Norm-Controllability of Nonlinear Systems

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***Abstract*—In this paper, we introduce and study the notion of norm-controllability for nonlinear systems. This property captures the responsiveness of a system with respect to applied inputs, which is quantified via the norm of an output. As a main contribution, we obtain several Lyapunov-like sufficient conditions for norm-controllability, some of which are based on higher-order derivatives of a Lyapunov-like function. Various aspects of the proposed concept and the sufficient conditions are illustrated by several examples, including a chemical reactor application. Furthermore, for the special case of linear systems, we establish connections between norm-controllability and standard controllability.**

***Index Terms*—Controllability, input-to-state stability (ISS), Lyapunov methods, nonlinear systems.**

I. Introduction

Controllability is one of the fundamental concepts in control theory. Usually, it is formulated as the ability to steer the state of a system from any point to any other point in any given time by an appropriate choice of the control input. For linear time-invariant systems, controllability can be easily checked via necessary and sufficient matrix rank conditions; furthermore, controllable modes can be identified with the help of the Kalman controllability decomposition (see, e.g., [4]). For nonlinear systems, similar decompositions have been studied, and for some special system classes—most notably systems affine in controls—controllability can be characterized in terms of the rank of a certain Lie algebra of vector fields (see, e.g, [5], [6]). However, for general nonlinear control systems our understanding of point-to-point controllability is much less complete compared to the linear case, and even in those settings where controllability tests are available they are more difficult to apply.

In this paper, we propose a new notion which can be viewed as a weaker/coarser version of the standard controllability. Namely, in contrast to point-to-point controllability, we look at the *norm* of the state and ask how it is affected by the applied inputs. In particular, we examine whether this norm can be made large by applying large enough inputs for sufficiently long time. When defining norm-controllability, we actually take a more general approach and consider the norm of an output which identifies directions of interest in the state space (and may in particular be the entire state). The definition (see Section III) is such that the size of the reachable set of the system (or its image under the output map) can be lower-bounded in terms of the norm of the applied inputs and the time horizon over which they were applied. We believe that this concept is very natural and relevant in many settings of practical interest. In economics, for example, it may be of interest to maximize the profit of a company, for which the effects of inputs such as the price of a certain product or the number of advertisements have to be analyzed. Another possible application context is in the process industry, where one wants, e.g., to determine whether and how increasing the amount of reagent yields a larger amount of product; such a chemical reactor example will be considered in Section V (see Example 6).

Another context of interest for norm-controllability is the case where the considered system input constitutes a disturbance. Then, it is interesting to obtain a lower bound for the effect of the worst-case disturbance on the system state or output, i.e., to obtain a lower bound for the reachable set at a given time. In this sense, the proposed concept of norm-controllability can also be viewed as complementary to the well-known concept of input-to-state-stability (ISS) as introduced in [7], and related notions involving outputs such as input-to-output stability (IOS) [8] or $\mathcal{L}_\infty$ stability (see, e.g., [9]). Namely, these concepts deal with the question whether bounded (small) inputs result in bounded (small) states or outputs, i.e., an *upper* bound for all possible system states or outputs at all times is sought in terms of a suitable norm of the input. In contrast, as stated above, norm-controllability yields a *lower* bound on the reachable set of the system (or its image under the output map) at each time in terms of the norm of the applied inputs. Besides the conceptual complementarity between norm-controllability and ISS (or related notions involving outputs), there is also a relation in terms of Lyapunov-like characterizations of these notions. While ISS has an equivalent characterization in terms of a Lyapunov function that decreases when the norm of the state is large compared to the norm of the input [10], our first main result (Theorem 1 in Section IV-A) formulates a similar Lyapunov-like sufficient condition for norm-controllability, but a Lyapunov function should now *increase* when the norm of the state is *small* compared to the norm of the input. However, while it is instructive to highlight the connections and similarities, most of the technical ideas employed in this paper are very different from those used in the above references.

We note also that our basic premise in this work is similar to that of the paper [11] which introduced and studied the concept of norm-observability. Instead of observability, usually

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defined as the ability to reconstruct the state of the system from measurements of the output (and of the input if one is present), norm-observability was defined in terms of being able to obtain from this data an upper bound on the norm of the state rather than the precise value of the state. For linear systems the two properties turn out to be equivalent, but for general nonlinear systems this is no longer the case and the latter, weaker notion has some interesting properties, as demonstrated in [11]. Again, the conceptual similarity notwithstanding, the technical developments presented here and in [11] are completely different and there does not appear to be any direct duality relationship between norm-controllability and norm-observability.

Besides introducing and discussing the notion of norm-controllability (see Section III), one of the main technical contributions of this paper is the development of several Lyapunov-like sufficient conditions (see Section IV). The first one (see Section IV-A) involves first-order derivatives of a Lyapunov-like function V and has first appeared in the conference paper [1]. While it is appealing due to its rather simple formulation, its applicability is in general restricted to systems with relative degree one. The two subsequent sufficient conditions (see Sections IV-B and C), whose formulation involves higher-order derivatives of V, resolve this issue and are applicable to systems with arbitrary relative degree; such conditions were first obtained in [2]. This generalization is nontrivial and relies on higher-order derivatives that are not classical ones but higher-order lower directional derivatives (see, e.g., [12], [13]). A simplified formulation of the sufficient conditions was recently proposed in[1] [3]. In the present paper, we adopt this simplified formulation and give a self-contained treatment of all results, including proofs which were partly missing in the conference versions. Furthermore, several novel results are presented in this paper, including various illustrative examples (see Sections V–VII), a further elaboration of norm-controllability for the special case of linear systems, where we also establish connections with the standard controllability (see Section VI), and two weaker versions of norm-controllability (see Section VII). These weaker versions capture cases where a system is only responsive to inputs of sufficiently large magnitude, or where the considered lower bounds are only valid after a certain amount of time; also for these cases, Lyapunov-like sufficient conditions are obtained.

II. Preliminaries and Setup

We consider nonlinear control systems of the form

$$\dot{x} = f(x,u), \qquad y = h(x), \qquad x(0) = x_0 \tag{1}$$

with state $x \in \mathbb{R}^n$, output $y \in \mathbb{R}^p$, and input $u \in U \subseteq \mathbb{R}^m$, where U is a closed set which specifies admissible input values. Suppose that $f \in C^{\bar{k}-1}$ for some $\bar{k} \geq 1$ and that the $(\bar{k}-1)$-st partial derivatives of f with respect to x are locally Lipschitz in (x,u). Admissible input signals $u(\cdot)$ to the system (1) satisfy $u(\cdot) \in L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U)$, where $L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U)$ denotes the set of all measurable and locally essentially bounded functions from $\mathbb{R}_{\geq 0}$ to U. We say that a set $\mathcal{B} \subseteq \mathbb{R}^n$ is *invariant* for system (1) under controls in a set $\bar{U} \subseteq U$ if for every $x_0 \in \mathcal{B}$ and every $u(\cdot) \in L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, \bar{U})$ the corresponding state trajectory satisfies $x(t) \in \mathcal{B}$ for all $t \geq 0$. We assume that the system (1) exhibits the *unboundedness observability* property (see [14] and the references therein), which means that for every trajectory of the system (1) with finite escape time t_{esc}, also the corresponding output becomes unbounded as $t \to t_{esc}$. This is a very reasonable assumption as one cannot expect to measure responsiveness of the system in terms of an output map h (as we will later do) if a finite escape time cannot be detected by this output map. We remark that, for example, all linear systems satisfy this assumption, as do all nonlinear systems with radially unbounded output maps.

For every $a, b > 0$, denote the set of all measurable and locally essentially bounded input signals whose norm does not exceed b on the time interval $[0, a]$ by

$$\mathcal{U}_{a,b} := \left\{ u(\cdot) : \|u\|_{[0,a]} \leq b,\ u(t) \in U\ \forall t \in [0,a] \right\} \tag{2}$$

where $\|\cdot\|_{[0,a]}$ is the essential supremum norm on the interval $[0, a]$. Let $\mathcal{R}^{\tau}\{x_0, \mathcal{U}\} \subseteq \mathbb{R}^n \cup \{\infty\}$ be the reachable set of the system (1) at time $\tau \geq 0$, starting at the initial condition $x(0) = x_0$ and applying input signals $u(\cdot)$ in some set $\mathcal{U} \subseteq L^{\infty}_{loc}(\mathbb{R}_{\geq 0}, U)$. The reachable set $\mathcal{R}^{\tau}\{x_0, \mathcal{U}\}$ contains ∞ if for some $u(\cdot) \in \mathcal{U}$ a finite escape time $t_{esc} \leq \tau$ exists. Define $R^{\tau}_h(x_0, \mathcal{U}) := \sup\{|h(x)| : x \in \mathcal{R}^{\tau}\{x_0, \mathcal{U}\}\}$ as the radius of the smallest ball in the output space centered at $y = 0$ which contains the image of the reachable set $\mathcal{R}^{\tau}\{x_0, \mathcal{U}\}$ under the output map h, or ∞ if this image is unbounded.

III. Norm-Controllability: Definition and Discussion

We are now in a position to define and discuss the notion of norm-controllability.

Definition 1: The system (1) is *norm-controllable from* x_0 *with gain function* γ if there exists a function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(\cdot, b)$ nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot)$ of class[2] $\mathcal{K}_{\infty}$ for each fixed $a > 0$, such that for all $a > 0$ and $b > 0$

$$R^{a}_h(x_0, \mathcal{U}_{a,b}) \geq \gamma(a, b) \tag{3}$$

where $\mathcal{U}_{a,b}$ is defined in (2). □

The above definition of norm-controllability[3] can be interpreted in the following way. It provides a measure for how large the norm of the output y can be made in terms of the maximal magnitude b of the applied inputs and the length a of the interval over which they are applied. This is captured via the gain function γ, which gives a lower bound on the radius of the smallest ball containing the image of the reachable set under the output map h, when inputs with magnitude at most b are applied over the time interval $[0, a]$. For each fixed time horizon a, $\gamma(a, \cdot)$ is required to be of class $\mathcal{K}_{\infty}$, which means that with inputs of increasing magnitude one should be able to also increase the norm of the output. Note that a necessary

[1]We remark that the sufficient conditions in [2], [3] missed the assumption expressed by the second bullet point in Theorems 2 and 3 of this paper, respectively, and hence were not correct as stated there.

[2]A function α: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{K}$ if α is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, it is of *class* $\mathcal{K}_{\infty}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$.

[3]We note that while Definition 1 is the same as the originally proposed definition of norm-controllability in [1], [2], the set $\mathcal{U}_{a,b}$ in (2) is defined slightly differently. This allows us to also restate the sufficient conditions later on in a simplified way.

condition for this to be satisfied is that the set U of admissible inputs is unbounded. On the other hand, for every fixed upper bound b on the input norm, increasing the time horizon a over which such inputs are applied should result in a non-decreasing magnitude of the output. Furthermore, in Definition 1 the initial state x_0 is taken as given and, when x_0 is changed, γ will in general also change. We also note the following subtlety: Definition 1 says that for each time $a > 0$ and each $b > 0$ there exists some trajectory $x(\cdot)$ such that $|h(x(a))| \geq \gamma(a, b)$, but not necessarily that there exists some trajectory satisfying $|h(x(a))| \geq \gamma(a, b)$ for all times $a > 0$, i.e., some (single) trajectory which can be made large for all times.

As mentioned in the Introduction, the concept of norm-controllability can be seen as somehow complementary to ISS (and related concepts involving outputs such as IOS and $\mathcal{L}_\infty$ stability). Namely, if a system is ISS, at each time a the norm of the system state can be upper-bounded in terms of the $\mathcal{L}_\infty$ norm of the input (plus some decaying term depending on the initial condition). In particular, there exist a function $\alpha \in \mathcal{K}_\infty$ and a function $\beta \in \mathcal{KL}$ such that for *all* $u(\cdot) \in \mathcal{U}_{a,b}$ and all $a, b > 0$

$$|x(a)| \leq \beta(|x_0|, a) + \alpha(b) \tag{4}$$

i.e., for *all* trajectories and all times one can upper-bound the norm of the system state via (4). This gives us an *upper* bound on the radius of the smallest ball which contains the reachable set, i.e., $R_h^a(x_0, \mathcal{U}_{a,b}) \leq \beta(|x_0|, a) + \alpha(b)$. On the other hand, norm-controllability gives a *lower* bound on $R_h^a(x_0, \mathcal{U}_{a,b})$ in terms of the gain function γ in (3), which (in case of $h(x) = x$) means that for each $a, b > 0$ there exists *some* $u(\cdot) \in \mathcal{U}_{a,b}$ such that $|x(a)| \geq \gamma(a, b)$. If a system is both norm-controllable (with $h(x) = x$) and ISS, then it follows from (4) that $\gamma(\cdot, b)$ is bounded for each $b > 0$. In this case, $\gamma(\infty, \cdot)$ gives a lower bound for the smallest possible ISS gain function α of the system. Similar considerations apply to related notions involving general outputs (different than the special choice $h(x) = x$) such as IOS [8] or $\mathcal{L}_\infty$ stability (see, e.g., [9]). Moreover, we remark that for a given initial condition x_0 and each $a, b > 0$, the function $\gamma(a, b)$ can be interpreted as a lower bound for the value function of a finite-horizon optimal control problem, where the utility function to be maximized is $|h(x(a))|$, subject to the constraints $\dot{x} = f(x, u)$, $u(t) \in U$ and $|u(t)| \leq b$ for all $0 \leq t \leq a$. Finally, we note some connections of norm-controllability to the concept of *excitation indices* as proposed in [15], which also studies the effect of inputs of norm less than b and for which certain Lyapunov-like conditions were obtained. This notion, however, is different from norm-controllability in that it is only formulated in the context of dissipative systems and focuses on the limiting behavior as $t \to \infty$.

IV. Sufficient Conditions for Norm-Controllability

In this section, we formulate several Lyapunov-like sufficient conditions for a system to be norm-controllable. The first one in Section IV-A is the simplest one and is based on first-order directional derivatives. However, this condition can in general only be satisfied for outputs with relative degree one. Hence, in Section IV-B, we subsequently develop a more general sufficient condition which is based on higher-order directional derivatives. Finally, in Section IV-C we present further extensions and relaxations of the derived sufficient conditions.

A. *Sufficient Condition Based on First-Order Directional Derivatives*

The first sufficient condition for norm-controllability uses the notion of lower directional derivatives, which we recall from [12], [16]. Namely, for a function $V : \mathbb{R}^n \to \mathbb{R}$, the lower directional derivative of V at a point $x \in \mathbb{R}^n$ in the direction of a vector $h_1 \in \mathbb{R}^n$ is defined as

$$V^{(1)}(x; h_1) := \liminf_{t \searrow 0, \bar{h}_1 \to h_1} \frac{1}{t} \left(V(x + t\bar{h}_1) - V(x) \right). \tag{5}$$

Note that at each point $x \in \mathbb{R}^n$ where V is continuously differentiable, it holds that $V^{(1)}(x; h_1) = L_{h_1} V(x) = (\partial V / \partial x)(x) h_1$.

