Addendum to "Stability of switched systems: a Lie-algebraic condition"

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The purpose of this note is to explain the overlap between our paper [1], published in this journal, and the earlier work [2] which appeared in the Russian literature.

The paper [1] studies switched linear systems of the form

$$\dot{x} = A_\sigma x \tag{1}$$

where $x \in \mathbb{R}^n$, $\{A_p, p \in \mathcal{P}\}$ is a compact set of $n \times n$ matrices, $\mathcal{P}$ is an index set, and $\sigma : [0, \infty) \to \mathcal{P}$ is a piecewise constant switching signal. The main result of [1] (Theorem 2) states that the system (1) is globally exponentially stable, uniformly over all switching signals, if all the matrices A_p, $p \in \mathcal{P}$ are Hurwitz and the Lie algebra generated by these matrices is solvable. (Hurwitzness of the matrices A_p, $p \in \mathcal{P}$, as well as of all their convex combinations, is also a necessary condition for such a stability property.)

The paper [2] studies bilinear control systems of the form

$$\dot{x} = \sum_{i=1}^{m} u_i A_i x \tag{2}$$

where $x \in \mathbb{R}^n$, $\{A_1, \ldots, A_m\}$ is a finite collection of matrices, and $u = \begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix}^T$ is the vector of measurable open-loop controls taking values in a compact subset U of $\mathbb{R}^m$. The main result of [2] is that when the Lie algebra generated by the matrices $A_1, \ldots, A_m$ is solvable, the system (2) is globally exponentially stable, uniformly over all controls, if and only if all the matrices $\sum_{i=1}^m u_i A_i$, $u \in U$ are Hurwitz.

For suitable choices of the control set U, the bilinear control system (2) becomes equivalent to the switched linear system (1) with $\mathcal{P} = \{1, \ldots, m\}$ or to the differential inclusion $\dot{x} \in \mathrm{co}\{A_1 x, \ldots, A_m x\}$ (where "co" stands for convex hull); see, e.g., [3] for details. Thus the system (2) is more general than (1), at least when $\{A_p, p \in \mathcal{P}\}$ is (in the linear span of) a finite set of matrices, and the main result of [1] can be deduced from that of [2].

The proofs given in [2] and [1] both rely on Lie's theorem, which states that matrices in a solvable Lie algebra can be simultaneously triangularized (over $\mathbb{C}$). After this step, the two proofs proceed differently. The argument in [2] establishes stability directly in the time domain. In [1] such a time-domain argument

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is only sketched, and an alternative proof is provided by constructing a common quadratic Lyapunov function for the systems $\dot{x} = A_p x$, $p \in \mathcal{P}$. (Such a Lyapunov function has independent interest.)

The work [2] appeared only in the Russian literature and did not use the term "switched system." For these reasons, we were unaware of it until recently. We regret this oversight and want to belatedly give proper credit to the author of [2]. We further note that subsequent work [4] by the same author also establishes a result essentially equivalent to Theorem 2 of [5] (the latter paper contains several other results not reported in [4]).

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References

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