{K}_\infty$ be arbitrary and satisfy

$$\chi_1 \circ \chi_2(s) < s \quad \forall s > 0. \tag{45}$$

Our triggering strategy is to update the control whenever $V_2(e) \ge \chi_2(V_1(x))$, which leads to the following closed-loop hybrid system[10]:

$$\begin{array}{lll} \dot{x} = f(x, k(x+e)), & \dot{e} = -f(x, k(x+e)), & (x, e) \in C \\ x^+ = x, & e^+ = 0, & (x, e) \in D \end{array} \tag{46}$$

where $C := \{(x, e) : V_2(e) \le \chi_2(V_1(x))\}$ and $D := \{(x, e) : V_2(e) \ge \chi_2(V_1(x))\}$.

Proposition V.9: Suppose that there exist Lyapunov functions V_1, V_2, a positive definite function α_1 and functions $\psi_{ij}, \chi_i \in \mathcal{K}_\infty$, $i, j \in \{1, 2\}$ such that (43)–(45) hold. Let V be defined via (9) with x, e in place of x_1, x_2. Then, there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that for the system (46) we have: $\psi_1(|(x, e)|) \le V(x, e) \le \psi_2(|(x, e)|)\ \forall x, e$; $\langle \nabla V(x, e), (f(x, k(x+e)), -f(x, k(x+e))) \rangle \le -\alpha_1(V(x, e))\ \forall (x, e) \in C$; and $V(x, 0) \le V(x, e)\ \forall (x, e) \in D$.

[10]This hybrid system follows from the same methodology used in [39]; however, [39] uses an alternative model and proof technique to establish its results.

Proof: First, for all $(x, e) \in D$ we have $V_1(x^+) = V_1(x)$. By this and (44), the conditions (15) and (17) of Assumption III.2 hold (with $Y \equiv 0$, $x_1 := x$, and $x_2 := e$). Consider an arbitrary $\bar{\chi}_2(s) > \chi_2(s)\ \forall s > 0$ and such that $\chi_1 \circ \bar{\chi}_2(s) < s\ \forall s > 0$; such a $\bar{\chi}_2$ always exists since the inequality (45) is strict. Then, we have that for $(x, e) \in C$ the following is *vacuously true*: $V_2(e) \ge \bar{\chi}_2(V_1(x)) \Rightarrow \langle \nabla V_2(e), -f(x, k(x+e)) \rangle \le -R(e)$, where $R(\cdot)$ can be arbitrary and, in particular, we can take $R(e) = \alpha_1(|e|)$. Moreover, for all e we have $V_2(e^+) = V_2(0) = 0$. Hence, the conditions (16) and (18) of Assumption III.2 hold (with arbitrary λ_2). By construction, the small-gain condition (item 4 of Assumption III.2) holds, and the result follows from Theorem III.3. □

To apply Corollary III.6, we need to check complete solutions that are either purely continuous or purely discrete (ignoring of course the trivial solution at the origin). Here we know that after a jump we must flow, since jumps reset e to 0. Thus, the only solutions that we need to analyze are purely continuous ones. However, in view of the ISS condition (44), the definition of C, and the small-gain condition (45), purely continuous behavior is possible only when both x and e converge to 0. Hence, V cannot stay constant and positive along any such solution. Finally, all solutions are complete because the properties of V in Theorem III.3 guarantee their boundedness and we have $C \cup D = \mathbb{R}^n$ by construction. We have arrived at the following result.

Corollary V.10: The closed-loop hybrid system (46) is GAS (with respect to the origin).

D. Quantized Feedback Control

This example is in some sense more specialized than the previous ones, because we will only work with linear dynamics. On the other hand, this additional structure will permit us to explicitly construct the Lyapunov functions V_1, V_2 (which will be quadratic) and derive expressions for the gain functions χ_1, χ_2 (which will be linear gains), instead of just assuming their existence.

Consider the linear time-invariant system $\dot{x} = Ax + Bu$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and A is a non-Hurwitz matrix. We assume that this system is stabilizable, so that there exist matrices $P = P^T > 0$ and K such that

$$(A + BK)^T P + P(A + BK) \le -I. \tag{47}$$

We denote by $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ the smallest and the largest eigenvalue of a symmetric matrix, respectively. By a *quantizer* we mean a piecewise constant function $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite or countable subset of $\mathbb{R}^n$. Following [23], we assume the existence of positive numbers M (the *range* of q, which can be a finite number or ∞ depending on whether $\mathcal{Q}$ is finite or countable) and Δ (the *quantization error bound*) satisfying

