How to Park a Car Blindfolded

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Abstract

In this paper, we present an algorithm that achieves global asymptotic stability for a broad class of driftless nonholonomic systems using limited data rate. Moreover, we prove in a constructive way that the minimum data-rate for this class of systems is zero, hence providing a data-rate theorem, which was an open problem in the literature [9]. Finally, we present the Dubin's car as an example of system in such a class.

1 Introduction

Recently there has been an increasing interest in the control and estimation of systems with limited data-rate [13, 10]. This is due to both theoretical and practical applications that are subject to communication constraints. Many of such applications are related to the control of networked systems, since they often have to share bandwidth. The smaller the bandwidth the stronger the constraints imposed on the data-rate available to the control task. A more concrete example are the multi-agents systems, which often need to use the same communication network [12].

This raises questions about what constraints the communication network imposes on the control tasks that can be solved [5]. It is well-known today that there exists a minimum data-rate below which there is no stabilizing controller for unstable linear systems [21, 1]. This type of result is refered to as data-rate theorem.

Unfortunately, however, most of the results concerning the control with minimum data-rate have been solved only for linear systems [9, 6] with some remarkable exceptions [14, 11]. For instance, the problem of minimum data-rate for exponential stabilization was addressed in [3]. In that paper a lower bound on the minimum data-rate for exponential stabilization, with a prescribed exponential rate of convergence α, was given. That result showed that there exist no control law using a smaller data-rate than the stabilization entropy that still drives the state to an equilibrium point at the prescribed rate α, in the sense that the norm of the difference between the trajectory and the equilibrium point decreases at the exponential rate α.

In a similar fashion, we prove a data-rate theorem for a class of driftless nonholonomic systems. This problem was first studied in [9], and, to the best of the authors' knowledge, has not received a satisfactory answer so far. The following claim was made in [9]: "Unfortunately, the data-rate theorem applies essentially only to linear systems, and while there has been some effort to extend the result to the nonlinear domain ..., treatment of nonlinear control systems which are not amenable to linear methods (e.g. wheeled vehicle kinematics, rigid body dynamics, etc.) remains largely unexplored", and an approximate answer was given in [9] for a path following problem. However, a general answer to the asymptotical stabilization problem is still missing.

The main contributions of this paper are to present an algorithm that globally asymptotically stabilizes this class of driftless non-holonomic control systems using a limited data-rate. Also, we show how to modify the same algorithm and prove constructively a data-rate theorem for this class of systems, solving at least partially, therefore, the open question asked in [9]. This data-rate theorem, which is stated in Theorem 2.1, claims that the minimum average data-rate for global asymptotic stabilization for this class of systems is zero. We interpret this zero data rate as the channel being almost always free rather than not sending any data.

Some definitions are in order. We denote by $\mathbb{R}$ the set of real numbers, $\mathbb{R}_{\geq 0}$ the set of non-negative reals, $\mathbb{N}$ the set of natural numbers without 0. For a function $f : I \subset \mathbb{R} \to Q$, where Q is any set and I is an interval, we define $f_{[a,b]}$ the restriction of f to $[a, b] \subset I$. For a set Q, $\#Q$ denotes its cardinality. We denote by $||f||_\infty$ the essential supremum norm of f, and $||x||$ the Euclidean norm of a vector. The sign function $\text{sign}(x)$ is 1 if $x \geq 0$ or -1 otherwise.

Denote by $Lie(X_1, \cdots, X_m)$ the Lie algebra generated by vector fields $\{X_1, \cdots, X_m\}$, with $X_i : \mathbb{R}^d \to \mathbb{R}^d$. Denote by $Lie^n(X_1, \cdots, X_m)$ the sub set of all Lie brackets up to length $n \in \mathbb{N}$. We call a system that is globally asymptotically stable a GAS system.

2 Problem Description

The goal of this Section is to introduce the control problem and necessary definitions. First, consider the following driftless control system

$$\dot{x}(t) = \sum_{i=1}^{m} g_i(x(t))u_i(t), \tag{1}$$
$$x(0) = x_0,$$

where $x_0 \in \mathbb{R}^d$. Also, let $u_i : \mathbb{R}_{\geq 0} \to U$, with $U \subset \mathbb{R}$, and let $u(t) = (u_1(t), \cdots, u_m(t))$. Consider further $\mathcal{U}$ that is the set of piecewise constant functions over U^m. In addition, denote by $\phi(\cdot, x_0, u_{[a,b]})$ the flow of (1) starting at x_0, under the action of the control law $u_{[a,b]}$. We recall the definition, for our class of systems, of Lie Algebra Rank Condition (LARC) [16]. LARC at the point $x_0 \in \mathbb{R}^d$ is the condtion that $\mathbb{R}^d = \text{span}\{X(x_0) : X \in \text{Lie}(f_1, \cdots, f_m)\}$. Also, recall that the non-holonomy degree is the maximal length of Lie brackets needed to generate $\mathbb{R}^d$, i.e., saying that system (1) has a non-holonomy degree equal $n \in \mathbb{N}$ means that $\mathbb{R}^d =$

span $\{X(x_0) : X \in \text{Lie}^n(f_1, \cdots, f_m),\}$ and $\mathbb{R}^d \neq$ span $\{X(x_0) : X \in \text{Lie}^{n-1}(f_1, \cdots, f_m),\}$, see Chapter 4 of [15] for instance. We are now in condition to make our main assumptions. We assume that:

1. System (1) satisfy the Lie Algerba Rank Condition (LARC) everywhere;

2. A global upper bound $\mathcal{D}$ on the non-holonomy degree is known;

3. The vector functions g_i, for $i = \{1, \cdots, m\}$ are smooth, i.e. $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$.

The first assumption ensure, through Chow's theorem [2], that the system (1) is controllable, and there exist control laws u that globally asymptotically stabilize (1) to 0. The third assumption is just a technical assumption that guarantees that we can differentiate the vector fields f_i. The second is a technical assumption which is instrumental, however, to our algorithm to work.

