Quasi-Integral-Input-to-State Stability for Switched Nonlinear Systems

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Abstract: In this paper we introduce and give sufficient conditions for the quasi-iISS property for switched nonlinear systems under dwell-time switching signals. Unlike previous works, our dwell-time bound does not rely on the knowledge of the state but it relies only on the system initial condition and the bound on the input energy. We prove, through a counterexample, that knowledge of the system initial state and bound on input energy is necessary for the estimation of a dwell-time that guarantees quasi-iISS for the switched system. An illustrative example is also included.

Keywords: Stability of nonlinear systems, Input-to-state stability, Switching stability and control, Stability of hybrid systems, Lyapunov methods

1. INTRODUCTION

Input-to-State Stability (ISS), Sontag and Wang (1995), and integral Input-to-State Stability (iISS), Angeli et al. (2000), probably are the most useful and most used state space based properties to characterize the behavior of nonlinear systems with inputs. The former provides a bound on the state obtained as the sum of a time decaying term depending on the initial condition of the system and a term that takes into account the supremum norm of the input. The difference with iISS lies in the fact that the latter considers, instead of the supremum norm of the input, the bound on the integral of the input.

These properties have been extensively adopted in several frameworks such as cascade systems Sontag and Teel (1995), Chaillet and Angeli (2008), time-delay systems Pepe and Jiang (2006), Yeganefar et al. (2007) and switched systems, Xie et al. (2001) Yang and Liberzon (2015). In particular, there exists a large literature characterizing ISS and iISS for switched systems because, in general, a switched system does not inherit properties of the individual subsystems. Some work has been done in the direction of characterizing ISS and iISS with respect to some Lyapunov conditions Mancilla-Aguilar and Garcia (2001), Feng and Zhang (2005), Haimovich and Mancilla-Aguilar (2018). For instance, the main contribution of Haimovich and Mancilla-Aguilar (2018) is to provide a characterization of iISS for switched systems with any set of admissible switching signals. More precisely, they prove that a switched system is iISS uniformly with respect to a given set of switching signals if and only if the system satisfies the uniformly bounded energy bounded state, Angeli et al. (2000), property and the zero-input global asymptotic stability property, both uniformly with respect to the given set of switching signals. Other works, instead, aim at finding some more practical conditions, i.e. *dwell-time* conditions, in order to assess the aforementioned properties for switched systems as in De Persis et al. (2003), Vu et al. (2005) and Müller and Liberzon (2012).

The problem of finding suitable dwell-time conditions that ensure ISS (iISS) property for a switched system may be stated as follows: given a switched system whose component subsystems are each ISS (iISS), find the sufficient conditions for the switching signal to preserve the ISS (iISS) property. For switched nonlinear systems with inputs, Xie *et. al.* showed that for dwell-time switching signals, a switched system is ISS if the individual systems are ISS, see Xie et al. (2001) and reference therein. For iISS systems instead, the solution proposed by De Persis *et. al.* relies on a "time-varying" or "adjustable" dwell-time that ensures the iISS property for the switched system. More precisely, in that paper, the estimation of the dwell-time is updated at each switching instant based on the value of the state at that time instant. However, it is not always possible to access the entire state, therefore an alternative dwell-time condition that does not rely on the knowledge of the state over time is needed. Differently from the case of ISS for switched system, in this paper we prove, through a counterexample, that it is not possible to achieve the iISS property for a switched system when no knowledge about the system's initial condition and a bound on the energy of the input is available.

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The aim of this work, in fact, is to provide a dwell-time condition relying only on the knowledge of the initial condition of the system and on the bound on the energy of the input, which guarantees some iISS-like condition. Specifically, the definition of *quasi*-iISS is here introduced, where quasi-iISS is intended as the iISS property holding for initial condition norm within a given bound and input characterized by finite energy with given bound. The main result presented in this paper provides a sufficient condition to achieve quasi-iISS for a switched system with a dwell-time switching signal. Moreover, since we are considering inputs with bounded energy, asymptotic convergence of the state for the switched system automatically follows from achieving quasi-iISS. Finally, we test our theoretical results with a numerical example.