$$|z| \le M \Rightarrow |q(z) - z| \le \Delta. \tag{48}$$

We assume that $q(x) = 0$ for x in some neighborhood of 0 (in order that the equilibrium at 0 be preserved under quantized control). It is well known that quantization errors in general destroy asymptotic stability, in the sense that the quantized feedback law $u = Kq(x)$ is no longer stabilizing. To overcome this problem, we will use quantized measurements of the form $q_\mu(x) := \mu q(x/\mu)$ for $\mu > 0$, as in [23]. The quantizer q_μ has range $M\mu$ and quantization error bound $\Delta\mu$. The "zoom" variable μ will be the discrete variable of the hybrid closed-loop system, initialized at some fixed value. The feedback law will be $u = Kq_\mu(x)$. We consider the following scheme for updating μ, which we refer to as the "quantization protocol":

$$\dot{\mu} = 0, \quad (x,\mu) \in C; \quad \mu^+ = \Omega\mu, \quad (x,\mu) \in D$$

where $C := \{(x,\mu) : |q_\mu(x)| \geq (\Theta + \Delta)\mu\}$, $D := \{(x,\mu) : |q_\mu(x)| \leq (\Theta + \Delta)\mu\}$, $\Omega \in (0,1)$, and Θ is a number satisfying $\Theta > \sqrt{\lambda_{\max}(P)}2\|PBK\|\Delta/\sqrt{\lambda_{\min}(P)}$. The overall closed-loop hybrid system then looks like (cf. the "natural decomposition" in Section V-A)

$$\begin{array}{lll} \dot{x} = Ax + BKq_\mu(x), & \dot{\mu} = 0, & (x,\mu) \in C \\ x^+ = x, & \mu^+ = \Omega\mu, & (x,\mu) \in D. \end{array} \tag{49}$$

The idea behind achieving asymptotic stability is to "zoom in", i.e., decrease μ to 0 in a suitable discrete fashion. To simplify the exposition, we will assume that the condition $|x| \leq M\mu$ always holds, i.e., x always remains within the range of q_μ. This is automatically true if M is infinite, and can be guaranteed by a proper initialization of μ if a bound on the initial state $x(0)$ is available. For finite M and completely unknown $x(0)$, this can be achieved by incorporating an initial "zooming-out" scheme and subsequently ensuring that the condition is never violated (see [23] for details). For a Lyapunov-based small-gain analysis of a quantization scheme that includes zoom-outs, see the recent work [40].

Lemma V.11: Consider the hybrid system (49). Let $V_1(x) := x^T Px$, with P and K from (47). Let $V_2(\mu) := \mu^2$. Pick two numbers ε_1 and ε_2 satisfying $0 < \varepsilon_1 < \varepsilon_2$ and

$$\Theta \geq \sqrt{\lambda_{\max}(P)\lambda_{\min}(P)}2\|PBK\|\Delta(1+\varepsilon_2)/\sqrt{\lambda_{\min}(P)}. \tag{50}$$

Then:

1) For all $(x,\mu) \in C$ we have

$$V_1(x) \geq \chi_1 V_2(\mu)$$
$$\Rightarrow \langle \nabla V_1(x), Ax + BKq_\mu(x)\rangle \leq -c_1 V_1(x), \tag{51}$$

$$V_2(\mu) \geq \chi_2 V_1(x) \Rightarrow \langle \nabla V_2(\mu), 0\rangle \leq -c_2 V_2 \tag{52}$$

where $\chi_1 := 4\lambda_{\max}(P)\|PBK\|^2\Delta^2(1+\varepsilon_1)^2$, $c_1 := \varepsilon_1/((1+\varepsilon_1)\lambda_{\max}(P))$, $\chi_2 := 1/(4\lambda_{\max}(P)\|PBK\|^2\Delta^2(1+\varepsilon_2)^2)$, and $c_2 > 0$ is arbitrary.

2) For all $(x,\mu) \in D$ we have $V_1(x^+) = V_1(x)$ and $V_2(\Omega\mu) = \Omega^2 V_2(\mu) < V_2(\mu)$.

Proof: Rewrite the right-hand side of the first equation in (49) as $Ax + BKq_\mu(x) = (A+BK)x + BK\mu(q(x/\mu) - (x/\mu))$. Using (47) and (48), we obtain $\langle \nabla V_1(x), Ax + BKq_\mu(x)\rangle \leq -|x|^2 + 2|x|\|PBK\|\Delta\mu$, which is easily seen to imply $|x| \geq 2\|PBK\|\Delta\mu(1+\varepsilon_1) \Rightarrow \langle \nabla V_1(x), Ax + BKq_\mu(x)\rangle \leq -\varepsilon_1|x|^2/(1+\varepsilon_1)$. In view of the bounds $\lambda_{\min}(P)|x|^2 \leq V_1(x) \leq \lambda_{\max}(P)|x|^2$ and the definitions of χ_1 and c_1 this yields (51). Next, use the same bounds again together with (50) and the definition of χ_2 to note that the condition $V_2(\mu) \geq \chi_2 V_1(x)$ implies $\mu \geq |x|/\Theta$ and hence $|q_\mu(x)| = |\mu q(x/\mu)| \leq |\mu(q(x/\mu) - (x/\mu))| + |x| \leq \Delta\mu + \Theta\mu$, which means that $(x,\mu) \in D$ by the definition of D. Thus, (52) is *vacuously true* for $(x,\mu) \in C$, and item 1 is established. Item 2 is obvious. □

The above lemma leads immediately to the following result.

