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Supervisory control of uncertain systems with quantized information

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SUMMARY

This work addresses the problem of stabilizing uncertain systems with quantized outputs using the supervisory control framework, in which a finite family of candidate controllers is employed together with an estimator-based switching logic to select the active controller at every time. For static quantizers, we provide a relationship between the quantization range and the quantization error bound that guarantees closed-loop stability. Such a condition also implies a lower bound on the number of information bits needed to guarantee stability of a supervisory control scheme with quantized information. For dynamic quantizers that can vary the quantization parameters in real time, we show that the closed loop can be asymptotically stabilized, provided that additional conditions on the quantization range and the quantization error bound are satisfied.

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1. INTRODUCTION

Control with limited information has attracted growing interest in the control research community lately, largely motivated by the *control over networks* paradigm. Unlike the classical control setting in which signals take values in a continuum and are available at every time, in networked control systems, information is limited in the sense that control and sensor signals are quantized/digitized before being sent over a communication channel, the information is only available at a certain rate and with delay, and there is a possibility of information loss during data transmission (see, e.g., [1] for a recent survey on networked control systems). In this paper, we focus on the scenario where the system output is quantized, which is frequently considered in the literature as a way to model communication constraints or limited sensing capabilities.

Most of the work concerning control with limited information deals with known plants (see the references in [1, 2]), and recently, work on the control of *uncertain systems with limited information* has been started [3]. While there are several aspects in control with limited information as outlined in the previous paragraph, dealing with both plant uncertainty and limited information at the same time is rather challenging. As a first step, we treat limited information as quantization only, and further, to make the problem manageable, we consider output quantization only (no input quantization can be thought of as the controller and the plant being co-located). Quantized control systems with known plants have been considered, for example, in [4–8]. In this paper, we consider the problem of *stabilizing uncertain systems* with output *quantization*. A similar problem, adaptive control with quantized input, has recently been studied by Hayakawa *et al*. in [3], where the authors provided a solution using a (static) logarithmic quantizer and a Lyapunov-based adaptive algorithm. Here, we

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consider output quantization. We use a different type of quantizers in a more general setting (the quantizer is characterized only by the quantization error bound and the quantization range without any specific structure) and a different adaptive control tool, namely supervisory control [9, 10]. We will cover the supervisory control framework in Section 3 (see also, e.g., [11, Chapter 6] and the references therein for further background on supervisory control and [12] for discussions on the advantages and the drawbacks of supervisory control compared with other adaptive schemes).

For static quantizers, we want to find a relationship between the quantization range and the quantization error bound to guarantee closed-loop stability. While it has been shown [10, Proposition 6; 17] that supervisory control is robust to measurement noise, extending this result to quantization is far from trivial because one needs to ensure that the information to be quantized does not exceed the quantization range. In this work, we give such a condition on the quantizer parameters with respect to the supervisory control scheme's design parameters to guarantee closed-loop stability.

To achieve asymptotic stability, we utilize the *dynamic quantizers* as in [13, 14], which have the capability of varying the quantization parameters in real time (in particular, the quantizer can zoom in and zoom out while keeping the number of alphabets fixed). In [13, 14], the authors have applied dynamic quantization to asymptotically stabilize *known plants* (see also [15] for performance analysis of dynamic quantization). We will show that the dynamic quantizer can also be used with unknown plants, in conjunction with supervisory control, to achieve asymptotic stability.

The contribution of our work is that we provide a new solution to the problem of stabilizing uncertain systems with quantized information using supervisory control and dynamic quantization. The implication of our work, in the context of control of uncertain systems and in addition to [3], is that to deal with plants with parametric uncertainty, one does not need sensors and actuators with infinite granularity and can use quantized information with finite, and possibly limited, bit rates in combination with an adaptive control scheme (which is supervisory control in our case). Moreover, asymptotic stability can be achieved with dynamic quantizers with finite quantization levels, which is different from and complements the logarithmic quantizers with infinite levels of quantization considered in [3].

The paper is organized as follows: In Section 2, we present the quantized control system. In Section 3, we present a supervisory control scheme for the quantized control system in Section 2, assuming that the parameter set is finite. In Section 4, we provide conditions for the quantizer's range and error bound in terms of the supervisory control design parameters in order to guarantee stability and asymptotic stability of the closed-loop system. In Section 5, we extend the results in Section 4 to the case with unmodeled dynamics. In Section 6, we discuss the case of continuum uncertainty sets. We provide an illustrative example with the discussion in Section 7. We also provide in Section 8 a discussion on the methodology to extend the results for linear plants in Section 4 to nonlinear ones, using the input-to-state stability (ISS) framework. Finally, in Section 9, we conclude and discuss future work.

2. QUANTIZED CONTROL SYSTEM

Consider an uncertain linear plant $\Gamma(p)$ parameterized by a parameter p and the true but unknown parameter denoted by p^*:

$$\Gamma(p^*): \begin{cases} \dot{x} = A_{p^*}x + B_{p^*}u \\ y = C_{p^*}x, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, and $y \in \mathbb{R}^{n_y}$ is the output. The parameter $p^* \in \mathbb{R}^{n_p}$ belongs to a known finite set $\mathcal{P} := \{p_1, \dots, p_m\}$, where m is the cardinality of $\mathcal{P}$.

Assumption 1
(A_p, B_p) is stabilizable and (A_p, C_p) is detectable for every $p \in \mathcal{P}$.

A (static) quantizer is a map $\mathrm{Q} : \mathbb{R}^{n_y} \to \{q_1, \dots, q_N\}$, where $q_1, \dots, q_N \in \mathbb{R}^{n_y}$ are quantization points, and Q has the following properties: (1) $|y| \leqslant M \Rightarrow |\mathrm{Q}(y) - y| \leqslant \Delta$ and (2)

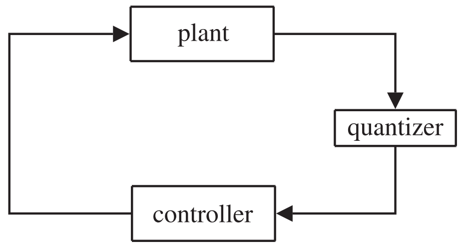

Figure 1. Quantized closed-loop system.

$|y| > M \Rightarrow |Q(y)| > M - \Delta$, where $|\cdot|$ denotes the Euclidean norm. The numbers M and Δ are known as the *range* and the *error bound* of the quantizer Q. A *dynamic quantizer* Q_ν, having an additional parameter ν which can be changed over time, is defined as

$$Q_\nu(y) := \nu Q(y/\nu), \tag{2}$$

where Q is a static quantizer with the range M and the error bound Δ. From the property (1) of the quantizer, we have

$$|y| \leqslant \nu M \Rightarrow |Q_\nu(y) - y| \leqslant \nu\Delta. \tag{3}$$

The parameter ν is known as a *zooming variable*: increasing ν corresponds to zooming out and essentially obtaining a new quantizer with a larger range and quantization error, whereas decreasing ν corresponds to zooming in and obtaining not only a quantizer with a smaller range but also a smaller quantization error.

Remark 1

Similarly to the setting in the work [14], here we consider a very broad class of quantizers, which is more general than the common bit digitizer, such as the q-bit quantizer $Q(y) = \lfloor y \times 2^q \rfloor / 2^q$. We do not require the quantized values to be evenly spaced. The quantizer considered here can be further relaxed by not requiring that the quantizer's value mapping is fixed. For the same value y, the value $Q(y)$ can be different at different times, as long as (3) holds. An example of such a quantizer is $Q(y,t) = \lfloor y \rfloor$ if $t \in [kT, kT + T/2)$ and $|y(t)| \leqslant M$, and $Q(y,t) = \lfloor y \rfloor + 1$ if $t \in [kT + T/2, kT + T)$ and $|y(t)| \leqslant M$, where $k = 0, 1, \ldots$ and $T > 0$, and $Q(y,t) = M$ if $|y(t)| > M$.

The quantized control system is depicted in Figure 1.

Assuming that the plant is unstable, the objective is to asymptotically stabilize the plant while the information available to the controller is $Q_\nu(y)$ instead of y. With a static quantizer, we would only achieve practical stability but we will later show that one can indeed achieve closed-loop asymptotic stability with dynamic quantization.

3. QUANTIZED SUPERVISORY CONTROL

Supervisory control [9, 12] employs multiple candidate controllers, and the choice of which controller to connect to the plant is orchestrated by an estimator-based supervisor (see Figure 2 for an illustration of the idea and for more detailed background on supervisory control, see, e.g., [11, Chapter 6] or [12] and the references therein).

We present one particular design of supervisory control for the linear plant (1), in which the controllers are state feedback and utilize the multi-estimator's state (more subsequent details). One can also have more general forms of dynamic controllers which do not use the multi-estimator state, provided that the multi-controller and the multi-estimator combination (known as the injected system, see (9) for more details) satisfy certain conditions [10].