2. PRELIMINARIES AND DEFINITIONS

For the sake of notation, we define here the following, Khalil (2002): a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *positive definite* if it is continuous, $\alpha(r) > 0 \ \forall r > 0$ and $\alpha(0) = 0$; it is of class $\mathcal{K}$ if it is positive definite and it is strictly increasing; it is of class $\mathcal{K}_\infty$ if it is a class $\mathcal{K}$ function and also $\alpha(r) \to \infty$ as $r \to \infty$; it is of class $\mathcal{L}$ if it is continuous, decreasing and $\alpha(r) \to 0$ as $r \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if for each fixed t, the function $\beta(s,t)$ is a class $\mathcal{K}$ function with respect to s, and for each fixed s the function $\beta(s,t)$ is of class $\mathcal{L}$ with respect to t. Moreover, in the rest of the paper we adopt the following notation: given a real valued matrix A, $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ indicate respectively the maximum and the minimum singular value for A. Finally, we will use the following notation for the maximum $a \vee b := \max\{a, b\}$.

Consider the family of systems

$$\dot{x} = f_p(x,u), \quad p \in \mathcal{P}, \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^l$ is measurable and locally essentially bounded, and $\mathcal{P}$ is a finite index set. For each $p \in \mathcal{P}$, f_p is locally Lipschitz and $f_p(0,0) = 0$. A *switched system* generated by the family of systems (1) and a *switching signal* σ is

$$\dot{x} = f_\sigma(x,u), \tag{2}$$

where $\sigma(\cdot) : [0,\infty) \to \mathcal{P}$ is a piecewise constant function, continuous from the left, specifying at every time the index of the active system. We denote by $x(\cdot, x_0, u, \sigma)$ the absolutely continuous solution of (2) with initial state $x(t_0) = x_0$, generated by the input u and a switching signal σ. When the context is clear, we will shorten it by $x(\cdot)$. We define the set $\mathcal{T}$ of admissible switching functions $\sigma(\cdot)$ as the set of piece-wise constant functions of time which exhibit a strictly increasing sequence of switching instants $T(\sigma) := \{t_1, ..., t_j, ...\}$ with finite number of switches in every finite interval. Moreover, a switching signal $\sigma(t)$ is said to have *dwell-time* τ_d if it holds

$$\tau_d \leq t_{i+1} - t_i, \qquad \forall t_i, t_{i+1} \in T(\sigma). \tag{3}$$

For ease of expression we will indicate with t_0 the initial time and we also assume $t_1 - t_0 \geq \tau_d$; that is, the dwell-time of the first mode is also lower-bounded by τ_d. In the following, we will denote with $\mathcal{T}_d(\tau_d) \subset \mathcal{T}$ the set of switching signals characterized by dwell-time τ_d. Finally, we will say that a system has certain properties *under slow switching* if those properties hold for large enough dwell-time.

For a switched system, the iISS property can be stated as follows:

Definition 1. System (2) is said to be *iISS under slow switching* if there exists $\tau_d > 0$ such that for each $\sigma \in \mathcal{T}_d(\tau_d)$ there exist functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that the solution $x(t)$ is defined for all $t \geq t_0$ and satisfies

$$|x(t, x_0, u, \sigma)| \leq \beta(|x_0|, t - t_0) + \gamma_1\left(\int_{t_0}^{t} \gamma_2(|u(s)|)ds\right), \tag{4}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$.

Trivially, when no switches occur, the iISS definition reduces to the classical one for single mode systems, see Angeli et al. (2000). Moreover, iISS can also be concluded by adoption of a Lyapunov function.

Lemma 1. Consider the system

$$\dot{x} = f(x,u),$$

with $f(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n$ locally Lipschitz function and $u \in \mathbb{R}^l$ measurable and locally essentially bounded input. Let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable function such that there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and a positive definite function ρ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^l$

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \tag{5}$$

$$\nabla V(x) \cdot f(x,u) \leq -\rho(V) + \chi(|u|), \tag{6}$$

then the system is iISS.

Note that condition (6) is different from the more usual one involving a Lyapunov function with time derivative

$$\nabla V(x) \cdot f(x,u) \leq -\tilde{\rho}(|x(t)|) + \chi(|u|), \tag{7}$$

where the argument of the positive definite function $\tilde{\rho}$ is $|x(t)|$ rather than V. Nevertheless, in (Angeli et al., 2000, Section IV) it is shown that the two conditions are equivalent and hence, without loss of generality, we can also use the one provided in (6).