Proposition V.12: All conditions of Assumption III.2 hold[11] for the system (49) and, hence, the conclusion of Theorem III.3 holds.

To conclude asymptotic stability, we can apply Corollary III.6. If $x(0) = 0$ then, since $\mu(0) > 0$, we will have a purely discrete solution along which $\mu \to 0$, hence V does not stay constant. It is not difficult to see that every $x(0) \neq 0$ and every $\mu(0) > 0$ give a solution that is neither purely continuous nor purely discrete. Indeed, after finitely many jumps μ becomes small enough so that $(x,\mu) \in C$ and flow must occur, and then due to (51) x will eventually become small enough so that $(x,\mu) \in D$ and a jump must occur. In fact, [1], [23], [25] contain results along these lines (see in particular Lemma IV.3 in [25]). Finally, it is clear that all solutions are complete because the dynamics are linear and $C \cup D = \mathbb{R}^n$. We have shown the following.

Corollary V.13: The closed-loop hybrid system (49) is GAS[12] (with respect to the origin).

The above quantization protocol has a clear geometric interpretation. We zoom in if the quantized measurements show that $|x| \leq (\Theta + 2\Delta)\mu$, which is guaranteed to happen whenever $|x| \leq \Theta\mu$. The condition (50) means that for each μ, the ball of radius $\Theta\mu$ around the origin contains the level set of V_1 superscribed around the ball of radius $2\|PBK\|\Delta\mu$, outside of which V_1 is known to decay (thus ensuring that the zoom-in will be triggered). Similar constructions were utilized in [1], [23], but previous analyses did not employ the small-gain argument.

VI. Conclusions

We proposed several constructions of strong and weak Lyapunov functions for feedback connections of hybrid systems satisfying a small-gain condition. A novel LaSalle theorem provided sufficient conditions that can be used in conjunction with the obtained weak Lyapunov functions to conclude GAS. We also presented constructions of ADT and RADT clocks that can be used to ensure that our assumptions hold. We illustrated our results in several design-oriented contexts: networked control, event-triggered control, and quantized feedback control.

[11] The vectors x and μ here play the roles of x_1 and x_2, respectively, in Assumption III.1.

[12] Recall that we required the condition $|x| \leq M\mu$ to hold for all times. When M is finite, this actually restricts the admissible initial conditions $(x(0),\mu(0))$. However, as shown in [23], if M large enough compared to Δ then an initial "zooming-out" scheme can be used to guarantee that the above requirement is fulfilled from some time onwards. Together with our analysis, this can be used to show *global* asymptotic stability of the resulting system.

REFERENCES

[1] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, pp. 1279–1289, 2000.

[2] C. Cai and A. R. Teel, "Characterizations of input-to-state stability for hybrid systems," *Syst. Control Lett.*, vol. 58, pp. 47–53, 2009.

[3] C. Cai and A. R. Teel, "Robust input-to-state stability for hybrid systems," *SIAM J. Contr. Optimiz.*, vol. 51, pp. 1651–1678, 2013.

[4] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems, part I: Existence is equivalent to robustness," *IEEE Trans. Autom. Control*, vol. 52, pp. 1264–1277, 2007.

[5] C. Cai, A. R. Teel, and R. Goebel, "Smooth Lyapunov functions for hybrid systems, part II: (Pre-)asymptotically stable compact sets," *IEEE Trans. Autom. Control*, vol. 53, pp. 734–748, 2008.

[6] D. Carnevale, A. R. Teel, and D. Nešić, "A Lyapunov proof of an improved maximum allowable transfer interval for networked control systems," *IEEE Trans. Autom. Control*, vol. 52, pp. 892–897, 2007.

[7] F. H. Clarke, *Optimization and Nonsmooth Analysis*. New York: Interscience, 1983.

[8] S. Dashkovskiy, B. Rüffer, and F. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM J. Control Optimiz.*, vol. 48, pp. 4089–4118, 2010.

[9] S. Dashkovskiy and M. Kosmykov, "Input-to-state stability of interconnected hybrid systems," *Automatica*, vol. 49, pp. 1068–1074, 2013.