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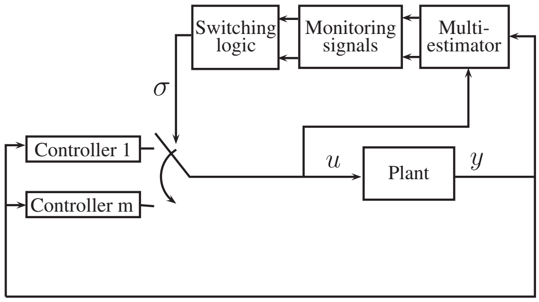

Figure 2. The supervisory control framework.

- *The multi-estimator*: A multi-estimator is a collection of dynamics, one for each fixed parameter $p \in \mathcal{P}$. The multi-estimator takes in the input u and produces a bank of outputs $y_p, p \in \mathcal{P}$. In this work, we use the following multi-estimator, whose state is $x_{\mathbb{E}} = (x_1^T, \dots, x_m^T)^T$ and whose dynamics are

$$\dot{x}_p = A_p x_p + B_p u + L_p(y_p - \mathrm{Q}_\nu(y)), \qquad (4)$$
$$y_p = C_p x_p, \quad p \in \mathcal{P},$$

where L_p is such that $A_p + L_p C_p$ is Hurwitz for every $p \in \mathcal{P}$. Let $\tilde{y}_p := y_p - \mathrm{Q}_\nu(y)$ be the output estimation errors with respect to the quantized output, $\bar{y}_e := y - \mathrm{Q}_\nu(y)$ be the actual output quantization error, and $\tilde{x}_p := x_p - x$ be the state estimation errors. The multi-estimator has the following property: there is $\hat{p} \in \mathcal{P}$ such that

$$|\tilde{x}_{\hat{p}}(t)| \leqslant c_e e^{-\lambda_e(t-t_0)}|\tilde{x}_{\hat{p}}(t_0)| + \gamma_e \int_0^t e^{-\lambda_e(t-s)}|\bar{y}_e(s)|\mathrm{d}s \qquad (5a)$$

$$|y_{\hat{p}} - y| \leqslant k_c|\tilde{x}_{\hat{p}}| \qquad (5b)$$

for all $t \geqslant t_0$, for all u, for some $c_e \geqslant 1, \{\gamma_e, \lambda_e\} > 0$, and $k_c := \max_{p\in\mathcal{P}} |C_p|$. This property is known as the matching property in supervisory control. The matching property (5a) is satisfied with $\hat{p} = p^*$ because $y_{p^*} - y = C_{p^*}(x_{p^*} - x)$ and because the dynamics of $\tilde{x}_{p^*}$ are $(\mathrm{d}/\mathrm{d}t)\tilde{x}_{p^*} = (A_{p^*} + L_{p^*}C_{p^*})\tilde{x}_{p^*} + L_{p^*}\bar{y}_e$, where $A_{p^*} + L_{p^*}C_{p^*}$ is Hurwitz by design. We do not know $\hat{p}$ but λ_e, c_e, and γ_e can be calculated as $\lambda_e := \min_{p\in\mathcal{P}} \lambda_p$, $c_e := \max_{p\in\mathcal{P}} c_p$ and $g_e := \max_{p\in\mathcal{P}} |L_p|c_e$, where c_p and λ_p are such that $|e^{(A_p+L_pC_p)t}| \leqslant c_p e^{-\lambda_p t}$ for all $t \geqslant 0$.
- *The multi-controller*: A family of *candidate feedback gains* $\{K_p\}$ is designed such that $A_p + B_p K_p$ is Hurwitz for every $p \in \mathcal{P}$. Then the *family of candidate controllers* is

$$u_p = K_p x_p \quad p \in \mathcal{P}. \qquad (6)$$

- *The monitoring signals*: The *monitoring signals* $\mu_p, p \in \mathcal{P}$ are the exponentially weighted norm of the output estimation errors plus an offset ε:

$$\mu_p = \varepsilon + \int_0^t e^{-\lambda(t-\tau)}\gamma|\tilde{y}_p(\tau)|\mathrm{d}\tau \qquad (7)$$

for some design parameters $\{\gamma, \varepsilon, \lambda\} > 0$. The monitoring signals can be implemented as the outputs of the filter $\gamma/(s+\lambda)$ with the inputs $|\tilde{y}_p|$ plus the offset ε.
- *The switching logic*: A switching logic produces a switching signal that indicates at every time the active controller. In this paper, we use the *scale-independent hysteresis switching logic* [16]:

$$\sigma(t) := \begin{cases} \underset{q\in\mathcal{P}}{\operatorname{argmin}}\, \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else,} \end{cases} \qquad (8)$$

where $h > 0$ is a *hysteresis constant* and h is a design parameter.

Overall, the control signal applied to the plant is

$$u(t) = u_\sigma(t) := K_{\sigma(t)} x_{\sigma(t)}(t),$$

where σ is generated by (8).

3.1. Design parameters

The design parameters $\gamma, \varepsilon, \lambda$, and h are chosen in the following way. First, one constructs the so-called injected systems by combining a fixed candidate controller with the multi-estimator, where the injected system with index q is

$$\begin{cases} \dot{x}_q &= (A_q + B_q K_q)x_q + L_q \tilde{y}_q, \\ \dot{x}_p &= (A_p + L_p C_p)x_p + B_p K_p x_q - L_p C_q x_q + L_p \tilde{y}_q, \quad p \neq q. \end{cases} \tag{9}$$

The injected system captures the switching mechanism of the closed-loop system (where the variable q will be switching among the index set $\mathcal{P}$) and plays a pivotal role in the analysis later (see, e.g., [12] for more details on inject systems in switched system analysis). The foregoing dynamics take the form

$$\dot{x}_{\mathbb{E}} = \mathbf{A}_q x_{\mathbb{E}} + \mathbf{B}_q \tilde{y}_q, \tag{10}$$

where the definitions of $\mathbf{A}_q$ and $\mathbf{B}_q$ are obvious from (9) (see Appendix A for the explicit formula of $\mathbf{A}_q$ and $\mathbf{B}_q$). It is clear from (9) that if $\tilde{y}_q = 0$, then $x_q \to 0$ by and then all $x_p \to 0$ for all p, which means that $\mathbf{A}_q$ is Hurwitz (because the system is linear). Because $\mathbf{A}_p$ is Hurwitz for all p, there exists a family of quadratic Lyapunov functions $V_p(x_{\mathbb{E}}) = x_{\mathbb{E}}^T P_p x_{\mathbb{E}}$, $P_p^T = P_p > 0$, $x_{\mathbb{E}} = (x_1^T, \dots, x_p^T)^T$, $p \in \mathcal{P}$, such that

$$\underline{a}|x_{\mathbb{E}}|^2 \leqslant V_p(x_{\mathbb{E}}) \leqslant \overline{a}|x_{\mathbb{E}}|^2 \tag{11a}$$

$$\frac{\partial V_p(x_{\mathbb{E}})}{\partial x}(\mathbf{A}_p x_{\mathbb{E}} + \mathbf{B}_p \tilde{y}_p) \leqslant -2\lambda_0 V_p(x_{\mathbb{E}}) + \gamma_0 |\tilde{y}_p|^2 \tag{11b}$$

for some constants $\{\underline{a}, \overline{a}, \lambda_0, \gamma_0\} > 0$ (the existence of such common constants for the family of Lyapunov functions is guaranteed because $\mathcal{P}$ is finite). There exists a number $\mu_V \geqslant 1$ such that

$$V_q(x) \leqslant \mu_V V_p(x) \quad \forall x \in \mathbb{R}^n, \forall p, q \in \mathcal{P}. \tag{12}$$

We can always pick $\mu_V = \overline{a}/\underline{a}$ but there may be other smaller μ_V satisfying (12) (for example, $\mu_V = 1$ if V_p is the same for all p even though $\overline{a}/\underline{a} > 1$).

Now, the design parameters $\gamma, \varepsilon, \lambda$, and h are chosen such that

$$0 < \lambda < \lambda_0, \tag{13}$$

$$\frac{\ln(1+h)}{\lambda m} \geqslant \frac{\ln \mu_V}{2(\lambda_0 - \lambda)}, \tag{14}$$

where λ_0 is as in (11b) and μ_V is as in (12).

4. STABILITY OF SUPERVISORY CONTROL WITH QUANTIZATION

4.1. Static quantization

The following theorem deals with the case where the quantizer is static (i.e., the zooming variable ν in Q_ν is a constant).

