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Input/output-to-state stability and state-norm estimators for switched nonlinear systems☆

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ABSTRACT

In this paper, the concepts of input/output-to-state stability (IOSS) and state-norm estimators are considered for switched nonlinear systems under average dwell-time switching signals. We show that when the average dwell-time is large enough, a switched system is IOSS if all of its constituent subsystems are IOSS. Moreover, under the same conditions, a non-switched state-norm estimator exists for the switched system. Furthermore, if some of the constituent subsystems are not IOSS, we show that still IOSS can be established for the switched system, if the activation time of the non-IOSS subsystems is not too big. Again, under the same conditions, a state-norm estimator exists for the switched system. However, in this case, the state-norm estimator is a switched system itself, consisting of two subsystems. We show that this state-norm estimator can be constructed such that its switching times are independent of the switching times of the switched system it is designed for.

1. Introduction

State estimation plays a central role in control theory. Namely, in many applications, the full system state cannot be measured, but only certain outputs are available. Yet, for controlling the system, often the full state x is needed. This problem can be addressed by designing an observer, which yields an estimate of the system state x, out of the observation of past inputs and outputs. For general nonlinear systems, however, and even more for more complex system classes like e.g. switched systems, the design of such an observer is a challenging task, far from being solved completely. On the other hand, for some control purposes, it may suffice to gain an estimate of the magnitude, i.e., the norm $|x|$, of the system state x (see Sontag and Wang (1997), Krichman, Sontag, and Wang (2001) and references therein). The notion of such a state-norm estimator as well as the intimately related theoretical concept of input/output-to-state stability (IOSS) was introduced in Sontag and Wang (1997) for continuous-time nonlinear systems. Loosely speaking, the IOSS property means that no matter what the initial state is, if the inputs and the observed outputs are small, then eventually the state of the system will also become small; the IOSS property can be seen as somewhat stronger than the zero-detectability property of linear systems. In Sontag and Wang (1997) and Krichman et al. (2001) it was shown that for continuous-time nonlinear systems, the existence of an appropriately defined state-norm estimator is equivalent to the system being IOSS (and also to the existence of an IOSS-Lyapunov function for the system). Furthermore, in Astolfi and Praly (2006) it was shown how an estimate of the norm $|x|$ can be exploited in constructing an observer, which in turn can be used for output feedback design to globally stabilize the system (Praly & Astolfi, 2005).

In this paper, we are interested in IOSS and state-norm estimation for switched systems. The study of the class of switched systems has attracted a lot of attention in recent years (see e.g. Liberzon (2003) and references therein). Switched systems arise in situations where several dynamical systems are present together with a switching signal specifying at each time the active system dynamics according to which the system state evolves. It is well-known that in general, switched systems do not necessarily inherit the properties of the subsystems they are comprised of.

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For example, a switched system consisting of asymptotically stable subsystems might become unstable (Liberzon, 2003) if certain switching laws are applied. Thus, when analyzing switched systems, different concepts in constraining the switching have been proposed, like e.g. dwell-time (Morse, 1996) or average dwell-time switching signals (Hespanha & Morse, 1999). On the other hand, it is also possible that a switched system exhibits some property like asymptotic stability even if some subsystems lack this property (Muñoz de la Peña & Christofides, 2008; Zhai, Hu, Yasuda, & Michel, 2001). This situation often appears in practice, e.g. in the context of networked control systems (Muñoz de la Peña & Christofides, 2008).

Considering the above, an interesting question is under what conditions IOSS can be established and how state-norm estimators can be constructed for switched systems. IOSS for switched differential inclusions (Mancilla-Aguilar, García, Sontag, & Wang, 2005) as well as state-norm estimators for switched systems (García & Mancilla-Aguilar, 2002) have been considered in the setting of arbitrary switching and a common IOSS-Lyapunov function. Furthermore, input-to-state stability (ISS), which can be seen as a special case of IOSS when no outputs are present, has been established for switched systems in Vu, Chatterjee, and Liberzon (2007); Xie, Wen, and Li (2001) for the situation where a common ISS-Lyapunov function does not exist, but still all of the subsystems are ISS. This was achieved using a dwell-time (Xie et al., 2001), respectively, average dwell-time (Vu et al., 2007) approach. In the recent work (Sanfelice, 2010), results on IOSS and state-norm estimators were established for a class of hybrid systems with a Lyapunov function satisfying an IOSS relation both along the flow and during the jumps.

In this work, we establish IOSS of switched nonlinear systems in the setting of multiple IOSS-Lyapunov functions and constrained switching. We consider both the cases where all of the constituent subsystems are IOSS as well as where some are not. In fact, our findings in the latter case also yield novel results on ISS (if no outputs are present) and asymptotic stability (if neither inputs nor outputs are present) of switched systems, when some of the subsystems lack the considered property. Furthermore, we show that under the same sufficient conditions under which IOSS can be established, a state-norm estimator exists for the switched system. For the case where all of the constituent subsystems are IOSS, we obtain a non-switched state-norm estimator, whereas in the case where also some non-IOSS subsystems are present, a switched state-norm estimator can be constructed, consisting of one stable and one unstable mode. It turns out that in the latter case, the switched state-norm estimator can be constructed in such a way that its switching times are independent of the switching times of the switched system it is designed for. This is a desirable property, as otherwise, the switching times of the switched system would have to be known *a priori*, or detected instantly.

The remainder of this paper is structured as follows. Section 2 introduces the notation and basic definitions used throughout the paper. Sections 3 and 4 contain the main results of the paper, which deal with establishing IOSS and constructing state-norm estimators for switched systems. Section 5 contains an illustrative example, highlighting the degrees of freedom in the construction and the difference between the proposed state-norm estimators. Section 6 concludes the paper.

2. Preliminaries

Consider a family of systems

$$\begin{aligned} \dot{x} &= f_p(x,u) \\ y &= h_p(x) \end{aligned} \quad p \in \mathcal{P} \tag{1}$$

where the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^l$ and $\mathcal{P}$ is an index set. For every $p \in \mathcal{P}$, $f_p(\cdot,\cdot)$ is locally Lipschitz, $h_p(\cdot)$ is continuous, $f_p(0,0) = 0$ and $h_p(0) = 0$. A *switched system*

$$\begin{aligned} \dot{x} &= f_\sigma(x,u) \\ y &= h_\sigma(x) \end{aligned} \tag{2}$$

is generated by the family of systems (1), an initial condition $x(t_0) = x_0$ with initial time $t_0 \geq 0$, and a switching signal $\sigma(\cdot)$, where $\sigma : [t_0,\infty) \to \mathcal{P}$ is a piecewise constant, right continuous function which specifies at each time t the index of the active system. Admissible input signals $u(\cdot)$ applied to the switched system (2) are measurable and locally bounded. In order to simplify notation, in the following we assume that the solution of the switched system (2) exists for all times. If this is not the case, but the solution is only defined on some finite interval $[t_0, t_{\max})$, all subsequent results are still valid for this interval.

According to Hespanha and Morse (1999) we say that a switching signal has *average dwell-time* τ_a if there exist numbers $N_0, \tau_a > 0$ such that

$$\forall T \geq t \geq t_0 : \quad N_\sigma(T,t) \leq N_0 + \frac{T-t}{\tau_a}, \tag{3}$$

where $N_\sigma(T,t)$ is the number of switches occurring in the interval $(t,T]$.

Denote the switching times in the interval $(t_0,t]$ by $\tau_1, \tau_2, \ldots, \tau_{N_\sigma(t,t_0)}$ (by convention, $\tau_0 := t_0$) and the index of the system that is active in the interval $[\tau_i, \tau_{i+1})$ by p_i.

The switched system (2) is *input/output-to-state stable (IOSS)* (Sontag & Wang, 1997) if there exist functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$[2] and $\beta \in \mathcal{KL}$[3] such that for each $t_0 \geq 0$, each $x_0 \in \mathbb{R}^n$ and each input $u(\cdot)$, the corresponding solution satisfies

$$|x(t)| \leq \beta(|x_0|, t-t_0) + \gamma_1(\|u\|_{[t_0,t]}) + \gamma_2(\|y\|_{[t_0,t]}) \tag{4}$$

for all $t \geq t_0$, where $\|\cdot\|_J$ denotes the supremum norm on an interval J. If no outputs are considered and equation (4) holds for $\gamma_2 \equiv 0$, then the system is said to be input-to-state stable (ISS). If also no inputs are present, then (4) reduces to global asymptotic stability.

In the following, the notion of a state-norm estimator will formally be introduced, which will be done in consistency with Sontag and Wang (1997).

Definition 1. Consider a system

$$\dot{z} = g(z,u,y) \tag{5}$$

whose inputs are the input u and the output y of the switched system (2), and g is locally Lipschitz. Denote by $z(\cdot)$ the solution trajectory of (5) starting at z_0 at time $t = t_0$. We say that (5) is a *state-norm estimator* for the switched system (2) if the following properties hold:

(1) The system (5) is ISS with respect to (u,y).
(2) There exist functions $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for each $t_0 \geq 0$, arbitrary initial states x_0 for (2) and z_0 for (5) and each input $u(\cdot)$,

$$|x(t)| \leq \beta(|x_0| + |z_0|, t-t_0) + \gamma(|z(t)|) \tag{6}$$

for all $t \geq t_0$. □

[2] A function α: $[0,\infty) \to [0,\infty)$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_\infty$.

