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Invertibility of switched nonlinear systems☆

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ABSTRACT

This article addresses the invertibility problem for switched nonlinear systems affine in controls. The problem is concerned with reconstructing the input and switching signal uniquely from given output and initial state. We extend the concept of switch-singular pairs, introduced recently, to nonlinear systems and develop a formula for checking if the given state and output form a switch-singular pair. A necessary and sufficient condition for the invertibility of switched nonlinear systems is given, which requires the invertibility of individual subsystems and the nonexistence of switch-singular pairs. When all the subsystems are invertible, we present an algorithm for finding switching signals and inputs that generate a given output in a finite interval when there is a finite number of such switching signals and inputs. Detailed examples are included to illustrate these newly developed concepts.

1. Introduction

Switched systems refer to dynamical systems with discrete switching events. Their evolution is described by a collection of dynamical subsystems, together with a switching signal, that specifies an active subsystem at each time instant. Examples include switching power converters, networks with switching topologies, and aircraft with different thrust modes. Also, switching control techniques, especially in the adaptive context, have been shown to achieve stability and improved transient response (see Liberzon, 2003, Chapter 6). Because of their utility in modeling and control design, switched systems have been a focus of ongoing research and several results related to stability, controllability, observability, and input-to-state stability of such systems have been published; see Liberzon (2003) for references. More recently, Vu and Liberzon (2008) introduced the problem of invertibility of switched linear systems. In this paper, we extend their methodology to study the problem of invertibility of continuous-time switched nonlinear systems, which is concerned with finding the conditions on the subsystems to guarantee unique recovery of the switching signal and the input from the initial state and the output. The problem statement is analogous to the classical invertibility problem for nonswitched systems. In fact, for every control system with an output, we have an input–output map and the question of left (resp. right) invertibility is, roughly speaking, that of the injectivity (surjectivity) of this map.

System invertibility problems are of great importance from a theoretical and practical viewpoint and have been studied extensively for fifty years, after being pioneered by Brockett and Mesarovic (1965). For nonswitched linear systems, the algebraic criterion for invertibility and the construction of inverse systems were given by Silverman (1969), and also by Sain and Massey (1969). The systematic study of invertibility for nonswitched nonlinear systems began with Hirschorn, who first studied the single-input single-output (SISO) case (see Hirschorn, 1979b), and then generalized Silverman's structure algorithm to multiple-input multiple-output (MIMO) nonlinear systems (see Hirschorn, 1979a). Singh (1981) then modified the algorithm to cover a larger class of systems. Isidori and Moog (1988) used this algorithm to calculate zero-output constrained dynamics and reduced inverse system dynamics. The algorithm is also closely related to the dynamic extension algorithm used to solve the dynamic state feedback input–output decoupling problem (see Nijmeijer & van der Schaft, 1990, Sections 8.2 and 11.3). Geometric methods have been studied by Nijmeijer (1982). A higher-level interpretation given by a linear-algebraic framework, which also establishes links between these algorithms and the geometric approach, is presented by Di Benedetto, Grizzle, and Moog (1989). We also recommend a useful survey on various invertibility techniques by Respondek (1990).

The problem of invertibility for switched linear systems was introduced very recently by Vu and Liberzon (2008) where the authors used Silverman's structure algorithm to formulate conditions

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for the invertibility of switched systems with continuous dynamics. The problem of invertibility for discrete-time switched linear systems has been discussed by Millerioux and Daafouz (2007) and Sundaram and Hadjicostis (2006) but there, the authors assume that the switching sequence is known and find the corresponding input. In this paper, we make no such assumption and adopt an approach similar to Vu and Liberzon (2008), to study the invertibility problem for continuous-time switched nonlinear systems, affine in controls, using Singh's nonlinear structure algorithm.[1] The concept of singular pairs, conceived by Vu and Liberzon (2008), is extended to nonlinear systems; although, in this paper, such pairs are termed as "switch-singular pairs" to avoid conflict with the singularities of individual nonlinear subsystems. Even though the form of the main result (invertibility of subsystems plus no switch-singular pairs) and the concepts presented in this article are essentially similar to those given by Vu and Liberzon (2008), the main contribution of this paper lies in the technical details of developing and checking the conditions for invertibility of nonlinear systems. In particular, the use of the nonlinear structure algorithm, possibility of finite escape times, and the existence of singularities in state space and output set require more careful analysis and technical rigor as compared to the linear case.

As is the case in the classical setting of nonswitched systems, we start with an output and an initial state, but here there is a set of dynamic models and we wish to recover the switching signal in addition to the input. In the context of hybrid systems, recovering the switching signal is equivalent to the mode identification for hybrid systems or the observability of the discrete state variable (location), which has been studied by Babaali and Pappas (2005), Vidal, Chiuso, and Soatto (2002) and Vidal, Chiuso, Soatto, and Sastry (2003). Hence, the inversion of switched systems can also be thought of doing the mode detection and input recovery simultaneously. Consequently, the basic idea for solving the invertibility problem is to first do the mode identification by utilizing the relationship among the outputs and the states of the subsystems, and then use the nonlinear structure algorithm for the corresponding subsystem to recover the input.