Theorem 1

Consider the uncertain system (1) with static output quantization, where the quantizer is as in (2) with a fixed ν, and with the supervisory control scheme described in Section 3. Suppose that the design parameters satisfy (13) and (14). Let $\varepsilon = \kappa_e \nu \Delta$ for some $\kappa_e > 0$. Suppose that at the initial time t_0, $|x_{\mathbb{E}}(t_0)| \leqslant \kappa_0 \nu \Delta$, $|\tilde{x}_p(t_0)| \leqslant \kappa_1 \nu \Delta$ for all $p \in \mathcal{P}$, and $|\mu_{\hat{p}}(t_0)| \leqslant \kappa_2 \nu \Delta$ for some constants $\{\kappa_0, \kappa_1\} > 0$ and $\kappa_2 \geqslant \kappa_e$. There exist constants $\kappa > k_c(\kappa_0 + \kappa_1)$ and $\kappa_x > \gamma_e/\lambda_e$ such that if

$$\kappa \Delta < M, \tag{15}$$

then all the closed-loop signals are bounded, and for every $\epsilon_x > 0$, $\exists T < \infty$ such that

$$|x(t)| \leqslant \kappa_x \nu \Delta + \epsilon_x \nu \Delta \quad \forall t \geqslant t_0 + T. \tag{16}$$

Proof

Let $T_{\max} := \sup\{t \in [t_0, \infty) : |y(t)| \leqslant \nu M\}$. At time t_0, in view of $y = y - y_p + y_p$ and of the bounds on $|x_p(t_0)|$ and $|\tilde{x}_p(t_0)|$, we have $|y(t_0)| \leqslant k_c \kappa_1 \nu \Delta + k_c \kappa_0 \nu \Delta < \kappa \nu \Delta < \nu M$ and so, $T_{\max} > t_0$.

Boundedness of a monitoring signal

From the definition of μ_p in (7), for arbitrary $t_0 \geqslant 0$, we have

$$\mu_p(t) = (1 - e^{-\lambda(t-t_0)})\varepsilon + e^{-\lambda(t-t_0)}\mu_p(t_0) + \gamma \int_{t_0}^{t} e^{-\lambda(t-s)} |\tilde{y}_p(s)| \mathrm{d}s \quad \forall p \in \mathcal{P}, t \geqslant t_0. \tag{17}$$

From the property (3) of a quantizer, we have

$$|\bar{y}_e(s)| = |y(s) - \mathrm{Q}_\nu(y(s))| \leqslant \nu \Delta \quad \forall s \in [t_0, T_{\max}). \tag{18}$$

Because $\tilde{y}_{\hat{p}} = y_{\hat{p}} - y + \bar{y}_e$, from (5) and (18), we obtain

$$|\tilde{y}_{\hat{p}}(s)| \leqslant k_c c_e e^{-\lambda_e(s-t_0)} |\tilde{x}_{\hat{p}}(t_0)| + (1 + k_c \gamma_e/\lambda_e)\nu \Delta \quad \forall s \in [t_0, T_{\max}). \tag{19}$$

From (17) and (19), we obtain that $\forall t \in [t_0, T_{\max})$,

$$\mu_{\hat{p}}(t) \leqslant \varepsilon + e^{-\lambda(t-t_0)}(\mu_{\hat{p}}(t_0) - \varepsilon) + \gamma k_c c_e f_e(t - t_0)|\tilde{x}_{\hat{p}}(t_0)| + a_3 \nu \Delta =: \bar{\mu}_{\hat{p}}(t - t_0), \tag{20}$$

where $a_3 := (1 + k_c \gamma_e/\lambda_e)\gamma/\lambda$, and

$$f_e(t - t_0) := \int_{t_0}^{t} e^{-\lambda(t-s)} e^{-\lambda_e(s-t_0)} \mathrm{d}s = \begin{cases} (e^{-\lambda_e(t-t_0)} - e^{-\lambda(t-t_0)})/(\lambda - \lambda_e) & \text{if } \lambda \neq 2\lambda_e \\ e^{-\lambda(t-t_0)}(t - t_0) & \text{if } \lambda = 2\lambda_e. \end{cases} \tag{21}$$

The function f_e is decreasing, and $f_e(0) = 1$ and $\lim_{t\to\infty} f_e(t) = 0$. For notational convenience, denote by $\mathcal{L}[b, a] : [0, \infty) \to [a, b]$ the class of functions such that if $f \in \mathcal{L}[a, b]$, then $f(0) = a$ and $\lim_{t\to\infty} f(t) = b$. Using this notation, $f_e \in \mathcal{L}[1, 0]$. It follows that the upper bound on the monitoring signal, the signal $\bar{\mu}_{\hat{p}}$, with the nominal index $\hat{p}$ satisfies $\bar{\mu}_{\hat{p}} \in \mathcal{L}[(\kappa_2 + \gamma k_c c_e \kappa_1 + a_3)\nu \Delta, (\kappa_e + a_3)\nu \Delta]$, in view of $\mu_{\hat{p}}(t_0) \leqslant \kappa_2 \nu \Delta$ and $|\tilde{x}_{\hat{p}}(t_0)| \leqslant \kappa_1 \nu \Delta$.

The switched injected system

(1) *The switching signal* σ: The hysteresis switching lemma [10, Lemma 1] (see also [17, Lemma 4.2]) with the scaled signals $\bar{\mu}_p(t) = e^{\lambda t}\mu_p(t)$, gives

$$N_\sigma(t, t_0) \leqslant N_0 + \frac{t - t_0}{\tau_a}, \tag{22}$$

where $N_\sigma(t, t_0)$ is the number of switches in $(t_0, t]$, and

$$N_0 := m + \frac{m}{\ln(1 + h)} \ln(\sup_{t \geqslant 0} \mu_{\hat{p}}(t)/\varepsilon) \tag{23a}$$

$$\tau_a := \ln(1 + h)/(m\lambda). \tag{23b}$$

From (20) and (23a), in view of $\bar{\mu}_{\hat{p}} \in \mathcal{L}[(\kappa_2 + \gamma k_c c_e \kappa_1 + a_3)\nu\Delta, (\kappa_e + a_3)\nu\Delta]$ and the fact that $\varepsilon = \kappa_e \nu \Delta$, we obtain

$$N_0 := m + \frac{m}{\ln(1+h)} \ln((\kappa_2 + \gamma k_c c_e \kappa_1 + a_3)/\kappa_e).$$

Because N_0 is bounded, the switching signal σ on the interval $(t_0, T_{\max})$ is an average dwell-time switching signal with the average dwell time τ_a.

(2) *The exponentially weighted integral norm of* $\tilde{y}_\sigma$: Also, from the hysteresis switching lemma, we have

$$\sum_{k=0}^{N_\sigma(t,t_0)} \bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k) \leqslant m((1+h)\bar{\mu}_\ell(t) - \min_{p\in\mathcal{P}} \bar{\mu}_p(t_0)), \tag{24}$$

where t_k is the switching times in (t_0, t). From (17), we obtain $\bar{\mu}_p(t) = e^{\lambda t}\varepsilon - e^{\lambda t_0}\varepsilon + e^{\lambda t_0}\mu_p(t_0) + \int_{t_0}^t e^{\lambda s}\gamma|\tilde{y}_p(s)|\mathrm{d}s$. We then have

$$\begin{aligned}\sum_{k=0}^{N_\sigma(t,t_0)} \bar{\mu}_{\sigma(t_k)}(t_{k+1}) - \bar{\mu}_{\sigma(t_k)}(t_k) &\geqslant \sum_{k=0}^{N_\sigma(t,t_0)} \int_{t_0}^{t_{k+1}} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|\mathrm{d}s - \int_{t_0}^{t_k} e^{\lambda s}\gamma|\tilde{y}_{\sigma(t_k)}(s)|\mathrm{d}s \\ &= \gamma\int_{t_0}^{t} e^{\lambda s}|\tilde{y}_{\sigma(s)}(s)|\mathrm{d}s.\end{aligned}$$

Dividing both sides of the foregoing inequality by $\gamma e^{\lambda t}$ and then combining with (24), we obtain the following inequality for the exponentially weighted integral norm of $\tilde{y}_\sigma$:

$$\int_{t_0}^{t} e^{-\lambda(t-\tau)}|\tilde{y}_\sigma(\tau)|\mathrm{d}\tau \leqslant \frac{m(1+h)}{\gamma}\mu_q(t) \quad \forall q \in \mathcal{P}. \tag{25}$$

Because the subsystems of the switched injected system are stable and because the condition and (14) hold (the switching is slow enough), we have that ([10, Corollary 4]; see also [17, Theorem 3.1])

$$|x_{\mathbb{E}}(t)|^2 \leqslant (\overline{a}/\underline{a})\mu_V^{1+N_0}|x_{\mathbb{E}}(t_0)|^2 e^{-2\lambda(t-t_0)} + \frac{1}{\underline{a}}\mu_V^{1+N_0}\gamma_0\int_{t_0}^{t} e^{-2\lambda(t-\tau)}|\tilde{y}_\sigma(\tau)|^2\mathrm{d}\tau \;\forall t \in [t_0, T_{\max}), \tag{26}$$

because the design parameters satisfy (13) and (14) and the switching signal has the average dwell time τ_a as in (23b). Using the Cauchy–Schwartz inequality $\int f^2(s)\mathrm{d}s \leqslant \left(\int f(s)\mathrm{d}s\right)^2$, we obtain from (20) and (26):

$$|x_{\mathbb{E}}(t)|^2 \leqslant c_1|x_{\mathbb{E}}(t_0)|^2 e^{-2\lambda(t-t_0)} + c_2\bar{\mu}_{\hat{p}}^2(t-t_0) \quad \forall t \in [t_0, T_{\max}), \tag{27}$$

where $c_1 := \mu_V^{1+N_0}\overline{a}/\underline{a}$, $c_2 := \mu_V^{1+N_0}(m(1+h)/\gamma)^2(\gamma_0/\underline{a})$, and $\bar{\mu}_{\hat{p}}$ is as in (20).