[10] C. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic Press, 1975.

[11] R. Goebel, R. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Syst. Mag.*, vol. 29, pp. 28–93, 2009.

[12] R. Goebel and A. R. Teel, "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, pp. 573–587, 2006.

[13] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, pp. 2735–2744, 2008.

[14] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th IEEE Conf. Decision and Control*, 1999, pp. 2655–2660.

[15] A. Isidori, *Nonlinear Control Systems II*. London, U.K.: Springer Verlag, 1999.

[16] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, pp. 1211–1215, 1996.

[17] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math. Control Signals Syst.*, vol. 7, pp. 95–120, 1994.

[18] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, pp. 857–869, 2001.

[19] I. Karafyllis and Z. P. Jiang, "A small-gain theorem for a large class of feedback systems with control applications," *SIAM J. Control Optimiz.*, vol. 46, pp. 1483–1517, 2007.

[20] I. Karafyllis and Z. P. Jiang, "A vector small-gain theorem for general non-linear control systems," *IMA J. Math. Contr. Inf.*, vol. 28, pp. 309–344, 2011.

[21] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1996.

[22] D. S. Laila and D. Nešić, "Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems," *IEEE Trans. Autom. Control*, vol. 48, pp. 1783–1788, 2003.

[23] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," *Automatica*, vol. 39, pp. 1543–1554, 2003.

[24] D. Liberzon and D. Nešić, "Stability analysis of hybrid systems via small-gain theorems," in *Proc. 9th Int. Workshop Hybrid Systems: Computation and Control*, 2006, pp. 421–435.

[25] D. Liberzon and D. Nešić, "Input-to-state stabilization of linear systems with quantized state measurements," *IEEE Trans. Autom. Control*, vol. 52, pp. 767–781, 2007.

[26] D. Liberzon, D. Nešić, and A. R. Teel, "Small-gain theorems of LaSalle type for hybrid systems," in *Proc. 51st IEEE Conf. Decision and Control*, 2012, pp. 6825–6830.

[27] T. Liu, D. J. Hill, and Z. P. Jiang, "Lyapunov formulation of the large-scale, ISS cyclic-small-gain theorem: The discrete-time case," *Syst. Control Lett.*, vol. 61, pp. 266–272, 2012.

[28] T. Liu, D. J. Hill, and Z. P. Jiang, "Small-gain based output-feedback controller design for a class of nonlinear systems with actuator dynamic quantization," *IEEE Trans. Autom. Control*, vol. 57, pp. 1326–1332, 2012.

[29] T. Liu, Z. P. Jiang, and D. J. Hill, "Lyapunov formulation of the ISS cyclic-small-gain theorem for hybrid dynamical networks," *Nonlinear Analysis: Hybrid Systems*, vol. 6, pp. 988–1001, 2012.

[30] I. Mareels and D. Hill, "Monotone stability of nonlinear feedback systems," *J. Math. Syst. Estim. Control*, vol. 2, pp. 275–291, 1992.

[31] S. Mitra, D. Liberzon, and N. Lynch, "Verifying average dwell time," *ACM Trans. Embedded Comp. Syst.*, vol. 8, pp. 1–37, 2008.

[32] D. Nešić and D. Liberzon, "A small-gain approach to stability analysis of hybrid systems," in *Proc. 44th IEEE Conf. Decision and Control*, 2005, pp. 5409–5414.

[33] D. Nešić and A. R. Teel, "Changing supply functions in input to state stable systems: The discrete-time case," *IEEE Trans. Autom. Control*, vol. 46, pp. 960–962, 2001.

[34] D. Nešić and A. R. Teel, "Input-output stability properties of networked control systems," *IEEE Trans. Autom. Control*, vol. 49, pp. 1650–1667, 2004.

[35] D. Nešić and A. R. Teel, "A Lyapunov-based small-gain theorem for hybrid ISS systems," in *Proc. 47th IEEE Conf. Decision and Control*, 2008, pp. 3380–3385.

[36] R. Sanfelice, "Interconnections of hybrid systems: Some challenges and recent results," *J. Nonl. Syst. Appl.*, pp. 111–121, 2011.

[37] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, pp. 435–443, 1989.

[38] E. D. Sontag and A. Teel, "Changing supply functions in input/state stable systems," *IEEE Trans. Autom. Control*, vol. 40, pp. 1476–1478, 1995.

[39] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Trans. Autom. Control*, vol. 52, pp. 1680–1685, 2007.

[40] A. R. Teel and D. Nešić, "Lyapunov functions for L_2 and input-to-state stability in a class of quantized control systems," in *Proc. 50th IEEE Conf. Decision and Control*, 2011, pp. 4542–4547.

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