Theorem 1: Suppose there exist a set $\bar{U} \subseteq U$ containing 0 in its closure, and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is invariant for system (1) under controls in $\bar{U}$. Furthermore, suppose there exist a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$ which is continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and such that $\partial V / \partial x$ is locally Lipschitz on $\mathbb{R}^n \setminus W$, and functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$ such that the following holds:

- For all $x \in \mathcal{B}$

$$\nu(|\omega(x)|) \leq |h(x)|, \tag{6}$$

$$\alpha_1(|\omega(x)|) \leq V(x) \leq \alpha_2(|\omega(x)|). \tag{7}$$

- For each $b > 0$ and each $x \in \mathcal{B}$ such that $|\omega(x)| \leq \rho(b)$, there exists some $u \in \bar{U}$ with $|u| \leq b$ such that

$$V^{(1)}(x; f(x, u)) \geq \chi(b). \tag{8}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function

$$\gamma(a, b) = \nu\left(\alpha_2^{-1}\left(\min\left\{a\chi(b) + V(x_0), \alpha_1(\rho(b))\right\}\right)\right). \tag{9}$$

Remark 1: If (8) holds not only if $|\omega(x)| \leq \rho(b)$ but rather for all $x \in \mathcal{B}$, then we can let $\rho \to \infty$ and γ in (9) simplifies to $\gamma(a, b) = \nu(\alpha_2^{-1}(a\chi(b) + V(x_0)))$. Note that in this case, $\gamma(a, \cdot)$ might not be of class $\mathcal{K}_\infty$, as $\gamma(a, 0) > 0$ if $V(x_0) > 0$. Nevertheless, $\gamma(a, \cdot)$ still satisfies all other properties of a class $\mathcal{K}_\infty$ function, i.e., is continuous, strictly increasing and unbounded. In particular, this means that the system is norm-controllable, but the function γ gives us a slightly stronger property than required by Definition 1. If a function γ satisfying $\gamma(a, 0) = 0$ is desirable, it can be obtained by replacing $V(x_0)$ with 0. □

Theorem 1 is a special case of Theorem 2 in Section IV-B, which is why we will only prove the latter. Nevertheless, we decided to state Theorem 1 separately as a result of its own interest and in order to highlight and discuss various aspects of our sufficient conditions with this simplest formulation, which we will do in the following. Furthermore, connections to typical ISS results are also most apparent for Theorem 1, compared to the more general formulation in Theorem 2.

Theorem 1 is stated so that norm-controllability is established on some invariant set $\mathcal{B}$. In particular, this includes as a special case $\mathcal{B} = \mathbb{R}^n$, in which case we can take $\bar{U} = U$. The condition expressed by (6), (7) means that V can be lower- and upper-bounded in terms of the norm of a function ω, which has to be "aligned" with the output map h in the sense given by (6). In the special case of $h(x) = \omega(x) = x$, this reduces to the usual condition that V is positive definite and radially unbounded. Note that (7) together with the requirement that ω is continuous implies that V is continuous for all x where $\omega(x)=0$. However, at these points we need to allow V to not be continuously differentiable because $V \in C^1$ together with (7) would imply that the gradient of V vanishes for all x where $\omega(x)=0$, and thus it would be impossible to satisfy (8) there. On the other hand, the fact that V need not be continuously differentiable for all x where $\omega(x)=0$ allows us to obtain a non-vanishing lower directional derivative at these points, and hence (8) can be satisfied. In the examples given in Section V, a typical choice will be $V(x)=|\omega(x)|$.

When the output map h is such that $h(x_0)=0$, norm-controllability captures the system's ability to "move away" from the initial state x_0. This could e.g. be of interest if one wants to know how far one can move away from an initial equilibrium state (x_0, u_0). In other settings, it makes sense to consider $h(x_0) \neq 0$, e.g. in a chemical process where initially already some product is available and hence the output should further be increased. This allows us, for fixed h, ω, and V, to vary the initial condition x_0, and the effect of this is given by the term $V(x_0)$ in (9). Note that one could also obtain a (more conservative) lower bound γ which is uniform in x_0 by replacing the term $V(x_0)$ in (9) with 0. Also, there might be several possible choices for the functions ω and V satisfying the conditions of Theorem 1. The degrees of freedom in the choice of ω and V can then be used to maximize the gain γ in (9). Example 4 will illustrate this point in more detail. Furthermore, if systems without outputs are considered, i.e., an output map h is not given *a priori*, we might first search for functions ω and V satisfying the relevant conditions of Theorem 1. Then, the system is norm-controllable for every *a posteriori* defined output map h satisfying (6). It is also useful to note that increasing the output dimension by appending extra variables to the output cannot destroy norm-controllability (it can only help attain it).

Besides the conceptual complementarity between norm-controllability and ISS as discussed in Section III, we emphasize that also the sufficient conditions of Theorem 1 are in some sense "dual" to the conditions in typical ISS results [10] (and related notions involving outputs such as IOS). Namely, a system is ISS if and only if there exist a continuously differentiable function V and $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \chi, \rho$ such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$, and $\dot{V} \leq -\chi(|x|)$ for all u and all x satisfying $|x| \geq \rho(|u|)$. This means that the decay rate of V can be upper-bounded in terms of $|x|$ if $|x|$ is large enough in comparison to $|u|$ (and this has to hold *for all* u). In contrast to this, for norm-controllability we require in (8) that *there exists* a u such that the growth rate of V can be lower-bounded in terms of b (the upper bound for the input norm) if $|\omega(x)|$ is small compared to b. Using this condition, in the proof of Theorem 1 (respectively, Theorem 2, see Section IV-B) we construct a specific piecewise constant control input which yields a trajectory with a suitably increasing output norm. Parts of this proof were inspired by [16] where the authors construct a piecewise constant control to asymptotically stabilize a system. However, we note that while it is instructive to highlight the connections and similarities between the sufficient condition for norm-controllability in Theorem 1 and typical Lyapunov-like conditions for ISS, most of the technical ideas employed here are different from those used in the above references.

B. *Sufficient Condition Based on Higher-Order Directional Derivatives*

The sufficient condition for norm-controllability presented in Theorem 1 is appealing due to its rather simple structure and its similarity to other Lyapunov-like results such as those for ISS. However, this condition can be rather restrictive and is in general not satisfied for systems whose output y has a relative degree greater than one (see Example 5 for a further discussion of this issue). The sufficient conditions presented in what follows resolve this issue by relaxing the conditions of Theorem 1, and can be used for systems with arbitrary relative degree. On the other hand, we note that verifying these relaxed conditions for a given system requires more complicated analysis than in the case of Theorem 1. The following sufficient conditions are based on higher-order lower directional derivatives, which we define in accordance with [12, Equation (3.4a)] (cf. also the earlier work [13]). Namely, if $V^{(1)}(x;h_1)$ exists, the second-order lower directional derivative of V at a point x in the directions h_1 and h_2 is defined as[4]

$$\begin{aligned} &V^{(2)}(x;h_1,h_2) \\ &\quad := \liminf_{t\searrow 0, \bar{h}_2\to h_2} \left(\frac{2}{t^2}\right) \\ &\qquad \times \left(V(x+th_1+t^2\bar{h}_2) - V(x) - tV^{(1)}(x;h_1)\right). \end{aligned}$$

In general, if the corresponding lower-order lower directional derivatives exist, for $k \geq 1$ define the kth-order lower directional derivative as

$$\begin{aligned} &V^{(k)}(x;h_1,\ldots,h_k) \\ &\quad := \liminf_{t\searrow 0, \bar{h}_k\to h_k} \left(\frac{k!}{t^k}\right) \\ &\qquad \times \Big(V(x+th_1+\cdots+t^k\bar{h}_k) - V(x) - tV^{(1)}(x;h_1) \\ &\qquad - \cdots - (1/(k-1)!)t^{k-1}V^{(k-1)}(x;h_1,\ldots,h_{k-1})\Big). \end{aligned} \tag{10}$$

Similar to Section IV-A, we will later consider lower directional derivatives of a function V along the solution $x(\cdot)$ of system (1). For[5] $k \leq \bar{k}$, we obtain the following expansion for the solution $x(\cdot)$ of system (1), starting at time t' at the point $x := x(t')$ and applying some *constant* input u:

$$x(t) = x + (\Delta t)h_1 + \cdots + (\Delta t)^k h_k + o\left((\Delta t)^k\right) \tag{11}$$

with $\Delta t := t - t'$ and

$$\begin{aligned} h_1 &:= \dot{x}(t') = f(x,u), \\ h_2 &:= (1/2)\ddot{x}(t') = (1/2)\partial f/\partial x|_{(x,u)} f(x,u), \\ &\vdots \\ h_k &:= (1/k!)x^{(k)}(t'). \end{aligned} \tag{12}$$

In order to facilitate notation, in the following we write

$$V^{(k)}\left(x; f(x,u)\right) := V^{(k)}(x;h_1,\ldots,h_k) \tag{13}$$

for the kth-order lower directional derivative of V at the point x along the solution of (1) when a *constant* input u is applied, i.e., with $h_1,\ldots,h_k$ given in (12). It is straightforward to verify that at every point where V is sufficiently smooth, $V^{(k)}(x;f(x,u))$

[4]In contrast to [12], [13], here we include the factor 2 (and later, in (10), the factor $k!$) into the definition of higher-order lower directional derivatives. We take this slightly different approach so that later, at each point where V is sufficiently smooth, these lower directional derivatives reduce to classical directional derivatives (without any extra factors as in [12], [13]).

[5]Recall that $f \in C^{\bar{k}-1}$ for some $\bar{k} \geq 1$.

reduces to $L_f^k V|_{(x,u)}$ (compare [12, Section 3] and [13, p.73]), where $L_f^k V|_{(x,u)}$ is the kth-order Lie derivative of V along the vector field f. Namely, if V is locally Lipschitz, the kth-order lower directional derivative reduces to the kth-order Dini derivative (i.e., in (10) the direction h_k is fixed (see [12, Proposition 3.4(a)]); furthermore, if $V \in C^k$, then the limit in (10) exists (compare [12, Proposition 3.4(b)]) which equals $L_f^k V$ [13, p. 73].

Theorem 2: Suppose there exist a set $\bar{U} \subseteq U$ containing 0 in its closure, and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is invariant for system (1) under controls in $\bar{U}$. Furthermore, suppose there exist a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$ which for some $1 \leq k \leq \bar{k}$ is k times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and such that the k-th order partial derivatives of V are locally Lipschitz on $\mathbb{R}^n \setminus W$, and functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$ such that (6)-(7) is satisfied and the following holds:

- For each $b > 0$ and each $x \in \mathcal{B}$ such that $|\omega(x)| \leq \rho(b)$, there exists some $u \in \bar{U}$ with $|u| \leq b$ such that
$$V^{(j)}(x; f(x,u)) \geq 0 \qquad j = 1, \ldots, k-1, \tag{14}$$
$$V^{(k)}(x; f(x,u)) \geq \chi(b). \tag{15}$$
- At least one of the following conditions is satisfied: (i) $\partial V^{(j)}(x; f(x,u))/\partial u = 0$ for all $j = 1, \ldots, k-1$ and all $x \in \mathcal{B} \setminus W$; (ii) $\bar{U} \cap (\cap_{x \in \mathcal{B}, |\omega(x)| \leq \rho(b)} U^b(x)) \neq \emptyset$ for all $b > 0$, where
$$U^b(x) := \{u \in U : |u| \leq b,\ (14) - (15) \text{ hold}\}. \tag{16}$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function
$$\gamma(a,b) = \nu\left(\alpha_2^{-1}\left(\min\left\{\frac{1}{k!}a^k \chi(b) + V(x_0), \alpha_1(\rho(b))\right\}\right)\right). \tag{17}$$

Remark 2: For the special case of $k = 1$, Theorem 1 is recovered. In particular, condition (i) of Theorem 2 is trivially satisfied if $k = 1$. □

Remark 3: Conditions (i) and (ii) in Theorem 2 are meant to ensure that derivatives of V up to order $k-1$ along solutions of (1) are continuous when applying a piecewise constant input, which will be needed in the proof. Condition (i) achieves this by requiring that derivatives up to order $k-1$ do not explicitly depend on u. On the other hand, condition (ii) implies that the set of "good" inputs (16) (in the sense that (14), (15) holds) has a non-empty intersection for all $x \in \mathcal{B}$ with $|\omega(x)| \leq \rho(b)$, i.e., there exists a common "good" input value u for all such x and hence a constant input signal can be used to increase the norm of the output. While condition (ii) might be restrictive for some systems, it is sufficient that either condition (i) or (ii) is satisfied, i.e., (ii) only has to hold if (i) is not true. Furthermore, depending on how the solutions of system (1) behave, in some cases we might be able to use a constant input (along a given trajectory) even if condition (ii) fails (see Example 7 for more details). □

Remark 4: When considering higher-order derivatives (i.e., $k \geq 2$), we do not necessarily need V to be not differentiable for all x where $\omega(x) = 0$ in order to be able to satisfy (14), (15), as was the case for $k = 1$. Nevertheless, allowing V to be not continuously differentiable everywhere still helps in finding V satisfying (14), (15); this is in particular true later for the situation of Theorem 3, where we generalize the results of Theorem 2 such that (14)-(15) can hold for flexible k. □

Proof of Theorem 2: In the following, we will develop two technical lemmas and then obtain the proof of Theorem 2 by combining them. Let $a, b > 0$ be arbitrary but fixed, and assume in the sequel that the hypotheses of Theorem 2 are satisfied. Furthermore, we assume that no $u(\cdot) \in \mathcal{U}_{a,b}$ leads to a finite escape time $t_{esc} \leq a$, for otherwise, by the unboundedness observability property, also $R_h^a(x_0, \mathcal{U}_{a,b}) = \infty$, and thus (3) is satisfied with γ as in (17) and we are done.

The idea of the proof is to construct a piecewise constant input signal $u(\cdot) \in \mathcal{U}_{a,b}$ such that when applying this input signal, the corresponding output trajectory satisfies $|y(a)| = |h(x(a))| \geq \gamma(a,b)$. The first lemma considers the initial phase and proves that V can be increased, and is in particular needed for the case where $\omega(x_0) = 0$, i.e., $x_0 \in W$. To this end, define the set
$$X_{b,\kappa} := \{x \in \mathcal{B} : \kappa \leq |\omega(x)| \leq \rho(b)\} \tag{18}$$
for κ satisfying $0 \leq \kappa \leq \rho(b)$. Recall the definition of $U^b(x)$ in (16) and note that the assumptions of Theorem 2 expressed by (14)-(15) imply that for each $b > 0$, $\bar{U} \cap U^b(x) \neq \emptyset$ for all $x \in X_{b,0}$ (and hence also for all $x \in X_{b,\kappa}$ for each $0 \leq \kappa \leq \rho(b)$, as $X_{b,\kappa} \subseteq X_{b,0}$).

Lemma 1: Assume that $x_0 \in X_{b,0}$ and $u_0 \in \bar{U} \cap U^b(x_0)$. Then there exists some $\tau > 0$ such that for all $t \in (0, \tau]$, it holds that $V(x(t)) > V(x_0)$ and hence in particular $V(x(t)) > 0$, where $x(\cdot)$ is the trajectory of the system (1) that results from applying the constant input u_0 during this time interval. □

Proof: See Appendix I. □

Next, we consider the situation where the state x is already away from the set W. Then, according to our assumptions, V is k times continuously differentiable with locally Lipschitz k-th order partial derivatives, which we can use to show that if some input u_i is "good" at some point x_i in the sense that $V^{(k)}$ is positive, it is also "good" for nearby x_i. Let $\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\} := \bigcup_{0 \leq t \leq a} \mathcal{R}^t\{x_0, \mathcal{U}_{a,b}\}$.