The condition on M and Δ

From (27), in view of $|x_{\mathbb{E}}(t_0)| \leqslant \kappa_0\nu\Delta$ and $\bar{\mu}_{\hat{p}} \in \mathcal{L}[(\kappa_2 + \gamma k_c c_e \kappa_1 + a_3)\nu\Delta, (\kappa_e + a_3)\nu\Delta]$, using the fact that $a^2 + b^2 \leqslant (a+b)^2$ for all $\{a, b\} \geqslant 0$, we have

$$|x_{\mathbb{E}}(t)|^2 \leqslant (\sqrt{c}_1\kappa_0 + \sqrt{c}_2(\kappa_2 + \gamma k_c c_e \kappa_1 + a_3))^2\nu^2\Delta^2 =: \bar{x}^2 \quad \forall t \in [t_0, T_{\max}). \tag{28}$$

Because $y = y_{\hat{p}} + (y - y_{\hat{p}})$, in view of $|y_{\hat{p}}| \leqslant \|C_{p^*}\||x_{\mathbb{E}}(t)| \leqslant k_c\bar{x}$, from (5) and (28), we have

$$|y(t)| \leqslant k_c\bar{x} + k_c c_e\kappa_1\nu\Delta + k_c\gamma_e/\lambda_e\nu\Delta =: \kappa\nu\Delta, \tag{29}$$

where $\kappa := (k_c\sqrt{c_1}\kappa_0 + k_c\sqrt{c_2}(\kappa_2 + \gamma k_c c_e\kappa_1 + a_3) + k_c c_e\kappa_1 + k_c\gamma_e/\lambda_e$. Clearly, $\kappa > k_c(\kappa_1 + \kappa_0)$ because $c_e \geqslant 1$ and $c_1 \geqslant 1$. Because $\nu M > \kappa\nu\Delta$ by the hypothesis of the theorem, from (29) and the definition of $T_{\max}$, we must have $T_{\max} = \infty$.

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Ultimate boundedness of the plant state

Let $\epsilon_x > 0$ be arbitrary. From (27), we have that

$$|x_{\mathbb{E}}(t)| \leqslant \sqrt{c_2}(\kappa_e + a_3)\nu\Delta + (\epsilon_x/2)\nu\Delta \quad \forall t \geqslant t_0 + T_1, \tag{30}$$

where T_1 is such that

$$\sqrt{c}_1\kappa_0 e^{-\lambda T_1} + \sqrt{c}_2 e^{-\lambda T_1}\kappa_2 + \sqrt{c}_2\gamma c_e f_e(T_1)\bar{y}_0\kappa_1 \leqslant \epsilon_x/2. \tag{31}$$

Such T_1 always exists because the left-hand side of the foregoing inequality goes to 0 as $T_1 \to \infty$. Let $T_2 > 0$ be such that

$$c_e\kappa_1 e^{-\lambda_e T_2} \leqslant \epsilon_x/2. \tag{32}$$

Then $|\tilde{x}_p(t)| \leqslant (\gamma_e/\lambda_e)\nu\Delta + (\epsilon_x/2)\nu\Delta$ for all $t \geqslant T_2$. Let $T := \max\{T_1, T_2\}$. In view of $x = \tilde{x}_{\hat{p}} + x_{\hat{p}}$, it follows that

$$|x(t)| \leqslant \sqrt{c_2}(\kappa_e + a_3)\nu\Delta + (\gamma_e/\lambda_e)\nu\Delta + \epsilon_x\nu\Delta =: \kappa_x\nu\Delta + \epsilon_x\nu\Delta \quad \forall t \geqslant t_0 + T. \tag{33}$$

□

Remark 2

The condition (15) on M and Δ implies a lower bound on the number of quantization bits. Suppose that each component of x has the same range M and is equally quantized into $2^{n_{\mathcal{Q}}}$ regions using $n_{\mathcal{Q}}$ quantization bits. Then in $\mathbb{R}^n$, $\Delta = \sqrt{n}M/2^{n_{\mathcal{Q}}+1}$. Then the condition (15) implies that $n_{\mathcal{Q}} \geqslant \log_2\lceil\sqrt{n}\kappa\rceil - 1$, which is the lower bound on the number of bits needed.

4.2. Dynamic quantization

If one has state contraction at time T, then asymptotic stability can be achieved by using a dynamic quantizer, varying the zooming variable ν as well as the parameter ε in the supervisory control scheme as $x_{\mathbb{E}}$ becomes closer to the origin. Unlike the case of known plants [14] where one only needs to worry about the contraction of the plant state x, here one needs to take into account the asymptotic behavior of other state variables coming from the supervisory control scheme such as μ_p and $\tilde{y}_p$. The following result says that using a dynamic quantizer together with a synchronously scaling ε, we can achieve closed-loop asymptotic stability.

Theorem 2

Consider the uncertain system (1) with static output quantization, where the quantizer is as in (2), and with the supervisory control scheme described in Section 3. Suppose that the design parameters satisfy (13) and (14). Let $\varepsilon(t) = \kappa_e\nu(t)\Delta$ for some $\kappa_e > 0$ and suppose that ν be a periodic scaling signal with period T_s and the scaling factor ρ such that

$$\nu(t) := \begin{cases} \nu(kT_s) & \text{if } t \in [kT_s, (k+1)T_s) \\ \rho\nu(kT_s) & \text{if } t = (k+1)T_s, \end{cases} \qquad k = 0, 1, \ldots. \tag{34}$$

Suppose that at the initial time t_0, $|x_{\mathbb{E}}(t_0)| < \kappa_0\nu(t_0)\Delta$, $|\tilde{x}_p(t_0)| \leqslant \kappa_1\nu(t_0)\Delta$ for all $p \in \mathcal{P}$ and $|\mu_{\hat{p}}(t_0)| \leqslant \kappa_2\nu(t_0)\Delta$ for some constants $\{\kappa_1, \kappa_2\} > 0$. Let the constants $\{\kappa, \kappa_x\} > 0$ be as in Theorem 1 and suppose that (15) holds. If further,

$$\kappa_x - \gamma_e/\lambda_e < \kappa_0, \tag{35a}$$

$$\gamma_e/\lambda_e < \kappa_1, \tag{35b}$$

$$\kappa_e + (1 + k_c\gamma_e/\lambda_e)(\gamma/\lambda) < \kappa_2, \tag{35c}$$

then one can find the scaling factor $\rho \in (0, 1)$ and the period $T_s < \infty$ for the zooming variable ν such that the plant state $|x(t)| \to 0$ as $t \to \infty$, and that all the closed-loop signals are bounded.

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Proof
Pick $\rho \in (0, 1)$ such that

$$\rho > \left\{ \frac{\kappa_x - \gamma_e/\lambda_e + \epsilon_s}{\kappa_0}, \frac{\gamma_e/\lambda_e + \epsilon_x}{\kappa_1}, \frac{\kappa_e + (1 + k_c\gamma_e/\lambda_e)(\gamma/\lambda) + \epsilon_s}{\kappa_2} \right\}$$

for some $\epsilon_s > 0$. The existence of such a constant ρ is guaranteed by the condition (35).