3. MOTIVATING EXAMPLE

The aim of this section is to provide a counterexample to prove that given a family of iISS systems, it is not possible to establish a dwell-time condition, independently of the system initial condition and of the input energy, that ensures the iISS property for the switched system. Consider the family of systems

$$\dot{x} = \frac{1}{1 + |x|^2} A_p x + u, \tag{8}$$

where $x = [x_1, x_2]^T \in \mathbb{R}^2$, $u = [u_1, u_2]^T \in \mathbb{R}^2$, $A_p \in \mathbb{R}^{2\times 2}$ are Hurwitz matrices, and $p \in \mathcal{P} = \{1, 2\}$. For this example, similarly to what is shown in (Liberzon, 2003, Section 3.3), let us consider matrices A_p of the form

$$A_1 = \begin{bmatrix} -0.1 & -1 \\ 2 & -0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & -2 \\ 1 & -0.1 \end{bmatrix}. \tag{9}$$

It is not difficult to show that both systems are iISS but not ISS. Let us consider for both systems the Lyapunov functions

$$V_p(x) = \sqrt{1 + x^T P_p x} - 1 \quad \forall p \in \{1, 2\}, \tag{10}$$

where P_p are symmetric positive definite matrices, solutions of the Lyapunov equation $A_p^T P_p + P_p A_p = -Q_p$ with Q_p being positive definite matrices. With such a choice of $V_p(x)$, by differentiating along the system trajectories it yields

$$\begin{aligned}\dot V_p(x) &\le -\frac{1}{2}\frac{x^T Q_p x}{(1+|x|^2)\sqrt{1+x^T P_p x}} + \kappa_p|u|,\\ &\le -\frac{1}{2}\frac{\underline{\sigma}(Q_p)|x|^2}{(1+|x|^2)\sqrt{1+\bar\sigma(P_p)|x|^2}} + \kappa_p|u|, \quad (11)\end{aligned}$$

where $\kappa_p := \bar\sigma(P_p)/\sqrt{\underline{\sigma}(P_p)}$. Then it is clear that (11) suits the iISS Lyapunov condition (7), thus proving that each system is iISS. It is straightforward to show that neither of the systems is ISS through selection of a bounded input that generates unstable solutions for each system.

Given $x(0) = x_0$, define the switching signal $\sigma(t) : \mathbb{R}_{\ge 0} \to \mathcal{P}$ as

$$\begin{cases}\sigma(t) = 1 & \text{if} \quad x_1(t)x_2(t) \ge 0,\\ \sigma(t) = 2 & \text{if} \quad x_1(t)x_2(t) < 0.\end{cases} \quad (12)$$

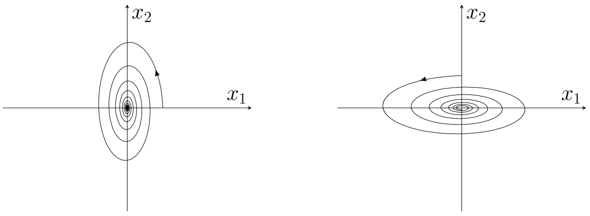

Fig. 1. Phase portraits for the systems $\dot x = \frac{1}{1+|x|^2}A_1 x$ and $\dot x = \frac{1}{1+|x|^2}A_2 x$

Let us initially consider the case $|u| \equiv 0$ and the polar coordinates transformation $x_1 = \rho\cos\theta$ and $x_2 = \rho\sin\theta$ which results in $\theta = \arctan x_2/x_1$. By differentiating both sides it yields

$$\dot\theta(t) = \begin{cases}\dfrac{2x_1^2 + x_2^2}{(1+|x|^2)|x|^2} & \text{when} \quad \sigma(t) = 1,\\ \dfrac{x_1^2 + 2x_2^2}{(1+|x|^2)|x|^2} & \text{when} \quad \sigma(t) = 2,\end{cases} \quad (13)$$

which is upper-bounded by $|\dot\theta(t)| \le 2/(1+|x|^2)$. Let us choose $x_0 = [x_{10},\ 0]^T$, with $x_{10} > 0$, then, as long as the system trajectory does not hit the x_2-axis, the system will behave as in the left plot of Figure 1. Hence, the first switch will occur after a time $\Delta T_1 := t_1 - t_0$ such that

$$\begin{aligned}\frac{\pi}{2} &= \int_0^{\Delta T_1} |\dot\theta(\tau)| d\tau\\ &\le \int_0^{\Delta T_1} \frac{2}{1+|x(\tau)|^2} d\tau\\ &\le \int_0^{\Delta T_1} \frac{2}{1+|x_0|^2} d\tau = \frac{2\Delta T_1}{1+|x_0|^2}, \quad (14)\end{aligned}$$

where the last inequality derives from the fact that the norm of the state at the first switching time has a higher value than at time zero. Picking switching instants as in (12) will produce unbounded trajectories. This translates to successive switching intervals being longer than ΔT_1. Choosing a dwell-time higher than ΔT_1 may seem to be a solution to achieve bounded trajectories. However, according to (14), ΔT_1 depends on the system initial state. This in turn implies that if the initial state was set to be further from the origin, then a larger dwell-time would be required to achieve iISS. This proves that, in order to achieve iISS for switched systems, the dwell-time condition cannot be independent of the initial state.