Lemma 2: Consider some time instant $0 \leq s < a$ with $x(s) \in \mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}$, and assume that $x(s) \in X_{b,\delta}$ for some $\delta > 0$; furthermore, pick some $u_s \in \bar{U} \cap U^b(x(s))$. Then, for each $0 < \varepsilon \leq 1$, there exists a number $\Delta(\varepsilon, \delta) > 0$ such that $V^{(k)}(x(t)) \geq (1-\varepsilon)\chi(b)$ and
$$V^{(j)}(x(t)) - V^{(j)}(x(s)) \geq \frac{1-\varepsilon}{(k-j)!}\chi(b)(t-s)^{k-j} \tag{19}$$
for all $j = 0, \ldots, k-1$ and all $t \in [s, s + \Delta(\varepsilon, \delta)] \cap [0, a]$, where $x(\cdot)$ is the trajectory that results from applying the constant input u_s during this time interval, and $V^{(j)}(x(t)) := V^{(j)}(x(t); f(x(t), u_s)) = L_f^j V|_{(x(t), u_s)}$ for $j = 1, \ldots, k-1$ and $V^{(0)}(x(t)) := V(x(t))$.

Proof: See Appendix II. □

Combining Lemmas 1 and 2, we are now able to prove Theorem 2. Fix an arbitrary $0 < \bar{\varepsilon} < 1$. Denote by Λ_b the sublevel set
$$\Lambda_b := \{x \in \mathbb{R}^n : V(x) \leq \alpha_1(\rho(b))\}. \tag{20}$$

We construct a desired input signal in a recursive fashion using the following algorithm. This input signal will by construction satisfy $u(t) \in \bar{U}$ for all $t \in [0, a]$; hence, the resulting state trajectory $x(\cdot)$ will remain in the set $\mathcal{B}$ in this time interval if $x_0 \in \mathcal{B}$. The trajectory $x(\cdot)$ is exemplarily illustrated in Fig. 1, which might be helpful in following the remainder of the proof.

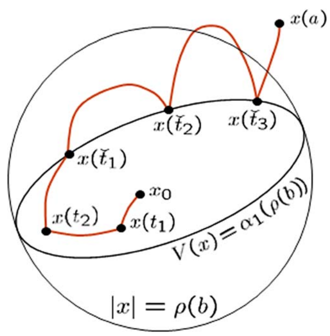

Fig. 1. Illustration of the state trajectory constructed in the proof of Theorem 2 for the case $\omega(x) = x$.

Step 0: Consider $x_0 \in \mathcal{B}$. If $x_0 \in \mathrm{int}(\Lambda_b)$, then by (7) we have $|\omega(x_0)| < \rho(b)$ and so $x_0 \in X_{b,0}$ according to (18). We can then pick some $u_0 \in \bar{U} \cap U^b(x_0)$, which exists as $\bar{U} \cap U^b(x_0) \neq \emptyset$, and apply Lemma 1 to find a time $\tau > 0$ such that the trajectory corresponding to the constant control $u \equiv u_0$ satisfies $V(x(t)) > V(x_0)$ and hence in particular also $V(x(t)) > 0$ for all $0 < t \leq \tau$. Pick some $t_1 \in (0, \min\{\tau, a\}]$; note that t_1 can be chosen arbitrarily small. Apply the constant input $u \equiv u_0$ on the interval $[0, t_1)$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (0, t_1)$, then denote this time t by $\check{t}_1$ and skip to Step 2, otherwise proceed to Step 1. If $x_0 \in \partial\Lambda_b$, let $t_1 = \check{t}_1 := 0$ and skip to Step 2. If already $x_0 \notin \Lambda_b$, then let $t_1 := 0$, pick some $u_1 \in \bar{U}$ with $|u_1| \leq b$ and apply the constant input $u \equiv u_1$ on the interval $[0, a)$ for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. Note that for each $b > 0$, such a u_1 exists, as by assumption $\bar{U}$ contains 0 in its closure, i.e., $\bar{U}$ contains input values u of arbitrarily small magnitude. If $x(t) \in \partial\Lambda_b$ for some $t \in [0, a)$, then denote this time t by $\check{t}_1$ and skip to Step 2, otherwise skip to Step 3.

Step 1: If[6] $x(t_1) \in \partial\Lambda_b$, then let $\check{t}_1 := t_1$ and skip to Step 2. Otherwise, $x(t_1) \in \mathrm{int}(\Lambda_b)$. Let $\bar{\delta} := \alpha_2^{-1}(V(x(t_1)))$ and note that $\bar{\delta} > 0$ according to the definition of t_1 in Step 0. From (7), the definition of Λ_b, and the fact that $x(t_1) \in \mathcal{B}$, it follows that $\bar{\delta} \leq |\omega(x(t_1))| < \rho(b)$, and hence $x(t_1) \in X_{b,\bar{\delta}}$ by (18). We can thus pick some $u_1 \in \bar{U} \cap U^b(x(t_1))$ and apply Lemma 2 with $s = t_1$, $u_s = u_1$, $\varepsilon = \bar{\varepsilon}$ and $\delta = \bar{\delta}$ to find a $\Delta(\bar{\varepsilon}, \bar{\delta})$ such that the trajectory corresponding to the constant control $u \equiv u_1$ on the interval $[t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$ satisfies (19) with $s = t_1$ and $\varepsilon = \bar{\varepsilon}$ on this interval. Apply the constant input $u \equiv u_1$ on this interval for as long as the resulting trajectory $x(\cdot)$ does not hit $\partial\Lambda_b$. If we have $x(t) \in \partial\Lambda_b$ for some $t \in (t_1, \min\{t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}), a\})$, then denote this time t by $\check{t}_1$ and skip to Step 2. If this does not happen but $t_1 + \Delta(\bar{\varepsilon}, \bar{\delta}) \geq a$, then skip to Step 3. Otherwise, let $t_2 := t_1 + \Delta(\bar{\varepsilon}, \bar{\delta})$. In this case, $x(t_2) \in \Lambda_b$ and, by Lemma 2, $V(x(t_2)) > V(x(t_1))$ according to (19) with $j = 0$ and $s = t_1$. So, we can check that $x(t_2) \in X_{b,\bar{\delta}}$ similarly to how we did it earlier for $x(t_1)$. Therefore, we can repeat Step 1 for the times t_2, t_3, ... (but without changing the value of $\bar{\delta}$); note that this sequence of time instants is finite, as $t_{i+1} - t_i = \Delta(\bar{\varepsilon}, \bar{\delta}) > 0$ for all i. In case condition (ii) in Theorem 2 is satisfied, we ensure additionally that the input values $u_1, u_2, \ldots$ are chosen to be the same.

[6]Even though $u(t_1)$ is not yet defined, $x(t_1)$ is defined by continuity as $\lim_{t \nearrow t_1} x(t)$.

Step 2: We have $x(\check{t}_1) \in \partial\Lambda_b$, i.e., $V(x(\check{t}_1)) = \alpha_1(\rho(b))$. If $\check{t}_1 = a$ then skip to Step 3. Otherwise, we can verify analogously to Step 1 that $x(\check{t}_1) \in X_{b,\check{\delta}}$ with $\check{\delta} := \alpha_2^{-1}(\alpha_1(\rho(b)))$, and hence $\bar{U} \cap U^b(x(\check{t}_1)) \neq \emptyset$. Pick some $\check{u}_1 \in \bar{U} \cap U^b(x(\check{t}_1))$. Apply the constant input $u \equiv \check{u}_1$ on the interval $[\check{t}_1, \check{t}_2)$ where $\check{t}_2 := \min\{\inf\{t : t > \check{t}_1,\, x(t) \in \partial\Lambda_b\}, a\}$. This interval is non-empty; in fact, $\check{t}_2 \geq \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}$ where $\Delta(\cdot,\cdot)$ comes from Lemma 2. Indeed, as $x(\check{t}_1) \in X_{b,\check{\delta}}$, we can apply Lemma 2 with $s = \check{t}_1$, $u_s = \check{u}_1$, $\varepsilon = 1/2$ (any other choice $0 < \varepsilon < 1$ would give a similar result), $\delta = \check{\delta}$ and $j = 0$ in order to conclude that $V(x(t)) - V(x(\check{t}_1)) > 0$ for all $t \in (\check{t}_1, \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}]$, which implies that $\check{t}_2 \geq \min\{\check{t}_1 + \Delta(1/2, \check{\delta}), a\}$. Moreover, if $\check{t}_2 < a$ then $x(\check{t}_2) \in \partial\Lambda_b$ and we can repeat Step 2 for the times $\check{t}_2$, $\check{t}_3$, and so on, and the above argument shows that this sequence of time instants is finite.

Step 3: We have now reached the time $t = a$ and we have constructed the following control input defined on the interval $[0, a)$, with the control values $u_i, \check{u}_j$ and the times $t_i, \check{t}_j$ as specified above (those times that are never defined are treated as ∞):

$$u(t) = \begin{cases} u_0 & 0 \leq t < \min\{t_1, \check{t}_1\} \\ u_i & t_i \leq t < \min\{t_{i+1}, \check{t}_1, a\}, \quad i = 1, 2, \ldots \\ \check{u}_j & \check{t}_j \leq t < \min\{\check{t}_{j+1}, a\}, \quad j = 1, 2, \ldots \end{cases} \tag{21}$$

This input, extended with the last value (u_0, u_i or $\check{u}_j$) at $t = a$ (and arbitrarily for $t > a$), satisfies $u \in \mathcal{U}_{a,b}$, as by construction, $|u(t)| \leq b$ for all $t \in [0, a]$. For each $0 < \bar{\varepsilon} < 1$, this input signal is piecewise constant in the interval $[0, a]$ with only finitely many different values u_i and $\check{u}_j$; this follows from the construction in Step 1 and the argument given in Step 2. Let $x(\cdot)$ denote the state trajectory resulting from the application of the control input $u(\cdot)$ to the system (1), and $V^{(j)}(x(t)) := V^{(j)}(x(t); f(x(t), u(t)))$ for $j = 1, \ldots, k$. The state trajectory $x(\cdot)$ has the following properties. First, consider the case where $t_1 > 0$ (recall from Step 0 that this corresponds to $x_0 \in \mathrm{int}(\Lambda_b)$). Using (19) with $j = k-1$ and $\varepsilon = \bar{\varepsilon}$, we obtain that $V^{(k-1)}(x(t_i^-)) \geq (1 - \bar{\varepsilon})(t_i - t_{i-1})\chi(b) + V^{(k-1)}(x(t_{i-1}))$ for all t_i with $i \geq 2$ which were defined in Step 1, and $V^{(k-1)}(x(\min\{\check{t}_1, a\}^-)) \geq (1 - \bar{\varepsilon})(\min\{\check{t}_1, a\} - t_r)\chi(b) + V^{(k-1)}(x(t_r))$, where t_r is the last time instant of the sequence $\{t_i\}$ defined in Step 1. Now if condition (i) is satisfied, we have that $V^{(k-1)}(x(t))$ is continuous also at the time instants $t = t_i$ where the control values switch (as $x(\cdot)$ is continuous), i.e., $V^{(k-1)}(x(t_i^-)) = V^{(k-1)}(x(t_i))$. On the other hand, if condition (ii) holds, then $u_1 = u_2 = \ldots$ by construction and hence again $V^{(k-1)}(x(t))$ is continuous also at the time instants $t = t_i$ (as $x(\cdot)$ is continuous). Hence it follows that for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$ we have $V^{(k-1)}(x(t)) \geq (1 - \bar{\varepsilon})(t - t_1)\chi(b) + V^{(k-1)}(x(t_1)) \geq (1 - \bar{\varepsilon})(t - t_1)\chi(b)$, where the second inequality follows from (14) with $j = k-1$, $x = x(t_1)$ and $u = u_1$. Now for each $t_i \leq t < t_{i+1}$, due to the fact that the constant control u_i is applied, it follows that $V^{(k-2)}(x(t)) = V^{(k-2)}(x(t_i)) + \int_{t_i}^{t} V^{(k-1)}(x(\tau))d\tau$. Using the same argumentation as above, it follows that $V^{(k-2)}(x(t))$ is also continuous at the time instants $t = t_i$, and hence we obtain $V^{(k-2)}(x(t)) \geq (1/2)(1 - \bar{\varepsilon})(t -$

$t_1)^2\chi(b) + V^{(k-2)}(x(t_1)) \geq (1/2)(1-\bar{\varepsilon})(t-t_1)^2\chi(b)$ for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$, where the second inequality again follows from (14) with $j = k-2$, $x = x(t_1)$ and $u = u_1$. Repeating the above $k-2$ times results in $V(x(t)) \geq (1/k!)(1-\bar{\varepsilon})(t-t_1)^k\chi(b) + V(x(t_1))$ for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$; furthermore, recall from Step 0 that $V(x(t_1)) > V(x_0)$ due to Lemma 1.

Next, if $\check{t}_1 < a$, then for $\check{t}_1 \leq t \leq a$ the construction of the input signal guarantees that $V(x(t)) \geq V(x(\check{t}_1)) = \alpha_1(\rho(b))$. Finally, if $t_1 = 0$ (recall from Step 0 that this corresponds to $x_0 \notin \operatorname{int}(\Lambda_b)$), then the preceding inequality $V(x(t)) \geq \alpha_1(\rho(b))$ is satisfied for all $0 \leq t \leq a$. Combining the above yields

$$V\left(x(a)\right) \geq \min\left\{\frac{1}{k!}(1-\bar{\varepsilon})(a-t_1)^k\chi(b) + V(x_0),\; \alpha_1\left(\rho(b)\right)\right\}. \quad (22)$$

Hence, using (7), we have

$$|\omega\left(x(a)\right)| \geq \alpha_2^{-1}\left(\min\left\{(1/k!)(1-\bar{\varepsilon})(a-t_1)^k\chi(b) + V(x_0), \alpha_1\left(\rho(b)\right)\right\}\right).$$

Finally, using (6), we obtain

$$|h\left(x(a)\right)| \geq \nu\left(\alpha_2^{-1}\left(\min\left\{(1/k!)(1-\bar{\varepsilon})(a-t_1)^k\chi(b) + V(x_0), \alpha_1\left(\rho(b)\right)\right\}\right)\right).$$

As $u(\cdot)$ is contained in $\mathcal{U}_{a,b}$ and as the above calculations hold for arbitrary $x_0 \in \mathcal{B}$, it follows that:

$$R_h^a(x_0, \mathcal{U}_{a,b}) \geq \nu\left(\alpha_2^{-1}\left(\min\left\{(1/k!)(1-\bar{\varepsilon})(a-t_1)^k\chi(b) + V(x_0), \alpha_1\left(\rho(b)\right)\right\}\right)\right) \quad (23)$$

for all $x_0 \in \mathcal{B}$. Note that (23) holds for every $0 < \bar{\varepsilon} \leq 1$, and according to Step 0, either $t_1 = 0$ or t_1 can be chosen arbitrarily small. Thus, as the left-hand side of (23) is independent of $\bar{\varepsilon}$ and t_1, we can let $\bar{\varepsilon} \to 0$ and $t_1 \to 0$ (in case t_1 is not 0) and arrive at the desired bound (3) with γ as defined in (17). The function γ satisfies the required properties of Definition 1, i.e., $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\gamma(a, \cdot) \in \mathcal{K}_\infty$ for each fixed $a > 0$. This concludes the proof of Theorem 2. □

Remark 5: The input signal $u(\cdot)$ constructed in the proof of Theorem 2 results in a monotonically increasing $V(x(\cdot))$, as long as $x(\cdot)$ does not leave the set Λ_b. However, note that this does not mean that our sufficient conditions only capture cases where the output $h(x)$ can be increased monotonically. Namely, h is related to V via (6), (7), and hence we can certainly have monotonic behavior of V but non-monotonic behavior of h. □

C. Extensions

In this section, we present two extensions of the sufficient conditions developed in Section IV-A and IV-B. This will allow us to treat some more advanced examples that the previous results cannot handle (see, e.g., Example 6 in Section V). We first show that in Theorem 2 (and hence also in Theorem 1), the assumption that the set $\mathcal{B}$ is invariant under controls in $\bar{U}$ can be relaxed. To this end, recall that for a given x, the set $U^b(x)$ was defined in (16) as the set of all "good" inputs, i.e., such that (14), (15) is satisfied. Now for each $b > 0$ and each $x \in \mathcal{B}$ such that $|\omega(x)| \leq \rho(b)$, denote by $\hat{U}^b(x)$ an arbitrary nonempty subset[7] of $U^b(x)$. Then, let $\widetilde{U}^b := \cup_{x\in\mathcal{B}, |\omega(x)|\leq\rho(b)}\hat{U}^b(x)$. If $\widetilde{U}^b = \emptyset$ for some $b' > 0$ (which can only happen if $|\omega(x)| > \rho(b')$ for all $x \in \mathcal{B}$), let $\widetilde{U}^b$ be an arbitrary nonempty subset of $\{u : |u| \leq b\}$. Now if $\mathcal{B}$ is invariant under controls in $\widetilde{U} := \cup_{b>0}\widetilde{U}^b$, we have the situation of Theorem 2 (with $\bar{U} = \widetilde{U}$); if not, consider the following.