Let T_1 be as in (31) where $\epsilon_x = 2\epsilon_s$. Let T_2 and T_3 be such that

$$c_e\kappa_1 e^{-\lambda_e T_2} \leqslant \epsilon_s \tag{36}$$

$$e^{-\lambda T_3'}\kappa_2 + \gamma k_c c_e f_e(T_3')\kappa_1 \leqslant \epsilon_s \tag{37}$$

Let $T_s := \max\{T_1, T_s, T_3\}$. In $[t_0, T_s)$, ν is constant and so,

$$|x_{\mathbb{E}}(t)| \leqslant (\sqrt{c_2}(\kappa_e + a_3) + \epsilon_s)\nu(t_0)\Delta = (\kappa_x - \gamma_e/\lambda_e + \epsilon_s)\nu\Delta < \kappa_0\rho\nu(t_0)\Delta \tag{38}$$

$$|\tilde{x}_{\hat{p}}| \leqslant (\gamma_e/\lambda_e + \epsilon_s)\nu\Delta < \kappa_1\rho\nu(t_0)\Delta \tag{39}$$

$$\mu_{\hat{p}}(t) \leqslant (\kappa_e + a_3 + \epsilon_s)\nu\Delta < \kappa_2\rho\nu(t_0)\Delta \tag{40}$$

for all $t \geqslant t_0 + T$. Therefore, if we reduce ν by a factor of ρ at the time $t_0 + T_s$, $\nu(t_0 + T_s) = \rho\nu(t_0)$, then (38) implies $|x_{\mathbb{E}}(t)| \leqslant \kappa_1\nu(t_0 + T_s)\Delta$ for all $t \geqslant t_0 + T_s$. Similarly, $\tilde{x}_{\hat{p}}(t)| \leqslant \kappa_1\nu(t_0 + T_s)\Delta$ and $\mu_{\hat{p}}(t) \leqslant \kappa_2\nu(t_0 + T_s)\Delta$ for all $t \geqslant t_0 + T_s$. Thus, at the time $t_0 + T_s$, we can repeat the same analysis as the analysis starting at t_0 and so on. It follows that we have

$$\begin{aligned} |x_{\mathbb{E}}(t)| &\leqslant \kappa_0\nu(t_0 + kT_s)\Delta \\ |\tilde{x}_{\hat{p}}| &\leqslant \kappa_1\nu(t_0 + kT_s)\Delta \quad \forall t \geqslant t_0 + kT_s. \\ \tilde{\mu}_{\hat{p}} &\leqslant \kappa_2\nu(t_0 + kT_s)\Delta \end{aligned}$$

Because $\nu(t_0 + kT_s) = \rho^k\nu(t_0) \to 0$ as $k \to \infty$, we have that $|x_{\mathbb{E}}(t)| \to 0$ as $t \to \infty$. In particular, $x_{\hat{p}} \to 0$. Similarly, $\tilde{x}_{\hat{p}} \to 0$. Because $x = -x_{\hat{p}} - \tilde{x}_{\hat{p}}$, we have that $|x| \to 0$ (and hence, $y \to 0$). Also, $y_p \to 0$ because $x_{\mathbb{E}} \to 0$. It follows that μ_p is bounded for all $p \in \mathcal{P}$. □

Remark 3
For a fixed κ_e, the condition (35) can always be satisfied for large enough κ_0, κ_1, and κ_2.

Remark 4
If the initial state $x(0)$ is such that $|x_{\mathbb{E}}(0)| > \kappa_1\nu(0)\Delta$, where $\nu(0)$ is the initial zooming value, then we can include a zooming-out stage at the beginning (see [14]) so that after a certain time t_0, we guarantee $|x(t_0)| < \kappa_1\nu(t_0)M$. This means increasing ν faster than the system can blow up (for any value of $p \in \mathcal{P}$ until the quantizer no longer saturates. For example, let λ_u be the largest real part of the unstable eigenvalues of the open loop (we can obtain this for all the parameters over the finite set $\mathcal{P}$), then the state is bounded by $|x_{\mathbb{E}}(t) \leqslant c_0|x_{\mathbb{E}}(0)|e^{\lambda_u t}$. Then if ν grows at the exponential rate $2\lambda_u$, that is, ν is double every period $T = \ln 2/(2\lambda_u)$, then there is t_0 such that $|x(t_0)| < \kappa_1\nu(t_0)M$.

Remark 5
We cannot guarantee that the switching signal stops on the true parameter p^* or stops at all, but the proof shows that the overall system achieves asymptotic stability as desired.

5. ROBUSTNESS

Suppose that there are additive unmodeled dynamics so the actual plant is

$$\begin{aligned} \dot{x} &= (A_{p^*} + \Delta_A)x + (B_{p^*} + \Delta_B)u \\ y &= (C_{p^*} + \Delta_C)x \end{aligned} \tag{41}$$

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where Δ_A, Δ_B, and Δ_C are matrices of appropriate dimensions. If Δ_A, Δ_B, and Δ_C are small enough in the sense of induced norm, then we still have the stability and asymptotic stability results in the previous section. For a matrix A, denote by $\|A\|$ the induced matrix norm of A.

Theorem 3

Consider the uncertain system (41) with static output quantization, where the quantizer is as in (2) with a fixed ν, and with the supervisory control scheme described in Section 3. Suppose that the design parameters satisfy (13) and (14). Let $\varepsilon = \kappa_e \nu(\Delta + \delta)$ for some $\{\kappa_e, \delta\} > 0$. Suppose that at the initial time t_0, $|x_{\mathbb{E}}(t_0)| \leqslant \kappa_0 \nu(\Delta + \delta_0)$, $|\tilde{x}_p(t_0)| \leqslant \kappa_1 \nu(\Delta + \delta_0)$ for all $p \in \mathcal{P}$, and $|\mu_{\hat{p}}(t_0)| \leqslant \kappa_2 \nu(\Delta + \delta_0)$ for some constants $\{\kappa_0, \kappa_1, \kappa_2\} > 0$. There exist constants $\kappa > k_c(\kappa_0 + \kappa_1)$ and $\kappa_x > \gamma_e/\lambda_e$, and $\delta > 0$ such that if

$$\kappa(\Delta + \delta_0) < M, \tag{42}$$

$$\max\{\|\Delta_A\|, \|\Delta_B\|, \|\Delta_C\|\} < \delta, \tag{43}$$

then all the closed-loop signals are bounded, and for every $\epsilon_x > 0$, $\exists T < \infty$ such that

$$|x(t)| \leqslant \kappa_x \nu \Delta + \epsilon_x \nu \Delta \quad \forall t \geqslant t_0 + T. \tag{44}$$

Proof

For the estimator with index p^*, the error dynamics of $\tilde{x}_{p^*}$ are

$$\dot{\tilde{x}}_{p^*} = (A_{p^*} + L_{p^*} C_{p^*})\tilde{x}_{p^*} - (\Delta_A + L_{p^*}\Delta_C)x - \Delta_B u + L_{p^*}(y - \mathrm{Q}_\nu(y)). \tag{45}$$

Because $A_{p^*} + L_{p^*} C_{p^*}$ is Hurwitz, there exists δ_1 such that if $\{\|\Delta_A\|, \Delta_C\|\} < \delta_1$, then $\bar{A} = A_{p^*} + L_{p^*} C_{p^*} + \Delta_A + L_{p^*}\Delta_C$ is still Hurwitz. Note that $x = -x_{p^*} - \tilde{x}_{p^*}$ and $u = K_{\sigma(t)} x_{\sigma(t)}$, and so, (45) implies that

$$|\dot{\tilde{x}}_{p^*}(t)| \leqslant c_e e^{-\lambda_e(t-t_0)}|\tilde{x}_{p^*}(t_0)| + \gamma_e \int_{t_0}^{t} e^{-\lambda_e(t-s)}(|\bar{y}_e(s)| + \delta_B |x_{\mathbb{E}}(s)|)\mathrm{d}s, \tag{46}$$

where $\delta_B \to 0$ as $\|\Delta B\| \to 0$ (recall that $\bar{y}_e = y - \mathrm{Q}_\nu(y)$). Also,

$$|y - y_{p^*}| \leqslant \bar{k}_c |\tilde{x}_{p^*}| + \delta_C |x_{\mathbb{E}}|, \tag{47}$$

where $\bar{k}_c = k_c + \|\Delta_C\|$, and $\delta_C \to 0$ as $\|\Delta_C\| \to 0$.

Let $\bar{\Delta} = \Delta + \delta_0$. Let $T_{\max} := \sup_{t \geqslant t_0}\{\delta_B |x_{\mathbb{E}}(t)| \leqslant \nu\delta_0, |y(t)| \leqslant \nu M, \delta_C |x_{\mathbb{E}}| < \nu\delta_0\}$. Because $|x_{\mathbb{E}}(t_0)| < \kappa_1 \nu(\Delta + \delta_0)$, if $\{\delta_C, \delta_B\} < \delta_0/(\kappa_1(\Delta + \delta_0)) =: \delta_2$, then $T_{\max} > t_0$. Then for all $t \in [t_0, T_{\max})$, we have

$$|\bar{y}_e(t)| + \delta_B |x_{\mathbb{E}}(t)| \leqslant \nu\Delta + \nu\delta_0 = \nu\bar{\Delta}. \tag{48}$$