For the case when subsystems are linear, Silverman's structure algorithm seems to be the most convenient tool to formulate invertibility conditions which leads to a simple and elegant rank test for checking the existence of switch-singular pairs, but in nonlinear systems it is hard to achieve such a level of generality. For this reason, we start with the SISO case to highlight the technical difficulties in moving from linear to nonlinear systems. Discussing the SISO case first also helps in understanding the concepts behind the formula derived for verification of switch-singular pairs.

The paper is organized as follows. Section 2 contains the definitions of invertibility and the formal problem statement. The main result on left invertibility is presented in Section 3. We then give a characterization of switch-singular pairs and the construction of inverse systems in Section 4. An algorithm for output generation is given in Section 5 along with an example. We conclude the article with some remarks on further research directions.

2. Preliminaries

In this section, we develop the required notations and provide some background on invertibility of nonswitched nonlinear systems. Based on that, we develop the definition for the invertibility of switched nonlinear systems followed by the formal problem statement to which we seek solution in the paper.

[1] A related problem is discussed by Chaib, Boutat, Banali, and Kratz (2007) but it doesn't follow the same theoretical approach we do, and instead uses a heuristic approach with the purpose of studying a specific application.

2.1. Nonswitched nonlinear systems

The dynamics of a square nonlinear system, affine in controls, are given by

$$\Gamma := \begin{cases} \dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \\ y = h(x) \end{cases} \tag{1}$$

where $x \in \mathbb{M}$, an n-dimensional real connected smooth manifold, for example $\mathbb{R}^n$; f, g_i are smooth vector fields on $\mathbb{M}$, and $h : \mathbb{M} \to \mathbb{R}^m$ is a smooth function. Admissible input signals are locally essentially bounded, Lebesgue measurable functions $u : [t_0, \infty) \to \mathbb{R}^m$. If the two inputs differ on a set of measure zero, i.e. $u_1(t) = u_2(t)$ almost everywhere (a.e.), then they are considered to be equal. We use the notation $u_{[t_0,T)}$ to denote the input u over the time interval $[t_0, T)$; and $\Gamma_{x_0}(u)$ denotes the state trajectory generated by (1) after applying the input u with initial condition x_0.

We start off by reviewing classical definitions of invertibility for such systems. For that, consider the input–output map $\mathrm{H}_{x_0} : \mathcal{U} \to \mathcal{Y}$ for some input function space $\mathcal{U}$ and the corresponding output function space $\mathcal{Y}$. H_{x_0} maps an input $u(\cdot)$ to the output $y(\cdot)$ generated by the system driven by $u(\cdot)$ with an initial condition x_0. Since the state trajectories of nonlinear systems may exhibit finite escape times, an input $u_{[t_0,\infty)}$ may not have a well defined image in the output space, over the interval $[t_0, \infty)$, under this map. For this reason, we only consider inputs over a finite interval $[t_0, T)$, which is the maximal interval of the existence of a solution, such that $\mathrm{H}_{x_0}(u_{[t_0,T)}) = y_{[t_0,T)}$ always exists and is well-defined.

Invertibility[2] of the dynamical system (1) basically refers to the injectivity of the map H_{x_0}. Before giving a formal definition, let us look at an example first.

Example 1. Consider a nonswitched nonlinear system with two inputs and two outputs,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_1 u_1 \\ x_3 u_1 \\ u_2 \end{pmatrix}, \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbb{M} = \mathbb{R}^3.$$

We then have

$$\dot{y}_1 = x_1 u_1, \tag{2a}$$

$$\ddot{y}_2 = \frac{x_3 \ddot{y}_1 - \dot{y}_1 \dot{y}_2 + \dot{y}_1 u_2}{x_1}. \tag{2b}$$

It follows that u_1 can be recovered uniquely from $\dot{y}_1$ if $x_1 \neq 0$, and u_2 can be recovered uniquely from $\ddot{y}_2$ if $\dot{y}_1 \neq 0$ and $x_1 \neq 0$. The point $x_1 = 0$ and $\dot{y}_1 = 0$ are the singularities in the state space and the output space, respectively. Let $\mathbb{M}^\alpha := \{x \in \mathbb{R}^3 \mid x_1 \neq 0\}$; $Y^s = \{z \in \mathbb{R}^2 \mid z_1 = 0\}$, and $\mathcal{Y}^s := \{y : [t_0, T) \to \mathbb{R}^2 \mid \dot{y}(t) \in Y^s$ for almost all $t \in [t_0, t_0 + \delta) \subseteq [t_0, T)$, where $\delta > 0$ is arbitrary$\}$. In words, $\mathcal{Y}^s$ includes those outputs which remain in a singular set for some time. The complement of $\mathcal{Y}^s$ is given by $\mathcal{Y}^\alpha := \{y : [t_0, T) \to \mathbb{R}^2 \mid \dot{y}(t) \notin Y_1^s$ for almost all $t \in [t_0, t_0 + \varepsilon)$ and some $\varepsilon > 0\}$. If the system is driven by a class of inputs u such that the resulting motion $\Gamma_{x_0}(u) \in \mathbb{M}^\alpha$ a.e. and $\mathrm{H}_{x_0}(u) \in \mathcal{Y}^\alpha$, then there is a one-to-one relation between the output and input signals provided their domains are restricted to $[t_0, t_0 + \varepsilon)$. In summary, the input can be recovered uniquely using the knowledge of output, its derivatives and possibly some states as long as the output and state trajectories do not hit some singularities. □

We now proceed to the formal definition of invertibility for nonswitched systems.