Proposition 1: Theorem 2 remains valid under the following modifications.

1) The assumption that a set $\bar{U}$ exists such that the set $\mathcal{B}$ is invariant under controls in $\bar{U}$ is replaced by the following: For each $b > 0$, there exists a set $H_b \subseteq \mathbb{R}^n$ with $H_b \cap \Lambda_b = \emptyset$ and Λ_b defined by (20) such that if $x_0 \in \mathcal{B}$ and $u(t) \in \widetilde{U}^b$ for all $t \geq 0$, then $x(t) \in \mathcal{B} \cup H_b$ for all $t \geq 0$.
2) Instead of only holding for all $x \in \mathcal{B}$, equations (6)-(7) hold for all $x \in \cup_{b>0}H_b \cup \mathcal{B}$.
3) In condition (ii) of Theorem 2, the set $U^b(x)$ is replaced by $\hat{U}^b(x)$.

Proof: See Appendix III. □

Condition 1) in Proposition 1 means that each trajectory $x(\cdot)$ cannot exit $\mathcal{B}$ before exiting Λ_b. In other words, when at some time instant t we have $x(t) \in \Lambda_b$, then also $x(t) \in \mathcal{B}$.

The second extension is to generalize the results of Theorem 2 to the case where the set $\mathcal{B}$ can be partitioned into several regions where (14), (15) holds for different k. This means that $\mathcal{B}$ can be written as $\mathcal{B} = \cup_{i=1}^{\ell}\mathcal{R}_i(b)$ for some $\ell \geq 1$ and sets $\mathcal{R}_i(b)$ (possibly depending on b).

Theorem 3: Suppose there exist a set $\bar{U} \subseteq U$ containing 0 in its closure, and a closed set $\mathcal{B} \subseteq \mathbb{R}^n$ which is invariant for system (1) under controls in $\bar{U}$. Furthermore, suppose there exist a constant $\ell \geq 1$ and for each $b > 0$ a partition $\mathcal{B} = \cup_{i=1}^{\ell}\mathcal{R}_i(b)$ with corresponding integer constants $1 \leq k_1 < k_2 < \ldots < k_\ell \leq \bar{k}$, a continuous function $\omega : \mathbb{R}^n \to \mathbb{R}^q$, $1 \leq q \leq n$, a function $V : \mathbb{R}^n \to \mathbb{R}$ which is k_ℓ times continuously differentiable on $\mathbb{R}^n \setminus W$ with $W := \{x \in \mathbb{R}^n : \omega(x) = 0\}$ and such that the k_ℓ-th order partial derivatives of V are locally Lipschitz on $\mathbb{R}^n \setminus W$, and functions $\alpha_1, \alpha_2, \chi, \rho, \nu \in \mathcal{K}_\infty$ such that (6), (7) is satisfied and the following holds:

- For each $b > 0$ and each $x \in \mathcal{R}_i(b)$ for some $1 \leq i \leq \ell$ satisfying $|\omega(x)| \leq \rho(b)$, there exists some $u \in \bar{U}$ with $|u| \leq b$ such that
$$V^{(j)}\left(x; f(x,u)\right) \geq 0 \qquad j = 1, \ldots, k_i - 1, \quad (24)$$
$$V^{(k_i)}\left(x; f(x,u)\right) \geq \chi_i(b). \quad (25)$$
- At least one of the following conditions is satisfied for all $b > 0$:
  (i) For all $1 \leq i \leq \ell$, $\partial V^{(j)}(x; f(x,u))/\partial u = 0$ for all $j = 1, \ldots, k_i - 1$ and all $x \in \mathcal{R}_i(b) \setminus W$;
  (ii) $\bar{U} \cap \left(\cap_{1\leq i\leq\ell} \cap_{x\in\mathcal{R}_i(b), |\omega(x)|\leq\rho(b)} U_i^b(x)\right) \neq \emptyset$, where
$$U_i^b(x) := \left\{u \in U : |u| \leq b,\ (24){-}(25) \text{ hold}\right\}. \quad (26)$$

Then the system (1) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function

$$\gamma(a,b) = \nu\Big(\alpha_2^{-1}\Big(\min\Big\{\Psi(a,b) + V(x_0), \alpha_1\left(\rho(b)\right)\Big\}\Big)\Big) \quad (27)$$

[7]Note that such a subset exists because $U^b(x) \neq \emptyset$ for all such x, as discussed in the paragraph below equation (18).

where

$$\Psi(a,b) = \min_{i\in\{1,\ldots,\ell\}} \frac{(k_\ell - k_i)!}{k_\ell!} a^{k_i}\chi_i(b). \tag{28}$$

Proof: See Appendix IV. □

Remark 6: In the special case of $\ell = 1$, Theorem 2 is recovered. □

Remark 7: Proposition 1 also applies to Theorem 3, i.e., the assumption of control-invariance of the set $\mathcal{B}$ can be relaxed. In this case, the following slight adaptation of Proposition 1 is necessary: the set $\widetilde{U}^b$ is now defined as $\widetilde{U}^b := \cup_{1\le i\le\ell}\cup_{x\in\mathcal{R}_i(b),|\omega(x)|\le\rho(b)} \hat{U}_i^b(x)$, where $\hat{U}_i^b(x)$ is an arbitrary nonempty subset of $U_i^b(x)$ such that condition (ii) of Theorem 3 is still satisfied with $U_i^b(x)$ replaced by $\hat{U}_i^b(x)$. □

V. Examples

In this Section, we illustrate the concept of norm-controllability as well as the presented sufficient conditions with several examples.

Example 1: Consider the system $\dot{x} = -x^3 + u$ with output $h(x) = x$, and take $\omega(x) = x$ and $V(x) = |x|$. For each $x \neq 0$ and each $b > 0$, by choosing u such that $|u| = b$ and $xu \geq 0$, we obtain $\dot{V} = -|x|^3 + \mathrm{sign}(x)u = -|x|^3 + \theta\mathrm{sign}(x)u + (1-\theta)\mathrm{sign}(x)u \geq (1-\theta)b =: \chi(b)$, for all $|x| \leq \sqrt[3]{\theta b} =: \rho(b)$ and arbitrary $0 < \theta < 1$. Here θ is just a parameter whose choice determines the functions χ and ρ. For $x = 0$, choosing u such that $|u| = b$ results in $V^{(1)}(0, f(0,u)) = b \geq \chi(b)$. Hence we can apply Theorem 1 with $\mathcal{B} = \mathbb{R}$ and[8] $\nu = \alpha_1 = \alpha_2 = \mathrm{id}$ to conclude that the considered system is norm-controllable from all $x_0 \in \mathbb{R}$ with gain function $\gamma_\theta(a,b) = \min\{(1-\theta)ab + |x_0|, \sqrt[3]{\theta b}\}$. As $R_h^a(x_0, \mathcal{U}_{a,b})$ is independent of θ, it follows that the considered system is norm-controllable from all $x_0 \in \mathbb{R}$ with gain function $\gamma(a,b) = \max_{0\le\theta\le 1}\gamma_\theta(a,b)$. Note that the latter conclusion also applies to all subsequent examples involving a parameter θ.

Example 2: Consider the system $\dot{x} = u/(1+|u|)$ with $h(x) = x$. This system is not norm-controllable. Indeed, it is easy to see that $|\dot{x}| \leq 1$ for all x and u. But this means that we cannot find a function $\gamma(\cdot,\cdot)$ which is a $\mathcal{K}_\infty$ function in the second argument such that (3) holds, as for a given time horizon a, the norm of the output cannot go to infinity as $b \to \infty$. On the other hand, one can see that for $a \to \infty$, we have $|h(x)| \to \infty$ for every constant control $u > 0$.

Example 3: Consider the system

$$\begin{aligned} \dot{x}_1 &= (1 + \sin(x_2 u))\,|u| - x_1 \\ \dot{x}_2 &= x_1 - \frac{1}{5}x_2 \\ y &= x_1 \end{aligned} \tag{29}$$

for which the set $\mathcal{B} := \{x : x_1 \geq 0\}$ is invariant under controls in $\bar{U} = \mathbb{R}$. Take $\omega(x) = x_1$ and $V(x) = |x_1|$. For each $b > 0$, choosing u such that $|u| = b$ and $\sin(x_2 u) \geq 0$ results in

$$\begin{aligned} V^{(1)}(x, f(x,u)) &= (1 + \sin(x_2 u))\,|u| - x_1 \\ &\geq b - x_1 \geq (1-\theta)b =: \chi(b) \end{aligned}$$

for all $x \in \mathcal{B}$ such that $x_1 \leq \theta b =: \rho(b)$ and arbitrary $0 < \theta < 1$. This means that for each $x \in \mathcal{B}$, the set of "good" inputs, defined by (16), is given as $U^b(x) = \{u : |u| = b, \sin(x_2 u) \geq 0\}$. We can now apply Theorem 1 to conclude that the system (29) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function

$$\gamma(a,b) = \min\{(1-\theta)ab + |x_1(0)|, \theta b\}. \tag{30}$$

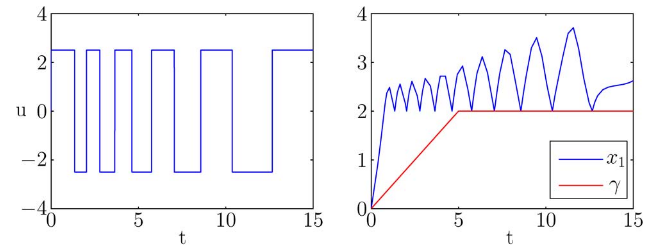

Fig. 2. Input and state trajectory together with lower bound γ for $b = 2.5$ in Example 3.

Fig. 2 shows simulated state and input trajectories with $x_0 = 0$, $b = 2.5$ and $\theta = 0.8$. The piecewise constant input signal was constructed as described in the proof of Theorem 2, i.e., new input values $u_i \in \bar{U} \cap U^b(x(t_i)) = U^b(x(t_i))$ and $\check{u}_j \in U^b(x(\check{t}_j))$ were chosen at the time instances t_i and $\check{t}_j$, respectively, and applied on the time interval $[t_i, t_{i+1})$ and $[\check{t}_j, \check{t}_{j+1})$, respectively. One can see that the input signal switches between the values b and $-b$ at certain time instances, and that during the second stage, i.e., once $|x_1| \geq \theta b$, a new control value is chosen at those time instances where $x_1 = \theta b$. □

Example 4: Consider the system

$$\dot{x} = f(x,u) = \begin{bmatrix} -x_1^3 + x_2 + u \\ -x_2 + x_1 + u \end{bmatrix}, \qquad h(x) = x. \tag{31}$$

As pointed out in Section IV-A, with this example we illustrate how different functions ω and V can be used to establish norm-controllability for system (31). To this end, consider the two functions $\omega_1(x) = x_1$ and $\omega_2(x) = x_2$, as well as $V_1(x) = |x_1|$ and $V_2(x) = |x_2|$. It holds that $|h(x)| \geq |\omega_i(x)|$ for $i = 1,2$; thus in both cases we can choose $\nu_i = \alpha_{1,i} = \alpha_{2,i} = \mathrm{id}$ in (6), (7). Furthermore, the positive orthant $\mathbb{R}^2_{\geq 0} := \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ is invariant under controls in $\bar{U} := \mathbb{R}_{\geq 0}$, which can be easily seen by noting that the vector field f points inside the positive orthant for all x on its boundary and all $u \geq 0$. Considering ω_1 and V_1, by similar calculations as in Example 1 one can show via Theorem 1 that the system (31) is norm-controllable from all $x_0 = [x_1(0)\ x_2(0)]^T \in \mathbb{R}^2_{\geq 0}$ with gain $\gamma_1(a,b) = \min\{(1-\theta)ab + |x_1(0)|, \sqrt[3]{\theta b}\}$ and arbitrary $0 < \theta < 1$. Similar calculations using ω_2 and V_2 yield that the system (31) is norm-controllable from all $x_0 \in \mathbb{R}^2_{\geq 0}$ with gain $\gamma_2(a,b) = \min\{(1-\theta)ab + |x_2(0)|, \theta b\}$ and arbitrary $0 < \theta < 1$. Hence we can conclude that the system (31) is norm-controllable from all $x_0 \in \mathbb{R}^2_{\geq 0}$ with gain $\gamma = \max\{\gamma_1, \gamma_2\}$, which shows how the possible degrees of freedom in the choice of the functions ω and V can be used to maximize the gain γ. Furthermore, by the choice of the functions ω_1, ω_2 and V_1, V_2, we also have proven norm-controllability of the system (31) for the *a posteriori* defined output maps $h_1(x) = x_1$ and $h_2(x) = x_2$.

Example 5: Consider the double integrator system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$ with output $h(x) = x_1$. The relative degree of this system is $r = 2$. Consider again the positive orthant $\mathcal{B}_1 := \mathbb{R}^2_{\geq 0}$ which is invariant under controls in $\bar{U} := \mathbb{R}_{\geq 0}$. Let $\omega(x) := x_1$ and $V(x) := |x_1|$. For all $x \in \mathcal{B}_1$ and $u \in \bar{U}$, we

[8] id denotes the identity function.

obtain $V^{(1)}(x; f(x,u)) = x_2$ (which means that condition (i) of Theorem 2 is satisfied) and $V^{(2)}(x; f(x,u)) = u$. Hence for each $b > 0$, by choosing $u = b$ we obtain that $V^{(2)}(x; f(x, u)) = b =: \chi(b)$. We can now apply Theorem 2 with $\mathcal{B}_1 := \mathbb{R}^2_{\geq 0}$, $\bar{U} := \mathbb{R}_{\geq 0}$, $k = 2$ and $\chi = \mathrm{id}$ together with Remark 1 to conclude that the system is norm-controllable from all $x_0 \in \mathcal{B}_1$ with gain function $\gamma(a,b) = (1/2)a^2 b$. Similar considerations also apply to the negative orthant $\mathcal{B}_2 := \mathbb{R}^2 \leq 0$ which is invariant under controls in $\bar{U} := \mathbb{R}_{\leq 0}$. Furthermore, we note that we necessarily need $k = 2$, i.e., there does not exist a function $V(x_1, x_2)$ satisfying the conditions of Theorem 1. To see this, first note that in order to satisfy the conditions (6), (7), we necessarily need that $V([0\ x_2]^T) \equiv 0$. For $x_1 = 0$, we obtain

$$\begin{aligned} V^{(1)}([0\ x_2]^T; f\left([0\ x_2]^T, u\right)) &= V^{(1)}\left([0\ x_2]^T; [x_2\ u]^T\right) \\ &= \liminf_{t\searrow 0, \bar{h}_1 \to [x_2\ u]^T} (1/t) V\left([0\ x_2]^T + t\bar{h}_1\right) \\ &\leq \liminf_{t \searrow 0} (1/t) V\left([t x_2\ x_2 + tu]^T\right). \end{aligned} \tag{32}$$

For $x_1 = x_2 = 0$, we obtain $V^{(1)}(x; f(x,u)) \leq 0$ independent of u, since $V([0\ x_2]^T) \equiv 0$ as discussed above (in fact, we can conclude that $V^{(1)}(x; f(x,u)) = 0$ for $x_1 = x_2 = 0$ because $V(x) \geq 0$ for all x). Hence we cannot find a function χ satisfying (8). We conjecture that a similar conclusion also holds for $x_1 \neq 0$, i.e., where V is differentiable. Furthermore, the above argument can be extended in a straightforward manner to SISO systems in normal form (see [17, p. 143]), and hence for such systems we conclude that in order to satisfy the conditions of Theorem 2 we need $k \geq r$, where r is the relative degree of the system. For MIMO systems in normal form [17, p. 224], an analogous statement holds with $r := \min\{r_1, \ldots, r_p\}$, where r_i is the smallest integer such that $y_i^{(r_i)}$ explicitly depends on some input component u_j.