In order to state our problem formally, consider the following definitions of coder-controller scheme and average data-rate [14]. Let $\{t_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of sampling times with $\lim_{n\to\infty} t_n \to \infty$. Also, let $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ be a sequence of alphabets with uniformly bounded cardinality, i.e. $\exists M > 0$, $\#\mathcal{C}_i < M, \forall i \in \mathbb{N}$. Furthermore, let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of functions such that $\gamma_n : \prod_{i=1}^{n-1} \mathcal{C}_i \times \mathbb{R}^{dn} \to \mathcal{C}_n$, where γ_n is called the *coder mapping at time n*. In a slightly more explicit way, we can write the coder mappings as

$$\begin{aligned} &\gamma_1 : x(t_1) \mapsto q_1, \\ &\gamma_n : (q_1, \cdots, q_{n-1}, x(t_1), \cdots, x(t_n)) \mapsto q_n, \end{aligned}$$

where $q_n \in \mathcal{C}_n$, for all $n \in \mathbb{N}$.

Additionally, let $\{\delta_n\}_{n\in\mathbb{N}}$ be a sequence of functions such that $\delta_n : \prod_{i=1}^{n} \mathcal{C}_i \times \mathbb{N} \to (U^m)^{[t_n, t_{n+1}]} \times \mathbb{N}$, where δ_n is called the *controller map at time n*, and $(U^m)^{[t_n, t_{n+1}]}$ is the set of functions from $[t_n, t_{n+1}]$ to U^m. As before, a more explicit way of writing the controller mappings is

$$\begin{aligned} &\delta_1 : (q_1, i) \mapsto (u_{[t_1, t_2]}, k) \\ &\delta_n : ((q_1, \cdots, q_n), i) \mapsto (u_{[t_n, t_{n+1}]}, k), \end{aligned}$$

where $k \in \mathbb{N}$ and $i \in \mathbb{N}$. Those parameters k and i are nonstandard and are instrumental in our approach. They work as counters that keep registered, in a sense explained later in Section 3, the last controller applied.

Finally, define the *coder-controller scheme* as the quadruple

$$\mathcal{S} = \left(\{t_n\}_{n\in\mathbb{N}}, \{\mathcal{C}_n\}_{n\in\mathbb{N}}, \{\gamma_n\}_{n\in\mathbb{N}}, \{\delta_n\}_{n\in\mathbb{N}}\right).$$

It is also necessary to define the concept of *average data-rate*. The average data-rate of the coder-controller scheme $\mathcal{S}$ is given by

$$b := \limsup_{j\to\infty} \frac{1}{t_j} \sum_{i=1}^{j} \log\left(\#\mathcal{C}_i\right). \tag{2}$$

In this paper $\mathcal{C}_i = \{-1, 1\}, \forall i \in \mathbb{N}$, and $U = [-1, 1]$.

The problem we want to solve is: consider a system described by equation (1), can we find a constructive algorithm that gives us a coder-controller scheme $\mathcal{S}$ such that system (1) is globally asymptotically stabilized with minimum average data-rate? The answer to this problem is summarized in the following

Theorem 2.1. *Consider system 1 satisfying assumptions 1, 2, and 3. Then, there exists a coder-controller scheme $\mathcal{S}$ that renders system (1) globally asymptotically stable with average data-rate* 0.

First, we will show how to stabilize system (1) with limited average data-rate constructively. Afterwards, we show how to adapt that solution so that GAS can be achieved with zero average data-rate by another constructive solution.

3 The Algorithm

In this Section we describe the algorithm that renders system (1) GAS with finite average data-rate. A high-level explanation of the algorithm's main idea is in order. First, note that system (1) can be rewritten in its integral form

$$x(t_n) = x(t_{n-1}) + \sum_{i=1}^{m} \int_{t_{n-1}}^{t_n} g_i(x(\tau))u_i(\tau)d\tau, \tag{3}$$
$$x(0) = x_0.$$

Define the parameter $\alpha_n := \left|\left|u_{[t_{n-1},t_n]}\right|\right|_\infty, \forall n \in \mathbb{N}$. Also, let $x_n := x(t_n), \forall n \in \mathbb{N}$. Furthermore, define $v_n := \sum_{i=1}^{m} \int_{t_{n-1}}^{t_n} \frac{g_i(x(\tau))u_i(\tau)}{\alpha_n} d\tau$. Note that the function $\frac{u_{i[t_{n-1},t_n]}}{\alpha_n}$ has its image on $[-1, 1]$ by definition of α_n, but it is an arbitrary piecewise constant function, otherwise. Therefore the following equation holds

$$x_n = x_{n-1} + \alpha_n v_n, \tag{4}$$

Now, consider $V : \mathbb{R}^d \to \mathbb{R}$ a convex, radially unbounded, and continuously differentiable, i.e. $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$, function with Lipschitz derivative around the origin. The idea is to choose a control law $u_{[t_{n-1},t_n]}$ such that v_n can be made into a decreasing direction for the function V departing from x_{n-1}. Note, however, that we choose the restriction $u_{[t_{n-1},t_n]}$ of the function u instead of choosing v_n directly. This extra step raises the question about the existence of such u and how to find it. We present an affirmative answer to the former and a method for the latter in Section 3.

In order to find this decreasing direction we will proceed as in the *compass search* procedure [4]. First, we need finite sets $\mathcal{V}_n \subset \mathbb{R}^d$, with $\#\mathcal{V}_n < M, \forall n \in \mathbb{N}$, for some $M > 0$, i.e., sets with uniformly bounded cardinality. Also, we require that the positive cone generated by $\mathcal{V}_n$ [4] is $\mathbb{R}^d$. This last condition means that $\forall x \in \mathbb{R}^d$, there exists $\alpha_i \in \mathbb{R}_{\geq 0}$, for $i = 1, \cdots, \#\mathcal{V}_n$, such that we have $x = \sum_{i=1}^{\#\mathcal{V}_n} \alpha_i v_{i_n}$, for $v_{i_n} \in \mathcal{V}_n$. We call a set with this last property a *positively* *spanning set* [4]. It is well known [4] that for a set that satisfies this condition, then for all nonzero $w \in \mathbb{R}^d$, there exists at least one $v_{i_n} \in \mathcal{V}_n$, such that $\langle w, v_{i_n} \rangle < 0$. If we take $w = \nabla V(x)$, for a nonstationary point $x \in \mathbb{R}^d$, then we have $\langle \nabla V(x), v_{i_n} \rangle < 0$, i.e., there exists a decreasing direction in $\mathcal{V}_n$.