[2] Throughout the paper, *invertibility* refers to the left invertibility.

Definition 1. Fix an output set $\mathcal{Y}$ and consider an arbitrary interval $[t_0, T)$. The system (1) is *invertible at a point* $x_0 := x(t_0) \in \mathbb{M}$ over $\mathcal{Y}$ if for every $y_{[t_0,T)} \in \mathcal{Y}$, the equality $\mathrm{H}_{x_0}(u_{1[t_0,T)}) = \mathrm{H}_{x_0}(u_{2[t_0,T)}) = y_{[t_0,T)}$ implies that $\exists\, \varepsilon > 0$ such that $u_{1[t_0,t_0+\varepsilon)} = u_{2[t_0,t_0+\varepsilon)}$. The system is *strongly invertible at a point* x_0 if it is invertible for each $x \in N(x_0)$, where N is some open neighborhood of x_0. The system is *strongly invertible* if there exists an open and dense submanifold $\mathbb{M}^\alpha$ such that $\forall x_0 \in \mathbb{M}^\alpha$, the system is strongly invertible at x_0. □

As illustrated in Example 1, a system is invertible at x_0 for the class of inputs $u(\cdot)$ such that along the trajectory of the system (1), the resulting motion $x(\cdot), y(\cdot)$ does not hit any singularities. It is entirely possible that the state trajectory or the output hits singularity at a time instant $t_0+\varepsilon$ with $0 < \varepsilon < T-t_0$, thus making it impossible to recover u uniquely beyond $t_0+\varepsilon$; this explains why we require distinct inputs over arbitrarily small time domains in Definition 1.

In the most general construction of inverse systems as the one given by Singh (1981), there exists a set of *singular outputs* $\mathcal{Y}^s$ such that the system is not invertible for $y \in \mathcal{Y}^s$; and its complement $\mathcal{Y}^\alpha := \mathcal{Y} \setminus \mathcal{Y}^s$ is the set of all outputs on which the system is strongly invertible. Also, in general, the inverses of nonlinear dynamical systems are not defined on the entire state space. If the vector fields $f(x), g(x)$ and the output function $h(x)$ are analytic, then the singular points are reduced to a closed and nowhere dense set comprising zeros of certain analytic functions. Under these assumptions, if the system is invertible then there exists an open and dense subset of $\mathbb{M}$ on which the dynamics of a nonlinear system are invertible; that subset is called the *inverse submanifold* and is denoted by $\mathbb{M}^\alpha$. All these notions will be developed formally in Section 4.

Using Definition 1, invertibility at x_0 is equivalent to saying that $u_{1[t_0,t_0+\varepsilon)} \neq u_{2[t_0,t_0+\varepsilon)}$ for all $\varepsilon \in (0, T-t_0)$ implies that $\mathrm{H}_{x_0}(u_{1[t_0,T)}) \neq \mathrm{H}_{x_0}(u_{2[t_0,T)})$. This notion was captured by Hirschorn (1979a). Our definition is essentially the same as one considered by Hirschorn in the sense that both notions address the injectivity of an input–output map. The difference lies in the fact that Hirschorn considered a class of *analytic* nonlinear systems with analytic inputs and $\mathcal{Y}^s = \emptyset$, an empty set. In that case, if the system is invertible and the state trajectory starts from a nonsingular set, it is possible to recover inputs on a small interval but because of analyticity, we continue to recover inputs uniquely even after hitting singularity; for if two analytic inputs are different on a subinterval then they are different everywhere, otherwise their difference (an analytic function) would have an infinite number of zeros on a finite interval. In this paper though, we consider non-analytic systems driven by inputs that are not necessarily analytic, so the input recovery can only be guaranteed over small time intervals only.

We will now generalize this notion of local invertibility to the switched systems.

2.2. Switched nonlinear systems

In the paper we will consider switched nonlinear systems, affine in controls, that have the following structure:

$$\Gamma_\sigma : \begin{cases} \dot{x} = f_\sigma(x) + G_\sigma(x)u = f_\sigma(x) + \displaystyle\sum_{i=1}^{m} (g_i)_\sigma(x) u_i, \\ y = h_\sigma(x) \end{cases} \tag{3}$$