[3] A function β: $[0,\infty) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$ if $\beta(\cdot,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$, and $\beta(r,t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

Definition 1 ensures that the norm of the switched system state at time t, $|x(t)|$, can be bounded above by the norm of the state-norm estimator at time t, $|z(t)|$, modulo a decaying term of the initial conditions of the switched system and the state-norm estimator. In this sense, the system (5) "estimates" the norm of the switched system (2), and thus it is called a state-norm estimator.

Remark 1. We will later enlarge the class of state-norm estimators by allowing the state-norm estimator to be a switched system itself. However, the properties (1) and (2) in Definition 1 such a state-norm estimator has to fulfill remain unchanged. □

3. Input/output-to-state properties of switched systems

In this section, we show under what Lyapunov-like conditions IOSS for the switched system (2) can be established. We start with the situation where all of the constituent subsystems are IOSS, before allowing some of the subsystems also to be not IOSS.

3.1. All subsystems IOSS

Theorem 1. *Consider the family of systems (1). Suppose there exist functions $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda_s > 0$, $\mu \geq 1$ such that for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ we have*

$$\alpha_1(|x|) \leq V_p(x) \leq \alpha_2(|x|) \tag{7}$$

$$|x| \geq \varphi_1(|u|) + \varphi_2(|h_p(x)|)$$
$$\Rightarrow \frac{\partial V_p}{\partial x} f_p(x, u) \leq -\lambda_s V_p(x) \tag{8}$$

$$V_p(x) \leq \mu V_q(x). \tag{9}$$

If σ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s}, \tag{10}$$

then the switched system (2) is IOSS.

In the following, the assumptions of Theorem 1 will be discussed shortly. First, note that conditions of the type (7)–(10) are quite common in the literature, when average dwell-time switching signals are considered. The existence of a function V_p satisfying (7)–(8) is a necessary and sufficient condition for the p-th subsystem to be IOSS (Krichman et al., 2001). Such a function V_p is called an exponential decay IOSS-Lyapunov function for the p-th subsystem (Krichman et al., 2001). Taking the right hand side of (8) as some negative multiple of V_p instead of just some negative definite function W_p is no loss of generality (Praly & Wang, 1996; Sontag & Wang, 1997). The most significant constraint on the set of possible IOSS-Lyapunov functions for the subsystems is given by condition (9). For example, this condition doesn't hold if V_p is quadratic and V_q is quartic for some $p, q \in \mathcal{P}$. This condition might seem to be somehow restrictive; however, it is quite common in the literature when dealing with average dwell-time switching signals, and it is a considerable relaxation to the case where a common Lyapunov function is required, i.e., where (9) has to hold for $\mu = 1$ (cf. also Remark 3). See also Vu et al. (2007) for a more detailed discussion and an example on how to further relax this assumption.

Proof of Theorem 1. Let $t_0 \geq 0$ be arbitrary. For $t \geq t_0$, define $\nu(t) := \varphi_1(\|u\|_{[t_0,t]}) + \varphi_2(\|y\|_{[t_0,t]})$ and $\xi(t) := \alpha_1^{-1}(\mu^{N_0}\alpha_2(\nu(t)))$, where N_0 comes from (3). Furthermore, define the ball around the origin $B_\nu(t) := \{x \mid |x| \leq \nu(t)\}$. Note that ν, and thus also ξ, are non-decreasing functions of time, and therefore the ball B_ν and B_ξ has non-decreasing volume.

If $|x(t)| \geq \nu(t) \geq \varphi_1(|u(t)|) + \varphi_2(|y(t)|)$ during some time interval $t \in [t', t'']$, then $|x(t)|$ can be bounded above (Hespanha & Morse, 1999) by

$$|x(t)| \leq \alpha_1^{-1}(\mu^{N_0} e^{-\lambda(t-t')}\alpha_2(|x(t')|)) := \beta(|x(t')|, t-t') \tag{11}$$

for some $\lambda \in (0, \lambda_s)$. To see why this is true, consider the function $W(t) := e^{\lambda_s t} V_{\sigma(t)}(x(t))$. On any interval $[\tau_i, \tau_{i+1}) \cap [t', t'']$, we have according to (8) $\dot{W}(t) \leq 0$. Using (9), we arrive at $W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i)$ and thus, for any $t \in [t', t'']$, we obtain $W(t) \leq \mu^{N_\sigma(t,t')} W(t')$ and therefore

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq e^{N_\sigma(t,t')\ln\mu - \lambda_s(t-t')} V_{\sigma(t')}(x(t')) \\ &\leq e^{N_0 \ln\mu} e^{(\frac{\ln\mu}{\tau_a} - \lambda_s)(t-t')} V_{\sigma(t')}(x(t')). \end{aligned} \tag{12}$$

If τ_a satisfies the condition (10), then $V_{\sigma(t)}(x(t))$ decays exponentially in the time interval $[t', t'']$, namely for every $t \in [t', t'']$, it is upper bounded by

$$V_{\sigma(t)}(x(t)) \leq e^{N_0 \ln\mu} e^{-\lambda(t-t')} V_{\sigma(t')}(x(t'))$$

with $\lambda := \lambda_s - \ln\mu/\tau_a \in (0, \lambda_s)$. Using (7), we arrive at (11).

Denote the first time when $x(t) \in B_\nu(t)$ by $\check{t}_1$, i.e., $\check{t}_1 := \inf\{t \geq t_0 : |x(t)| \leq \nu(t)\}$. For $t_0 \leq t \leq \check{t}_1$, we get

$$|x(t)| \leq \beta(|x_0|, t-t_0), \tag{13}$$

according to (11). If $\check{t}_1 = \infty$, which only can happen if $\nu(t) \equiv 0$, i.e., both the input u as well as the output y are equivalent to zero for all times, then (4) is established and thus the switched system (2) is IOSS. Hence in the following we assume that $\check{t}_1 < \infty$.

For $t > \check{t}_1$, $|x(t)|$ can be bounded above in terms of $\nu(t)$. Namely, let $\hat{t}_1 := \inf\{t > \check{t}_1 : |x(t)| > \nu(t)\}$. If this is an empty set, let $\hat{t}_1 := \infty$. Clearly, for all $t \in [\check{t}_1, \hat{t}_1)$, it holds that $|x(t)| \leq \nu(t) \leq \xi(t)$. For the case that $\hat{t}_1 < \infty$, due to continuity of $x(\cdot)$ and monotonicity of $\nu(\cdot)$ it holds that $|x(\hat{t}_1)| = \nu(\hat{t}_1)$. Furthermore, for all $\tau > \hat{t}_1$, if $|x(\tau)| > \nu(\tau)$ define

$$\hat{t} := \sup\{t < \tau : |x(t)| \leq \nu(t)\}, \tag{14}$$

which can be interpreted as the previous exit time of the trajectory $x(\cdot)$ from the ball B_ν. Again, due to the same argument as above, one obtains that $|x(\hat{t})| = \nu(\hat{t})$. But then, according to (11), it holds that

$$\begin{aligned} |x(\tau)| &\leq \beta(\nu(\hat{t}), \tau - \hat{t}) = \alpha_1^{-1}(\mu^{N_0} e^{-\lambda(\tau-\hat{t})}\alpha_2(\nu(\hat{t}))) \\ &\leq \alpha_1^{-1}(\mu^{N_0}\alpha_2(\nu(\hat{t}))) = \xi(\hat{t}) \leq \xi(\tau), \end{aligned} \tag{15}$$

where the last inequality follows from the monotonicity of $\xi(\cdot)$.

Summarizing the above, for all $t \geq \check{t}_1$ it holds that

$$\begin{aligned} |x(t)| &\leq \xi(t) \\ &= \alpha_1^{-1}(\mu^{N_0}\alpha_2(\varphi_1(\|u\|_{[t_0,t]}) + \varphi_2(\|y\|_{[t_0,t]}))) \\ &\leq \alpha_1^{-1}(\mu^{N_0}\alpha_2(2\varphi_1(\|u\|_{[t_0,t]}))) \\ &\quad + \alpha_1^{-1}(\mu^{N_0}\alpha_2(2\varphi_2(\|y\|_{[t_0,t]}))) \\ &=: \gamma_1(\|u\|_{[t_0,t]}) + \gamma_2(\|y\|_{[t_0,t]}). \end{aligned} \tag{16}$$

Combining (13) and (16) we arrive at

$$|x(t)| \leq \beta(|x_0|, t-t_0) + \gamma_1(\|u\|_{[t_0,t]}) + \gamma_2(\|y\|_{[t_0,t]})$$

for all $t \geq t_0$. But as $t_0 \geq 0$ was arbitrary, this means according to (4) that the switched system (2) is IOSS. □

Remark 2. Theorem 1 recovers as special cases results on ISS (if no outputs are considered (Vu et al., 2007)) and global asymptotic stability (if neither inputs nor outputs are considered (Hespanha & Morse, 1999)) for switched systems.