Furthermore, a similar reasoning can be adopted for the case of nonzero input. In fact, for the addressed nonlinear system, it is possible to design a nonzero input trajectory $u(t)$ with large but finite integral and only a very small support near $t = 0$ that moves the system state to $[x_{10},\ 0]^T$. This would take the system to the previous configuration for which, even with zero input, the switched system presents unbounded dynamics. Therefore, the dwell-time condition required to achieve iISS for switched systems cannot be chosen independently of the input energy either.

4. QUASI-IISS FOR SWITCHED SYSTEMS

Given the above reasoning, a new notion related to iISS naturally arises, thus we here introduce the notion of quasi-iISS.

Definition 2. (quasi-iISS System). The switched system (2) is said to be *quasi-iISS under slow switching* if there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}$ such that for each $\delta_1, \delta_2 > 0$ there exists $\tau_d > 0$ so that for any $\sigma \in \mathcal{T}_d(\tau_d)$, there exist functions $\beta \in \mathcal{KL}$ such that the estimation (4) holds for all $t \ge t_0 \ge 0$, $|x_0| \le \delta_1$ and $\int_{t_0}^{\infty} \gamma_2(|u(\tau)|)d\tau \le \delta_2$.

Note that, in the definition of quasi-iISS, the functions γ_1 and γ_2 are independent of all other quantifiers. In fact, only the function β is dependent on the switching signal σ and hence it is implicitly affected by δ_1 and δ_2.

In the following we shall prove the existence of a dwell-time estimate (depending on the bound on the initial conditions and on bound on the energy of the input) that guarantees the quasi-iISS property for a family of switching iISS systems.

4.1 Main Result

As a preliminary step, let us state two lemmas that will help us in the formulation of the main result.

Lemma 2. Let $\rho : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a continuous positive definite function. Then there exist functions $\rho_1 \in \mathcal{K}$ and $\rho_2 \in \mathcal{L}$ such that

$$\rho(r) \ge \rho_1(r)\rho_2(r). \quad (15)$$

Lemma 3. Consider a positive definite function ρ, a continuous and differentiable function $y : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and a non-decreasing continuous function $z : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$. If

$$\dot y(t) \le -\rho\left((y(t) + z(t)) \vee 0\right)$$

for all $t \ge 0$, then there exists a $\mathcal{KL}$ function $\bar\beta$ such that

$$y(t) \le \bar\beta(y(0), t) \vee z(t) \quad (16)$$

where $\bar\beta(s,t)$ is the solution to the initial value ODE problem $\dot v = -\rho_1(v)\rho_2(2v)$, $v(0) = s$ where ρ_1 and ρ_2 are obtained from ρ as in Lemma 2.

The reader is referred respectively to (Angeli et al., 2000, Lemma IV.1) and (Angeli et al., 2000, Lemma IV.2) for the proof of these Lemmas.

Theorem 4. Consider the switched system (2) and suppose that each mode is iISS, that is to say that there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and a positive definite function ρ such that for each mode $p \in \mathcal{P}$ there exists a differentiable function $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|), \tag{17}$$

$$\nabla V_p(x) \cdot f_p(x, u) \leq -\rho(V_p) + \chi(|u|), \tag{18}$$

for all $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}^l$. Assume further that it holds

$$V_p(x) \leq \mu V_q(x) \quad \forall p, q \in \mathcal{P}, \quad \forall x \in \mathbb{R}^n \tag{19}$$

with $\mu > 1$. If there exists positive $\lambda < 1$ such that for each positive D, E there exists $\tau_d \geq 0$ such that

$$\bar{\beta}((2\mu - 1)s, \tau_d) \leq \lambda s \quad \forall s \in [0, D \vee E], \tag{20}$$

where $\bar{\beta} \in \mathcal{KL}$ is constructed from ρ via Lemma 3, then system (2) is quasi-iISS under slow switching. In particular, there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}$, and $\beta \in \mathcal{KL}$ depending on the switching signal $\sigma \in \mathcal{T}_d(\tau_d)$, such that estimation (4) holds for all $t \geq t_0$, $|x_0| \leq \alpha_1^{-1}(D)$, $\int_{t_0}^{\infty} \gamma_2(|u(\tau)|)d\tau \leq E$.