From (46)–(48), in view of $\tilde{y}_{p^*} = y_{p^*} - y + \bar{y}_e$, we have

$$\begin{aligned} |\tilde{y}_{p^*}(t)| &\leqslant \bar{k}_c c_e e^{-\lambda_e(t-t_0)} + \bar{k}_c(\gamma_e/\lambda_e)\nu\bar{\Delta} + \delta_C|x_{\mathbb{E}}| + \nu\Delta \\ &\leqslant \bar{k}_c c_e e^{-\lambda_e(t-t_0)} + (1 + \bar{k}_c(\gamma_e/\lambda_e))\nu\bar{\Delta} \end{aligned} \tag{49}$$

for all $t \in [t_0, T_{\max}$ in view of $\delta_C|x_{\mathbb{E}}| \leqslant \nu\delta_0$. Repeat the analysis in the proof of Theorem 1 from (19) but replace Δ in that proof by $\bar{\Delta}$ everywhere. All the other constants in the proof of Theorem 1 do not change. We then have that $|y(t)| \leqslant \kappa\bar{\Delta}$ and $|x_{\mathbb{E}}(t)| \leqslant (\sqrt{c_1}\kappa_0 + \sqrt{c_2}(\kappa_2 + \gamma k_c c_e \kappa_1 + \kappa_e + a_3)\nu(\Delta + \delta_0) =: \bar{x}$ for all $t \in [t_0, T_{\max})$. Because $\delta_B \to 0$ as $\|\Delta_B\| \to 0$ and $\delta_C \to 0$ as $\|\Delta_C\| \to 0$, let $\delta_2 > 0$ be the number such that if $\{\|\Delta_B\|, \|\Delta_C\|\} < \delta_2$, then $\delta_B \leqslant \nu\delta_0/\bar{x}$ and $\delta_C \leqslant \nu\delta_0/\bar{x}$. Let $\delta := \min\{\delta_1, \delta_2\}$. Because $\kappa\bar{\Delta} < M$ and $\max\{\|\Delta_A\|, \|\Delta_B\|, \|\Delta_C\|\} < \delta$ by the theorem's hypothesis, we have that $T_{\max} = \infty$ by the definition of $T_{\max}$. From here, proving ultimate boundedness is the same as in the proof of Theorem 1. □

The following asymptotic stability result for the case of unmodeled dynamics is straightforward from Theorems 2 and 4; the proof is exactly the same as the proof of Theorem 2.

Theorem 4
Consider the uncertain system (41) with static output quantization, where the quantizer is as in (2) with a fixed ν, and with the supervisory control scheme described in Section 3. Suppose that the design parameters satisfy (13) and (14). Let $\varepsilon(t) = \kappa_e \nu(t)(\Delta + \delta_0)$ for some $\{\delta_0, \kappa_e\} > 0$, and suppose that ν be a periodic scaling signal with period T_s and the scaling factor ρ as in (34). Suppose that at the initial time t_0, $|x_{\mathbb{E}}(t_0)| < \kappa_0 \nu(t_0)(\Delta + \delta_0)$, $|\tilde{x}_p(t_0)| \leqslant \kappa_1 \nu(t_0)(\Delta + \delta_0)$ for all $p \in \mathcal{P}$, and $|\mu_{\hat{p}}(t_0)| \leqslant \kappa_2 \nu(t_0)(\Delta + \delta_0)$ for some constants $\{\kappa_1, \kappa_2\} > 0$. Let the constants $\{\kappa, \kappa_x, \delta\} > 0$ be as in Theorem 4, and suppose that (42), (43), and (35) hold true, then one can find the scaling factor $\rho \in (0, 1)$ and the period $T_s < \infty$ for the zooming variable ν such that the plant state $|x(t)| \to 0$ as $t \to \infty$, and that all the closed-loop signals are bounded.

6. CONTINUUM UNCERTAINTY SET

So far, we have assumed that the set $\mathcal{P}$ is finite. Now, assume that the uncertain parameter p^* belongs to an unknown continuum uncertainty set by $\Omega \subseteq \mathbb{R}^{n_p}$. By continuum set, we mean that for any $p \in \Omega$, there exists an open ball centered at p that lies completely inside Ω (note that the set Ω can be disconnected, containing two or more disjointed continuum sets).

We divide Ω into a finite number of subsets Ω_i such that $\bigcup_{i \in \mathcal{P}} \Omega_i = \Omega$, $i \in \{1, \ldots, m\} =: \mathcal{Q}$. The existence of Ω_i and a finite set $\mathcal{Q}$ is guaranteed if, for example, Ω is compact. How to divide Ω into Ω_i and what is the number of subsets are interesting research questions of their own and are not pursued here (see [18]). Intuitively, we want Ω_i small in some sense (such as each Ω can be enclosed in a ball of small radius). For every subset Ω_i, $i \in \mathcal{Q}$, pick a nominal value $p_i \in \Omega_i$. By this procedure, we obtain a finite family of nominal plants parameterized by $p \in \mathcal{P}$, $\{P(p), p \in \mathcal{P}\}$, where $\mathcal{P} = \{p_i, i \in \mathcal{Q}\}$. The difference between the case with a continuum uncertain set Ω and the case with a finite uncertainty set $\mathcal{P}$ in Section 2 is that we may not have exact matching, that is, it could be that $p^* \notin \mathcal{P}$. However, if the nominal value in the set containing p^* is close to p^* in some sense, then the quantized supervisory control scheme still works, in view of the robustness property proved in Section 5.

Assumption 2
Suppose that there exists $\hat{p} \in \mathcal{P}$ such that the dynamics of the plant $P(\hat{p})$ can be written as

$$\begin{aligned} \dot{x} &= (A_{p^*} + \Delta_A)x + (B_{p^*} + \Delta_B)u \\ y &= (C_{p^*} + \Delta_C)x \end{aligned} \tag{50}$$

and $\max\{\|\Delta_A\|, \|\Delta_B\|, \|\Delta_C\|\} < \delta$ for the same δ as in (43).

The aforementioned robustness assumption is not very restrictive in the case of linear systems: if the system matrices are continuous with respect to p, then δ can be as small as possible if $|p^* - \hat{p}|$ is small enough (due to robustness and structural stability property of LTI systems). If Assumption 2 holds, in view of Section 5, all the reasoning and the results for linear systems with finite set $\mathcal{P}$ in Section 5 hold for the continuum uncertainty set Ω without any modification; in particular, if Assumption 2 holds and the set Ω is compact, the statements of all the theorems in Section 5 remain unchanged.

7. ILLUSTRATIVE EXAMPLE

We illustrate the methodology represented in this paper in a simple example in order to shed light into our methodology and the quantized supervisory control approach. Consider the following uncertain plant

$$\dot{x} = x + bu, \quad y = \mathrm{Q}_\nu(2x) \tag{51}$$

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where $b \in \{-1, 1\} =: \mathcal{P}$. Not knowing the sign of b (the sign of high frequency gain) makes the adaptive problem challenging in general. We design the multi-estimator with $L_1 = L_2 = -4.5$,

$$\begin{aligned} \dot{x}_1 &= x_1 + u - 4.5(2x_1 - \mathrm{Q}_\nu(2x)) \\ \dot{x}_2 &= x_2 - u - 4.5(2x_2 - \mathrm{Q}_\nu(2x)) \\ y_1 &= 2x_1 \\ y_2 &= 2x_2 \end{aligned}$$

and the multi-controller

$$\begin{aligned} u_1 &= -3x_1 \\ u_2 &= 3x_2, \end{aligned}$$

which places the closed-loop pole of the individual controller at -2 and the pole of the individual estimator at -8. We calculate the system matrices of the injected systems

$$A_1 = \begin{bmatrix} -2 & 0 \\ 12 & -8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -8 & 12 \\ 0 & -2 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} -4.5 \\ -4.5 \end{bmatrix}$$

Using LMI, we find that $\lambda_0 = 0.25$, $\mu_V = 2$, $\gamma_0 = 5$, $\overline{a} = 0.1663$, and $\underline{a} = 0.0433$, where $P_1 = \begin{bmatrix} 0.1344 & -0.0539 \\ -0.0539 & 0.0753 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0.0753 & -0.0539 \\ -0.0539 & 0.1344 \end{bmatrix}$. Choose the design parameters $h = 1.1$, $\lambda = 0.125$ that satisfy the conditions (13) and (14). Choose $\gamma = 0.01$. The numerical values of the bounds critically depend on these design parameters (see Appendix B for a discussion on how to select the design parameters).

We consider the floor function as the quantizer Q:

$$\mathrm{Q}(y) := \begin{cases} \nu \left\lfloor \dfrac{y}{\nu} \right\rfloor & \text{if } |y| \leqslant M \\ M & \text{else,} \end{cases} \tag{52}$$

which has the quantization error bound $\Delta = 1$.

Static quantizer

Let $\kappa_0 = 0.1, \kappa_1 = 2, \kappa_2 = \kappa_e = 0.1$. Using the formula in the proof of Theorem 1, we calculate that $\kappa = 22.77 \times 10^3$ and $\kappa_x = 9.91 \times 10^3$. For $\nu = 1$, the calculated upper bound on x is $\bar{x} = 11.38 \times 10^3$ and the calculated lower bound on νM is $\kappa \nu \Delta = 22.77 \times 10^3$ versus the simulated values of 51.1 and 102.2, respectively. The actual performance of the supervisory control in this example is much better than expected. The better performance is partly due to the 'nice' quantizer (52) (we also observe that the state converges to the constant value $\nu\Delta/2$ for this quantizer and that switching stops in finite time) but mainly, our bounds are conservative because they do not assume any structure on the quantizer and the characterization of the switching signal is conservative. It is noted that conservativeness of performance of supervisory control is currently a drawback in all supervisory control work, not just our work here, and improving the analysis of supervisory control is one of the ongoing research efforts in this area.