Remark 3. If (9) holds for $\mu = 1$, then the condition (10) which the average dwell-time has to satisfy in order that the system is IOSS reduces to $\tau_a > 0$, which means that the system is IOSS for arbitrarily small average dwell time. Actually, $\mu = 1$ in condition (9) implies the existence of a common IOSS-Lyapunov function for the switched system (2), and thus it is in fact IOSS for arbitrary switching (see also Mancilla-Aguilar et al., 2005). □

Remark 4. In the proof of Theorem 1, one major difference compared to the non-switched case is the proceeding after the time $\check{t}_1$. Namely, if we denote the index of the subsystem active at this time by p_1^* and if we define the level set $\Omega_p(t) := \{x \mid V_p(x) \leq \alpha_2(\nu(t))\}$, then the solution $x(t)$ couldn't leave the level set $\Omega_{p_1^*}(t)$ again if no switching occurred for $t > \check{t}_1$, because $\dot{V}_{p_1^*}$ is negative on its boundary. Thus in this case, we could conclude the proof by simply noting that $|x(t)| \leq \alpha_1^{-1}(\alpha_2(\nu(t)))$ for all $t > \check{t}_1$. Due to switching, however, $x(t)$ can leave the level set $\Omega_{p_1^*}(t)$ again and thus we have to proceed with the proof as shown above. □

3.2. Some subsystems not IOSS

In the following, the previous analysis will be extended to the case where not all subsystems of the family (1) are IOSS, i.e., (8) doesn't hold for all $p \in \mathcal{P}$, but only for a subset $\mathcal{P}_s$ of $\mathcal{P}$.

Let $\mathcal{P} = \mathcal{P}_s \cup \mathcal{P}_u$ such that $\mathcal{P}_s \cap \mathcal{P}_u = \emptyset$. Denote by $T^u(t, \tau)$ the total activation time of the systems in $\mathcal{P}_u$ and by $T^s(t, \tau)$ the total activation time of the systems in $\mathcal{P}_s$ during the time interval $[\tau, t)$, where $t_0 \leq \tau \leq t$. Clearly,

$$T^s(t, \tau) = t - \tau - T^u(t, \tau). \tag{17}$$

Theorem 2. *Consider the family of systems (1). Suppose there exist functions $\alpha_1, \alpha_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda_s, \lambda_u > 0$, $\mu \geq 1$ such that (7) and (9) hold for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ and furthermore, the following holds:*

$$|x| \geq \varphi_1(|u|) + \varphi_2(|h_p(x)|) \Rightarrow \begin{cases} \dfrac{\partial V_p}{\partial x} f_p(x, u) \leq -\lambda_s V_p(x) & \forall p \in \mathcal{P}_s \\ \dfrac{\partial V_p}{\partial x} f_p(x, u) \leq \lambda_u V_p(x) & \forall p \in \mathcal{P}_u. \end{cases} \tag{18}$$

If there exist constants $\rho, T_0 \geq 0$ such that

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u} \tag{19}$$

$$\forall t \geq \tau \geq t_0 : \quad T^u(t, \tau) \leq T_0 + \rho(t - \tau) \tag{20}$$

and if σ is a switching signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s(1 - \rho) - \lambda_u \rho}, \tag{21}$$

then the switched system (2) is IOSS.

Remark 5. The existence of smooth and proper functions V_p satisfying the second part of (18), i.e., the condition for the subsystems in $\mathcal{P}_u$, is equivalent to the fact that these subsystems exhibit the unboundedness observability property (Angeli & Sontag, 1999), which means that any unboundedness in the state (i.e., any finite escape time) can be detected by the output. This is a very reasonable assumption, as one cannot expect to obtain the IOSS property for the switched system if for some subsystems an unbounded state cannot be "observed". Furthermore, note that we impose the additional condition (7) on the functions V_p (which in particular implies that $V(0)$ has to be 0), which was not part of the equivalent characterization of the unboundedness observability property in Angeli and Sontag (1999). □

Remark 6. The conditions (19)–(20) constrain the activation time T^u of the systems in $\mathcal{P}_u$ in the interval $[\tau, t)$ to a certain fraction of this interval (plus some offset T_0). Note that fulfillment of (19)–(20) with $T_0 = 0$ implies that the systems in $\mathcal{P}_u$ are not active at all, as (20) is required to hold for any interval $[\tau, t]$ and $\rho < 1$.

In order to prove Theorem 2, we need the following technical lemma.

Lemma 1. *Suppose the assumptions of Theorem 2 hold and on some interval $[t', t'']$ we have $|x(t)| \geq \varphi_1(|u(t)|) + \varphi_2(|y(t)|)$. Then the trajectory of the switched system (2) satisfies*

$$|x(t)| \leq \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} e^{-\lambda(t-t')} \alpha_2(|x(t')|)). \tag{22}$$

for all $t \in [t', t'']$ with $\lambda \in (0, \lambda_s - (\lambda_s + \lambda_u)\rho)$.

Proof of Lemma 1. Consider the function $W(t) := e^{\lambda_s t} V_{\sigma(t)}(x(t))$. On any interval $[\tau_i, \tau_{i+1}) \cap [t', t'']$ we have according to (18)

$$\dot{W}(t) \leq 0 \quad \text{if } p_i \in \mathcal{P}_s$$
$$\dot{W}(t) \leq (\lambda_s + \lambda_u) W(t) \quad \text{if } p_i \in \mathcal{P}_u.$$

Using (9), we thus arrive at

$$W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i)$$

if $p_i \in \mathcal{P}_s$ and

$$W(\tau_{i+1}) \leq \mu W(\tau_{i+1}^-) \leq \mu W(\tau_i) e^{(\lambda_s+\lambda_u)\ (\tau_{i+1}-\tau_i)}$$

if $p_i \in \mathcal{P}_u$. Thus, for any $t \in [t', t'']$ we obtain

$$W(t) \leq \mu^{N_\sigma(t,t')} W(t') e^{(\lambda_s+\lambda_u)T^u(t,t')}$$

and therefore, using (20),

$$\begin{aligned} V_{\sigma(t)}(x(t)) &\leq e^{N_\sigma(t,t')\ln\mu + (\lambda_s+\lambda_u)T^u(t,t') - \lambda_s(t-t')} V_{\sigma(t')}(x(t')) \\ &\leq e^{N_0 \ln\mu + (\lambda_s+\lambda_u)T_0} e^{(\frac{\ln\mu}{\tau_a} + (\lambda_s+\lambda_u)\rho - \lambda_s)(t-t')} \\ &\quad \times V_{\sigma(t')}(x(t')). \end{aligned} \tag{23}$$

We conclude that if ρ and τ_a satisfy the conditions (19) and (21), respectively, then $V_{\sigma(t)}(x(t))$ decays exponentially, namely it is upper bounded by

$$V_{\sigma(t)}(x(t)) \leq e^{N_0 \ln\mu + (\lambda_s+\lambda_u)T_0} e^{-\lambda(t-t')} V_{\sigma(t')}(x(t'))$$

with $\lambda := \lambda_s - (\lambda_s + \lambda_u)\rho - \ln\mu/\tau_a \in (0, \lambda_s - (\lambda_s + \lambda_u)\rho)$. Finally, using (7), we arrive at (22), which completes the proof of Lemma 1. □

Proof of Theorem 2. The proof of Theorem 2 follows the lines of the proof of Theorem 1. Define $\nu(t)$ as well as $\check{t}_1$ as in the proof of Theorem 1. Furthermore, define $\xi(t) := \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} \alpha_2(\nu(t)))$.

According to Lemma 1 we obtain that for $t_0 \leq t \leq \check{t}_1$,

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} e^{-\lambda(t-t_0)} \alpha_2(|x_0|)) \\ &:= \beta(|x_0|, t - t_0) \end{aligned} \tag{24}$$

for some $\lambda \in (0, \lambda_s - (\lambda_s + \lambda_u)\rho)$.

Analogous to Theorem 1, we obtain that for all $t \geq \check{t}_1$ it holds that $|x(t)| \leq \xi(t)$. Namely, for each $\tau > \check{t}_1$ such that $|x(\tau)| > \nu(\tau)$, define the previous exit time $\hat{t}$ of the trajectory $x(\cdot)$ from the ball B_ν as in (14). Then, using Lemma 1 on the interval $[\hat{t}, \tau]$, we obtain the following inequality analogous to (15):

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} e^{-\lambda(\tau-\hat{t})} \alpha_2(\nu(\hat{t}))) \\ &\leq \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} \alpha_2(\nu(\hat{t}))) = \xi(\hat{t}) \leq \xi(\tau). \end{aligned}$$

Hence we conclude that for all $t \geq t_0$

$$|x(t)| \leq \beta(|x_0|, t-t_0) + \xi(t) \leq \beta(|x_0|, t-t_0) + \gamma_1(\|u\|_{[t_0,t]}) + \gamma_2(\|y\|_{[t_0,t]}),$$

where

$$\gamma_1(r) := \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0}\alpha_2(2\varphi_1(r)))$$
$$\gamma_2(r) := \alpha_1^{-1}(\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0}\alpha_2(2\varphi_2(r))),$$

which means according to (4) that the switched system (2) is IOSS, as $t_0 \geq 0$ was arbitrary. □

Remark 7. Theorem 2 includes as special cases novel results on ISS (if no outputs are considered) and global asymptotic stability (if neither inputs nor outputs are considered) for switched systems where some of the subsystems lack the considered property. In fact, if systems with no inputs and outputs are considered, Lemma 1 can be evoked for the interval $[t_0, \infty)$ to prove global asymptotic stability for the switched system. □

4. State-norm estimators for switched systems

In this section, we address the question of existence and construction of state-norm estimators for switched systems. For continuous-time non-switched systems, it was proved in Sontag and Wang (1997) that the existence of a state-norm estimator as defined in Definition 1 implies that the system is IOSS. This is also true for switched systems, as the proof works in the exact same way as for continuous-time non-switched systems. In fact, it was shown in Krichman et al. (2001) that for non-switched systems, the converse is also true, i.e., that a state-norm estimator exists if the system is IOSS. This was done by showing that the system being IOSS implies the existence of an (exponential decay) IOSS-Lyapunov function, which in turn implies the existence of a state-norm estimator. In García and Mancilla-Aguilar (2002), the equivalence between the IOSS property and the existence of a state-norm estimator was established for switched systems for the situation where a common IOSS-Lyapunov function exists. On the other hand, within our setup we cannot establish such an equivalence relationship as we consider switched systems where no common IOSS-Lyapunov function exists, and some of the subsystems might not even be IOSS at all. Nevertheless, it turns out that under the same sufficient conditions under which IOSS could be established in the previous section, a state-norm estimator also exists for such a switched system. Thus what follows can be seen as an alternative way of establishing IOSS for the switched system (2), which yields the nice "intermediate" result of obtaining a state-norm estimator for the considered switched system.