It is not difficult to verify that condition (20) is more conservative than the dwell-time estimation obtained for switched systems with ISS subsystems in Müller and Liberzon (2012). However, the set of time-invariant iISS systems includes the class of time-invariant ISS systems as a subset, hence the dwell-time condition provided in Theorem 4 is applicable to a larger class of systems.

Proof. Similarly to Angeli et al. (2000) and Hespanha et al. (2008), we begin by defining a function $v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $v(t) = V_{\sigma(t)}(x(t))$, then from (18) and (19)

$$\dot{v}(t) \leq -\rho(v) + \chi(|u|) \qquad \forall t \notin T(\sigma), \tag{21}$$

$$v(t_k^+) \leq \mu v(t_k) \qquad \forall t_k \in T(\sigma), \tag{22}$$

with the initial condition $v(t_0) = V_{\sigma(t_0)}(x(t_0))$. Let $z(t)$ be the (nonnegative and non-decreasing) continuous solution to

$$\dot{z}(t) = \chi(|u|), \tag{23}$$

with the initial condition set as $z(t_0) = 0$. Define $y(t) := v(t) - z(t)$. Then $y(t)$ satisfies $y(t_0) = V_{\sigma(t_0)}(x(t_0))$ and

$$\dot{y}(t) \leq -\rho(v(t)) = -\rho(y(t) + z(t)) \qquad \forall t \notin T(\sigma), \tag{24}$$

$$\begin{aligned} y(t_k^+) &\leq \mu v(t_k) - z(t_k) \\ &= \mu y(t_k) + (\mu - 1)z(t_k) \qquad \forall t_k \in T(\sigma). \end{aligned} \tag{25}$$

For a function $\bar{\beta}(r, t) \in \mathcal{KL}$, we define its *extension* as the function that takes values $\bar{\beta}(r, t)$ for all $r \geq 0$ and $t \geq 0$, and 0 for all $r < 0$ and $t \geq 0$. With a slight abuse of notation, we keep the same symbol for a class $\mathcal{KL}$ function and its extension. While the only purpose of introducing extensions of class $\mathcal{KL}$ functions is to allow negative first argument in $\bar{\beta}$, readers can safely treat the extension as having the same properties of class $\mathcal{KL}$ functions in the rest of this proof. It follows from Lemma 3 that there exists a function $\bar{\beta}(r, t)$ such that, in each interval $[t_i, t_{i+1})$, $y(t)$ satisfies

$$y(t_{i+1}) \leq \bar{\beta}(y(t_i^+), t_{i+1} - t_i) \vee z(t_{i+1}). \tag{26}$$

Replacing $y(t_i^+)$ as in (25) we obtain

$$\begin{aligned} y(t_{i+1}) \leq& \bar{\beta}(\mu y(t_i) + (\mu - 1)z(t_i), t_{i+1} - t_i) \vee z(t_{i+1}) \\ \leq& \bar{\beta}((2\mu - 1)y(t_i), t_{i+1} - t_i) \\ &\vee \bar{\beta}((2\mu - 1)z(t_i), t_{i+1} - t_i) \\ &\vee z(t_{i+1}), \end{aligned} \tag{27}$$

where in the last inequality we used the property $\Gamma(k_1 r_1 + k_2 r_2, w) \leq \Gamma((k_1 + k_2)r_1, w) \vee \Gamma((k_1 + k_2)r_2, w)$ for any class $\mathcal{KL}$ function Γ, non-negative constants k_1, k_2 and constants r_1, r_2. Let us consider the case $i = 0$ first: $y(t_0) = v(x(t_0)) = V_{\sigma(t_0)}(x(t_0)) \leq D$ and $z(t_0) \leq E$, hence we can apply assumption (20) to the first two terms of (27) and obtain

$$\begin{aligned} y(t_1) &\leq \lambda y(t_0) \vee \lambda z(t_0) \vee z(t_1) \\ &\leq \lambda y(t_0) \vee z(t_1), \end{aligned} \tag{28}$$

where the second inequality results from $z(t)$ being (by definition) nonnegative and non-decreasing and λ being strictly less than one. Note that, from (28), $y(t_1) < D \vee E$, thus (20) is applicable for each iteration by induction. Then, applying this reasoning recursively results in

$$y(t_k) \leq \lambda^k y(t_0) \vee z(t_k). \tag{29}$$

Furthermore, from (25)