A simulation with the static quantizer $\nu = 1$ and $M = 1000$ is plotted in Figure 3. We also ran the simulation for various ν from 0.01 to 100 and found the empirical peak output values for each ν; the result is plotted in Figure 4. An interesting observation from Figure 4 is that the output peak values are not monotonic in ν: a bigger ν, corresponding to a coarser quantization, can result in a smaller output peak value.

7.1. *Dynamic quantizer*

We simulate the supervisory control scheme with a dynamic quantizer with the re-scaling period $T = 5$ and the scaling factor $\rho = 0.6$. To illustrate the usage of dynamic quantization, we start off

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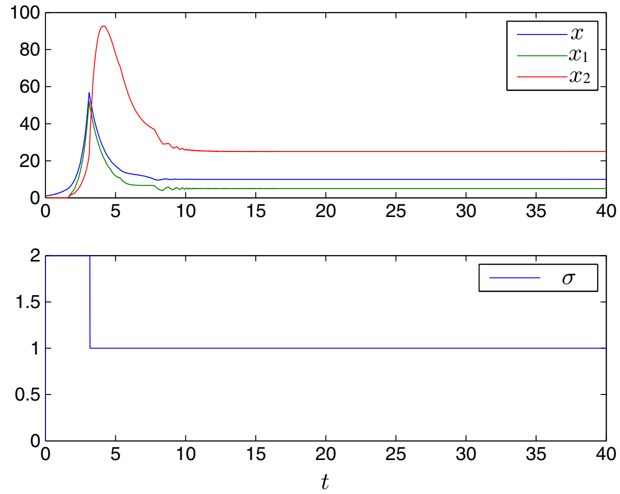

Figure 3. Simulation with the static quantizer $\nu = 1$ and $M = 1000$.

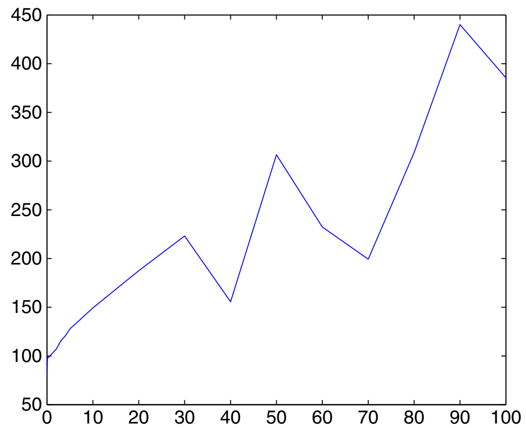

Figure 4. Output peak values $y_{\max}$ versus the quantization zoom ν.

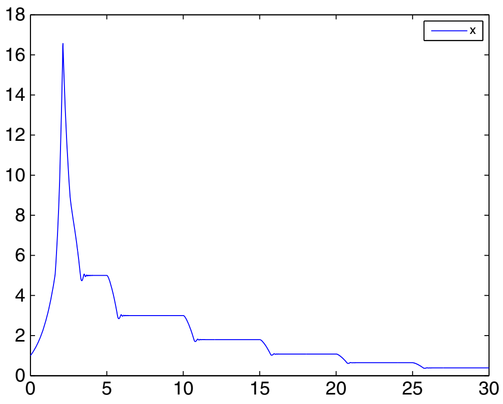

Figure 5. Dynamic quantization state x versus time t.

with large quantization error $\nu(0) = 10$ and with $\kappa_2 = \kappa_e = 0.01$, $\lambda = 0.05$, and $h = 0.289$. Simulation shows convergence to zero (Figure 5). Similarly to the static quantizer case, our calculated number in this case is also conservative compared with the simulation.

8. A REMARK ON THE NONLINEAR CASE

We discuss in this section how the results for linear plants in the previous section could be generalized to certain classes of nonlinear plants by using the ISS framework (see [19] for the background on the ISS framework). The methodology is as follows:

1. Setting up the nonlinear switching supervisory framework with quantized output.
2. Performing the analysis of the closed loop system using injected system analysis.
3. Establishing condition on the quantizer's range M and the zooming variable ν to ensure practical stability.
4. Using dynamic zooming to obtain asymptotic stability.

The nonlinear supervisory framework with quantized output:

Consider the following uncertain nonlinear plant

$$\Gamma(p^*)\begin{cases} \dot{x} = f(x,u,p^*) \\ y = h(x,p^*) \end{cases}, p^* \in \mathcal{P} \tag{53}$$

parameterized by an unknown parameter $p^* \in \mathcal{P}$, $\mathcal{P}$ is a finite set, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $y \in \mathbb{R}^\ell$ is the output. Assuming that the plant is open-loop unstable and the information available to the controller is the quantized output, $Q_\nu(y)$, as in (2), the goal is to achieve closed-loop stability and asymptotic stability (see also Figure 1 for the control scheme).

The nonlinear supervisory control structure is similar to the linear case in Section 3 with the difference that the multi-estimator can now have nonlinear forms.

- *The multi-estimator*: The nonlinear multi-estimator takes the form

$$\begin{cases} \dot{\hat{x}}_E = F(\hat{x}_E, \mathrm{Q}_\nu(y), u), \\ \hat{y}_p = H_p(\hat{x}_p), \end{cases} \qquad p \in \mathcal{P}, \tag{54}$$

$x_E = (x_1, \ldots, x_{p_m})$, with the property that there is $\hat{p} \in \mathcal{P}$ such that

$$|\tilde{x}_{\hat{p}}(t)| \leqslant \beta_e(|\tilde{x}_{\hat{p}}(t_0)|, t - t_0) + \int_{t_0}^{t} e^{-\lambda_e(t-s)}\gamma_e(|\bar{y}_e(s)|)\mathrm{d}s \tag{55a}$$

$$|y_{\hat{p}} - y| \leqslant k_c|\tilde{x}_{\hat{p}}| \tag{55b}$$

for all $t \geqslant t_0$, for all u, and for some $\lambda_s > 0$, $\beta_e \in \mathcal{KL}$, $\gamma_e \in \mathcal{K}$.

- *The multi-controller*: A family of *candidate controller* is designed such that the controller with index p stabilizes the plant with the same parameter,

$$\begin{cases} \dot{x}_C = G(x_C, Q_\nu(y), u) \\ u_p = r_p(x_C) \end{cases} \qquad p \in \mathcal{P}. \tag{56}$$

- *The monitoring signals and the switching logic*: The monitoring signals $\mu_p, p \in \mathcal{P}$ are the same as in (7), and the switching logic is the hysteresis switching logic (8).

The switched injected system is the combination of the multi-estimator and the multi-controller, as in the case of linear systems, and is

$$\begin{aligned} \dot{x}_{CE} &= \begin{bmatrix} g_p(x_{\mathbb{C}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p, r_p(x_{\mathbb{C}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p)) \\ F(x_{\mathbb{E}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p, r_p(x_{\mathbb{C}}, H_p(x_{\mathbb{E}}) - \tilde{y}_p)) \end{bmatrix} \\ &=: f_p(x_{\mathbb{CE}}, \tilde{y}_p), \end{aligned} \tag{57}$$

where $x_{\mathbb{CE}} := \begin{pmatrix} x_{\mathbb{C}} \\ x_{\mathbb{E}} \end{pmatrix}$ is the state of the injected system.

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The matching condition as in (5) for the nonlinear multi-estimator can be generalized using the class $\mathcal{K}$ and class $\mathcal{KL}$ functions[‡]. The following is an assumption on the injected systems (57) (see also Remark 6).

Assumption 3
There exist continuously differentiable functions $V_p : \mathbb{R}^n \to [0, \infty)$, $p \in \mathcal{P}$, class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \gamma$, and numbers $\lambda_0 > 0$ such that $\forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$, and $\forall p, q \in \mathcal{P}$, we have

$$\alpha_1(|\xi|) \leqslant V_p(\xi) \leqslant \alpha_2(|\xi|), \tag{58}$$

$$\frac{\partial V_p}{\partial \xi} f_p(\xi, \eta) \leqslant -\lambda_0 V_p(\xi) + \gamma(|\eta|), \tag{59}$$

$$V_p(\xi) \leqslant \mu_V V_q(\xi). \tag{60}$$

Remark 6
If every subsystem is ISS, then for every $p \in \mathcal{P}$ there exist class $\mathcal{K}_\infty$ functions $\alpha_{1,p}, \alpha_{2,p}, \gamma_p$, numbers $\lambda_{\circ,p} > 0$, and ISS-Lyapunov functions V_p satisfying

$$\alpha_{1,p}(|\xi|) \leqslant V_p(\xi) \leqslant \alpha_{2,p}(|\xi|),$$

$$\frac{\partial V_p}{\partial \xi} f_p(\xi) \leqslant -\lambda_{\circ,p} V_p(\xi) + \alpha_2 \gamma_p(|\eta|),$$

$\forall \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^\ell$ (see [21,22]). If the set $\mathcal{P}$ is finite, then (58) and (59) are trivially satisfied. Also, if the set $\mathcal{P}$ is compact, and suitable continuity assumptions on $\left\{\alpha_{1,p}, \alpha_{2,p}, \alpha_2\gamma_p\right\}_{p\in\mathcal{P}}$ and $\left\{\lambda_{\circ,p}\right\}_{p\in\mathcal{P}}$ with respect to p hold, then (58) and (59) follow. We shall henceforth stipulate that our collection of ISS-Lyapunov functions $\{V_p\}_p \in \mathcal{P}$ satisfies (58) and (59). The set of possible ISS-Lyapunov functions is restricted by the condition (60). This inequality does not hold, for example, if V_p is quadratic for one value of p and quartic for another. If $\mu_V = 1$, the relation (60) implies that $V = V_p,\ p \in \mathcal{P}$ is a common ISS-Lyapunov function for the family of the subsystems. In this case, the switched system is ISS for *arbitrary switching* (also called *uniformly input-to-state stable* [23]).