4.1. State-norm estimators: all subsystems IOSS

As in Section 3, we start with the situation where all subsystems are IOSS. For the construction of a state-norm estimator, we need a slightly different characterization of the IOSS property than equation (8). Namely, in Krichman et al. (2001); Sontag and Wang (1995) it was shown that (8) is equivalent to

$$\frac{\partial V_p}{\partial x} f_p(x,u) \leq -\lambda_s V_p(x) + \chi_1(|u|) + \chi_2(|h_p(x)|), \tag{25}$$

for some $\chi_1, \chi_2 \in \mathcal{K}_\infty$ and $\lambda_s > 0$. Furthermore, if (8) holds for some $\lambda_s > 0$, then (25) holds with the same value of λ_s.[4]

We are now in a position to state the following theorem concerning state-norm estimators for switched systems whose subsystems are all IOSS.

[4] On the other hand, when going in the other direction, i.e., from (25) to (8), in general λ_s needs to be decreased. Nevertheless, with a slight abuse of notation, we continue to use the same symbol λ_s in (25) as in (8) for convenience.

Theorem 3. *Consider the family of systems (1). Suppose there exist functions $\alpha_1, \alpha_2, \chi_1, \chi_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda_s > 0$, $\mu \geq 1$ such that for all $x \in \mathbb{R}^n$ and all $p, q \in \mathcal{P}$ the conditions (7), (9) and (25) are satisfied. Furthermore, suppose that σ is a switching signal with average dwell-time τ_a satisfying (10). Then there exists a (non-switched) state-norm estimator for the switched system (2). A possible choice for such a state-norm estimator is*

$$\dot z(t) = g(z,u,y) = -\lambda_s^* z(t) + \chi_1(|u(t)|) + \chi_2(|y(t)|), \quad z_0 \geq 0 \tag{26}$$

for some $\lambda_s^* \in (0, \lambda_s)$.[5]

Remark 8. In Theorem 3 (and also the following theorems), for technical reasons in the proofs, we restrict the initial condition of the state-norm estimator to be nonnegative, whereas in Definition 1 we allow (in consistency with Sontag and Wang (1997)) the initial condition of the state-norm estimator to be arbitrary. However, as we design the state-norm estimator and thus can choose any initial condition we want, this is not a major restriction. □

Proof of Theorem 3. Consider as a candidate for a state-norm estimator the system (26) with $\lambda_s^* \in (0, \lambda_s)$. In the following, we have to verify that (26) satisfies the two properties of Definition 1, namely that it is ISS with respect to the inputs (u, y) and that (6) holds. It is easy to see that (26) is ISS with respect to the inputs (u, y), as it is a linear, exponentially stable system driven by these inputs. Thus it remains to show that (6) holds.

Note that as $\chi_1(|u|) + \chi_2(|y|) \geq 0$, we have $\dot z(t) \geq -\lambda_s^* z(t)$ and thus, as $z_0 \geq 0$,

$$z(t) \geq e^{-\lambda_s^*(t-t_0)} z_0 \geq 0 \tag{27}$$

for all $t \geq t_0$. Furthermore, for all $t_0 \leq \tau_i \leq t$ we get

$$z(\tau_i) \leq e^{\lambda_s^*(t-\tau_i)} z(t). \tag{28}$$

Now consider the function $W(t) := V_{\sigma(t)}(x(t)) - z(t)$. Using (25)–(27), we obtain that in any interval $[\tau_i, \tau_{i+1})$,

$$\dot W = \dot V_{p_i} - \dot z \leq -\lambda_s V_{p_i} + \lambda_s^* z \leq -\lambda_s V_{p_i} + \lambda_s z = -\lambda_s W$$

and thus

$$\begin{aligned} W(\tau_{i+1}) &= V_{\sigma(\tau_{i+1})}(x(\tau_{i+1})) - z(\tau_{i+1}) \\ &\leq \mu V_{\sigma(\tau_i)}(x(\tau_{i+1}^-)) - z(\tau_{i+1}) \\ &= \mu W(\tau_{i+1}^-) + (\mu - 1) z(\tau_{i+1}) \\ &\leq \mu W(\tau_i) e^{-\lambda_s(\tau_{i+1}-\tau_i)} + (\mu-1) z(\tau_{i+1}). \end{aligned} \tag{29}$$

Iterating (29) from $i = 0$ to $i = N_\sigma(t, t_0)$ and using (28), we arrive at

$$W(t) \leq \mu^{N_\sigma(t,t_0)} \Bigg(e^{-\lambda_s(t-t_0)} W(t_0) + (\mu-1) \sum_{k=1}^{N_\sigma(t,t_0)} \mu^{-k} e^{-\lambda_s(t-\tau_k)} z(\tau_k) \Bigg)$$

[5] In Definition 1, we required g to be locally Lipschitz for the reason of existence and uniqueness of solutions. However, g defined in (26) might not be locally Lipschitz in u and y as χ_1 and χ_2 are not necessarily locally Lipschitz. Nevertheless, with $w := \chi_1(|u|) + \chi_2(|y|)$, g defined in (26) is locally Lipschitz as a function of (z, w), and hence a unique solution to the system (26) exists on the same interval as for the switched system (2). Similar considerations apply to the state-norm estimators proposed in Theorems 4 and 5.

$$\le e^{N_\sigma(t,t_0)\ln\mu-\lambda_s(t-t_0)}W(t_0) + (\mu-1)z(t)\sum_{k=1}^{N_\sigma(t,t_0)} e^{(N_\sigma(t,t_0)-k)\ln\mu-(\lambda_s-\lambda_s^*)(t-\tau_k)}. \quad (30)$$

Since $N_\sigma(t,t_0)-k = N_\sigma(t,\tau_k)$, we get, using (3),

$$\begin{aligned}(N_\sigma(t,t_0)-k)\ln\mu-(\lambda_s-\lambda_s^*)(t-\tau_k) &\le N_\sigma(t,\tau_k)\ln\mu-(\lambda_s-\lambda_s^*)(t-\tau_k)\\ &\le \left(N_0+\frac{t-\tau_k}{\tau_a}\right)\ln\mu-(\lambda_s-\lambda_s^*)(t-\tau_k)\\ &\le N_0\ln\mu-\lambda(t-\tau_k) \quad (31)\end{aligned}$$

with $\lambda := \lambda_s-\lambda_s^*-\ln\mu/\tau_a \in (0,\lambda_s-\lambda_s^*)$ if the average dwell time τ_a satisfies the bound

$$\tau_a > \frac{\ln\mu}{\lambda_s-\lambda_s^*}. \quad (32)$$

Note that as we can choose λ_s^* arbitrarily close to 0, we can choose it small enough such that for any average dwell time τ_a satisfying (10), condition (32) is also satisfied.

The average dwell-time property (3) furthermore implies that

$$t-\tau_k \ge (N_\sigma(t,t_0)-k-N_0)\tau_a. \quad (33)$$

Combining (31) and (33) we arrive at

$$\sum_{k=1}^{N_\sigma(t,t_0)} e^{(N_\sigma(t,t_0)-k)\ln\mu-(\lambda_s-\lambda_s^*)(t-\tau_k)} \le e^{N_0(\ln\mu+\lambda\tau_a)}\sum_{k=1}^{N_\sigma(t,t_0)} e^{-\lambda\tau_a(N_\sigma(t,t_0)-k)} =: a_1.$$

Applying the index shift $i := N_\sigma(t,t_0)-k$ we obtain

$$\begin{aligned}a_1 &= e^{N_0(\ln\mu+\lambda\tau_a)}\sum_{i=0}^{N_\sigma(t,t_0)-1} e^{-\lambda\tau_a i}\\ &\le e^{N_0(\ln\mu+\lambda\tau_a)}\sum_{i=0}^{\infty} e^{-\lambda\tau_a i}\\ &= e^{N_0(\ln\mu+\lambda\tau_a)}\frac{1}{1-e^{-\lambda\tau_a}} =: a_2. \quad (34)\end{aligned}$$

Thus, by virtue of (30), we get

$$\begin{aligned}W(t) &\le e^{N_\sigma(t,t_0)\ln\mu-\lambda_s(t-t_0)}W(t_0)+(\mu-1)a_2z(t)\\ &= e^{(N_0+\frac{t-t_0}{\tau_a})\ln\mu-\lambda_s(t-t_0)}W(t_0)+(\mu-1)a_2z(t)\\ &\le \mu^{N_0}e^{-\lambda'(t-t_0)}W(t_0)+(\mu-1)a_2z(t)\end{aligned}$$

with $\lambda' := \lambda_s-\ln\mu/\tau_a \in (\lambda_s^*,\lambda_s)$ if τ_a satisfies (32). This leads to

$$\begin{aligned}V_{\sigma(t)}(x(t)) &\le (1+a_2(\mu-1))z(t) + \mu^{N_0}e^{-\lambda'(t-t_0)}(V_{\sigma(t_0)}(x_0)-z_0)\\ &\le (1+a_2(\mu-1))|z(t)| + 2\mu^{N_0}e^{-\lambda'(t-t_0)}\alpha_2(|x_0|+|z_0|),\end{aligned}$$

if we assume without loss of generality that $\alpha_2(r)\ge r$ for all $r\ge 0$. Using (7) again, we finally arrive at

$$\begin{aligned}|x(t)| &\le \alpha_1^{-1}(2(1+a_2(\mu-1))|z(t)|) + \alpha_1^{-1}(4\mu^{N_0}e^{-\lambda'(t-t_0)}\alpha_2(|x_0|+|z_0|))\\ &=: \gamma(|z(t)|)+\beta(|x_0|+|z_0|,t-t_0), \quad (35)\end{aligned}$$

which means that our state-norm estimator candidate (26) satisfies the condition (6). □