$$y(t_k^+) \leq \mu \lambda^k y(t_0) + (2\mu - 1)z(t_k). \tag{30}$$

Consider now any time instant t belonging to the interval $[t_k, t_{k+1})$. Then, we obtain

$$\begin{aligned} y(t) \leq& \bar{\beta}(y(t_k^+), t - t_k) \vee z(t) \\ \leq& \bar{\beta}(\mu \lambda^k y(t_0) + (2\mu - 1)z(t_k), t - t_k) \vee z(t) \\ \leq& \bar{\beta}(2\mu \lambda^k y(t_0), t - t_k) \vee \bar{\beta}(2(2\mu - 1)z(t_k), t - t_k) \\ &\vee z(t) \\ \leq& \bar{\beta}(2\mu \lambda^k y(t_0), 0) \vee \bar{\beta}(2(2\mu - 1)z(t_k), 0) \\ &\vee z(t) \qquad \forall t \in [t_k, t_{k+1}) \end{aligned} \tag{31}$$

where in the third inequality we used the property $\Gamma(r_1 + r_2, w) \leq \Gamma(2r_1, w) \vee \Gamma(2r_2, w)$ for any class $\mathcal{KL}$ function Γ and any nonnegative constants r_1 and r_2. For each $t \in [t_0, \infty)$ let $N(t) \in [0, \infty)$ be the (finite) number of switches up to time t. Two scenarios may occur: $N(t)$ may be either bounded or unbounded. The first case translates into the existence of a time instant $\bar{t} > 0$ such that $\sigma(t) \equiv p$ $\forall t \geq \bar{t}$ for some $p \in \mathcal{P}$, and iISS trivially follows. In the case that $N(t) \to \infty$ as $t \to \infty$, note that $\bar{\beta}(2\mu\lambda^{N(t)} s, 0)$ can be easily upper-bounded by a class $\mathcal{KL}$ function, let us name it $\tilde{\beta}(s, t - t_0)$. Hence, for any time t

$$\begin{aligned} y(t) &\leq \tilde{\beta}(y(t_0), t - t_0) \vee \bar{\beta}(2(2\mu - 1)z(t_{N(t)}), 0) \vee z(t), \\ &\leq \tilde{\beta}(y(t_0), t - t_0) \vee \bar{\beta}(2(2\mu - 1)z(t), 0) \vee z(t). \end{aligned} \tag{32}$$

Collecting the above facts, we obtain for each $p \in \mathcal{P}$

$$\begin{aligned} V_{\sigma(t)}(x(t)) &= v(t) = y(t) + z(t) \\ &\leq \tilde{\beta}(y(t_0), t - t_0) + \alpha(z), \end{aligned} \tag{33}$$

where $\alpha(z) := 2z(t) + \bar{\beta}(2(2\mu - 1)z(t), 0)$ is a class $\mathcal{K}$ function. Then, using the inequalities in (17) it holds

$$
\begin{aligned}
|x(t)| &\le \alpha_1^{-1}\left(\tilde{\beta}(y(t_0),t-t_0)+\alpha(z(t))\right)\\
&=\alpha_1^{-1}\left(\tilde{\beta}(V_{\sigma(t_0)}(x(t_0)),t-t_0)+\alpha\left(\int_{t_0}^{t}\chi(|u(\tau)|)d\tau\right)\right)\\
&\le \alpha_1^{-1}\left(\tilde{\beta}\left(\alpha_2(x(t_0)),t-t_0\right)+\alpha\left(\int_{t_0}^{t}\chi(|u(\tau)|)d\tau\right)\right)\\
&\le \alpha_1^{-1}\left(2\tilde{\beta}\left(\alpha_2(x(t_0)),t-t_0\right)\right)+\\
&\quad \alpha_1^{-1}\left(2\alpha\left(\int_{t_0}^{t}\chi(|u(\tau)|)d\tau\right)\right) \qquad (34)
\end{aligned}
$$

proving quasi-iISS with δ_1 and δ_2 from Definition 2 being respectively $\alpha_1^{-1}(D)$ and E and with $\beta(s,t) = \alpha_1^{-1}(2\tilde{\beta}(\alpha_2(s),t))$, $\gamma_1(r) = \alpha_1^{-1}(2\alpha(r))$ and $\gamma_2(r) = \chi(r)$.

Note that the above result applies to switched systems with bounded input energy, thus Theorem 4 also ensures asymptotic convergence of the state. This claim follows directly from (Sontag, 1998, Proposition 6). Hence, Theorem 4 implies the following corollary.