Also, notice that, for driftless systems, one can start from the initial state $\bar{x}_n = \phi\left(t_n, x_{n-1}, u_{[t_{n-1},t_n]}\right)$ and go back to x_{n-1} by applying $u_{[t_n,t_n+\Delta_n]}(t) = -u_{[t_n,t_{n-1}]}(2t_n - t)$, for $t \in [t_n, t_n + \Delta_n]$, where $\Delta_n = t_n - t_{n-1}$. That means that

$$\phi\left(\Delta_n, \bar{x}_n, -u_{[t_n,t_n+\Delta_n]}\right) = x_{n-1},$$

this property is known as *strong reversibility*, see for example chapter 4 of [15], and is well known to be satisfied by driftless systems. From this, we have that if we fail in our attempt to decrease the value of V by applying $u_{[t_{n-1},t_n]}$, we can go back to the previous point x_{n-1} by applying another control. Otherwise we can set $x_n = \bar{x}_n$ and continue iterating.

Nonetheless, finding a decreasing direction is not sufficient; the step size α_n needs to be small. That can be solved by decreasing the step size after checking all directions on $\mathcal{V}_n$ and getting only unsuccessful iterations. A typical way of doing so is to take $\alpha_{n+1} = \frac{\alpha_n}{2}$. Furthermore, if the decrease in the function value between two consecutive iterations is not large enough we might get stuck. To disentangle this issue, consider a non-decreasing continuous function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, with $\lim_{t \to 0^+} \frac{\rho(t)}{t} = 0$. This function ρ is called a *forcing function* [4]. We declare the iteration successful if $V(\phi(t_n, x_{n-1}, u_{[t_{n-1},t_n]})) - V(x_{n-1}) + \rho(\alpha_n)) < 0$. In this way we enforce a sufficient decrease condition.

In order to generate the aforementioned directions $v_{i_n} \in \mathcal{V}_n$, we need to pick control laws that generate approximations to all Lie brackets up to order $\mathcal{D}$. That's done by means of the usual binary valued piecewise constant functions described in [7]. We know, by assumption (1), that the Lie brackets of $\{\pm f_i\}_{i=1}^m$ positively span $\mathbb{R}^d$, therefore, for a good enough approximation of those brackets we should also positively span $\mathbb{R}^d$. Now, consider $L_{\mathcal{D}}^{\alpha}$, for $\alpha > 0$ to be the ordered set of all nonzero scaled Lie brackets of $\{f_i\}_{i=1}^m$ up to order $\mathcal{D}$ with sign, e.g. for $\mathcal{D} \geq 3$ it contains $\pm\alpha f_i$, $\pm\alpha^2[f_i, f_j]$, and $\pm\alpha^3[f_i, [f_j, f_k]]$, for all $i, j, k \in \{1, \cdots, m\}$. The parameter α is just a positive scaling factor. Also, the order relation in $L_{\mathcal{D}}^{\alpha}$ can be chosen arbitrarily.

Consider $T_p > 0$ a fixed sampling time. We denote by $u_j^{\alpha} : \left[0, \frac{T_p}{2}\right] \to U$, the control law that generates the approximations to the j-th element of $L_{\mathcal{D}}^{\alpha}$ divided by α, i.e. these controllers ensure that, at the end of iteration n, $\bar{x}_n = x_{n-1} + \alpha^p X_p + o(\alpha^p)$, where X_p is a Lie bracket of $\{f_i\}_{i=1}^m$ order $p-1$. Using the notation from equation (4) we can identify $v_n = \alpha^{p-1} X_p + \frac{o(\alpha^p)}{\alpha}$. Therefore, in this sense we can use the controls $u_j^{\alpha_n}$ to generate directions that approximate scaled Lie brackets of $\{f_i\}_{i=1}^m$ at the iteration n. A final remark is needed, if $p = 1$, then $o(\alpha^p) = 0$, because we can generate the directions spanned by the vector fields $\{f_i\}_{i=1}^m$ exactly.

The coder-controller scheme adopted to solve the problem is the following. The sequence of sampling times is given by $t_n = nT_p$, where $n \in \mathbb{N}$. The coder map at time n is

$$\gamma_n\left(q_1,\cdots,q_{n-1},x(t_1),\cdots,x(t_n)\right)=\\ \operatorname{sign}\left(V(x(t_n))-V(x(t_{n-1}))+\rho(\alpha_n)\right),$$

notice that α_n is a function of $(q_n)_{n\in\mathbb{N}}$, where $q_i \in \mathcal{C}_i$. Here we need to impose further a technical assumption on the forcing function $\rho(\cdot)$. We require that ρ satisfies $\lim_{t\downarrow 0}\frac{\rho(t)}{t^{\mathcal{D}}}$. The reasons for that will be made clear at the end of Section 5. Define $\mathcal{L}=\#L^{\alpha}_{\mathcal{D}}$, and notice that it is constant for any $\alpha>0$.

The control map at time n is

$$\delta_n(q_n,i)=\begin{cases}\left(-u_{i-1}^{\alpha_{n-1}}(t_n-\cdot)\wedge u_i^{\alpha_n}(\cdot-t_n),i+1\right) & \text{(a)}\\ \left(-u_{\mathcal{L}}^{\alpha_{n-1}}(t_n-\cdot)\wedge u_1^{\alpha_n}(\cdot-t_n),1\right) & \text{(b)}\\ \left(0\wedge u_1^{\alpha_n}(\cdot-t_n),1\right) & \text{(c)}\end{cases}$$

where condition (a) is $q_n=1$ and $i\in\{1,\cdots,\mathcal{L}-1\}$, condition (b) is $q_n=1$ and $i=\mathcal{L}$, and condition (c) is $q_n=-1$. Also, we denote by $0:\left[0,\frac{T_p}{2}\right]\to\{0\}$ the zero function. Finally, $f\wedge g$ represents the concatenation of two functions in the following way: given $f:\left[0,\frac{T_p}{2}\right]\to U$ and $g:\left[0,\frac{T_p}{2}\right]\to U$, then $f\wedge g:[0,T_p]\to U$ is defined by

$$(f\wedge g)(t)=\begin{cases}f\left(t+\frac{T_p}{2}\right) & t\in\left[0,\frac{T_p}{2}\right]\\ g\left(t-\frac{T_p}{2}\right) & t\in\left[\frac{T_p}{2},T_p\right]\end{cases}.$$

This control law simply states that if there was not a sufficient decrease in the cost V, then one should apply another control to undo the effect of the previously applied control $u_i(\cdot)$ and try a new one afterwards. Otherwise, a new control law should be applied. Furthermore, notice that the parameter i is just a counter that keeps track of which control law was previously applied and is not transmitted through the network. This discussion is depicted in the block diagram in figure 1.