Remark 9. The construction of a state-norm estimator as shown above might suggest designing a state-norm estimator which exhibits a jump by a factor of μ in its state at the switching times of the switched system. In this case, the derivation would simplify significantly (in particular (30)–(34)). This idea was also used in Sanfelice (2010), where the state-norm estimator is a hybrid system which exhibits jumps at certain time instances. In this paper, we do not consider this possibility for the following reasons: we obtain a state-norm estimator whose trajectory is absolutely continuous (even continuously differentiable), and furthermore, for our state-norm estimator we do not need to know the switching times of σ. □

Remark 10. If a state-norm estimator is constructed as proposed in Theorem 3, a degree of freedom in the design is the choice of λ_s^*. The only restriction is that condition (32) has to be satisfied, which, as stated in the proof, is always possible and gives an upper bound for the values λ_s^* can take. Choosing λ_s^* as large as possible would be desirable as the state-norm estimator (26) then has a better convergence rate. However, if λ_s^* is chosen close to its largest possible value, i.e., such that (32) is only barely satisfied, then (31) is only valid for λ very close to zero. According to (34), this leads to a large value for a_2, which in turn implies that the gain γ in (35), with which $|x|$ can be bounded in terms of $|z|$, also becomes large, which is not desirable. Thus a tradeoff for a good choice of λ_s^* has to be found. This will be illustrated in Section 5.1 with an example. □

4.2. State-norm estimators: some subsystems not IOSS

In the following, we will consider the case where some of the subsystems of the family (1) are not IOSS. Again, we use a slightly different but equivalent formulation of condition (18). For the subsystems in $\mathcal{P}_s$, we again use (25), which, as stated above, is an equivalent formulation of the IOSS property (Krichman et al., 2001; Sontag & Wang, 1995). For the subsystems in $\mathcal{P}_u$, we use

$$\frac{\partial V_p}{\partial x}f_p(x,u) \le \lambda_u V_p(x)+\chi_1(|u|)+\chi_2(|h_p(x)|), \quad (36)$$

for some $\chi_1,\chi_2 \in \mathcal{K}_\infty$ and $\lambda_u > 0$, which is an equivalent characterization of the unboundedness observability property (Angeli & Sontag, 1999; Sontag & Wang, 1995). Again, if (18) is satisfied for some $\lambda_u > 0$, then also (36) holds with the same value of λ_u.[6]

4.2.1. Known switching times

In the following theorem, we show that under the same conditions as in Theorem 2, a state-norm estimator can be constructed if the exact switching times between an IOSS and a non-IOSS subsystem of (2) are known.

Theorem 4. *Consider the family of systems (1). Suppose there exist functions $\alpha_1,\alpha_2,\chi_1,\chi_2 \in \mathcal{K}_\infty$, continuously differentiable functions $V_p : \mathbb{R}^n \to \mathbb{R}$ and constants $\lambda_s,\lambda_u > 0$, $\mu \ge 1$ such that for all $x \in \mathbb{R}^n$, (7) and (9) hold for all $p,q \in \mathcal{P}$, (25) for all $p \in \mathcal{P}_s$ and (36) for all $p \in \mathcal{P}_u$. Furthermore, suppose that σ is a switching signal such that (19)–(21) are satisfied. Then there exists a switched state-norm estimator $\dot z = g_\zeta(z,u,y)$ for the switched system (2), consisting of two subsystems, where $\zeta : [0,\infty) \to \{0,1\}$ is a switching signal whose switching times are those switching times of*

[6] Similar to the discussion for λ_s, when going in the other direction, i.e., from (36) to (18), in general λ_u needs to be increased. Again, with a slight abuse of notation, we continue to use the same symbol λ_u in (36) as in (18) for convenience.

σ where a switch from a system in $\mathcal{P}_s$ to a system in $\mathcal{P}_u$ or vice versa occurs. A possible choice for the two subsystems is

$$\begin{aligned}\dot{z} &= g_0(z,u,y) = -\lambda_s^* z(t) + \chi_1(|u(t)|) + \chi_2(|y(t)|)\\ \dot{z} &= g_1(z,u,y) = \lambda_u^* z(t) + \chi_1(|u(t)|) + \chi_2(|y(t)|)\end{aligned} \tag{37}$$

with an appropriate choice of $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$.

Proof. See Appendix A. □

Remark 11. Similar considerations as in Remark 10 apply to the choice of $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$, if a state-norm estimator is constructed as proposed in Theorem 4. Namely, a tradeoff between a good convergence rate of the state-norm estimator and a tighter gain γ, with which $|x|$ can be bounded in terms of $|z|$, has to be found. □

4.2.2. Unknown switching times

The construction of the state-norm estimator in Theorem 4 requires the exact knowledge of the switching times of the considered switched system (2), at least of those switching times where a switch from a subsystem in $\mathcal{P}_s$ to a subsystem in $\mathcal{P}_u$ or vice versa occurs. This is a very restrictive assumption, as the switching signal would have to be known *a priori* or switches would somehow have to be detected instantly. Thus, one would like to have some robustness in the construction of the state-norm estimator with respect to the knowledge of the switching times. Even more desirable would be the case where a state-norm estimator can be constructed with a switching signal that is independent of the switching times of the switched system the state-norm estimator is designed for. Then, the only knowledge needed about the switching signal σ of the switched system would be that it satisfies some average dwell-time condition, but knowledge about the (exact) switching times would not be needed.

In the following, we show that under the same conditions as in Theorem 4 (and thus, under the same conditions as in Theorem 2), a state-norm estimator can be constructed whose switching times are independent of the switching times of σ. For the proof of this result, we exploit that a state-norm estimator as proposed in Theorem 4, i.e., with (exact) knowledge of the switching times of σ, exists; however, this knowledge is not needed for designing the switching signal ζ' of the proposed state-norm estimator.

Theorem 5. *Suppose the conditions of Theorem 4 are satisfied. Then there exists a switched state-norm estimator*

$$\dot{w} = g_{\zeta'(t)}(w, u, y), \qquad w_0 \geq 0 \tag{38}$$

for the switched system (2), consisting of two subsystems, where $\zeta' : [0,\infty) \to \{0,1\}$ is a switching signal whose switching times are independent of the switching times of σ. As in Theorem 4, a possible choice for the two subsystems of the state-norm estimator is given by (37). Furthermore, a possible choice for the switching signal ζ' is given by

$$\zeta'(t) = \begin{cases} 0 & \forall t \in [k\tau_a^w, k\tau_a^w + (1-\rho^w)\tau_a^w) \\ 1 & \forall t \in [k\tau_a^w + (1-\rho^w)\tau_a^w, (k+1)\tau_a^w) \end{cases} \tag{39}$$

with $k = 0, 1, 2, \ldots$, where the constants $\tau_a^w > 0$ and $\rho^w > 0$ are chosen such that

$$\rho < \rho^w < \frac{\lambda_s^*}{\lambda_s^* + \lambda_u^*}, \tag{40}$$

$$\rho^w \tau_a^w \geq T_0 + \rho \tau_a^w. \tag{41}$$

Proof. See Appendix B. □

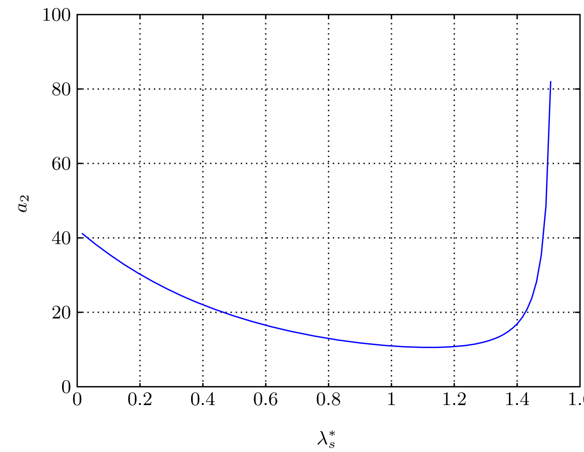

Fig. 1. Influence of λ_s^* on a_2.

Remark 12. If a state-norm estimator is constructed as proposed in Theorem 5, a further degree of freedom lies in the choice of the switching signal ζ'. In particular, if the switching signal ζ' is chosen to be of the form (39), this translates into a degree of freedom in the choice of the parameters τ_a^w and ρ^w. Namely, in order to obtain an as small as possible activation time of the unstable subsystem of the state-norm estimator (38), according to (39), ρ^w should be chosen as small as possible. On the other hand, a choice of ρ^w close to its minimum possible value (i.e., close to ρ according to (40)), which in turn implies that a larger value of τ_a^w is needed according to (41), would result in a larger gain γ' with which $|x|$ can be bounded in terms of $|z|$ (see the proof of Theorem 5 for details). Hence, similar to the choice of λ_s^* and λ_u^*, a tradeoff for a good choice of τ_a^w and ρ^w has to be found.