Example 6: Consider an isothermal continuous stirred tank reactor (CSTR) in which an irreversible, second-order[9] reaction from reagent A to product B takes place [18]:

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{q}{V}\left(C_{A_i} - C_A\right) - kC_A^2 \\ \frac{dC_B}{dt} &= -\frac{q}{V} C_B + kC_A^2 \end{aligned} \tag{33}$$

where C_A and C_B denote the concentrations of species A and B (in $[mol/m^3]$), respectively, V is the volume of the reactor (in $[m^3]$), q is the flow rate of the inlet and outlet stream (in $[m^3/s]$), k is the reaction rate (in $[1/s]$), and C_{A_i} is the concentration of A in the inlet stream, which can be interpreted as the input. Using $x_1 := C_A$, $x_2 := C_B$, $c := q/V$ and $u := C_{A_i}$, one obtains

$$\begin{aligned} \dot{x}_1 &= -cx_1 - kx_1^2 + cu =: f_1(x,u) \\ \dot{x}_2 &= kx_1^2 - cx_2 =: f_2(x). \end{aligned} \tag{34}$$

The physically meaningful states and inputs are $x \in \mathbb{R}^2_{\geq 0}$, $u \in \mathbb{R}_{\geq 0}$, i.e., nonnegative concentrations of the two species in the reactor and the inlet stream, respectively. We are interested in the amount of product B per time unit, i.e. in the output $y = h(x) = qx_2$. Taking $\omega(x) = x_2$ and $V(x) = |\omega(x)|$, one obtains that for all $x \in \mathbb{R}^2_{\geq 0}$ and $u \in \mathbb{R}_{\geq 0}$, we have $V^{(1)}(x; f(x,u)) = kx_1^2 - cx_2$, $V^{(2)}(x; f(x,u)) = -kx_1^2(3c + 2kx_1) + c^2 x_2 + 2kcx_1 u$, and $V^{(3)}(x; f(x,u)) = (-6kcx_1 - 6k^2 x_1^2 + 2kcu) f_1(x,u) + c^2 f_2(x)$. Now consider the region

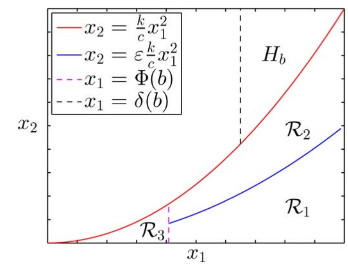

Fig. 3. Partition of the invariant region $\mathcal{B}$ and the set H_b in Example 6.

$$\mathcal{B} := \left\{x : 0 \leq x_2 \leq (k/c)x_1^2\right\} \subset \mathbb{R}^2_{\geq 0}. \tag{35}$$

Note that $V^{(1)}(x; f(x,u)) \geq 0$ for all $x \in \mathcal{B}$. Fix some arbitrary $0 < \varepsilon, \theta < 1$, and for $b \geq 0$ define $\varphi_1(b;\varepsilon) := (-(3-\varepsilon)c + \sqrt{(3-\varepsilon)^2 c^2 + 16ck\theta b})/(4k)$, $\varphi_2(b) := \min\{cb/(8(c+k)), \sqrt{cb/(8(c+k))}\}$ and $\Phi(b) := \min\{\varphi_1(b;0), \varphi_2(b)\}$. Now consider the following partition of $\mathcal{B}$, which is also exemplarily depicted in Fig. 3: $\mathcal{R}_1(b) := \{x \in \mathcal{B} : 0 \leq x_2 \leq \varepsilon(k/c)x_1^2, x_1 \geq \Phi(b)\}$, $\mathcal{R}_2(b) := \{x \in \mathcal{B} : (\varepsilon k/c)x_1^2 \leq x_2 \leq (k/c)x_1^2, x_1 \geq \Phi(b)\}$, and $\mathcal{R}_3(b) := \{x \in \mathcal{B} : 0 \leq x_2 \leq (k/c)x_1^2, x_1 \leq \Phi(b)\}$. For all $b > 0$ and all $x \in \mathcal{R}_1(b)$, by choosing $u = b$ we obtain that $V^{(1)}(x; f(x,u)) \geq (1-\varepsilon)kx_1^2 \geq (1-\varepsilon)k\Phi^2(b) =: \chi_1(b)$. For all $b > 0$ and all $x \in \mathcal{R}_2(b)$, by choosing $u = b$ we obtain that $V^{(2)}(x; f(x,u)) \geq 2(1-\theta)kcb\Phi(b) =: \chi_2(b)$, for all $x_1 \leq \varphi_1(b;\varepsilon)$, which holds if $x_2 \leq (\varepsilon k/c)\varphi_1^2(b;\varepsilon) =: \rho(b)$. Finally, for all $b > 0$ and all $x \in \mathcal{R}_3(b)$, by choosing again $u = b$ we obtain that $V^{(2)}(x; f(x,u)) \geq 0$ and that $V^{(3)}(x; f(x,u)) \geq kc^2 b^2 =: \chi_3(b)$.

In order to be able to apply Theorem 3, it remains to show that either condition (i) or (ii) is satisfied and the set $\mathcal{B}$ can be rendered invariant, where for the latter we will use the relaxed form given by Proposition 1 (compare Remark 7). Due to the above, for each $1 \leq i \leq 3$ and each $x \in \mathcal{R}_i(b)$ with $|\omega(x)| \leq \rho(b)$, we have $U_i^b(x) = \{b\}$, and hence $\cap_{1 \leq i \leq \ell} \cap_{x \in \mathcal{R}_i(b), |\omega(x)| \leq \rho(b)} U_i^b(x) = \{b\} \neq 0$, i.e., condition (ii) of Theorem 3 is satisfied. Furthermore, also $\widetilde{U}^b = \{b\}$. Now note that for all x such that $x_2 = 0$ and $x_1 \geq 0$, $f(x,u)$ points inside $\mathcal{B}$ for all $u \in \mathbb{R}_{\geq 0}$, and hence no trajectory can leave the set $\mathcal{B}$ there. At the other boundary, i.e., for all x such that $x_2 = (k/c)x_1^2$, $f(x,u)$ points outside $\mathcal{B}$ only if $x_1 \geq (-c + \sqrt{c^2 + 4cku})/(2k) =: \delta(u)$. However, for each $b > 0$, if $x_1(\tau) \geq \delta(b)$ for some $\tau \geq 0$, then it follows from (34) that also $x_1(t) \geq \delta(b)$ for all $t \geq \tau$ in case that $u(t) \in \widetilde{U}^b$ (i.e., $u(t) = b$) for all $t \geq \tau$. Hence for each $b > 0$, we define the set $H_b := \{x : x_1 \geq \delta(b), x_2 \geq (k/c)x_1^2\}$ (see also Fig. 3). Furthermore, it is straightforward to verify that for each $b > 0$, $H_b \cap \Lambda_b = \emptyset$, where $\Lambda_b = \{x : |x_2| \leq \rho(b)\}$ according to (20).

Summarizing the above, we can apply Theorem 3 with $\ell = 3$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $\alpha_1 = \alpha_2 = \mathrm{id}$ and $\nu = q\mathrm{id}$ together

[9]Similar conclusions as in the following can also be reached in the case of a first-order reaction, i.e., with C_A^2 replaced by C_A in both places in (33), and with the invariant set $\mathcal{B}$ in (35) below redefined using x_1 instead of x_1^2.

with Proposition 1 (and Remark 7) to conclude that the system (34) is norm-controllable from all $x_0 \in \mathcal{B}$ with gain function

$$\gamma(a,b) = q \min\left\{\Psi(a,b) + x_2(0), \rho(b)\right\} \tag{36}$$

with Ψ defined in (28). An interpretation of this fact is as follows. If $x_2 \leq (k/c)x_1^2$, then a sufficiently large amount of reagent A compared to the amount of product B is present in the reactor in order that the amount of product B can be increased. On the other hand, if $x_2 > (k/c)x_1^2$, then already too much product B is inside the reactor so that its amount will first decrease (due to the outlet stream), no matter how large the concentration of A in the inlet stream (i.e., the input u) is, and hence the conditions of Theorem 3 (in particular (24)) cannot be satisfied there with $V(x) = |x_2|$. However, this does not imply that (24) cannot be satisfied there for some different function V (cf. Remark 5). In fact, one can show by somewhat tedious direct calculations that the system is still norm-controllable in the latter case (see the technical report [19]). The reason for this is that while the amount of product B inside the reactor will first decrease (due to the outlet stream) if $x_2 > (k/c)x_1^2$, the time during which it decreases goes to zero as b, i.e., the concentration of A in the inlet stream, increases. Hence still for each fixed time $a > 0$, the amount of product can be made large by using a large concentration of A in the inlet stream.

VI. Norm-Controllability for Linear Systems

In this section, we further elaborate the property of norm-controllability for linear systems. We show how global norm-controllability for special linear output maps corresponding to real left eigenvectors of the system matrix can be established, and how this can be related to the standard controllability notion. Furthermore, we show how invariant sets can be used in order to establish norm-controllability for a larger class of output maps h.

Consider the linear system

$$\dot{x} = Ax + Bu \tag{37}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and admissible input set $U := \mathbb{R}^m$. Denote the controllable subspace of the system (37) by[10] $\mathcal{S} := \mathrm{span}[B, AB, \ldots, A^{n-1}B]$. Furthermore, let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A (with algebraic and geometric multiplicity possibly greater than 1) and $\ell_1^T, \ldots$ the corresponding left eigenvectors of A. Consider the scalar linear function $\omega(x) := c^T x$ for some $c \in \mathbb{R}^n$, and take $V(x) := |c^T x|$. For all x where $\omega(x) \neq 0$ we obtain

$$\begin{aligned}\dot{V} &= \mathrm{sign}(c^T x) c^T (Ax + Bu)\\ &= \mathrm{sign}(c^T x)\tilde{c}^T x + \mathrm{sign}(c^T x) c^T Bu\end{aligned} \tag{38}$$

with $\tilde{c}^T := c^T A$. Furthermore, for all x where $\omega(x) = 0$ we obtain, using the definition (5) of the lower directional derivative

$$\begin{aligned}V^{(1)}(x; f(x,u)) &= \liminf_{t\searrow 0, \bar{h}_1 \to Ax+Bu} \frac{1}{t}\left|c^T t\bar{h}_1\right|\\ &= \left|c^T(Ax+Bu)\right| = |\tilde{c}^T x + c^T Bu|.\end{aligned} \tag{39}$$

[10] For a matrix $X \in \mathbb{R}^{n\times m}$, denote by span(X) the subspace of $\mathbb{R}^n$ spanned by the columns of X.

A. Global Norm-Controllability of Linear Systems

Proposition 2: A linear system (37) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with output map $h(x) = \ell_i^T x$, for each real left eigenvector ℓ_i of A which is not orthogonal to $\mathcal{S}$. □

Proof: Consider a real left eigenvector ℓ_i of A which is not orthogonal to $\mathcal{S}$. Then, for some $0 \leq k \leq n-1$, it holds that $\ell_i^T A^k B \neq 0$. But from this it follows that also $\ell_i^T B \neq 0$, as

$$\ell_i^T A^k B = \lambda_i \ell_i^T A^{k-1} B = \ldots = \lambda_i^k \ell_i^T B \neq 0.$$

Now consider the linear function $\omega(x) := \ell_i^T x$ and take $V(x) := |\ell_i^T x|$. For each $b > 0$ and all x where $\omega(x) \neq 0$, (38) with $c = \ell_i$ yields

$$\begin{aligned}\dot{V} &= \lambda_i |\omega(x)| + \mathrm{sign}\left(\ell_i^T x\right)\ell_i^T Bu\\ &\geq (1-\theta)\left|\ell_i^T B\right| b =: \chi(b)\end{aligned} \tag{40}$$

for all $|\omega(x)| \leq \left|\frac{\theta \ell_i^T B}{\lambda_i}\right| b =: \rho(b)$ and arbitrary $0 < \theta < 1$ if we choose $u = \mathrm{sign}(\ell_i^T x) b B^T \ell_i / |B^T \ell_i|$. Furthermore, for each $b > 0$ and all x where $\omega(x) = 0$, (39) yields

$$V^{(1)}(x; f(x,u)) = \left|\lambda_i \ell_i^T x + \ell_i^T Bu\right| = \left|\ell_i^T Bu\right| \geq \chi(b) \tag{41}$$

if we choose $u = \pm b B^T \ell_i / |B^T \ell_i|$. Hence we can apply Theorem 1 with $\mathcal{B} = \mathbb{R}^n$ and $\alpha_1 = \alpha_2 = \mu := \mathrm{id}$ to conclude that the system (37) with output map $h(x) := \omega(x) = \ell_i^T x$ is norm-controllable from all $x_0 \in \mathbb{R}^n$, which concludes the proof of Proposition 2. □

Remark 8: In order to fulfill the assumptions of Theorem 1 globally (i.e., with $\mathcal{B} = \mathbb{R}^n$) with $V(x) = |c^T x|$, we necessarily need that $c = \ell_i$ for some real left eigenvector ℓ_i^T of A being not orthogonal to $\mathcal{S}$. Indeed, if $c \neq \ell_i$, then $\tilde{c}^T \neq \lambda_i \ell_i^T$, and hence we can always find some $x \in \mathbb{R}^n$ such that $|c^T x| \leq \rho(b)$, but $|\tilde{c}^T x|$ is arbitrarily large and $\mathrm{sign}(c^T x \tilde{c}^T x) = -1$, i.e., the right hand side of (38) is negative, and hence (8) cannot hold with $V(x) = |c^T x|$. Furthermore, if $c = \ell_i$ for some real left eigenvector of A which is orthogonal to $\mathcal{S}$, i.e., $c^T B = 0$, then both the right-hand sides of (38) and (39) are independent of u (in particular, the latter is zero), and hence again (8) cannot hold. □

For every linear system (37) which is controllable, the controllable subspace is $\mathcal{S} = \mathbb{R}^n$, and hence none of the left eigenvectors of A are orthogonal to $\mathcal{S}$. According to Proposition 2, this means that the system is norm-controllable for each output map $h(x) = \ell_i^T x$ with ℓ_i being a real left eigenvector of A. In fact, it turns out that also the converse is true, given that all eigenvalues of A are real:

Proposition 3: A linear system (37) with real eigenvalues of A is controllable if and only if it is norm-controllable from all $x_0 \in \mathbb{R}^n$ with output map $h(x) = \ell_i^T x$, for all left eigenvectors ℓ_i^T of A. □

Proof: Necessity follows from Proposition 2 by noting that eigenvectors corresponding to real eigenvalues are real. To show sufficiency, note that if the system (37) is norm-controllable from all $x_0 \in \mathbb{R}^n$ with output map $h(x) = \ell_i^T x$, for some left eigenvector ℓ_i^T of A, then $\ell_i^T B \neq 0$. Indeed, suppose $\ell_i^T B = 0$ for some left eigenvector ℓ_i^T. Then $\dot{y} = \ell_i^T(Ax + Bu) = \ell_i^T Ax = \lambda_i \ell_i^T x = \lambda_i y$, where λ_i is the eigenvalue of A corresponding to ℓ_i^T. But this means that the output y evolves independently of the input u, and hence the system cannot be norm-controllable with this output map, as claimed.