where $\sigma : [t_0, T) \to \mathcal{P}$ is the switching signal that indicates the active subsystem at every time, $\mathcal{P}$ is some finite index set, and f_p, G_p, h_p, where $p \in \mathcal{P}$, define the dynamics of individual subsystems. The state space $\mathbb{M}$ is a connected real smooth manifold of dimension n, for example $\mathbb{R}^n$; f_p, $(g_i)_p$ are real smooth vector fields on $\mathbb{M}$, and $h_p : \mathbb{M} \to \mathbb{R}^m$ is a smooth function. A switching signal is a piecewise constant and everywhere right-continuous function that has a finite number of discontinuities at τ_i, which we call *switching times*, on every bounded time interval. Denote by $\sigma^p_{[t_0,T)}$ the constant switching signal over the interval $[t_0, T)$ such that $\sigma^p(t) := p \in \mathcal{P}$, $\forall t \in [t_0, T)$. We assume that all the subsystems are equidimensional, they live in the same state space $\mathbb{M}$, and that there is no state jump at switching times. For any initial state x_0, switching signal $\sigma(\cdot)$, and any admissible input $u(\cdot)$, a solution of (3) always exists (in Carathéodory sense) and is unique, provided the flow of every subsystem is well-defined for the time interval during which it is active, i.e., the state trajectories do not blow up in finite time. In fact, this assumption results in absolutely continuous state trajectories (see Sontag, 1998). Denote by $[t_0, T)$ the maximal interval of existence of solution, so that the outputs are well-defined on $[t_0, T)$. Since the switching signals are right-continuous, the outputs are also right-continuous (note that, in general, $h_i(x) \neq h_j(x)$, for $i \neq j$) and whenever we take the derivative of an output, we assume it is the right derivative. For $p \in \mathcal{P}$, denote by $\Gamma_{p,x_0}(u)$ the trajectory of the corresponding subsystem with the initial state x_0 and the input u, and the corresponding output by $\Gamma^O_{p,x_0}(u)$.

We will use $\mathcal{F}^{pc}$ to denote the space of *piecewise right-continuous functions*[3] and $\mathcal{F}^n$ to denote the subset of $\mathcal{F}^{pc}$ whose elements are n times differentiable between two consecutive discontinuities. Likewise, $\mathcal{F}^{AC}$ denotes the subset of $\mathcal{F}^{pc}$ whose elements are absolutely continuous between two consecutive discontinuities. Finally, we use $\oplus$ for the concatenation of two signals.

In case of switched systems (3), the map H_{x_0} has an augmented domain; that is, now we have a (switching signal $\times$ input)-output map $\mathrm{H}_{x_0} : \mathcal{S} \times \mathcal{U} \to \mathcal{Y}$, where $\mathcal{S}$ is a switching signal set, $\mathcal{U}$ is the input space, and $\mathcal{Y}$ is the output space. Let us first extend the definition of invertibility to switched systems.

Definition 2. Fix an output set $\mathcal{Y}$ and consider an arbitrary interval $[t_0, T)$. A *switched system is invertible at a point* x_0 over $\mathcal{Y}$ if for every $y_{[t_0,T)} \in \mathcal{Y}$, the equality $\mathrm{H}_{x_0}(\sigma_{1[t_0,T)}, u_{1[t_0,T)}) = \mathrm{H}_{x_0}(\sigma_{2[t_0,T)}, u_{2[t_0,T)}) = y_{[t_0,T)}$ implies that $\exists\, \varepsilon > 0$ such that $\sigma_{1[t_0,t_0+\varepsilon)} = \sigma_{2[t_0,t_0+\varepsilon)}$ and $u_{1[t_0,t_0+\varepsilon)} = u_{2[t_0,t_0+\varepsilon)}$. A *switched system is strongly invertible at a point* x_0 if it is invertible at each $x \in N(x_0)$, where N is some open neighborhood of x_0. A *switched system is strongly invertible* if there exists an open and dense submanifold $\mathbb{M}^\alpha$ of $\mathbb{M}$ such that $\forall x_0 \in \mathbb{M}^\alpha$, the system is strongly invertible at x_0. □

For linear switched systems, as discussed by Vu and Liberzon (2008), all the notions in Definition 2 coincide and a system is termed invertible if the input and switching signal could be recovered uniquely for all x_0.

The invertibility property formulated in Definition 2 may fail to hold in two ways: (a) either because there exist two different inputs u_1 and u_2 that yield the same output or (b) because there exist two different switching signals $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$ that yield the same output. The first case refers to the notion of classical invertibility as already explained in Definition 1 and Section 2.1. To address the second possibility, we need the concept of switch-singular pairs which refers to the ability of more than one subsystem to produce a segment of the desired output starting from the same initial condition. The formal definition is given below:

[3] By piecewise right-continuous functions, we mean that there is a finite number of jump discontinuities in any finite interval; the function is continuous in between any two consecutive discontinuities; and the function is continuous from the right at discontinuities. To avoid excessive rigidness, we will use the term "piecewise continuous" throughout the paper, and it is understood that "piecewise continuous" means "piecewise right-continuous".

Definition 3. Consider $x_0 \in \mathbb{M}$ and $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ on some time interval $[t_0, T)$, where $p, q \in \mathcal{P}, p \neq q$. The pair (x_0, y) is a *switch-singular pair* of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 and $\varepsilon > 0$ such that $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\varepsilon)}) = \Gamma^O_{q,x_0}(u_{2[t_0,t_0+\varepsilon)}) = y_{[t_0,t_0+\varepsilon)}$. □

If all subsystems are linear, $x_0 = 0$ and $y \equiv 0$ always form a switch-singular pair regardless of the dynamics of the subsystems. This is because $u \equiv 0$ and any switching signal will produce $y \equiv 0$, that is, $\mathrm{H}_0(\sigma, 0) = 0\,\forall\sigma$, and therefore H_0 is not injective if the zero function belongs to $\mathcal{Y}$. In nonlinear systems, this is not the case in general and all switch-singular pairs are solely determined by the subsystem dynamics. As stated earlier and will be formally proved below, the switched system is not invertible if $\mathcal{Y}$ contains outputs that form switch-singular pairs with x_0. Thus, if there exist any switch-singular pairs, we have to restrict the output set $\mathcal{Y}$, instead of letting $\mathcal{Y}$ be the set of all possible concatenations of nonsingular output trajectories.