Recall that a plant is input-output-to-state (IOSS) (see, e.g., [19]) if the state x of the (open-loop) plant satisfies the following property

$$|x(t)| \leqslant \beta(|x(t_0)|, t - t_0) + \gamma_u(\|u\|_{[t_0,t)}) + \gamma_y(\|y\|_{[t_0,t)}) \tag{61}$$

for all $t \geqslant t_0$ for some $\beta \in \mathcal{KL}, \gamma_u, \gamma_y \in \mathcal{K}_\infty$.

For nonlinear systems, if the matching Assumption 3 holds and the plant is IOSS, we would expect that there exist functions $\{\kappa, \kappa_x\} \in \mathcal{K}_\infty$ such that if

$$\kappa(\Delta) < M, \tag{62}$$

then all the closed-loop signals are bounded, and for every $\epsilon_x > 0$, $\exists T < \infty$ such that

$$|x(t)| \leqslant \kappa_x(\nu\Delta) + \epsilon_x \nu\Delta \quad \forall t \geqslant t_0 + T. \tag{63}$$

The analysis would follow similar steps as in the linear case but utilize the ISS framework to relate between Lyapunov functions and states, similarly to the approach in [17]. In [17], we establish robustness of supervisory control to bounded noise, and here one can think of the quantization errors playing a similar role as a noise in the proof in [17]. We also expect that under some additional strict (nonlinear) conditions on the quantizer range M and the quantizer error bound Δ, one can use linear zooming with period T_s and the scaling factor ρ as in (34) to obtain closed-loop asymptotic stability. There could also be a case for nonlinear zooming which needs further exploration. We have not

‡Recall that (see, e.g., [20]) a continuous function $\alpha : [0, \infty) \to [0, \infty)$ is of class $\mathcal{K}$ if α is strictly increasing, and $\alpha(0) = 0$, and further, $\alpha \in \mathcal{K}_\infty$ if $\alpha(r) \to \infty$ as $r \to \infty$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for every fixed t, and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for every fixed r.

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yet finalized the nonlinear results but still included the methodology for the nonlinear case here for a complete picture of the problem of supervisory control with quantized information and also for showing the flexibility of the approach beyond the linear cases.

9. CONCLUSIONS

In this paper, we have treated the problem of stabilizing uncertain systems with output quantization using supervisory control. For a static quantizer, we provided a condition between the quantization range and the quantization error bound to guarantee closed-loop stability. With a dynamic quantizer, we provided a zooming strategy on the quantization zooming variable ν and on the parameter ε of the supervisory control scheme to achieve asymptotic stability for the closed loop. Future research aims to obtain tighter bounds for various quantities in the quantized supervisory control scheme. Other direction is to consider other types of limited information, such as sampling, delay, or package loss, or a combination of those with quantization. In this direction, it may be fruitful to combine the approach in this paper with the result in [24].

APPENDIX

A. Formula for $\mathbf{A}_q$ *and* $\mathbf{B}_q$

Pick an ordering of the set $\mathcal{P}$, $\mathcal{P} := \{p_1, \ldots, p_m\}$. The formula for $\mathbf{A}_q$ and $\mathbf{B}_q$, $q \in \mathcal{P}$, is

$$\mathbf{A}_q|_{q=p_j} := \bigoplus_{i=1}^{m}(A_{p_i} + L_{p_i}C_{p_i}) + \begin{bmatrix} 0_{mn\times(j-1)n} & B_{p_1}K_{p_j} - L_{p_1}C_{p_j} & 0_{mn\times(m-j-1)n} \\ \vdots & \vdots & \vdots \\ 0_{mn\times(j-1)n} & B_{p_m}K_{p_j} - L_{p_m}C_{p_j} & 0_{mn\times(m-j-1)n} \end{bmatrix} \tag{64}$$

$$\mathbf{B}_q := \begin{bmatrix} L_{p_1} \\ \vdots \\ L_{p_m} \end{bmatrix} \quad \forall q \in \mathcal{P},$$

where $\bigoplus$ is the Kronecker product.

B. Choosing the design parameters

The value of various constants used in the design of a supervisory control is critical to ensure small bounds. Unfortunately, how to chose the design parameters in supervisory control to minimize the bounds is still an open question. The design parameters are chosen heuristically and the design procedure is trial and error, but here, we discuss the effect of various constants and provide some guidelines and numerical tools for obtaining smaller bounds.

- Usually we want small μ, small γ_0, λ_0. If one chooses the positive definite matrices P_p for the Lyapunov functions $V_p(x) = x^T P_p x$ of the p-th injected system using the Lyapunov equation $\mathbf{A}_p^T P_p + P_p\mathbf{A}_p = -Q$ for some positive definite Q, then it is likely that the common λ_0 will be small and the common γ_0 will be large, and the Lyapunov gain μ_V among V_p will be large. To find small μ and γ_0 and large λ_0, one can use LMI to solve for P_i in the following way: we write (11b) as follows:

$$\dot{V}_p = x_{\mathbb{E}}^T(\mathbf{A}_p^T P_p + P_p\mathbf{A}_p)x_{\mathbb{E}} + x_{\mathbb{E}}^T P_p\mathbf{B}_p\tilde{y}_p + \tilde{y}_p^T\mathbf{B}_p^T P_p \leqslant -\lambda_0 x_{\mathbb{E}}^T P_p x_{\mathbb{E}} + \gamma_0\tilde{y}_p^T\tilde{y}_p$$

$$\Leftrightarrow \begin{bmatrix} x_{\mathbb{E}} & \tilde{y}_p \end{bmatrix}\begin{bmatrix} \mathbf{A}_p^T P_p + P_p\mathbf{A}_p + \lambda_0 P_p & P_p\mathbf{B}_p \\ \mathbf{B}_p^T P_p & -\gamma_0 I \end{bmatrix}\begin{bmatrix} x_{\mathbb{E}} \\ \tilde{y}_p \end{bmatrix} \leqslant 0.$$

Because the foregoing inequality is true for all $x_{\mathbb{E}}$ and all $\tilde{y}_p$, it is equivalent to

$$\begin{bmatrix} \mathbf{A}_p^T P_p + P_p \mathbf{A}_p + \lambda_0 P_p & P_p \mathbf{B}_p \\ \mathbf{B}_p^T P_p & -\gamma_0 I \end{bmatrix} \leqslant 0.$$

We then have the following LMIs:

$$\begin{bmatrix} \mathbf{A}_p^T P_p + P_p \mathbf{A}_p + \lambda_0 P_p & P_p \mathbf{B}_p \\ \mathbf{B}_p^T P_p & -\gamma_0 I \end{bmatrix} \leqslant 0$$
$$P_p > 0 \quad p = 1, \ldots, m. \tag{65}$$

The aforementioned set of LMIs can be solved numerically for given $\mathbf{A}_p, \mathbf{B}_p, \lambda_0, \gamma$. For our analysis in this work, we also want the ratio $\overline{a}/\underline{a}$ small, where $\overline{a}$ and $\underline{a}$ are as in (11a). This can be achieved by adding the following LMIs into (65)

$$P_p \geqslant \underline{a} I, \quad P_p \leqslant \overline{a} I. \tag{66}$$

One then iteratively try various sets of ν, λ_0, γ_0 while the LMI (65) is still feasible.

- We want h small because small h allows for fast detection; a larger h means that the estimator could stay on the wrong controller for longer time, resulting in a larger empirical bound. However, a smaller h implies a larger N_0 and hence, a larger c_1 and c_2 and a larger calculated bound. Thus, there is a trade-off here for h and one can try to tune h starting from the smallest value.
- We want γ small because that means small N_0 and hence, smaller calculated bounds. However, a larger γ means larger monitoring signals μ_p and that in turn implies faster switching (or detection). So there is also a trade-off here between for γ.

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