Corollary 5. Let the assumptions of Theorem 4 hold, then the solution of the switching system (2) satisfies

$$\lim_{t\to\infty} x(t) = 0. \qquad (35)$$

Note that it is not always possible to obtain a dwell-time estimation from (20). This is for instance the case when $\bar{\beta}(s,t) = s/(1+st)$ for which the dwell-time is not defined for $s = 0$. In fact, applying condition (20) for this function results in $(2\mu-1)/(1+(2\mu-1)s\tau_d) \le \lambda$ which has no solution for τ_d if $s = 0$. The next subsection will give more insight about the sufficient conditions to obtain a dwell-time estimation.

4.2 Further characterization of dwell-time

Theorem 1 provides an estimate of the dwell-time necessary to achieve asymptotic convergence of the state for the switched system. However, this result relies on assumption (20) which may be not straightforward to verify. The following lemma and corollary present further insight into the link between the nonlinear decreasing rate of the iISS Lyapunov functions in (6) and the aforementioned assumption. More precisely, they provide a constructive way to verify quasi-iISS under slow switching for a switched system directly from (6).

Lemma 6. Let $\bar{\beta}(s,t) \in \mathcal{KL}$ and $\mu \ge 1$. The following two statements are equivalent:

(1)
$$\limsup_{t\to\infty}\limsup_{s\to 0^+}\frac{\bar{\beta}((2\mu-1)s,t)}{s} < 1, \qquad (36)$$

(2) $\exists\lambda < 1$ such that $\forall b > 0, \exists\tau_d \ge 0$ such that
$$\bar{\beta}((2\mu-1)s,\tau_d) \le \lambda s \qquad \forall s\in[0,b] \qquad (37)$$

Proof. The proof of this lemma will be found in the full version of the paper.

Note that Lemma 6 still holds for the extension of the function $\bar{\beta}(r,t)$.

Corollary 7. Consider a set of positive definite functions V_p characterized by (5) and satisfying

$$\dot{V}_p(x) \le -\rho(V_p)+\chi(|u|) \qquad \forall x\in\mathbb{R}^n, \quad \forall u\in\mathbb{R}^l, \qquad (38)$$
$$V_p(x) \le \mu V_q(x) \qquad \forall p,q\in\mathcal{P}, \quad \forall x\in\mathbb{R}^n. \qquad (39)$$

If ρ is differentiable at 0 with $\rho'(0) > 0$, then the switched system is quasi-iISS under slow switching.

Proof. The proof of this corollary will be found in the full version of the paper.

5. NUMERICAL EXAMPLE

In order to test the effectiveness of the theoretical results presented in Sections 3 and 4, a numerical example is provided. Consider the nonlinear switched system defined as the family of systems in (8) with matrices A_1 and A_2 as in (9) and switching signal (12). The iISS Lyapunov functions $V_p(x)$ are selected as in Section 3 with P_p being solution of $A_p^T P_p + P_p A_p = -\delta I$ with δ being a scaling factor set to 10^{-2}. Note that given the symmetry of the addressed problem, it holds $\bar{\sigma}(P_1) = \bar{\sigma}(P_2) =: M = 0.075$ and $\underline{\sigma}(P_1) = \underline{\sigma}(P_2) =: m = 0.037$. Let us assume the set of all possible initial states to be included in the set $\Omega := \left\{x : |x(0)| \le \alpha_1^{-1}(D)\right\}$ where $\alpha_1(|x|) = \sqrt{1+m|x|}-1$ and $D = 0.040$. The initial state $x(0) = [1 \quad 0]^T$ belongs to this set, and $\sigma(0)$ is set to 1. The system input is chosen as an exogenous finite energy signal $u(t) = [z(t), \ \ 0]^T$ with $z(t) := z_0 e^{-\eta t}$. For this numerical example we assigned to z_0 and η respectively the values 1 and 10. Note that γ_2 function in (4) is the same as χ function (6) and hence for our numerical example $\gamma_2(|u|) = \kappa_p|u|$ as in (11). Therefore the input energy is defined as

$$\int_0^\infty \kappa z_0 e^{-\eta t}dt = \frac{\kappa z_0}{\eta} = 0.038, \qquad (40)$$

where κ is the maximum between $\kappa_1 = \bar{\sigma}(P_1)/\sqrt{\underline{\sigma}(P_1)}$ and $\kappa_2 = \bar{\sigma}(P_2)/\sqrt{\underline{\sigma}(P_2)}$ thus $\kappa = M/\sqrt{m}$. It follows that $s = D \vee E = 0.040$. Moreover, the parameter μ from (19) is selected as