Next, we use the concept of switch-singular pairs to study the invertibility problem of switched systems. Since Definition 2 contains different variants of invertibility, we start off with the weakest of them all, i.e., invertibility of a switched system at a point. In particular, we are interested in solving the following fundamental problem: *Find a suitable set $\mathcal{Y}$ and a condition on the subsystems such that the system is invertible at x_0 over $\mathcal{Y}$.* An abstract characterization of the set $\mathcal{Y}$ and constraints on subsystem dynamics which guarantee invertibility are given in Section 3 under Theorem 1; Corollaries 1 and 2 then characterize the set $\mathcal{Y}$ more explicitly (depending on the required variant of invertibility). Later in Section 4, we give mathematical formulae (Lemmas 1 through 5) for checking the abstract conditions given in Section 3.

3. Characterization of invertibility

In this section, we describe the output set $\mathcal{Y}$ used in Definition 2 and give conditions on the subsystem dynamics so that the switched system is invertible for some sets $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{Y}$. Restricting the outputs to lie in $\mathcal{Y}$ implies a set of restrictions on the set of allowable inputs, but an explicit characterization of such inputs is not always obtainable. That is why we do not explicitly specify what the input sets $\mathcal{U}$ and $\mathcal{S}$ are, but instead specify the set $\mathcal{Y}$ and then $\mathcal{U}$ will be the corresponding set which, together with $\mathcal{S}$, generates $\mathcal{Y}$.

For all $p \in \mathcal{P}$, let $\mathcal{Y}_p$ be the set of smooth outputs[4] that can be generated by Γ_p, and let $\mathcal{Y}^{\mathrm{all}}$ be the set of all the possible concatenations of all elements of $\mathcal{Y}_p$, $\forall p \in \mathcal{P}$. Due to the existence of certain singular outputs (for which the system is not invertible), we seek invertibility at a fixed point x_0 over a subset $\mathcal{Y}^\alpha \subseteq \mathcal{Y}^{\mathrm{all}}$.

Theorem 1. *Consider the switched system (3) and an output set $\mathcal{Y}^\alpha \subseteq \mathcal{Y}^{\mathrm{all}}$. The switched system is invertible at $x_0 \in \mathbb{M}$ over $\mathcal{Y}^\alpha$ if and only if each subsystem Γ_p is invertible at x_0 over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$ and for all $y \in \mathcal{Y}^\alpha$, the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q, p, q \in \mathcal{P}$.*

Proof (*Necessity*)**.** We show that if any of the subsystems is not invertible at x_0 or if there exist switch-singular pairs (x_0, y), then the switched system is not invertible.

Suppose that a subsystem $\Gamma_p, p \in \mathcal{P}$, is not invertible at x_0 over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$, then there exists $y_{[t_0,T)} \in \mathcal{Y}^\alpha \cap \mathcal{Y}_p$ such that $\Gamma^O_{p,x_0}(u_{1[t_0,T)}) = \Gamma^O_{p,x_0}(u_{2[t_0,T)}) = y_{[t_0,T)}$ for some u_1, u_2 and $\forall\varepsilon \in (0, T - t_0), u_1 \neq u_2$ on $[t_0, t_0 + \varepsilon)$. This implies that $\mathrm{H}_{x_0}(\sigma^p_{[t_0,T)}, u_{1[t_0,T)}) = \mathrm{H}_{x_0}(\sigma^p_{[t_0,T)}, u_{2[t_0,T)}) = y_{[t_0,T)}$ and thus, Definition 2 implies that the switched system is not invertible at x_0 over $\mathcal{Y}^\alpha$.

For necessity of the second condition, suppose that $\exists y \in \mathcal{Y}^\alpha \cap \mathcal{Y}_p \cap \mathcal{Y}_q$, so that (x_0, y) is a switch-singular pair of Γ_p, Γ_q, $p \neq q$. This means that both subsystems, even though invertible at x_0, can produce this output over the interval $[t_0, t_0 + \varepsilon) \subset [t_0, T), \forall\varepsilon > 0$. Consequently, $\exists u_{1[t_0,T)}, u_{2[t_0,T)}$ (possibly same) such that $\Gamma^O_{p,x_0}(u_{1[t_0,T)}) = \Gamma^O_{q,x_0}(u_{2[t_0,T)}) = y_{[t_0,T)}$. Hence, we have $\mathrm{H}_{x_0}(\sigma^p_{[t_0,T)}, u_{1[t_0,T)}) = \mathrm{H}_{x_0}(\sigma^q_{[t_0,T)}, u_{2[t_0,T)}) = y_{[t_0,T)}$; that is, the preimage of y is not unique as $\sigma^p \neq \sigma^q$ on $[t_0 + t_0 + \varepsilon), \forall\varepsilon \in (0, T - t_0)$. This implies that the switched system is not invertible at x_0 for given $\mathcal{Y}^\alpha$.