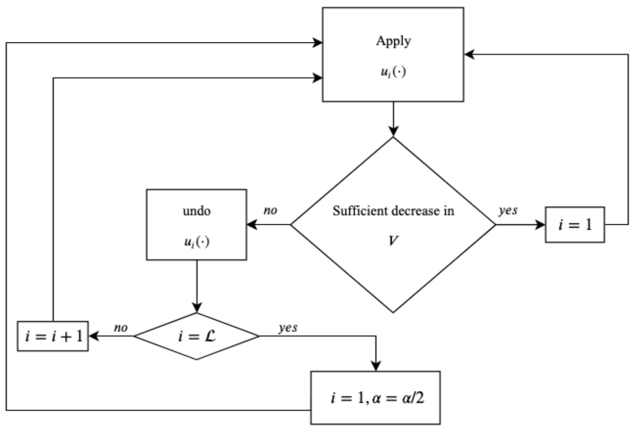

Figure 1: Block diagram with the algorithm's idea

In this way, as we will justify in Section 5, we can drive the state to the origin using constantly 1 bit per sample. Therefore, the average data-rate is $b=1$. Now, one could modify the algorithm by only transmitting data if $n = 2^k$, for $k \in \mathbb{N}$. The control law applied whenever data is not transmitted should be $u = 0$ constantly. In this manner our algorithm still globally asymptotically stabilizes system (1), but with an average data-rate $b = 0$.

4 Example

Consider the equations of the Dubin's car [8]

$$\begin{aligned} \dot{x}_1 &= u_1 \cos(\theta) \\ \dot{x}_2 &= u_1 \sin(\theta) \\ \dot{\theta} &= u_2 \end{aligned} \tag{5}$$

In this example, the states x_1 and x_2 represent the $x - y$ coordinates of a unicycle, while θ is the angle that it makes with respect to the x-axis measured in a counterclokwise manner. Consider the function $V(x_1, x_2, \theta) = x_1^2 + x_2^2 + \theta^2$. Note that it is convex, radially unbounded, and its gradient ∇V is Lipschitz near 0 (in fact it is analytic, but our analysis only needs the assumption of Lipschitzness). Also, consider the forcing function $\rho(t) = t^2$. Furthermore, notice that the vector fields

$$f_1(x_1, x_2, \theta) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad f_2(x_1, x_2, \theta) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix},$$

$$[f_1, f_2](x_1, x_2, \theta) = \begin{bmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{bmatrix},$$

span $\mathbb{R}^d$, therefore $\pm f_1, \pm f_2, \pm[f_1, f_2]$ positively span $\mathbb{R}^d$. Finally, consider the functions $u_1^\alpha(t) = (\alpha, 0)$ for $t \in \left[0, \frac{T_p}{2}\right]$, $u_2^\alpha(t) = (0, \alpha)$ for $t \in \left[0, \frac{T_p}{2}\right]$,

$$u_3^\alpha(t) = \begin{cases} (\alpha, 0) & t \in \left[0, \frac{T_p}{8}\right) \\ (0, \alpha) & t \in \left[\frac{T_p}{8}, \frac{T_p}{4}\right) \\ (-\alpha, 0) & t \in \left[\frac{T_p}{4}, \frac{3T_p}{8}\right) \\ (0, -\alpha) & t \in \left[\frac{3T_p}{8}, \frac{T_p}{2}\right] \end{cases},$$

and the functions $u_4^\alpha = -u_1^\alpha$, $u_5^\alpha = -u_2^\alpha$, and $u_6^\alpha = -u_3^\alpha$. Writing the solution of equation (5) we get that $x(t_n) = x(t_{n-1}) + \alpha v_n$ by reinterpreting it using equation (4). Therefore, we can see that we can generate the vectors $v_n = \pm f_1$, $v_n = \pm f_2$, and $v_n = \pm\alpha[f_1, f_2] + \frac{o(\alpha^2)}{\alpha}$. One can prove that, for α small enough, this set of directions v_n positively span $\mathbb{R}^d$. In this way, we can globally asymptotically stabilize the Dubin's car to the origin with minimum data-rate equal to 0.

We simulated the algorithm using the parameters $\alpha_0 = 0.5$, $x_1(0) = 1$, $x_2(0) = 1$, and $\theta(0) = 0.5$ rad. Furthermore, we performed simulations with $T_p = 1$ and $T_p = 2$ time units. The evolution of the cost function value after each sucessful iteration is presented in figure 2, while the evolution of the state of the system for $T_p = 1$ and $T_p = 2$ are depicted in figures 3 and 4, respectively.