Next, since $\ell_i^T B \neq 0$ holds for all left eigenvectors ℓ_i^T of A, the matrix $[\lambda I - A\ B]$ has full row rank, as $\ell^T(\lambda I - A) \neq 0$ for all ℓ^T which are not a left eigenvector of A. But this means according to the Hautus test that the system (37) is controllable. □

Furthermore, for each scalar linear function $\omega(x) = c^T x$ there exists a $\mathcal{K}_\infty$-function ν such that (6) is satisfied for the output map $h = \mathrm{id}$ (e.g., we can take $\nu(r) = r/|c|$). Hence we can state the following corollary of Proposition 2:

Corollary 1: Every controllable linear system with at least one real eigenvalue of A and output $y = x$ is norm-controllable from every $x_0 \in \mathbb{R}^n$. □

The following example shows that also for linear systems with complex eigenvalues one might be able to establish global norm-controllability. However, as discussed in Remark 8, in this case a different function than $V(x) = |c^T x|$ is needed. Also, as suggested in Remark 3, we use a slight extension of Theorem 2, taking into account information on the behavior of the solution of the system.

Example 7: Consider the linear harmonic oscillator $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + u$ with output $y = x$. This system has two complex eigenvalues $\pm i$. Taking $\omega(x) = x$ and $V(x) = (1/2)(x_1^2 + x_2^2)$ (and hence $\alpha_1(r) = \alpha_2(r) = (1/2)r^2$), we obtain $V^{(1)}(x; f(x,u)) = x_2 u$ and $V^{(2)}(x; f(x,u)) = -ux_1 + u^2$. Choosing u such that $|u| = b$ and $x_2 u \geq 0$, it follows that $V^{(1)}(x; f(x,u)) \geq 0$ and $V^{(2)}(x; f(x,u)) \geq (1-\theta)b^2 =: \chi(b)$ for all $|x_1| \leq \theta b =: \rho(b)$ with $0 < \theta < 1$ arbitrary, where the latter is implied by $|x| \leq \rho(b)$. This means that for all $|x| \leq \rho(b)$, the set $U^b(x)$ as defined in (16) is given as $U^b(x) = \{b\}$ if $x_2 > 0$, $U^b(x) = \{-b\}$ if $x_2 < 0$ and $U^b(x) = \{\pm b\}$ if $x_2 = 0$. But then, neither condition (i) nor condition (ii) of Theorem 2 are satisfied. However, the solutions of the considered system are such that for fixed x_0, even as condition (ii) is not satisfied, we can still apply some *constant* input u until $V(x(t)) = \alpha_1(\rho(b))$, i.e., the input values u_i as chosen in Step 1 of the proof of Theorem 2 can be the same (compare Remark 3). This is the case because $x_2(\cdot)$ is either nonnegative or nonpositive (depending on the initial condition) as long as $V(x(t)) \leq \alpha_1(\rho(b))$. But then, $V^{(1)}(x(t); f(x(t), u(t)))$ is continuous during that time, which is sufficient for proving Theorem 2. Hence the conclusions of Theorem 2 with $k=2$, $\nu = \mathrm{id}$ and $\alpha_1(r) = \alpha_2(r) = (1/2)r^2$ still hold, i.e., the system is globally norm-controllable with gain function γ given by (17). □

B. Norm-Controllability of Linear Systems on Invariant Sets

In the previous subsection, we have established global norm-controllability of a linear system whose output corresponds to a real left eigenvector of A not orthogonal to B. In this section, we will consider the case of more general linear output maps and establish norm-controllability on invariant subsets $\mathcal{B}$ of $\mathbb{R}^n$. We first consider the case where $\mathcal{B}$ is the controllable subspace, i.e., $\mathcal{B} = \mathcal{S}$. Without loss of generality, consider the linear system (37) in the Kalman controllability decomposition form (see, e.g., [4])

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + \tilde{B}u, \qquad \dot{x}_2 = A_{22}x_2. \tag{42}$$

Herein, $\dim(x_1) = \mathrm{rank}[B, AB, \ldots, A^{n-1}B] =: n_1$, and $\dim(x_2) = n - n_1$.

Proposition 4: A linear system (37) with $B \neq 0$ is norm-controllable from all $x_0 \in \mathcal{S}$ with output map $h(x) = [\tilde{\ell}_i^T\ 0]x$, for each real left eigenvector $\tilde{\ell}_i^T$ of A_{11}. □

Proof: As by assumption $B \neq 0$, we have $n_1 > 0$. Furthermore, $\mathrm{span}[\tilde{B}, A_{11}\tilde{B}, \ldots, A_{11}^{n_1-1}\tilde{B}] = \mathbb{R}^{n_1}$. Now consider some real left eigenvector $\tilde{\ell}_i^T$ of A_{11} with corresponding eigenvalue λ_i. Note that $\tilde{\ell}_i^T \tilde{B} \neq 0$. Indeed, if this were not true and $\tilde{\ell}_i^T \tilde{B} = 0$, then

$$\tilde{\ell}_i^T A_{11}^{n_1-1}\tilde{B} = \lambda_i \tilde{\ell}_i^T A_{11}^{n_1-2}\tilde{B} = \cdots = \lambda_i^{n_1-1}\tilde{\ell}_i^T \tilde{B} = 0.$$

Thus $\tilde{\ell}_i \perp \mathrm{span}[\tilde{B}, A_{11}\tilde{B}, \ldots, A_{11}^{n_1-1}\tilde{B}] = \mathbb{R}^{n_1}$, which is a contradiction as $\tilde{\ell}_i \in \mathbb{R}^{n_1}$. Now consider $\omega(x) = [\tilde{\ell}_i^T\ 0]x$ for some i, and take $V(x) = |\omega(x)|$. Then for all x where $\omega(x) \neq 0$, (38) with $c = [\tilde{\ell}_i^T\ 0]^T$ yields

$$\dot{V} = \lambda_i\,|\omega(x)| + \mathrm{sign}\,(\omega(x))\,\tilde{\ell}_i^T A_{12}x_2 + \mathrm{sign}\,(\omega(x))\,\tilde{\omega}_i^T \tilde{B}u. \tag{43}$$

If we start with an initial condition on the controllable subspace, i.e., $x_2(0) = 0$, then, as the controllable subspace is invariant, the term involving A_{12} in (43) vanishes for all times, and thus for each $b > 0$ the growth of V can be bounded from below by a function $\chi(b)$ analogous to (40). By similar calculations as in (41) one can also establish this property for all x where $\omega(x) = 0$. Thus one can apply Theorem 1 with $\mathcal{B} = \mathcal{S}$ to conclude that the linear system (37) with output map $h(x) := \omega(x) = [\tilde{\ell}_i^T\ 0]x$ is norm-controllable from all $x_0 \in \mathcal{S}$, which concludes the proof of Proposition 4. □

For general linear output maps $h(x) = c^T x$ and invariant sets $\mathcal{B}$ different from the controllable subspace $\mathcal{S}$, one can sometimes establish norm-controllability with the help of Theorems 1–3, similarly to the general (nonlinear) case, as illustrated by the following example.

Example 8: Consider the linear system

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{44}$$

which is given in the Kalman decomposition form (42). The system has a double eigenvalue $\lambda = -1$ with corresponding left eigenvector $\ell^T = [0\ 1]$, which is orthogonal to the controllable subspace $S = \mathrm{span}([1\ 0]^T)$. Now consider the output map $h(x) := c^T x$ with $c = [c_1\ \ c_2]^T$ and take $\omega(x) := c^T x$ and $V(x) := |c^T x|$. As discussed in Section VI-A, if $c_1 = 0$ then the system is not norm-controllable, as $c^T = [0\ c_2]$ (with $c_2 \neq 0$) is a left eigenvector which is orthogonal to the controllable subspace $S = \mathrm{span}([1\ 0]^T)$. Hence in the following we consider the case $c_1 \neq 0$; without loss of generality, assume that $c_1 > 0$. If this were not the case, just consider $-h(x)$ as output. Now consider the sets $\mathcal{B}_1 := \{x \in \mathbb{R}^2 : c^T x \geq 0, x_2 \geq 0\}$ and $\mathcal{B}_2 := \{x \in \mathbb{R}^2 : c^T x \leq 0, x_2 \leq 0\}$. It is straightforward to show that $\mathcal{B}_1$ and $\mathcal{B}_2$ are invariant under controls in $\bar{U} = \mathbb{R}_{\geq 0}$ (respectively, $\bar{U} = \mathbb{R}_{\leq 0}$).

Now for all x where $\omega(x) \neq 0$, (38) yields $\dot{V} = -|\omega(x)| + \mathrm{sign}(c^T x)c_1 x_2 + \mathrm{sign}(c^T x)c_1 u$. For all $x \in \mathcal{B}_1 \cup \mathcal{B}_2$, we have $\mathrm{sign}(c^T x)c_1 x_2 \geq 0$, and hence it follows that $\dot{V} \geq (1-\theta)|c_1|b =: \chi(b)$ for all $|\omega(x)| \leq \theta|c_1|b =: \rho(b)$ and arbitrary $0 < \theta < 1$ if we choose $u = b$ if $x \in \mathcal{B}_1$, respectively $u = -b$ if $x \in \mathcal{B}_2$. Furthermore, for all x where $\omega(x) = 0$, (39) yields $V^{(1)}(x; f(x,u)) = |c_1(x_2+u)| \geq \chi(b)$ if we choose u as specified above. Hence we can apply Theorem 1 with $\nu = \alpha_1 = \alpha_2 = \mathrm{id}$ to conclude that the system (44) with output map $h(x) = c^T x$ is norm-controllable from all $x_0 \in \mathcal{B}_1 \cup \mathcal{B}_2$ with gain $\gamma(a,b) = \min\{(1-\theta)|c_1|ab + |c^T x_0|, \theta|c_1|b\}$. The gain function γ can be interpreted as follows. If we consider "normalized" outputs $h(x) = c^T x$ with $|c| = 1$, then the gain γ becomes maximal if the output is aligned with the controllable subspace, i.e., if $c = [1\ 0]^T$. When the output is "rotated", the gain decreases until the system is not norm-controllable at all anymore (corresponding to a gain $\gamma = 0$), which happens if the output is orthogonal to the controllable subspace, i.e., if $c = [0\ 1]^T$.

VII. Weaker Versions of Norm-Controllability

In this Section, we discuss some relaxations (weaker versions) of norm-controllability. In the definition of norm-controllability (see Definition 1), we required the function $\gamma(a,\cdot)$ to be of class $\mathcal{K}_\infty$. One possible relaxation could be to only require this function to be nondecreasing and unbounded.

Definition 2: The system (1) is *weakly norm-controllable in b from x_0 with gain function γ* if there exists a function $\gamma: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\gamma(\cdot, b)$ nondecreasing for each fixed $b > 0$ and $\gamma(a,\cdot)$ nondecreasing and unbounded for each fixed $a > 0$, such that for all $a > 0$ and $b > 0$

$$R_h^a(x_0, \mathcal{U}_{a,b}) \geq \gamma(a,b) \tag{45}$$

where $\mathcal{U}_{a,b}$ is defined in (2). □

Such a weaker notion of norm-controllability is interesting in such cases where the system is only responsive to large, but not to small, inputs, which will be illustrated with a simple example below (see Example 9). In particular, as we require $\gamma(a,\cdot)$ to be nondecreasing and unbounded, it can be zero on some finite interval $[0, \hat{b}]$ for some $\hat{b} \geq 0$. Sufficient conditions for this weaker notion of norm-controllability can be obtained in the same way as for our original definition of norm-controllability:

Proposition 5: Let the conditions of Theorem 1 (respectively, Theorem 2 or Theorem 3) be satisfied, but with $\chi: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ in (8) (respectively, $\chi: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ in (15) or $\chi_i: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $1 \leq i \leq \ell$, in (25)) only being nondecreasing and unbounded instead of $\mathcal{K}_\infty$. Then the system (1) is weakly norm-controllable in b from all $x_0 \in \mathcal{B}$ with gain function γ given by (9) (respectively, (17) or (27)) with $V(x_0)$ replaced by[11] $V(x_0)I(\chi(b))$ (respectively, $V(x_0)\min_{1\leq i\leq \ell} I(\chi_i(b))$ in case of Theorem 3). □

Proof: For each $b > 0$ such that $\chi(b) > 0$ (respectively, $\chi_i(b) > 0$ for all $1 \leq i \leq \ell$ in case of Theorem 3), the construction of a piecewise constant input signal satisfying $|y(a)| = |h(x(a))| \geq \gamma(a,b)$ works as in the proof of Theorem 1 (respectively, Theorem 2 or Theorem 3). As χ (respectively, χ_i for all $1 \leq i \leq \ell$ in case of Theorem 3) is required to be nonnegative, nondecreasing and unbounded, it follows that $\chi(b) = 0$ (respectively, $\chi_i(b) = 0$ for some $1 \leq i \leq \ell$) only on some finite interval $[0, \hat{b}]$ for some $\hat{b} \geq 0$. For these values of b, existence of a piecewise constant input signal satisfying $|y(a)| = |h(x(a))| \geq \gamma(a,b)$ as constructed in the proof of Theorem 1 (respectively, Theorem 2 or Theorem 3) is not guaranteed anymore (in particular, Lemmas 1 and 2 do not hold anymore if $\chi(b) = 0$). Nevertheless, for such values of b, the gain function γ as given in the proposition satisfies $\gamma = 0$, and hence (45) is trivially satisfied. The proof is concluded by noting that the gain function γ as given in the proposition satisfies the required properties. In particular, $\gamma(a,\cdot)$ is nondecreasing and unbounded for each fixed $a > 0$. □

Example 9: Consider the system (1) with output $h(x) = x$ and $f(x,u) = 0$ for $|u| \leq 1$, $f(x,u) = |u| - 1$ for $1 < |u| \leq 2$, $f(x,u) = 3 - |u|$ for $2 < |u| \leq 3$, $f(x,u) = 0$ for $3 < |u| \leq 4$, and $f(x,u) = |u| - 4$ for $4 < |u|$; note that $\mathbb{R}_{\geq 0}$ is invariant under controls in $\bar{U} = \mathbb{R}_{\geq 0}$. This system is not norm-controllable according to Definition 1, as $f(x,u) \equiv 0$ for all (x,u) such that $|u| \leq 1$, i.e., the system is not responsive to inputs of small magnitude. On the other hand, for each $b > 0$ and each $x \in \mathbb{R}_{\geq 0}$, (8) is satisfied with $V(x) = |x|$ and $\chi(b) = 0$ for $0 \leq b \leq 1$, $\chi(b) = b - 1$ for $1 < b \leq 2$, $\chi(b) = 1$ for $2 < b \leq 5$, and $\chi(b) = b - 4$ for $5 < b$. Note that for each $x \in \mathbb{R}_{\geq 0}$, the set of "good" input values, i.e., such that (8) holds with χ as given above, is $U^b(x) = [0, b]$ for $0 \leq b \leq 1$, $U^b(x) = \{b\}$ for $1 < b \leq 2$, $U^b(x) = \{2\}$ for $2 < |u| \leq 5$, and $U^b(x) = \{b\}$ for $5 < b$. Hence we can apply Proposition 5 together with Remark 1 (using $\nu = \alpha_1 = \alpha_2 = \mathrm{id}$) to conclude that the system is weakly norm-controllable in b from all $x_0 \in \mathbb{R}_{\geq 0}$ with gain function $\gamma(a,b) = a\chi(b) + x_0 I(\chi(b))$. □

A second possible relaxation (weaker version) of norm-controllability could be with respect to the time horizon a. Namely, instead of requring $\gamma(a,\cdot)$ to be of class $\mathcal{K}_\infty$ for each $a > 0$ as in Definition 1, we now only require that $\bar{\gamma}(b) := \gamma(a + \underline{a}(x_0,b), b)$ be of class $\mathcal{K}_\infty$ for each $a > 0$. Herein, $\underline{a}(x_0,b) \geq 0$ can be interpreted as some "dead zone", i.e., the norm of the output can only be increased after some time $\underline{a}(x_0,b) \geq 0$, which possibly depends on the initial condition x_0 of the system as well as the upper bound b for the input norm.