Sufficiency: Suppose that for the given $x_0 \in \mathbb{M}$, there exist some inputs u_1, u_2 and switching signals σ_1, σ_2 such that $\mathrm{H}_{x_0}(\sigma_1, u_1) = \mathrm{H}_{x_0}(\sigma_2, u_2) = y \in \mathcal{Y}^\alpha$ over $[t_0, T)$. Since (x_0, y) is not a switch-singular pair, there exists ε_1 such that $\sigma_1(t) = \sigma_2(t) = p, \forall t \in [t_0, t_0 + \varepsilon_1)$[5] and $y_{[t_0,t_0+\varepsilon_1)} \in \mathcal{Y}_p$. Since Γ_p is invertible at x_0, $\exists\varepsilon_2 < \varepsilon_1$ such that $u_{1[t_0,t_0+\varepsilon_2)} = u_{2[t_0,t_0+\varepsilon_2)}$ and $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\varepsilon_2)}) = \Gamma^O_{p,x_0}(u_{2[t_0,t_0+\varepsilon_2)}) = y_{[t_0,t_0+\varepsilon_2)}$. Letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, it then follows from Definition 2 that the switched system is invertible at x_0 over $\mathcal{Y}^\alpha$. □

In the proof of the sufficiency part, the switched system is strongly invertible at x_0 for the signals whose domain is restricted to the interval $[t_0, t_0 + \varepsilon)$, where $t_0 + \varepsilon$ is the time instant at which the state trajectory or the output enters the singular set. If the output y loses continuity over the interval $[t_0, t_0 + \varepsilon)$ because of switching, then $(\sigma_{[t_0,t_0+\varepsilon)}, u_{[t_0,t_0+\varepsilon)}) = (\sigma_{[t_0,\tau_1)}, u_{[t_0,\tau_1)}) \oplus \cdots \oplus (\sigma_{[\tau_k,t_0+\varepsilon)}, u_{[\tau_k,t_0+\varepsilon)})$, where k is the total number of switches in the interval $[t_0, t_0+\varepsilon)$ and $\tau_i,\ i = 1, \ldots, k$, are the switching instants.

Let us now consider a refinement of Theorem 1 by characterizing the set $\mathcal{Y}^\alpha$. For all $p \in \mathcal{P}$, let $\mathcal{Y}^s_p$ be the set of singular outputs of Γ_p for which Γ_p is not invertible (see Example 1 and Section 4.2, or Singh (1981)), and let $\mathcal{Y}^\alpha_p = \mathcal{Y}_p \setminus \mathcal{Y}^s_p$ be the set of outputs on which Γ_p is invertible at x_0. Define $\mathcal{Y}^s := \cup_{p\in\mathcal{P}} \mathcal{Y}^s_p$ as the collection of all singular outputs and let $\mathcal{Y}^{\mathrm{all}}$ be the set of outputs generated by all the possible concatenations of all elements of $\mathcal{Y}_p$, $\forall p \in \mathcal{P}$; Finally, define $\overline{\mathcal{Y}}^\alpha := \mathcal{Y}^{\mathrm{all}} \setminus \mathcal{Y}^s$ as a set of outputs over which we seek invertibility. We now have the following modified version of Theorem 1.

Corollary 1. *The switched system is invertible at x_0 over the set $\overline{\mathcal{Y}}^\alpha$ if and only if the pairs (x_0, y) are not switch-singular pairs of Γ_p and Γ_q, for all $y \in \overline{\mathcal{Y}}^\alpha$, for all $p \neq q,\ p, q \in \mathcal{P}$.*

Proof. By the application of Theorem 1, the desired result is obtained by showing that Γ_p, $\forall p \in \mathcal{P}$, is invertible at x_0 over the set $\overline{\mathcal{Y}}^\alpha \cap \mathcal{Y}_p$. By construction, $\mathcal{Y}_p = \mathcal{Y}^\alpha_p \cup \mathcal{Y}^s_p$ and $\overline{\mathcal{Y}}^\alpha \cap \mathcal{Y}^s_p = \emptyset$; using these two equalities, it is easy to see that $\overline{\mathcal{Y}}^\alpha \cap \mathcal{Y}_p \subseteq \mathcal{Y}^\alpha_p$. As each subsystem Γ_p is invertible at x_0 over $\mathcal{Y}^\alpha_p$, it follows that each subsystem Γ_p is, in particular, invertible at x_0 over the output set $\overline{\mathcal{Y}}^\alpha \cap \mathcal{Y}_p$. □

Corollary 2. *Consider the switched system (3) and an output set $\mathcal{Y}^\alpha \subseteq \mathcal{Y}$. The switched system is strongly invertible at $x_0 \in \mathbb{M}$ over $\mathcal{Y}^\alpha$ if and only if each subsystem Γ_p is strongly invertible at x_0*

[4] This assumption can be relaxed depending upon the system under consideration, see Remarks 4 and 5 in Section 4 for details.

[5] This argument can also be proved in another way: it will be shown later that the points in state space that form switch-singular pairs are actually a zero set of smooth nonlinear equations. Thus, if x_0 does not form a switch-singular pair with y then there exists a neighborhood $N(x_0)$ such that $\forall x \in N(x_0)$, (x, y) is not a switch-singular pair. As there are no switch-singular pairs in $N(x_0)$, $\exists\varepsilon_1 > 0$ such that $\sigma_{1[t_0,t_0+\varepsilon_1)} = \sigma_{2[t_0,t_0+\varepsilon_1)}$.

over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$ and there exists a neighborhood $\overline{N}(x_0)$ such that for all $x \in \overline{N}(x_0), y \in \mathcal{Y}^\alpha$, the pairs (x, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q, p, q \in \mathcal{P}$.