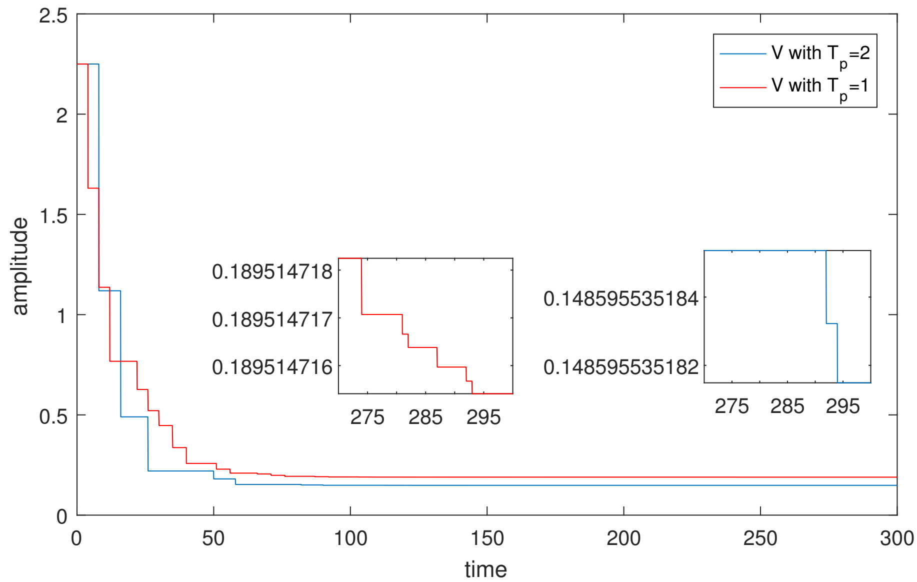

Figure 2: Evolution of the cost value

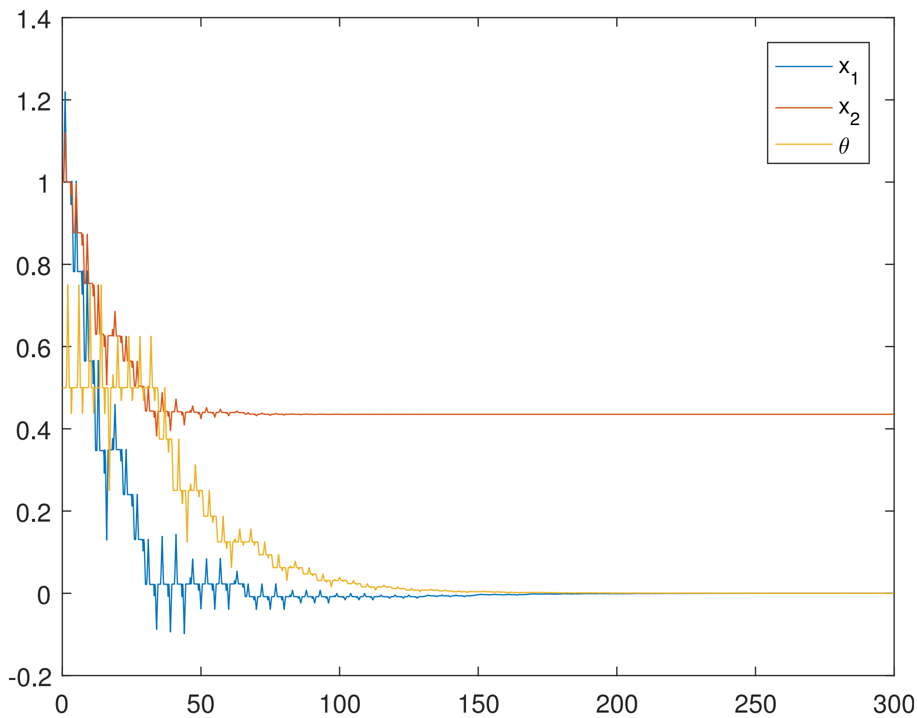

Figure 3: Evolution of the state for $T_p = 1$

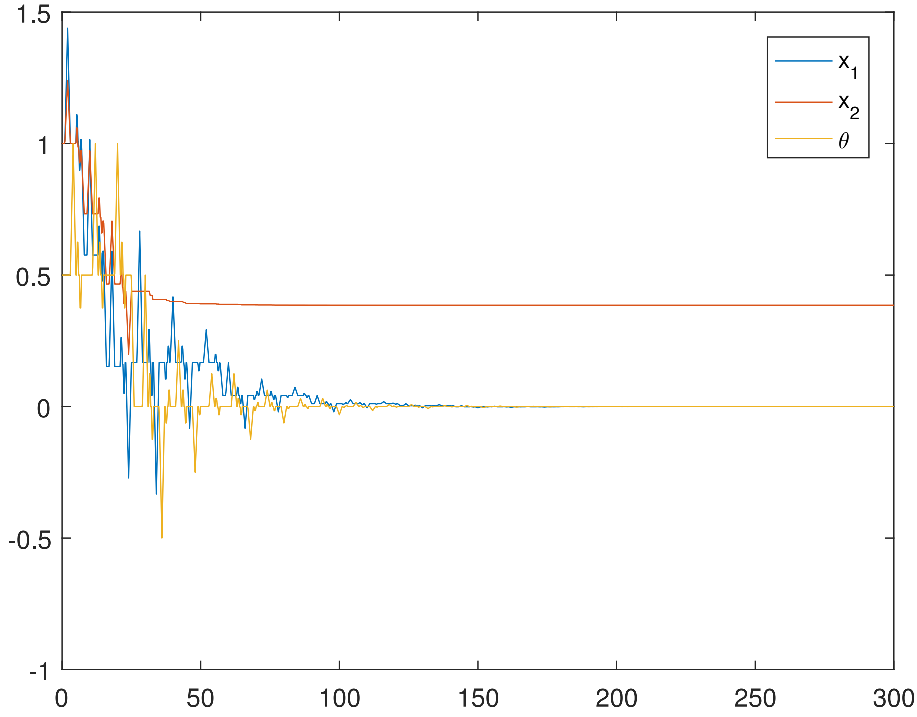

Figure 4: Evolution of the state for $T_p = 2$

It is important to notice that by decreasing the sampling time T_p we increase the average data-rate. In fact, the average data-rate for $T_p = 1$ is $b = 1$ bit/unit of time, while for $T_p = 2$ it is $b = 0.5$ bits/unit of time.

Figure 2 shows that the cost sharply decreases at first for both values of T_p, then it reaches a plateau. This effect has two main causes. The first is related to the local convergence of direct search methods. Since our algorithm essencially applies the compass search algorithm (which is a direct search method [4]), it inherits its bad local convergence features. The second cause can be explained by figure 3 and 4. Those figures show a rapid convergence of x_1 and θ to zero, nonetheless, when θ approaches zero, so does $\sin(\theta)$, making the dynamics of x_2 slow. This latter cause explains why there is no simple relation between the average data-rate and performance for this algorithm, given that a faster convergence of θ to zero makes the convergence of x_2 slower. Finally, it is worth mentioning that, even though the cost function value seemingly stagnates, it keeps going down as can be seen in the zoomed parts in figure 2.