Definition 3: The system (1) is *weakly norm-controllable in a from x_0 with gain function γ* if there exists a function $\gamma: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and for each $b \geq 0$ a constant $\underline{a}(x_0,b) \geq 0$ such that $\gamma(\cdot, b)$ is nondecreasing for each fixed $b > 0$ and $\bar{\gamma}(b) := \gamma(a + \underline{a}(x_0,b), b)$ is of class $\mathcal{K}_\infty$ for each fixed $a > 0$, and for each $a > 0$ and $b > 0$

$$R_h^a(x_0, \mathcal{U}_{a,b}) \geq \gamma(a,b)$$

where $\mathcal{U}_{a,b}$ is defined in (2). □

Proposition 6: Suppose that the conditions of Theorem 1 (respectively, Theorem 2 or Theorem 3) are satisfied. Furthermore, assume that for each $x_0 \in \mathbb{R}^n$ and each $b > 0$ there exists a constant $\underline{a}(x_0,b)$ such that $\mathcal{R}^{\underline{a}(x_0,b)}\{x_0, \mathcal{U}_{\underline{a}(x_0,b),b}\} \cap \mathcal{B} \neq \emptyset$. Then the system (1) is weakly norm-controllable in a from all $x_0 \in \mathbb{R}^n$ with gain function $\tilde{\gamma}(a,b) := \gamma(\max\{0, a - \underline{a}(x_0,b)\}, b)$, where γ is given by (9) (respectively, (17) or (27)) with $V(x_0)$ replaced by 0. □

Proof: For each $x_0 \in \mathbb{R}^n$ and each $b > 0$, by assumption there exists an input $\tilde{u}(\cdot) \in \mathcal{U}_{\underline{a}(x_0,b),b}$ such that the resulting state trajectory satisfies $x(\underline{a}(x_0,b)) \in \mathcal{B}$. By Theorem 1 (respectively, Theorem 2 or Theorem 3), the system is

[11] The indicator function $I: \mathbb{R}_{\geq 0} \to \{0,1\}$ is defined as $I(s) = 0$ if $s = 0$ and $I(s) = 1$ otherwise.

norm-controllable from $x(\underline{a}(x_0,b))$ with gain function γ given by (9) (respectively, (17) or (27)) with $V(x_0)$ replaced by 0 (or by $V(x(\underline{a}(x_0,b)))$, respectively). Hence it follows that for all $a \geq \underline{a}(x_0,b)$, we have $R_h^a(x_0,\mathcal{U}_{a,b}) \geq \tilde{\gamma}(a,b)$, with $\tilde{\gamma}$ as specified in the proposition. For $a < \underline{a}(x_0,b)$, this bound is trivially satisfied as $\tilde{\gamma}(a,b) = 0$ for $a < \underline{a}(x_0,b)$, because γ given by (9) (respectively, (17) or (27)) with $V(x_0)$ replaced by 0 has the property that $\gamma(0,b) = 0$ for all $b > 0$. As $x_0 \in \mathbb{R}^n$ and $b > 0$ were arbitrary, the proposition is established by noting that $\tilde{\gamma}$ satisfies the required properties of Definition 3, i.e., is $\tilde{\gamma}(\cdot,b)$ is nondecreasing for each fixed $b > 0$ and $\bar{\gamma}(b) := \tilde{\gamma}(a+\underline{a}(x_0,b),b) = \gamma(a,b)$ is of class $\mathcal{K}_\infty$ for each $a > 0$ (defining $\underline{a}(x_0,0) = 0$). □

Example 10: A simple example to illustrate weak norm-controllability in a is the system (1) with $y = x$ and $f(x,u) = 1$ for $x < 0$ and $f(x,u) = 1 + x|u|$ for $x \geq 0$. It is easy to verify (using, e.g., Theorem 1 with $V(x) = |x|$ together with Remark 1) that this system is norm-controllable from all $x_0 \in \mathcal{B} := \{x \in \mathbb{R} : x \geq 1\}$ with gain function $\gamma(a,b) = ab + V(x_0)$. If $x_0 < 1$, for each input signal $u(\cdot)$ there exists some $t' \leq -x_0 + 1$ such that the solution satisfies $x(t) \geq 1$ for all $t \geq t'$. Hence, defining $\underline{a}(x_0,b) := \max\{0, -x_0+1\}$, it follows from Proposition 6 that the system is weakly norm-controllable in a from all $x_0 \in \mathbb{R}$ with gain function $\tilde{\gamma}(a,b) := \max\{0, a - \underline{a}(x_0,b)\}b$.

Remark 9: Note that weak norm-controllability in a according to Definition 3 is different (and strictly weaker) than the situation encountered in Example 6 for initial conditions $x(0) \notin \mathcal{B}$. There, while $|y|$ was initially decreasing, we still could find a function γ satisfying the requirements of Definition 1. On the other hand, in Example 10, for initial conditions $x_0 < 0$ the norm of the output will necessarily decrease to zero and then increase again; in particular, if $x_0 < 0$ then we have $y(\tau) = 0$ for $\tau = -x_0$ independently of b. Hence the system is not globally norm-controllable, but only weakly norm-controllable in a. □

VIII. Conclusion

In this work, we introduced and studied the notion of norm-controllability for general nonlinear systems. We obtained several Lyapunov-like sufficient conditions for norm-controllability and illustrated their applicability as well as various aspects of the proposed concept with several examples. Two weaker variants of norm-controllability, along with sufficient conditions for verifying them, were proposed as well. We expect that besides its theoretical interest, the notion of norm-controllability can be interesting in various application-related contexts, such as process engineering (which we motivated with a simple CSTR example) or economics; studying more sophisticated application examples will hence be a topic of future research. Also, it could be interesting to formulate and study a slightly stronger variant of norm-controllability which guarantees the existence of individual trajectories that stay large for all times. Moreover, it would be worthwhile to explore possible relationships of our sufficient conditions using higher-order directional derivatives with stability results using higher-order derivatives of Lyapunov functions such as, e.g., [20], and to obtain more definitive results for (global) norm-controllability of linear systems with general output maps. Finally, Theorems 1–3 in general only provide sufficient conditions for norm-controllability; studying necessary conditions is hence another interesting future research direction.

Appendix I
Proof of Lemma 1

Let $h(t) := 1/t^k(x(t) - x_0 - th_1 - \cdots - t^{k-1}h_{k-1})$ for $t > 0$, where $h_1,\ldots,h_{k-1}$ are defined as in (12) with $t' = 0$, $x = x_0$, and $u = u_0$. Note that h is continuous in t and $\lim_{t\searrow 0} h(t) =: h_k$ according to (11). Furthermore, for $t > 0$, define the function

$$g(t) := k!/t^k\Big(V\left(x_0 + th_1 + \cdots + t^k h(t)\right) - V(x_0) - tV^{(1)}(x_0;h_1) - \cdots - t^{k-1}V^{(k-1)}(x_0;h_1,\ldots,h_{k-1})\Big).$$

Consider $g_- := \liminf_{t\searrow 0} g(t)$. By the definitions of g and $V^{(k)}$, it holds that

$$g_- = \liminf_{t\searrow 0} g(t) \geq V^{(k)}\left(x_0; f(x_0,u_0)\right) \overset{(15)}{\geq} \chi(b).$$

The first inequality holds because in the definition of $V^{(k)}$ in (10), the infimum over all $\bar{h}_k$ with $\bar{h}_k \to h_k$ is taken, while in g_- the specific choice $\bar{h}_k = h(t) \to h_k$ is used. Thus, by definition of the (one-sided) limit inferior, for every $\varepsilon > 0$ there exists a $\tau > 0$ such that for all $0 < t \leq \tau$, it holds that

$$g(t) \geq g_- - \varepsilon \geq \chi(b) - \varepsilon. \tag{46}$$

Choosing ε small enough so that $\chi(b) > \varepsilon$ (which can be done for each fixed $b > 0$) we obtain

$$\begin{aligned} V\left(x(t)\right) &= V\left(x_0 + th_1 + \cdots + t^k h(t)\right) \\ &= (1/k!)g(t)t^k + V(x_0) + tV^{(1)}(x_0;h_1) \\ &\quad + \cdots + t^{k-1}V^{(k-1)}(x_0;h_1,\ldots,h_{k-1}) \\ &\overset{(46),(14)}{\geq} (1/k!)(\chi(b)-\varepsilon)t^k + V(x_0) > V(x_0) \geq 0 \end{aligned}$$

as claimed. □

Appendix II
Proof of Lemma 2

In order to prove Lemma 2, we first need a simple auxiliary result.

Lemma 3: For each $\kappa > 0$ there exist constants M and N such that

$$|f(x,u)| \leq M, \tag{47}$$

$$\left|V^{(k)}\left(x'; f(x',u)\right) - V^{(k)}\left(x; f(x,u)\right)\right| \leq N|x - x'| \tag{48}$$

for all $x, x' \in \overline{\mathcal{R}^{\leq a}\{x_0,\mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$ and $|u| \leq b$, where $X_{\infty,\kappa} := \{x \in \mathcal{B} : \kappa \leq |\omega(x)|\}$.

Proof of Lemma 3: As stated earlier, we assume that for each $u(\cdot) \in \mathcal{U}_{a,b}$ there is no finite escape time $t_{esc} \leq a$. This implies that the set $\overline{\mathcal{R}^{\leq a}\{x_0,\mathcal{U}_{a,b}\}}$ is compact [21, Proposition 5.1]. As $\mathcal{B}$ is closed, also $X_{\infty,\kappa}$ is closed due to continuity of ω, and hence $\overline{\mathcal{R}^{\leq a}\{x_0,\mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$ is compact, for each $\kappa > 0$. Furthermore, also the set $\{u : |u| \leq b\}$ is compact.

From the fact that $V \in C^k$ for all $x \in \mathbb{R}^n \setminus W$ and the k-th order partial derivatives of V are locally Lipschitz there, and furthermore $f \in C^{k-1}$ and the $(k-1)$-st partial derivatives of f with respect to x are locally Lipschitz in (x, u), it follows that $V^{(k)}(x; f(x,u)) = L_f^k V|_{(x,u)}$ is Lipschitz in (x, u) on the set $(\overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}) \times \{u : |u| \leq b\}$, for each $\kappa > 0$. This follows from the fact that $L_f^k V|_{(x,u)}$ consists of sums of products of locally Lipschitz functions, and sums and products of Lipschitz functions on a compact set are Lipschitz on that set. Hence there exist constants M and N satisfying (47), (48) for each $x, x' \in \overline{\mathcal{R}^{\leq a}\{x_0, \mathcal{U}_{a,b}\}} \cap X_{\infty,\kappa}$ and each $|u| \leq b$. □

Proof of Lemma 2: Let $\tau_0 := \inf\{\tau : \tau \geq s, V(x(\tau)) = (1/2)V(x(s))\} \in (s, \infty]$, where $x(\cdot)$ is the trajectory resulting from the application of the constant control input u_s for all $\tau \geq s$. Note that due to continuity of V and $x(\cdot)$, we have $\tau_0 > s$. Hence, as $u_s \in \bar{U}$ and $\mathcal{B}$ is invariant under controls in $\bar{U}$, from (7) it follows that $x(t) \in X_{\infty,\delta'}$ with $\delta' = \alpha_2^{-1}((1/2)\alpha_1(\delta))$ for all $t \in [s, \tau_0)$ (respectively for all $t \in [s, \tau_0]$ if $\tau_0 < \infty$). Now consider some $s \leq s' \leq \tau_0$. For all $t \in [s, s'] \cap [0, a]$, the trajectory $x(\cdot)$ satisfies

$$x(t) = x(s) + \int_s^t f(x(\tau), u_s)\, d\tau. \quad (49)$$

Furthermore, Lemma 3 (with $\kappa = \delta'$) provides a constant M such that from (49) it follows that for each $t \in [s, s'] \cap [0, a]$

$$|x(t) - x(s)| \leq M(t - s). \quad (50)$$

With this we obtain that for each $t \in [s, s'] \cap [0, a]$

$$\begin{aligned} V^{(k)}(x(t)) &\overset{(48)}{\geq} V^{(k)}(x(s)) - N|x(t) - x(s)| \\ &\overset{(50)}{\geq} V^{(k)}(x(s)) - NM(t-s) \\ &\overset{(15)}{\geq} \chi(b) - NM(t-s) \geq (1-\varepsilon)\chi(|b|) \end{aligned} \quad (51)$$

for every $0 < \varepsilon \leq 1$ if $s' \leq \min\{\tau_0, s + \Delta(\varepsilon, \delta)\}$ with

$$\Delta(\varepsilon, \delta) := \frac{\varepsilon \chi(b)}{NM}.$$

The dependence of Δ on δ is due to the fact that the constants M and N possibly depend on δ' and hence on δ. As the constant input u_s is applied, we have that $V^{(k-1)}(x(t)) - V^{(k-1)}(x(s)) = \int_s^t V^{(k)}(x(\tau))d\tau$, i.e.

$$V^{(k-1)}(x(t)) - V^{(k-1)}(x(s)) \geq (1-\varepsilon)\chi(b)(t-s) \quad (52)$$

for all $t \in [s, \min\{\tau_0, s + \Delta(\varepsilon, \delta)\}] \cap [0, a]$. Furthermore, for $j = k-2, \ldots, 0$, integrating (52) $k - j - 1$ times from s to t while using (14) with $x = x(s)$ and $u = u_s$ in order to get rid of the terms $V^{(j+1)}(x(s)), \ldots, V^{(k-1)}(x(s))$ yields

$$V^{(j)}(x(t)) - V^{(j)}(x(s)) \geq \frac{1-\varepsilon}{(k-j)!}\chi(b)(t-s)^{k-j} \quad (53)$$

for all $t \in [s, \min\{\tau_0, s + \Delta(\varepsilon, \delta)\}] \cap [0, a]$. Hence Lemma 2 is established if we can show that $\tau_0 \geq s + \Delta(\varepsilon, \delta)$, which will be done by contradiction. Namely, suppose that $\tau_0 < s + \Delta(\varepsilon, \delta)$. But then (53) is valid for all $t \in [s, \tau_0]$, and from there it follows (with $j = 0$) that $V(x(\tau_0)) > V(x(s))$, which contradicts the definition of τ_0. □

Appendix III
Proof of Proposition 1

Let $a, b > 0$ be arbitrary but fixed. The construction of a piecewise constant input signal works as in the proof of Theorem 2, with the following small modifications.