Proof (*Necessity*)**.** If the switched system is strongly invertible at x_0, then $\exists N(x_0)$ such that the switched system is invertible at every $x \in N(x_0)$ over $\mathcal{Y}^\alpha$. Let $\overline{N}(x_0) := N(x_0)$. By Theorem 1, each subsystem is invertible at every $x \in N(x_0)$, hence strongly invertible at x_0, and there does not exist any switch-singular pairs (x, y), for all $x \in \overline{N}(x_0), y \in \mathcal{Y}^\alpha$.

Sufficiency: If each subsystem is strongly invertible at x_0, i.e., $\exists N_p(x_0)$ such that Γ_p is invertible at every $x \in N_p(x_0)$, then $N^\alpha := \bigcap_{p\in\mathcal{P}} N_p$ is an open set on which all subsystems are invertible. If we define $N := N^\alpha \cap \overline{N}$, then the switched system is invertible at every $x \in N(x_0)$ over $\mathcal{Y}^\alpha$ and hence by Theorem 1, strongly invertible at x_0. □

For the strong invertibility of the switched system on an open and dense subset, assume that the vector fields f_p, $(g_i)_p$ and the output function h_p are analytic. Under these assumptions, if a subsystem Γ_p is strongly invertible, then $\mathbb{M}_p^\alpha$ denotes the inverse submanifold of Γ_p.

Corollary 3. *The switched system (3) is strongly invertible, with inverse submanifold $\mathbb{M}^\alpha \subseteq \mathbb{M}$, over an output set $\mathcal{Y}^\alpha \subseteq \mathcal{Y}$ if and only if each subsystem is strongly invertible over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$ and the subsystem dynamics are such that the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q for all $p \neq q, p, q \in \mathcal{P}$, for every $x_0 \in \mathbb{M}^\alpha$, and every $y \in \mathcal{Y}^\alpha$.*

Proof (*Necessity*)**.** If the switched system is strongly invertible, then it is strongly invertible at every $x_0 \in \mathbb{M}^\alpha$ over $\mathcal{Y}^\alpha$. By Corollary 2, each subsystem is strongly invertible at every $x_0 \in \mathbb{M}^\alpha$, and hence strongly invertible with inverse submanifold $\mathbb{M}^\alpha$. Furthermore, there does not exist any switch-singular pairs (x_0, y), $\forall x_0 \in \mathbb{M}^\alpha, y \in \mathcal{Y}^\alpha$.

Sufficiency: Under the given hypothesis, there exists an inverse submanifold $\mathbb{M}_p^\alpha$ such that Γ_p is strongly invertible at every $x_0 \in \mathbb{M}_p^\alpha$ over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$, for all $p \in \mathcal{P}$. Define $\mathbb{M}^\alpha := \bigcap_{p\in\mathcal{P}} \mathbb{M}_p^\alpha$, then $\mathbb{M}^\alpha$ is an open and dense subset of $\mathbb{M}$ because it is a finite intersection of open and dense subsets. Under relative topology, $\mathbb{M}^\alpha$ is a submanifold. Since each subsystem Γ_p is strongly invertible at every $x_0 \in \mathbb{M}^\alpha$ over $\mathcal{Y}^\alpha \cap \mathcal{Y}_p$ and there exist no switch-singular pairs, application of Corollary 2 implies that the switched system is strongly invertible at every $x_0 \in \mathbb{M}^\alpha$ over $\mathcal{Y}^\alpha$. □

In essence, Theorem 1, and the related corollaries state that the invertibility of subsystems in a certain sense implies the invertibility of the switched system in a similar sense provided there are no switch-singular pairs between the states and the outputs considered. Before concluding this section, a couple of remarks are in order.

Remark 1. For the switched system (3), if all the subsystems are globally invertible in addition to the hypothesis of Corollary 3, that is, $\mathbb{M}^\alpha = \mathbb{M}$ and $\mathcal{Y}^s = \emptyset$, then it is possible to recover the inputs and switching signals uniquely over the time interval $[t_0, T)$. Also note that T may be arbitrarily large if the state trajectories do not exhibit finite escape time. □

Remark 2. If a subsystem has more inputs than outputs, then it cannot be (left) invertible. On the other hand, if it has more outputs than inputs, then some outputs are redundant (as far as the task of recovering the input is concerned). Thus, the case of input and output dimensions being equal is, perhaps, the most interesting case. □

4. Checking invertibility

In this section, we address the computational aspect of the concepts introduced in previous sections and develop algebraic criteria for checking the invertibility of switched systems. The first condition in Theorem 1 asks for invertibility of subsystems and is verified by the structure algorithm. To put everything into perspective, we provide appropriate background related to the invertibility of nonswitched systems and use it to develop the concept of functional reproducibility. To check if (x_0, y) is a switch-singular pair, we develop a formula using the functional reproducibility criteria of nonswitched systems. After verifying the invertibility of subsystems and nonexistence of switch-singular pairs, we will be able to construct a switched inverse system that recovers the original input and switching signal uniquely.