We also present the results of this simulation, for the case where $T_p = 2$, in video format. We created two videos with this simulation, one with the state space evolution [17] and another with the top view of a car parking, i.e. the evolution in the $x - y$ plane [19]. It is important to remark that whenever a decrease happens, the blue dot remains in place for $\frac{T_p}{2}$ units of time. That occurs because we apply $u = 0$ for that duration of time, as explained in Section 3.

Furthermore, one way of mitigating the aforedescribed second cause for the slow convergence is by changing the cost function. In our case, we can give a smaller weight to the term that depends on θ to reduce its relative importance in the total cost. We choose $\tilde{V}(x_1, x_2, \theta) = 20(x_1^2 + x_2^2) + 0.1\theta^2$ for our simulations.

Also, we keep all other parameters the same, i.e., $\alpha_0 = 0.5$, $x_1(0) = 1$, $x_2(0) = 1$, and $\theta(0) = 0.5$ rad. Once more we run the simulations with $T_p = 1$ and $T_p = 2$.

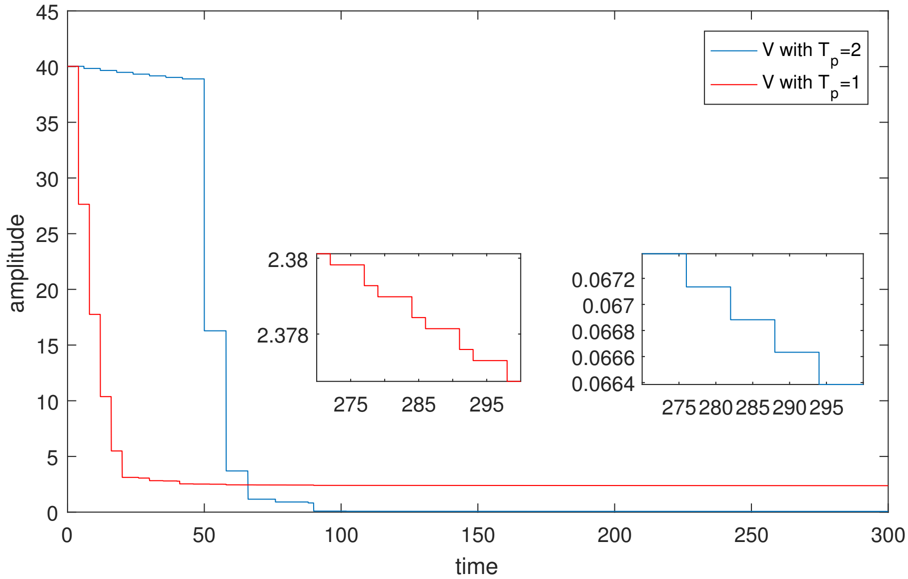

Figure 5: Evolution of the cost value

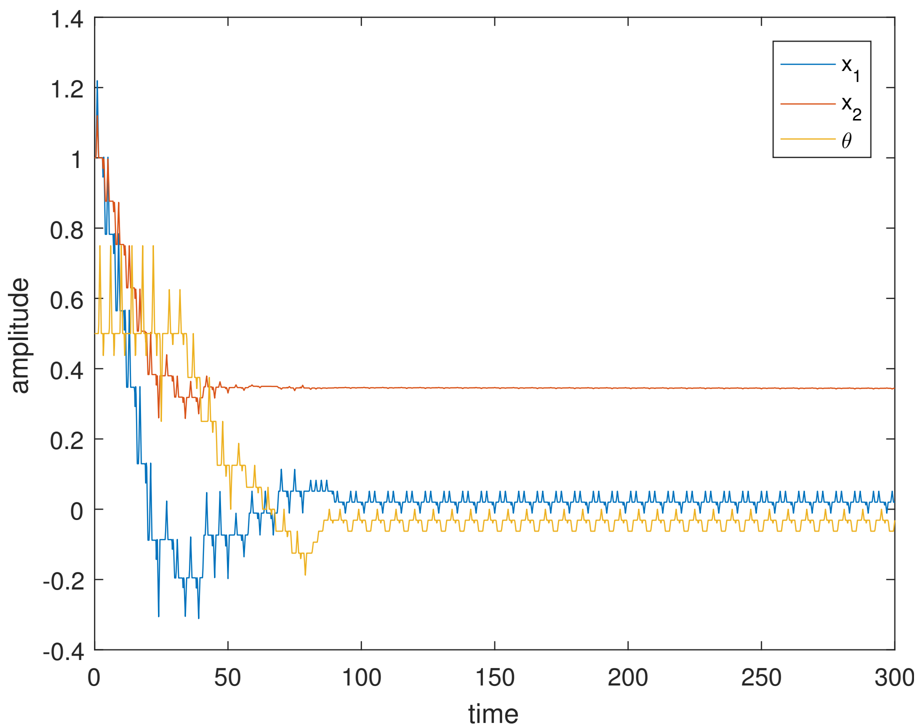

Figure 6: Evolution of the state for $T_p = 1$

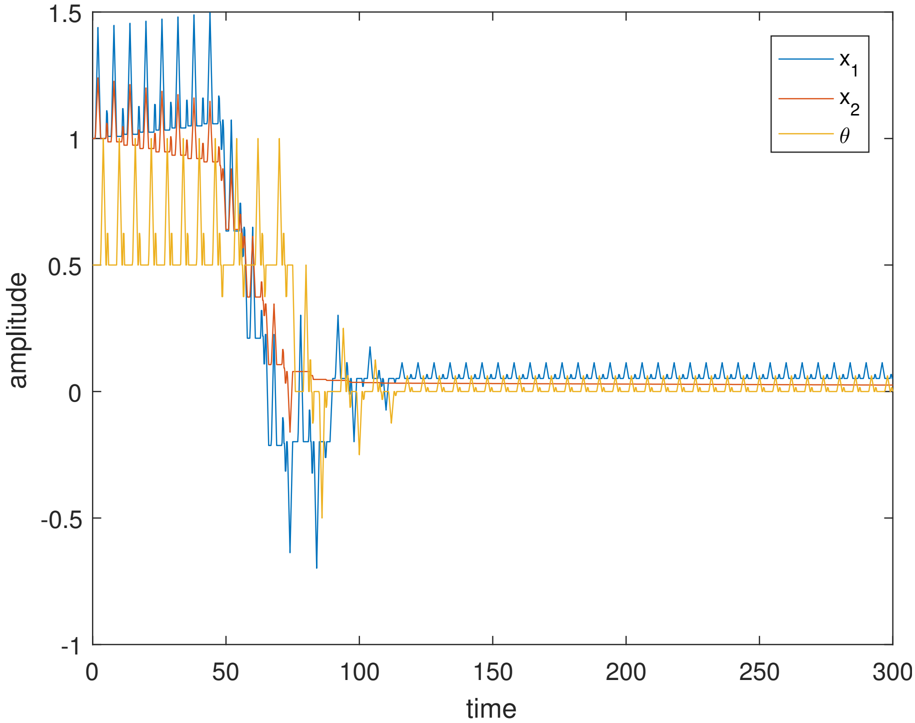

Figure 7: Evolution of the state for $T_p = 2$

Figure 5 shows the evolution of the new cost function value. It is important to remark that, since the cost functions are different, we cannot compare figure 2 and figure 5 directly. In order to see the improvement we compare figure 4 and figure 7. Those two figures show that the state is closer to zero than before. Moreover, state x_2, which was the previous cause for the plateau, is much closer to zero, as desired. Nonetheless, figure 6 does not show a visible improvement to figure 3, which strengthens the claim that the relationship between the average data-rate and performance is very complex. Lastly, we also present this simulation, for $T_p = 2$, in two videos [18], with the state space evolution and [20] with the top view of the car parking.

5 Analysis

Our mathematical analysis is based on the analysis of directional direct-search methods in optimization [4], considering equation (4), and considering the algorithm described in Section 2. In essence, we will show that, for a fixed $\alpha > 0$ and a departing point $x_n \in \mathbb{R}^d$, the set of controls $\{u_k^\alpha\}_{k=1}^{\mathcal{L}}$ aforementioned, generates positively spanning sets $P(x_n)$ from the dynamics, i.e.

$$P(x_n) = \left\{ x \in \mathbb{R}^d : x = \phi\left(\frac{T_p}{2}, x_n, u_k^\alpha\right) - x_n, \forall k = 1, \cdots, \mathcal{L} \right\},$$

is positively spanning. Then, we will use modifications of some theorems from [4] to show that the iterates x_k, generated by our algorithm, converge to some x^* such that $||\nabla V(x^*)|| = 0$. First, we state some assumptions that will be used frequently in our analysis.