5. Example

The goal of this section is to illustrate the degrees of freedom in the construction of a state-norm estimator and the difference between the state-norm estimators proposed in Sections 4.2.1 and 4.2.2, respectively.

Consider the family of subsystems

$$\begin{aligned}\dot{x} &= f_p(x,u) = \begin{bmatrix} a_{1p}x_1 + b_{1p}\sin(x_1 - x_2) + c_{1p}u \\ a_{2p}x_2 + b_{2p}\sin(x_2 - x_1) + c_{2p}u \end{bmatrix}\\ h_p(x) &= x_1 - x_2\end{aligned} \tag{42}$$

with $p \in \mathcal{P} := \{1, 2\}$, subject to an average dwell-time switching signal with $N_0 = 2$ and $\tau_a = 1$.

5.1. Choice of parameters

First, we illustrate the degrees of freedom in the choice of the parameters of the state-norm estimator, as pointed out in Remark 10. For clarity of presentation, we only consider the situation where both subsystems are IOSS. Nevertheless, similar considerations apply to the case where some subsystems are not IOSS (cf. Remark 11).

Let $a_{11} = a_{12} = a_{21} = a_{22} = -1$, $b_{11} = b_{12} = b_{22} = 1$, $b_{21} = 0.8$, $c_{11} = c_{21} = 0$ and $c_{11} = c_{21} = 0.5$. Taking $V_1(x) = 1/2(x_1^2 + 1.25x_2^2)$ and $V_2(x) = 1/2(x_1^2 + x_2^2)$ as IOSS-Lyapunov functions for the two subsystems, it is straightforward to verify that (25) holds with $\lambda_s = 7/4$, $\chi_1(r) = 9/8r^2$ and $\chi_2(r) = r$. Furthermore, (9) is satisfied with $\mu = 1.25$. In order to satisfy (32), λ_s^* has to be smaller than 1.51.

Fig. 1 illustrates the influence of the choice of λ_s^* on the value of a_2 (34), which is the tunable parameter in the gain γ (35). The

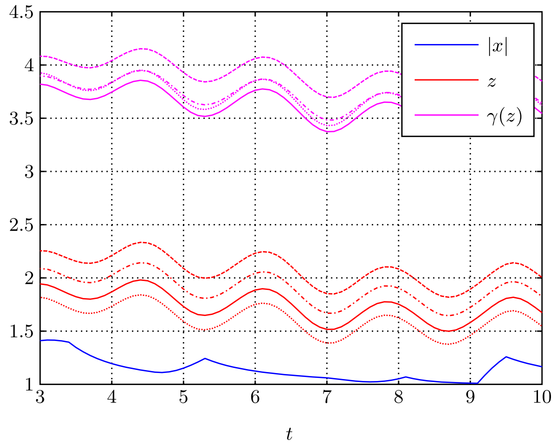

Fig. 2. State-norm estimators with different values of λ_s^*. Dashed lines: $\lambda_s^* = 1.03$; dashed–dotted lines: $\lambda_s^* = 1.13$; solid lines: $\lambda_s^* = 1.23$; dotted lines: $\lambda_s^* = 1.33$.

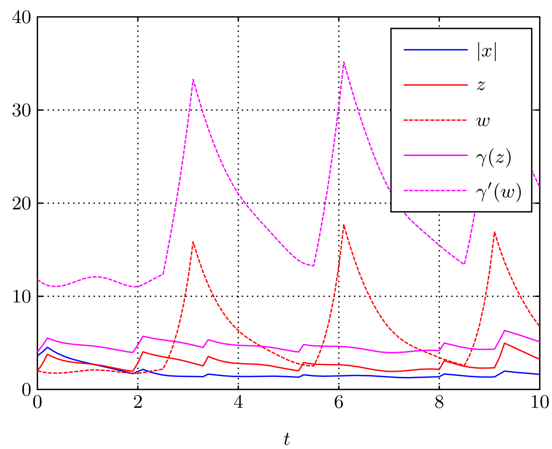

Fig. 3. Comparison of the two state-norm estimators of Theorems 4 and 5. Solid lines: state-norm estimator with exact switching times; dashed lines: state-norm estimator with independent switching times.

minimum is attained for $\lambda_s^* \approx 1.13$. For larger values of λ_s^*, which would be desirable in order to obtain a better convergence rate of the estimator, a_2 and thus the gain γ increases rapidly. Thus a good tradeoff has to be found, which is illustrated by Fig. 2, where simulation results for different values of λ_s^* are given. Both z (red lines) and $\gamma(z)$ (magenta lines), as well as $|x|$ (blue line) are shown after the transient phase. The best estimate $\gamma(z)$ is obtained for $\lambda_s^* = 1.23$, which is smaller than the largest possible value 1.51, but larger than the 1.13, for which the minimum value of a_2 is attained.

5.2. Comparison of state-norm estimators

In this section, we compare the two state-norm estimators proposed in Theorems 4 and 5, respectively, when not all subsystems are IOSS. To this end, consider again the family of two subsystems (42), and let the parameter values be as in Section 5.1, except for $a_{12} = a_{22} = 1$. With these parameters, subsystem 1 is IOSS whereas subsystem 2 is not. Taking V_1 and V_2 as above, it is straightforward to verify that (25) and (36) are satisfied with $\lambda_s = 7/4$, $\lambda_u = 13/6$, $\chi_1(r) = 3/2r^2$ and $\chi_2(r) = r$. Fig. 3 shows simulation results for the two different estimators with $\lambda_s^* = 1.3$ and $\lambda_u^* = 2.7$. The switching signal σ is such that (19)–(21) is satisfied with $\rho = 0.1$ and $T_0 = 0.3$, and the parameters ρ^w and τ_a^w are chosen as $\rho^w = 0.2$ and $\tau_a^w = 3$ and satisfy (40)–(41).

One can see that while the state-norm estimator proposed in Theorem 5 uses less information about the switched system (no knowledge of the switching times is needed), it also gives more conservative estimates for $|x|$, due to the worst-case estimates used in the proof. However, our simulation results show that in practice, the state-norm estimator with independent switching times can also be used with a significantly less conservative gain.

6. Conclusions

In this paper, we established IOSS for switched nonlinear systems under average dwell-time switching signals, both when all of the constituent subsystems are IOSS as well when some of the subsystems lack this property. Furthermore, we showed that under the same sufficient conditions, a state-norm estimator exists for the switched system. In the case where some of the subsystems are not IOSS, the state-norm estimator is a switched system itself, consisting of two subsystems with one stable and one unstable mode. We showed that in this case, the switching times of the state-norm estimator can be chosen independently of the switching times of the switched system.

Appendix A. Proof of Theorem 4

Consider as a candidate for a switched state-norm estimator the system

$$\dot{z} = g_{\zeta(t)}(z, u, y), \qquad z_0 \geq 0, \tag{A.1}$$

where the switching signal $\zeta(t)$ is defined by

$$\zeta(t) = \begin{cases} 0 & \text{if } \sigma(t) \in \mathcal{P}_s \\ 1 & \text{if } \sigma(t) \in \mathcal{P}_u \end{cases} \tag{A.2}$$

and $g_i, i \in \{0, 1\}$ is the family of two systems (37) with $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$.

This means that the system (A.1) consists of a subsystem $\dot{z} = g_0$, which is ISS with respect to the inputs (u, y) and which is active whenever one of the IOSS subsystems of (2) is active, and an unstable subsystem $\dot{z} = g_1$, which is active whenever one of the non-IOSS subsystems of (2) is active. Thus the switching times of s coincide with those switching times of σ where a switch from a system in $\mathcal{P}_s$ to a system in $\mathcal{P}_u$ or vice versa occurs, and the activation time of g_1 in any interval $[\tau, t)$, denoted by $T_z^u(t, \tau)$, is equal to the activation time $T^u(t, \tau)$ of the non-IOSS systems of the switched system (2) in this interval.

In the following, we have to verify that our state-norm estimator candidate (A.1) satisfies the two properties of Definition 1, namely that it is ISS with respect to the inputs (u, y) and that (6) holds.

In order to establish the first property, we can apply Theorem 2, i.e., we have to show that the conditions of Theorem 2 are satisfied for the state-norm estimator candidate (A.1). Choosing e.g. $V_0(z) = V_1(z) =: V(z) = \frac{1}{2}z^2$, it is straightforward to verify that this is the case if $T_z^u(t, \tau)$ satisfies (20) with $\rho < \frac{\lambda_s^*}{\lambda_s^* + \lambda_u^*}$, but with no further condition on the average dwell time τ_a^z of the switching signal s, as (21) yields $\tau_a^z > 0$.

It remains to show that our state-norm estimator candidate (A.1) satisfies the second property of Definition 1, i.e., that (6) holds.