$$\mu \le \sqrt{M/m} \qquad (41)$$

which yields $\mu = 1.41$. Finally, let us pick $\lambda = 0.9$. This is all the information needed to compute an estimate of the dwell-time that guarantees the iISS property of the switched system. Following the steps of Theorem 4, we obtain $\dot{V}_{\sigma(t)}(x(t))$ in a similar form as (11) and, by straightforward computation, we attain

$$\dot{v}(t) \le -\rho(v)+\kappa|u|, \qquad (42)$$

which corresponds to equation (21), where $\rho(v)$ is given by

$$\rho(v) := \frac{1}{2}\delta\frac{m}{M}\frac{(v+1)^2-1}{(v+1)\left[(v+1)^2+m-1\right]}. \qquad (43)$$

Note that this positive definite function satisfies $\rho'(0) = 2/m > 0$ as required by corollary 7. Then, it is possible to choose functions $\rho_1(v) \in \mathcal{K}_\infty$ and $\rho_2(v) \in \mathcal{L}$ respectively as

$$\rho_1(v) = \frac{1}{2}\delta\frac{m}{M}\left[(v+1)^2-1\right], \qquad (44)$$

$$\rho_2(v) = \frac{1}{(v+1)\left[(v+1)^2+m-1\right]}. \qquad (45)$$

Then, we can compute the function $\bar{\beta}$ from condition (20), using Lemma 3, as the solution of $\dot{\nu}(t) = -\rho_1(\nu)\rho_2(2\nu)$ with initial condition $\nu(0) = (2\mu-1)s$ with $s\in[0, D\vee E]$. The analytical solution of this differential equation is rather difficult to obtain, hence, for this case, a numerical result will be provided. Condition (20) results in a dwell-time $\tau_d = 38.20$. This estimation is obviously not the tightest that would guarantee iISS (hence asymptotic convergence of the state to the origin) for the switched system, but it is the tightest to the best of the authors' ability. System trajectory can be observed in Figure 2.

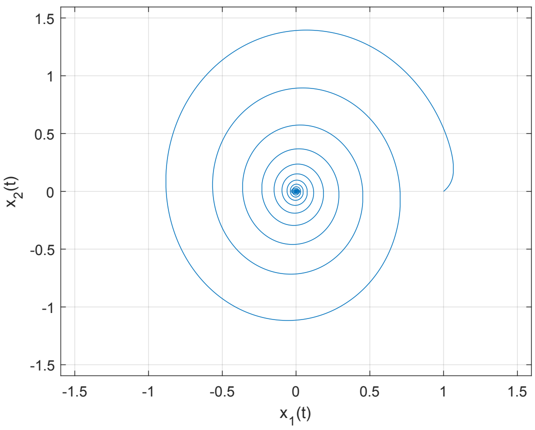

Fig. 2. State trajectory of the switched system

It must be noted that, selecting different values of the scaling factor δ yields different values of the dwell-time estimation. In fact, it is trivial to verify that a choice of a bigger δ results in a larger s, therefore in a larger distance between $\nu(0)$ and λs but, at the same time, it speeds up the rate of convergence of $\dot{\nu}$. Hence, the best choice of δ, that guarantees the smallest possible dwell-time, results from a trade-off analysis between these two factors.

6. CONCLUSION

In this paper, we have provided a constructive way to prove quasi-iISS for a switched system where each individual subsystem is iISS. Unlike previous works, we achieved this result by exploiting the knowledge of the initial condition and of the input energy bound only. We showed the necessity of knowing the initial condition and the input energy bounds by a counterexample. Moreover, given the bounded energy of the input, the dwell-time condition provided is actually sufficient to guarantee asymptotic convergence of the state to the origin. Finally, we provided a numerical example to show the effectiveness of the proposed strategy. The authors would like to acknowledge the reviewers for their remark on the lack of uniformity of the quasi-iISS property with respect to the switching signal. Uniformity of quasi-iISS is actually guaranteed with the same assumptions as in Theorem 4. However, a different approach is needed in order to prove uniformity and it will appear in an extended work. Future research will attempt to refine the dwell-time analysis in order to achieve a less conservative estimation. Moreover, the existence of global dwell-time and of average dwell-time conditions for iISS switched systems will be investigated. Furthermore, it may be of interest to study the case of systems with measurement noises and unmodeled dynamics.

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