(i) The level set $L(x_0) = \{x \in \mathbb{R}^d : V(x) \leq V(x_0)\}$ is compact;

(ii) If for some real constant $\alpha > 0$, the step size at iteration k, α_k, is such that $\alpha_k > \alpha$, for all $k \in \mathbb{N}$. Then the algorithm visits only a finite number of points;

(iii) Let $\xi_1, \xi_2 > 0$ be some fixed constants. The positive spanning sets $\mathcal{V}_k$ used in the algorithm are chosen from the set

$$\left\{\mathcal{V}_k \text{ positively span } \mathbb{R}^d : \text{cm}(\mathcal{V}_k) > \xi_1,\ ||\bar{d}|| \leq \xi_2, \forall \bar{d} \in \mathcal{V}_k\right\}$$

Recall that the cosine measure of a finite positively spanning set $\mathcal{V} \subset \mathbb{R}^d$ is $\text{cm}(\mathcal{V}) = \min_{v,||v||=1} \max_{d\in\mathcal{V}} \frac{\langle d,v\rangle}{||d||}$.

(iv) The gradient ∇V is Lipschitz continuous in an open set containing $L(x_0)$.

Notice that items (i) and (iv) are satisfied if we choose V to be $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$, convex, radially unbounded, and ∇V to be Lipschitz continuous around the origin. Those assumptions also ensure that V has a unique global minimum, which will be chosen to be 0 without loss of generality. The assumption (iii) is ensured by our previous assumption 1 on system (1). To see that, notice that $\mathcal{V}_k$ satisfies $||\bar{d}|| \leq \xi_2$ for some $\xi_2 > 0$ by construction. Also, the cosine measure is lower bounded by $\xi_1 > 0$ because of the fact that LARC holds everywhere. Additionally, assumption (ii) is guaranteed by the introduction of the forcing function and Theorem 5.1 adapted from [4].

Now, we state and prove a modification of theorem 2.8 from section 2 of [4] that will be instrumental in our argument to show that the state converges to the origin asymptotically, proving GAS.

Theorem 5.1. *Let $P \subset \mathbb{R}^d$ be a positively spanning set, $V : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ and convex function, and $\alpha > 0$ be given. Assume that ∇V is Lipschitz continuous in a neighborhood of the minimum of V with Lipschitz constant $\nu > 0$. Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a forcing function. If $V(x) \leq V(x + \alpha d) - \rho(\alpha)$, for all $d \in P$, then*

$$||\nabla V(x)|| \leq \frac{\nu}{2} \text{cm}(P)^{-1} \max_{d\in P} ||d||\alpha + \text{cm}(P)^{-1} \frac{\rho(\alpha)}{\min_{d\in P} ||d||\alpha} \tag{6}$$

Proof. Notice that $\text{cm}(P) \leq \max_{d\in\mathcal{V}} \frac{\langle d,v\rangle}{||d||}$, for any v with unit norm. Choose $v = -\frac{\nabla V(x)}{||\nabla V(x)||}$ and let $d \in P$ be a vector for which

$$\text{cm}(P)||\nabla V(x)||||d|| \leq -\langle \nabla V(x), d\rangle. \tag{7}$$

Now, from the fundamental theorem of calculus and the fact that $V(x) \leq V(x + \alpha d) - \rho(\alpha)$ we get for all $d \in P$, that

$$\rho(\alpha) \leq V(x + \alpha d) - V(x) = \int_0^1 \langle \nabla V(x + t\alpha d), \alpha d\rangle dt. \tag{8}$$