As $\chi_1(|u|) + \chi_2(|y|) \geq 0$, we have

$$g_0(z, u, y) \geq -\lambda_s^* z(t)$$
$$g_1(z, u, y) \geq \lambda_u^* z(t)$$

and thus, as $z_0 \geq 0$,

$$z(t) \geq e^{-\lambda_s^* T^s(t,t_0) + \lambda_u^* T^u(t,t_0)} z_0 \geq 0 \tag{A.3}$$

for all $t \geq t_0$. Furthermore, for all $t_0 \leq \tau_i \leq t$ we get

$$z(\tau_i) \leq e^{\lambda_s^* T^s(t,\tau_i)-\lambda_u^* T^u(t,\tau_i)} z(t). \tag{A.4}$$

Now consider the function $W(t) := V_{\sigma(t)}(x(t)) - z(t)$. Following the lines of the proof of Theorem 3, we get that for any interval $[\tau_i, \tau_{i+1})$,

$$\begin{aligned} &W(\tau_{i+1}) \leq \mu W(\tau_i) e^{-\lambda_s(\tau_{i+1}-\tau_i)} + (\mu-1) z(\tau_{i+1}) \\ &\quad \text{if } \zeta(t) = 0 \text{ in } [\tau_i, \tau_{i+1}) \\ &W(\tau_{i+1}) \leq \mu W(\tau_i) e^{\lambda_u(\tau_{i+1}-\tau_i)} + (\mu-1) z(\tau_{i+1}) \\ &\quad \text{if } \zeta(t) = 1 \text{ in } [\tau_i, \tau_{i+1}). \end{aligned}$$

Iterating this from $i = 0$ to $i = N_\sigma(t, t_0)$ and using (A.4), (20) and (17) we arrive at

$$\begin{aligned} W(t) \leq\ & \mu^{N_\sigma(t,t_0)} e^{-\lambda_s T^s(t,t_0)+\lambda_u T^u(t,t_0)} W(t_0) \\ &+ (\mu-1) z(t) \sum_{k=1}^{N_\sigma(t,t_0)} (e^{(N_\sigma(t,t_0)-k)\ln\mu} \\ &\times e^{-(\lambda_s-\lambda_s^*)T^s(t,\tau_k)-(\lambda_u^*-\lambda_u)T^u(t,\tau_k)}) \\ \leq\ & e^{(\lambda_s+\lambda_u)T_0} e^{N_\sigma(t,t_0)\ln\mu-(\lambda_s-\rho(\lambda_s+\lambda_u))\,(t-t_0)} W(t_0) \\ &+ (\mu-1)\varepsilon_1 z(t) \sum_{k=1}^{N_\sigma(t,t_0)} (e^{(N_\sigma(t,t_0)-k)\ln\mu} \\ &\times e^{-(\lambda_s-\lambda_s^*-\varepsilon_2)\,(t-\tau_k)}), \end{aligned} \tag{A.5}$$

with

$$\varepsilon_1 = \begin{cases} 1 & \text{if } \lambda_s + \lambda_u - \lambda_s^* - \lambda_u^* \leq 0 \\ e^{(\lambda_s+\lambda_u-\lambda_s^*-\lambda_u^*)T_0} & \text{else} \end{cases}$$

and

$$\varepsilon_2 = \begin{cases} 0 & \text{if } \lambda_s + \lambda_u - \lambda_s^* - \lambda_u^* \leq 0 \\ \rho(\lambda_s + \lambda_u - \lambda_s^* - \lambda_u^*) & \text{else.} \end{cases}$$

Using the average dwell-time property (3) and proceeding as in the proof of Theorem 3, we arrive at

$$\begin{aligned} &\sum_{k=1}^{N_\sigma(t,t_0)} e^{(N_\sigma(t,t_0)-k)\ln\mu-(\lambda_s-\lambda_s^*-\varepsilon_2)\,(t-\tau_k)} \\ &\quad \leq e^{N_0(\ln\mu+\lambda\tau_a)} \frac{1}{1-e^{-\lambda\tau_a}} =: b \end{aligned} \tag{A.6}$$

with $\lambda := \lambda_s - \lambda_s^* - \varepsilon_2 - \ln\mu/\tau_a \in (0, \lambda_s - \lambda_s^* - \varepsilon_2)$ if the average dwell time τ_a satisfies the bound

$$\tau_a > \frac{\ln\mu}{(\lambda_s - \lambda_s^* - \varepsilon_2)}. \tag{A.7}$$

In case of $\varepsilon_2 > 0$, the above average dwell-time condition (A.7) is well defined if we choose λ_s^* and λ_u^* such that

$$\rho < \frac{\lambda_s^*}{\lambda_s^* + \lambda_u^*} < \frac{\lambda_s}{\lambda_s + \lambda_u}. \tag{A.8}$$

Furthermore, note that for any average dwell-time τ_a satisfying (21) and any ρ satisfying (19), we can choose $\lambda_s^* \in (0, \lambda_s)$ and $\lambda_u^* \geq \lambda_u$ such that the conditions (A.7) and (A.8) are also satisfied.

Combining (A.5) and (A.6), we get

$$\begin{aligned} W(t) \leq\ & \mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} W(t_0) e^{-\lambda'(t-t_0)} + b(\mu-1)\varepsilon_1 z(t) \\ =:\ & \mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} W(t_0) e^{-\lambda'(t-t_0)} + b_1 z(t) \end{aligned} \tag{A.9}$$

with $\lambda' := \lambda_s - \rho(\lambda_s + \lambda_u) - \ln\mu/\tau_a \in (0, \lambda_s - \rho(\lambda_s + \lambda_u))$ if the average dwell-time τ_a satisfies the bound (21). This leads to

$$\begin{aligned} V_{\sigma(t)}(x(t)) \leq\ & (1+b_1) z(t) \\ &+ \mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} e^{-\lambda'(t-t_0)} (V_{\sigma(t_0)}(x_0) - z_0) \\ \leq\ & (1+b_1)|z(t)| \\ &+ 2\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} e^{-\lambda'(t-t_0)} \alpha_2(|x_0| + |z_0|), \end{aligned}$$

if we assume without loss of generality that $\alpha_2(r) \geq r$ for all $r \geq 0$. Using (7), we finally arrive at

$$\begin{aligned} |x(t)| \leq\ & \alpha_1^{-1}(2(1+b_1)|z(t)|) \\ &+ \alpha_1^{-1}(4\mu^{N_0} e^{(\lambda_s+\lambda_u)T_0} e^{-\lambda'(t-t_0)} \alpha_2(|x_0| + |z_0|)) \\ =:\ & \gamma(|z(t)|) + \beta(|x_0| + |z_0|, t - t_0), \end{aligned} \tag{A.10}$$

which means that our state-norm estimator candidate (A.1) satisfies the condition (6). □

Appendix B. Proof of Theorem 5

Consider again the switched state-norm estimator (A.1) designed in the proof of Theorem 4 and its state $z(t)$. The idea of this proof is that if we can design a candidate state-norm estimator $\dot{w} = g_{\zeta'(t)}(w, u, y)$ such that for all $t \geq t_0$

$$|z(t)| \leq c|w(t)| \tag{B.1}$$

for some constant $c \geq 1$, then the system $\dot{w} = g_{\zeta'(t)}(w, u, y)$ is also a state-norm estimator for the switched system (2). Furthermore, the gain γ' with which $|x|$ can be bounded in terms of w is then given by $\gamma'(w) = \gamma(cw)$, where γ is the gain of the state-norm estimator (A.1), given by (A.10).

Consider the following candidate for a state-norm estimator with switching times independent of the switching times of σ:

$$\dot{w} = g_{\zeta'(t)}(w, u, y), \qquad w_0 \geq 0 \tag{B.2}$$

where g_i, $i \in \{0, 1\}$ is the family of two systems (37) and the switching signal $\zeta'(t)$ is defined by (39). The choice of the constants τ_a^w and ρ^w in (40)–(41) means that in any interval of length τ_a^w, the period of time during which $w(t)$ is unstable (namely $\rho^w \tau_a^w$ according to (39)) is greater than or equal to the maximum unstable time of $z(t)$ ($T_0 + \rho\tau_a^w$ according to (20)).

It is straightforward to verify that the activation time of g_1 in any interval $[\tau, t)$, denoted by $T_w^u(t, \tau)$, satisfies

$$T_w^u(t, \tau) \leq T_0^w + \rho^w(t - \tau) \tag{B.3}$$

with $T_0^w = \rho^w(1-\rho^w)\tau_a^w$. Using the same argument as in the proof of Theorem 4, we see that according to Theorem 2, the candidate state-norm estimator (B.2) is ISS with respect to the inputs (u, y), i.e., the first property of Definition 1 is satisfied.

In the following, we will prove by induction that (B.1) holds, which implies that the second property of Definition 1 is also satisfied and thus the candidate (B.2) is a state-norm estimator for the switched system (2). Note that we can write (B.1) without the absolute values, as $z(t)$ as well as $w(t)$ are greater than or equal to zero for all $t \geq t_0$.

As an initial step, note that we can choose z_0 and w_0 such that $z_0 \leq w_0$. Now assume that

$$z(k\tau_a^w) \leq w(k\tau_a^w) \tag{B.4}$$

for some integer $k \geq 0$. If we can show that also

$$z((k+1)\tau_a^w) \leq w((k+1)\tau_a^w) \tag{B.5}$$

and that

$$z(t) \leq cw(t) \qquad k\tau_a^w \leq t \leq (k+1)\tau_a^w \tag{B.6}$$

for some $c \geq 1$, then we can conclude that (B.1) holds.

Let $t_1 := k\tau_a^w$, $t_2 := k\tau_a^w + (1-\rho^w)\tau_a^w$, and $t_3 := (k+1)\tau_a^w$. Furthermore, let $\bar{\chi}(t) := \chi_1(|u(t)|) + \chi_2(|y(t)|)$.

First, consider the interval $[t_1, t_2)$. During this interval, $w(t)$ is stable, whereas $z(t)$ might switch between its stable and its unstable mode.