Step 0: If $x_0 \in \mathrm{int}(\Lambda_b)$: The control value u_0 is now chosen such that $u_0 \in \hat{U}^b(x_0) \subseteq \widetilde{U}^b$. Note that for this choice of u_0, Lemma 1 is still valid as $\hat{U}^b(x_0) \subseteq U^b(x_0)$. If $x_0 \notin \Lambda_b$: Pick some $u_1 \in \widetilde{U}^b$.

Step 1: In Step 1, the control values u_i at time instants t_i are defined at points where $x(t_i) \in \mathrm{int}(\Lambda_b)$. Recursively, as up to time t_i only input values in $\widetilde{U}^b$ have been applied to the system, by assumption we have that $x(t_i) \in \mathcal{B} \cup H_b$. But as $H_b \cap \Lambda_b = \emptyset$, it holds that $x(t_i) \in \mathcal{B}$. Now pick some $u_i \in \hat{U}^b(x(t_i)) \subseteq \widetilde{U}^b$. Note that for this choice of u_i, Lemma 2 is still valid as $\hat{U}^b(x(t_i)) \subseteq U^b(x(t_i))$ and $x(\cdot) \in \mathcal{B}$ as long as $x(\cdot) \in \Lambda_b$. Again, in case that condition (ii) of Theorem 2 holds, one can choose the input values $u_1, u_2, \ldots$ to be the same (which is possible thanks to condition 3) in the proposition).

Step 2: As was the case in Step 1, one can recursively show that as up to time $\check{t}_j$ only input values in $\widetilde{U}^b$ have been applied to the system, by assumption we have that $x(\check{t}_j) \in \mathcal{B} \cup H_b$. But as $H_b \cap \Lambda_b = \emptyset$, it holds that $x(\check{t}_j) \in \mathcal{B}$. Pick some $\check{u}_j \in \hat{U}^b(x(\check{t}_j)) \subseteq \widetilde{U}^b$. Then, again Lemma 2 is still valid for this choice of $\check{u}_j$.

Step 3: Summarizing the above, the piecewise constant input signal (21) with the given modified values of u_0, u_i and $\check{u}_j$ again satisfies (22), as Lemmas 1 and 2 are still valid as discussed above. From here, the rest of the proof follows along the lines of the proof of Theorem 2, thanks to condition 2) of the proposition. □

Appendix IV
Proof of Theorem 3

In order to prove Theorem 3, we need the following auxiliary result.

Lemma 4: Let $a_0, a_1, \ldots, a_\ell \geq 0$ for some $\ell \geq 1$ with $a_\ell > 0$. Define $\mathcal{P} := \{p : 0 \leq p \leq \ell, a_p > 0\}$ and $b_p(s) := a_p(s - t_1)^p$ for all $p \in \mathcal{P}$ and some $t_1 \geq 0$. Then for each $t \geq t_1$ it holds that

$$\int_{t_1}^{t} \min_{p \in \mathcal{P}} b_p(s) ds \geq \frac{1}{\ell + 1} \min_{p \in \mathcal{P}} \left\{ (p+1) \int_{t_1}^{t} b_p(s) ds \right\}. \quad (54)$$

Proof: Let $p_{\min} := \min\{p : p \in \mathcal{P}\}$. In case that $p_{\min} = \ell$, inequality (54) is trivially satisfied with equality. Hence in the following we assume that $p_{\min} < \ell$ and thus $\mathcal{P}$ contains at least two elements. According to the definition of the functions b_p, for each pair of elements $p, q \in \mathcal{P}$ with $p > q$ there exists an $\bar{s} > t_1$ such that $b_p(s) < b_q(s)$ for all $t_1 < s < \bar{s}$, $b_p(\bar{s}) = b_q(\bar{s})$, and $b_p(s) > b_q(s)$ for all $s > \bar{s}$. Hence

$$\begin{aligned} \int_{t_1}^{\bar{s}} b_p(s) ds &= \frac{1}{p+1} a_p (\bar{s} - t_1)^{p+1} \\ &= \frac{1}{p+1} a_q (\bar{s} - t_1)^{q+1} = \frac{q+1}{p+1} \int_{t_1}^{\bar{s}} b_q(s) ds. \end{aligned} \quad (55)$$

Furthermore, the above implies that there exist some $s'' \geq s' > t_1$ such that $\min_{p\in\mathcal{P}} b_p(s) = b_\ell(s)$ for all $t_1 \leq s \leq s'$ and $\min_{p\in\mathcal{P}} b_p(s) = b_{p_{\min}}(s)$ for all $s \geq s''$. Let $\tau_0 := t_1$ and $p_0 := \ell$. Define recursively

$$\tau_i := \min\{s : s > \tau_{i-1}, \exists p \in \mathcal{P}, 0 \leq p \leq p_{i-1} - 1, b_{p_{i-1}}(s) = b_p(s)\}$$

$$p_i := \min\left\{p : 0 \leq p \leq p_{i-1} - 1, b_p(\tau_i) = b_{p_{i-1}}(\tau_i)\right\}$$

for all $i = 1, \ldots, i_{\max}$, where $i_{\max} := \min\{i \geq 1 : p_i = p_{\min}\}$. Note that according to the above considerations, the time instances τ_i are well defined and $i_{\max} \leq |\mathcal{P}|$, where $|\mathcal{P}|$ denotes the number of elements in $\mathcal{P}$. With this, for each $t \geq t_1$, define $i'(t) := \max\{i \geq 0 : \tau_i \leq t\}$. Then, by using the fact that the sequence $\{p_i\}$ is decreasing by construction, the left hand side of (54) is equal to

$$\begin{aligned}
&\int_{t_1}^{\tau_1} b_{p_0}(s)ds + \int_{\tau_1}^{\tau_2} b_{p_1}(s)ds + \cdots + \int_{\tau_{i'(t)}}^{t} b_{p_{i'(t)}}(s)ds \\
&\overset{(55)}{\geq} \frac{p_1+1}{p_0+1}\int_{t_1}^{\tau_2} b_{p_1}(s)ds + \cdots + \int_{\tau_{i'(t)}}^{t} b_{p_{i'(t)}}(s)ds \\
&\overset{(55)}{\geq} \cdots \overset{(55)}{\geq} \frac{p_{i'(t)}+1}{p_0+1}\int_{t_1}^{t} b_{p_{i'(t)}}(s)ds \\
&\geq \frac{1}{\ell+1}\min_{p\in\mathcal{P}}\left\{(p+1)\int_{t_1}^{t} b_p(s)ds\right\}
\end{aligned}$$

which concludes the proof of Lemma 4. □

Proof of Theorem 3: Let $a, b > 0$ and $0 < \bar{\varepsilon} < 1$ be arbitrary but fixed, and assume in the sequel that the hypotheses of Theorem 3 are satisfied. Furthermore, for each $x \in \mathcal{B}$, define $\underline{i}(x) := \min\{1 \leq i \leq \ell : x \in \mathcal{R}_i(b)\}$. A piecewise constant input signal (21) is now constructed as in Steps 0–2 of the proof of Theorem 2, except that $U^b(x)$ is replaced by $U^b_{\underline{i}(x)}(x)$ in all places. Step 3 of the proof of Theorem 2 is replaced with the following.

As shown in the proof of Theorem 2, the input signal $u(\cdot)$ consists of only finitely many different values u_i and $\check{u}_j$. Let $\{t_i\}$ be the sequence of time instants defined in Step 1 (i.e., where a new control value u_i is chosen), with t_s being the last time instant of this sequence (i.e., such that $t_s \leq \min\{\check{t}_1, a\}$). Let the subsequence $\{\tau_1, \ldots, \tau_r\}$ of $\{t_i\}$ consist of those values t_i, $2 \leq i \leq s$, such that $\underline{i}(x(t_i)) \neq \underline{i}(x(t_{i-1}))$, i.e., $\{\tau_i\}$ consists of those time instants where the conditions of Theorem 3 are applied in a different region $\mathcal{R}_i(b)$ than at the previous time instant t_{i-1}. The state trajectory $x(\cdot)$ resulting from application of the constructed control input (21) satisfies the following properties. First, we consider the situation where $t_1 > 0$ (recall from Step 0 that this corresponds to $x_0 \in \mathrm{int}(\Lambda_b)$) and $\underline{i}(x(t_1)) = \ell$. Using (19) with $k = k_\ell$, $j = k_\ell - 1$, $\chi = \chi_\ell$ and $\varepsilon = \bar{\varepsilon}$, we obtain that $V^{(k_\ell-1)}(x(t_i^-)) \geq (1-\bar{\varepsilon})(t_i - t_{i-1})\chi_\ell(b) + V^{(k_\ell-1)}(x(t_{i-1}))$ for all $t_2 \leq t_i \leq \tau_1$. By the same argument as in Step 3 of the proof of Theorem 2 (i.e., using either condition (i) or (ii) of the Theorem), it follows that $V^{(k_\ell-1)}(x(t))$ is continuous also at the time instants $t = t_i$, and hence $V^{(k_\ell-1)}(x(t)) \geq (1-\bar{\varepsilon})(t - t_1)\chi_\ell(b) + V^{(k_\ell-1)}(x(t_1))$ for all $t_1 \leq t < \min\{\tau_1, \check{t}_1, a\}$. Again, using the same procedure as in Step 3 of the proof of Theorem 2 (i.e., piecewise integrating this inequality and using the fact that $V^{(k)}(x(t))$ for $k < k_\ell$ is continuous also at $t = t_i$), it follows that

$$V^{(k_{\ell-1})}(x(t)) \geq \frac{(1-\bar{\varepsilon})}{(k_\ell - k_{\ell-1})!}(t-t_1)^{k_\ell - k_{\ell-1}}\chi_\ell(b) + V^{(k_{\ell-1})}(x(t_1)) \quad (56)$$

for all $t_1 \leq t < \min\{\tau_1, \check{t}_1, a\}$. Now consider the case where $\underline{i}(x(\tau_1)) = \ell - 1$. By construction of the input values u_i in Step 1, from Lemma 2 it follows that

$$V^{(k_{\ell-1})}(x(t)) \geq (1-\bar{\varepsilon})\chi_{\ell-1}(b) \quad (57)$$

for all $\tau_1 \leq t < \min\{\tau_2, \check{t}_1, a\}$. If $\underline{i}(x(\tau_2)) = \ell$, i.e., the solution $x(\cdot)$ has entered $\mathcal{R}_\ell$ again, then analogously to the above we conclude that for all $\tau_2 \leq t < \min\{\tau_3, \check{t}_1, a\}$, (56) is satisfied with t_1 replaced by τ_2. In particular then, as $V^{(k_{\ell-1})}(x(t))$ is also continuous at $t = \tau_2$, we have that $V^{(k_{\ell-1})}(x(t)) \geq V^{(k_{\ell-1})}(x(\tau_2)) \geq (1-\bar{\varepsilon})\chi_{\ell-1}(b)$, for all $\tau_2 \leq t < \min\{\tau_3, \check{t}_1, a\}$. Using this argument recursively, it follows that (57) is satisfied for all $\tau_1 \leq t < \min\{\tau_m, \check{t}_1, a\}$, where $m := \inf_{k\geq 3, \underline{i}(x(\tau_k))\notin\{\ell,\ell-1\}} k$. Combining this with the fact that (56) is satisfied for all $t_1 \leq t < \min\{\tau_1, \check{t}_1, a\}$, it follows that

$$\begin{aligned}
&V^{(k_{\ell-1})}(x(t)) \\
&\quad\geq \min_{i\in\{\ell-1,\ell\}} \frac{(1-\bar{\varepsilon})(k_\ell - k_i)!}{(k_\ell - k_{\ell-1})!}(t-t_1)^{k_i - k_{\ell-1}}\chi_i(b) \\
&\quad=: \bar{\varphi}(t, t_1)
\end{aligned} \quad (58)$$

for all $t_1 \leq t < \min\{\tau_m, \check{t}_1, a\}$. Note that this inequality is not only valid in case that $\underline{i}(x(t_1)) = \ell$, but also in case that $\underline{i}(x(t_1)) = \ell - 1$.

We now apply the same argumentation as above to the region $\mathcal{R}_\ell \cup \mathcal{R}_{\ell-1} \cup \mathcal{R}_{\ell-2}$. Namely, assume that $\underline{i}(x(\tau_m)) = \ell - 2$; then it follows that

$$V^{(k_{\ell-2})}(x(t)) \geq (1-\bar{\varepsilon})\chi_{\ell-2}(b)$$

for all $\tau_m \leq t < \min\{\tau_n, \check{t}_1, a\}$, where $n := \inf_{k\geq m, \underline{i}(x(\tau_k))\notin\{\ell,\ell-1,\ell-2\}} k$. Combining this with the fact that (58) is valid for all $t_1 \leq t < \tau_m$, by piecewise integrating (58) $r := k_{\ell-2} - k_{\ell-1}$ times from t_1 to t, using again the fact that $V^{(k)}(x(t))$ for $k < k_{\ell-1}$ is continuous also at the respective time instants $t = t_i$, and applying (24) with $x = x(t_1)$, $u = u_1$ and $j = k_{\ell-1} - 1, \ldots, k_{\ell-2}$, one obtains that

$$\begin{aligned}
&V^{(k_{\ell-2})}(x(t)) \\
&\quad\geq \min\left\{\int_{t_1}^{t}\ldots\int_{t_1}^{s_2}\bar{\varphi}(s_1, t_1)ds_1\ldots ds_r, (1-\bar{\varepsilon})\chi_{\ell-2}(b)\right\}
\end{aligned}$$

for all $t_1 \leq t < \min\{\tau_n, \check{t}_1, a\}$. Using Lemma 4 $k_{\ell-2} - k_{\ell-1}$ times, this results in

$$\begin{aligned}
&V^{(k_{\ell-2})}(x(t)) \\
&\quad\geq \min_{i\in\{\ell-2,\ldots,\ell\}} \frac{(1-\bar{\varepsilon})(k_\ell - k_i)!}{(k_\ell - k_{\ell-2})!}(t-t_1)^{k_i - k_{\ell-2}}\chi_i(b)
\end{aligned} \quad (59)$$

for all $t_1 \leq t < \min\{\tau_n, \check{t}_1, a\}$. Again, this inequality is also satisfied if $\underline{i}(x(t_1)) = \ell - 2$, i.e., for arbitrary values $\underline{i}(x(\tau_i)) \in \{\ell, \ell-1, \ell-2\}$ for all $1 \leq i \leq n$. Applying the above argument repeatedly down to $i=1$, it follows that (59) with $\ell - 2$ replaced by 1 is satisfied for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$ and arbitrary values $\underline{i}(x(\tau_i)) \in \{1, \ldots, \ell\}$ for all $1 \leq i \leq r$. Integrating this inequality another k_1 times while using Lemma 4 and (24) with $x = x(t_1)$, $u = u(t_1)$ and $j = k_1 - 1, \ldots, 1$, we obtain that $V(x(t)) \geq (1-\bar{\varepsilon})\Psi(t-t_1, b) + V(t_1)$ for all $t_1 \leq t \leq \min\{\check{t}_1, a\}$ and for all $x_0 \in \text{int}(\Lambda_b)$, with Ψ given in (28). From here, the rest of the proof follows as in Step 3 of the proof of Theorem 2. □

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