Take now the inequality (7), rewrite its right side in integral form, and multiply both sides by $\alpha > 0$. Then

$$\mathrm{cm}(P)||\nabla V(x)|| ||d||\alpha \leq \int_0^1 -\langle \nabla V(x), \alpha d\rangle dt. \tag{9}$$

Adding (8) and (9), we conclude that

$$\mathrm{cm}(P)||\nabla V(x)|| ||d||\alpha \leq \int_0^1 \left(\langle \nabla V(x + t\alpha d) - \nabla V(x), \alpha d\rangle\right) dt - \rho(\alpha).$$

Finally, using the Lipschitz assumption, isolating the term $||\nabla V(x)||$ on the left, and majorizing the terms that depend on $||d||$, we conclude that

$$||\nabla V(x)|| \leq \frac{\nu}{2}\mathrm{cm}(P)^{-1} \max_{d\in P} ||d||\alpha + \mathrm{cm}(P)^{-1} \frac{\rho(\alpha)}{\min_{d\in P} ||d||\alpha}$$

□

Note that the condition $V(x) \leq V(x + \alpha d) - \rho(\alpha)$ of theorem 5.1 is satisfied whenever we are in an unsuccessful iteration. Moreover, this theorem implies that if the step size of our algorithm $\alpha_k \downarrow 0$, and there exist $\xi_1 > 0$ and $\xi_2 > 0$ such that $\mathrm{cm}(P) > \xi_1$ and $\max_{d\in P} ||d|| < \xi_2$, then $||\nabla V(x_k)|| \downarrow 0$. Since V is convex and differentiable with minimum at 0, $x_k \to 0$. We now proceed to prove that our algorithm implies that $\alpha_k \downarrow 0$. We rely on the next result that is, essentially, theorem 7.11 from [4] with only slight modifications in notation and is transcribed here for completeness.

Lemma 5.1. *Assume we only accept a new iterate if* $V(x_{k+1}) \leq V(x_k) - \rho(\alpha_k)$. *Also, let assumption (i) hold. Finally, assume that* $\exists \alpha > 0$, *such that* $\alpha_k > \alpha$, *for all* k. *Then the algorithm visits only a finite number of points.*

Proof. Since ρ is increasing, we have that $0 < \rho(\alpha) \leq \rho(\alpha_k)$ for all $k \in \mathbb{N}$. Suppose that there exists an inifnite sequence of successful iterates. From $V(x_{k+1}) \leq V(x_k) - \rho(\alpha_k)$ we get, for all successful iterations, that $V(x_{k+1}) \leq V(x_k) - \rho(\alpha_k) \leq V(x_k) - \rho(\alpha)$. Consequently, the sequence $(V(x_n))_{n\in\mathbb{N}}$ converges to $-\infty$, which contradicts assumption (i). □

Our next lemma was left as an exercise in chapter 7 of [4] and is based on their theorem 7.1. It shows that assumption (ii) imply that α_k goes to zero as desired.

Lemma 5.2. *Assume we only accept a new iterate if* $V(x_{k+1}) \leq V(x_k) - \rho(\alpha_k)$. *Also, let assumption (ii) hold. Then, the sequence of step size parameters satisfies*

$$\liminf_{k\to\infty} \alpha_k = 0$$

Proof. Assume that there exists $\alpha > 0$ such that $\alpha_k > \alpha$ for all k. However, x_k is updated only if $V(x_{k+1}) < V(x_k) - \rho(\alpha_k)$, therefore, there exists $\bar{K} \in \mathbb{N}$, such that $x_k = x_{\bar{k}}$, for all $k \geq \bar{k}$.

Hence, all iterations after $\bar{k}$ are unsuccessful and, by our α_k update law, it follows that $\liminf_{k\to\infty} \alpha_k = 0$, which contradicts the existence of such $\alpha > 0$. □

A corollary of this lemma shows that there exist a subsequence $(k_i)_{i\in\mathbb{N}}$ of unsuccessful iterates, such that $\lim_{i\to\infty} \alpha_i = 0$ and $\lim_{i\to\infty} x_{k_i} = x^*$ for some $x^* \in \mathbb{R}^d$. This follows from the previous lemma and the fact that α_k is decreasing, thus convergent to zero, and from the fact that $(x_k)_{k\in\mathbb{N}}$ is inside a compact set by assumption (i).

Denote by κ the minimum of the norm of the nonzero Lie brackets of $\{f_i\}_{i=1}^m$ up to order $\mathcal{D}$ evaluated at the iterate x_k. Notice that $\kappa > 0$, given that the aforedescribed set is always finite. We can conclude now, by invoking theorem 5.1, that

$$||\nabla V(x_k)|| \leq \frac{\nu}{2}\mathrm{cm}(P(x_k))^{-1} \max_{d\in P(x_k)} ||d||\alpha_k + \mathrm{cm}(P(x_k))^{-1}\frac{\rho(\alpha_k)}{\min_{d\in P(x_k)} ||d||\alpha_k},$$

hence by assumption (iii) and the fact that $\frac{\rho(\alpha_k)}{\min_{d\in P(x_k)} ||d||\alpha_k} \leq \frac{\rho(\alpha_k)}{\kappa(\alpha_k)^{\mathcal{D}}} \downarrow 0$, for $\alpha_k \downarrow 0$, we conclude that $||\nabla V(x_k)|| \downarrow 0$. Therefore, $(x_k)_{k\in\mathbb{N}}$ converges to 0 proving GAS.

6 Conclusion

In this paper we presented a limited information control algorithm that globally asymptoticaly stabilizes a subclass of non-holonomic driftless affine systems. In addition, we proved that the minimum average data-rate for achieving GAS for this class is 0. Also, we presented the Dubin's car as an example of system for which our method works. Furthermore, a mathematical analysis of the algorithm was presented.

One line of investigation for future works is extending the constructive method presented here to other classes of control systems. One example of such a problem is to modify this algorithm to solve the same problem for the control with limited information of affine systems with drift. This adaptation is not trivial given that the drift makes it impossible, in general, for us to go back in case we chose a non-decreasing direction. Also, the problem of providing a data-rate theorems for more general classes of systems in a constructive manner is another possible extension of this work.

Finally, an in depth analysis of the performance of our algorithm is missing. It would show how the average data-rate affects the convergence rate of the state.

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