If during some interval $[t', t'')$, where $t_1 \le t' \le t'' \le t_2$, both w and z are stable and $z(t') \le \varepsilon w(t')$ for some $\varepsilon \ge 1$, then

$$\frac{d}{dt}(z-w) = g_0(z,u,y_\sigma) - g_0(w,u,y_\sigma) = -\lambda_s^*(z-w)$$

and thus

$$\begin{aligned} z(t'') - w(t'') &= e^{-\lambda_s^*(t''-t')}(z(t') - w(t')) \\ &\le e^{-\lambda_s^*(t''-t')}(\varepsilon - 1)w(t') \le (\varepsilon-1)w(t''). \end{aligned}$$

Thus we obtain

$$z(t'') \le \varepsilon w(t''). \tag{B.7}$$

If during some interval $[t'', t''')$, where $t_1 \le t'' \le t''' \le t_2$, w is stable and z is unstable and $z(t'') \le \kappa w(t'')$ for some $\kappa \ge 1$, then

$$\begin{aligned} w(t''') &= e^{-\lambda_s^*(t'''-t'')}\left(w(t'') + \int_{t''}^{t'''} e^{\lambda_s^*(s-t'')}\bar{\chi}(s)ds\right) \\ &\ge e^{-\lambda_s^*(t'''-t'')}\left(w(t'') + \int_{t''}^{t'''} \bar{\chi}(s)ds\right) \end{aligned}$$

and

$$\begin{aligned} z(t''') &= e^{\lambda_u^*(t'''-t'')}\left(z(t'') + \int_{t''}^{t'''} e^{-\lambda_u^*(s-t'')}\bar{\chi}(s)ds\right) \\ &\le e^{\lambda_u^*(t'''-t'')}\left(\kappa w(t'') + \int_{t''}^{t'''} \bar{\chi}(s)ds\right) \\ &= e^{(\lambda_u^*+\lambda_s^*)\,(t'''-t'')}\kappa e^{-\lambda_s^*(t'''-t'')} \\ &\quad \times \left(w(t'') + \frac{1}{\kappa}\int_{t''}^{t'''} \bar{\chi}(s)ds\right) \\ &\le \kappa e^{(\lambda_u^*+\lambda_s^*)\,(t'''-t'')} w(t'''). \end{aligned} \tag{B.8}$$

Thus, as the interval $[t_1, t_2)$ consists of disjoint subintervals where z is alternating between stable and unstable, and as according to (B.4) $z(t_1) \le w(t_1)$, by iterating (B.7) and (B.8) we arrive at

$$z(t_2) \le e^{(\lambda_u^*+\lambda_s^*)T^u(t_2,t_1)} w(t_2). \tag{B.9}$$

Second, consider the interval $[t_2, t_3)$. During this interval, $w(t)$ is unstable, whereas $z(t)$ might switch between its stable and its unstable mode. Yet, as according to (20), $T^u(t_3, t_1) \le T_0 + \rho(t_3 - t_1) = T_0 + \rho\tau_a^w$, the time in the interval $[t_2, t_3)$ where z is unstable is bounded above by

$$T^u(t_3, t_2) \le T_0 + \rho\tau_a^w - T^u(t_2, t_1) \tag{B.10}$$

and thus

$$\begin{aligned} T^s(t_3, t_2) &= t_3 - t_2 - T^u(t_3, t_2) \\ &\ge \rho^w\tau_a^w - (T_0 + \rho\tau_a^w - T^u(t_2, t_1)). \end{aligned} \tag{B.11}$$

Suppose there are l switching times of σ during the interval (t_2, t_3), and assume without loss of generality that these are the switching times $\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{r+l}$, where r is some nonnegative integer. Furthermore, for convenience, let $\tau_r := t_2$ and $\tau_{r+l+1} := t_3$. Let $\mathcal{K}_1 := \{k \in \{0,\ldots,l\} : \zeta(t) = 0 \text{ for } t \in [\tau_{r+k}, \tau_{r+k+1})\}$ and $\mathcal{K}_2 := \{k \in \{0,\ldots,l\} : \zeta(t) = 1 \text{ for } t \in [\tau_{r+k}, \tau_{r+k+1})\}$. Then

$$\begin{aligned} w(t_3) &= e^{\lambda_u^*(t_3-t_2)}\left(w(t_2) + \int_{t_2}^{t_3} e^{-\lambda_u^*(s-t_2)}\bar{\chi}(s)ds\right) \\ &\ge e^{\lambda_u^*(t_3-t_2)}w(t_2) \\ &\quad + \sum_{k\in\mathcal{K}_1} e^{\lambda_u^*(t_3-\tau_{r+k+1})}\int_{\tau_{r+k}}^{\tau_{r+k+1}} \bar{\chi}(s)ds \\ &\quad + \sum_{k\in\mathcal{K}_2} e^{\lambda_u^*(t_3-\tau_{r+k})}\int_{\tau_{r+k}}^{\tau_{r+k+1}} e^{-\lambda_u^*(s-\tau_{r+k})}\bar{\chi}(s)ds \\ &=: e^{\lambda_u^*(t_3-t_2)}w(t_2) + a_1 + a_2 \end{aligned} \tag{B.12}$$

and

$$\begin{aligned} z(t_3) &= e^{-\lambda_s^*T^s(t_3,t_2)+\lambda_u^*T^u(t_3,t_2)}z(t_2) \\ &\quad + \sum_{k\in\mathcal{K}_1}\Bigg(e^{-\lambda_s^*T^s(t_3,\tau_{r+k+1})+\lambda_u^*T^u(t_3,\tau_{r+k+1})-\lambda_s^*(\tau_{r+k+1}-\tau_{r+k})} \\ &\quad \times \int_{\tau_{r+k}}^{\tau_{r+k+1}} e^{\lambda_s^*(s-\tau_{r+k})}\bar{\chi}(s)ds\Bigg) \\ &\quad + \sum_{k\in\mathcal{K}_2}\Bigg(e^{-\lambda_s^*T^s(t_3,\tau_{r+k+1})+\lambda_u^*T^u(t_3,\tau_{r+k+1})+\lambda_u^*(\tau_{r+k+1}-\tau_{r+k})} \\ &\quad \times \int_{\tau_{r+k}}^{\tau_{r+k+1}} e^{-\lambda_u^*(s-\tau_{r+k})}\bar{\chi}(s)ds\Bigg) \\ &\le e^{-\lambda_s^*T^s(t_3,t_2)+\lambda_u^*T^u(t_3,t_2)}z(t_2) + a_1 + a_2. \end{aligned} \tag{B.13}$$

Using (B.9)–(B.13), we arrive at

$$\begin{aligned} w(t_3) - z(t_3) &\ge e^{\lambda_u^*(t_3-t_2)}w(t_2) - e^{-\lambda_s^*T^s(t_3,t_2)+\lambda_u^*T^u(t_3,t_2)}z(t_2) \\ &\ge e^{\lambda_u^*(t_3-t_2)}w(t_2) - e^{-\lambda_s^*(\rho^w\tau_a^w-(T_0+\rho\tau_a^w-T^u(t_2,t_1)))} \\ &\quad \times e^{\lambda_u^*(T_0+\rho\tau_a^w-T^u(t_2,t_1))}z(t_2) \\ &\ge (e^{\lambda_u^*\rho^w\tau_a^w} - e^{(\lambda_u^*+\lambda_s^*)\,(T_0+\rho\tau_a^w)-\lambda_s^*\rho^w\tau_a^w})w(t_2). \end{aligned} \tag{B.14}$$

As $w(t_2) \ge 0$, the right-hand side of (B.14) is nonnegative if and only if

$$\lambda_u^*\rho^w\tau_a^w \ge (\lambda_u^*+\lambda_s^*)\,(T_0+\rho\tau_a^w) - \lambda_s^*\rho^w\tau_a^w$$
$$\Updownarrow$$
$$\rho^w\tau_a^w \ge T_0 + \rho\tau_a^w$$

which is true according to our choice of ρ^w and τ_a^w in (40)–(41).

Thus (B.5) and (B.6) are satisfied, and the constant c in (B.6) can be calculated from (B.9) and the fact that $T^u(t_2, t_1) \le T_0 + \rho(t_2 - t_1) = T_0 + \rho(1-\rho^w)\tau_a^w$ as

$$c = e^{(\lambda_u^*+\lambda_s^*)\,(T_0+\rho(1-\rho^w)\tau_a^w)}. \tag{B.15}$$

Note that we calculated c from (B.9) (i.e., the worst-case ratio z/w can be calculated at time t_2) as during the time interval $[t_2, t_3)$, the ratio z/w decreases. Namely, as during the time interval $[t_2, t_3)$ $w(t)$ is unstable and $z(t)$ switches between its stable and unstable mode,

$$\frac{d}{dt}(z-w) \le \lambda_u^*(z-w).$$

Thus, if $z(t_2) = \varepsilon w(t_2)$ for some ε, we obtain that for any $t \in [t_2, t_3)$,

$$\begin{aligned} z(t) - w(t) &= e^{\lambda_u^*(t-t_2)}(z(t_2) - w(t_2)) \\ &\le e^{-\lambda_s^*(t-t_2)}(\varepsilon-1)w(t_2) \le (\varepsilon-1)w(t) \end{aligned}$$

and thus

$$z(t) \le \varepsilon w(t),$$

i.e., the ratio $\frac{z(t)}{w(t)}$ is smaller than or equal to the ratio $\frac{z(t_2)}{w(t_2)}$.

Therefore, we conclude that the system (B.2) is also a state-norm estimator for the switched system (2) and by construction of the switching signal ζ', its switching times are independent of the switching times of the switching signal σ. $\square$

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