4.1. Single-input single-output (SISO) systems

We start off with the case when all the subsystems are SISO because it gives more insight into computations and helps understand the concepts which we will later generalize to multivariable systems. To this end, consider a SISO nonlinear system affine in controls (1) with $m = 1$ and assume it has a relative degree r at x_0 (see Isidori, 1995), i.e., $\exists$ a neighborhood $N(x_0)$ such that $L_g L_f^k h(x) = 0$, $\forall x \in N(x_0), k = 0, \dots, r-1$ and $L_g L_f^{r-1} h(x_0) \neq 0$, where $L_f^k h(x) = \frac{\partial (L_f^{k-1} h(x))}{\partial x} f(x)$ and $L_f^0 h(x) = h(x)$.

To check if the subsystem is invertible or not, following Hirschorn (1979b), we first derive an explicit expression for the input u in terms of the output y by computing the derivatives of y as follows:

$$y(t) = h(x(t)) \tag{4a}$$

$$\dot{y}(t) = L_f h(x(t)) \tag{4b}$$

$$\vdots$$

$$y^{(r)}(t) = L_f^r h(x(t)) + L_g L_f^{r-1} h(x(t)) u(t). \tag{4c}$$

From the last equation, we can derive an expression for $u(t)$:

$$u(t) = -\frac{L_f^r h(x(t))}{L_g L_f^{r-1} h(x(t))} + \frac{1}{L_g L_f^{r-1} h(x(t))} y^{(r)}(t). \tag{5}$$

Hence, u can be determined explicitly in terms of the measured output y, and state x. On substituting the expression for u from (5) in Eq. (1), one gets the dynamics for the inverse system:

$$\dot{z} = f(z) + g(z)\left(-\frac{L_f^r h(z)}{L_g L_f^{r-1} h(z)} + \frac{1}{L_g L_f^{r-1} h(z)} y^{(r)}\right),$$

$$u = -\frac{L_f^r h(z)}{L_g L_f^{r-1} h(z)} + \frac{1}{L_g L_f^{r-1} h(z)} y^{(r)}. \tag{6}$$

The dynamics of this inverse subsystem evolve on the set $\mathbb{M}^\alpha := \{z \in \mathbb{M} \mid L_g L_f^{r-1} h(z) \neq 0\}$. $\mathbb{M}^\alpha$ is open and dense if f, g, h are analytic. Since the inverse system dynamics are driven by $y^{(r)}(\cdot)$ which satisfies Eq. (4c), it is not hard to see that the state trajectories of the inverse system satisfy the differential equation of the original system (1) where the input has just been replaced by a function of y. So if the inverse system is initialized with the same initial condition as that of the plant, then both of the systems follow exactly the same trajectory. This discussion motivates the following result:

Lemma 1. *A SISO system is strongly invertible at x_0 if the system has a finite relative degree r at x_0.*

Remark 3. The condition given in Lemma 1 for strong invertibility at a point x_0 is only sufficient, and not necessary. As an example, consider $\dot{x} = 1 + xu$, $y = x$, $x \in \mathbb{R}$, $x_0 = 0$; no relative degree at x_0, but the system is strongly invertible at x_0 because the trajectory immediately leaves the singularity. In general, this occurs when the first function of the sequence $L_g h(x), L_g L_f h(x), \dots, L_g L_f^k h(x)$ which is not identically zero (in a neighborhood of x_0) has a zero exactly at the point $x = x_0$. A result somewhat similar to Lemma 1 appears in Hirschorn (1979b, Theorem 2.1), where the author gives a necessary and sufficient condition for strong invertibility of a SISO system but considers only analytic systems with a slightly different notion of relative degree. $\square$

Remark 4. For SISO systems, the input u appears in the r-th derivative of the output (4). Thus the differentiability/smoothness of u will not affect the existence of first $r-1$ derivatives of y. If $u : [t_0, T) \to \mathbb{R}$ is a locally essentially bounded, Lebesgue measurable function, then $y^{(r)}(\cdot)$ exists almost everywhere and $y^{(r-1)}(\cdot)$ is absolutely continuous (see Sontag, 1998). So for SISO nonlinear nonswitched systems, $\mathcal{U}$ is defined as the space of functions which are locally essentially bounded and Lebesgue measurable; and $\mathcal{Y}^\alpha$ is the set of corresponding outputs. $\square$

We now turn to the concept of functional reproducibility, which in broad terms means the ability to follow a given reference signal. This concept will help us study the existence of switch-singular pairs. We look at the conditions under which a system can produce the desired output y_d over some interval $[t_0, T)$ starting from a particular initial state x_0. To be precise, given the system (1) with $m = 1$ and initial state x_0, we want to find out if there exists a control u such that $\Gamma_{x_0}^O(u) = y_d$. The following result was given by Hirschorn (1979b):

Lemma 2. *If the system (1), with $m = 1$ and $x(t_0) = x_0$, has a relative degree $r < \infty$ at x_0, then there exists a control input u such that $\Gamma_{x_0}^O(u) = y_d$ if and only if*

$$y_d^{(k)}(t_0) = L_f^k h(x_0) \quad \forall k = 0, 1, \dots, r-1. \tag{7}$$

This result is easy to comprehend by looking at the expressions for the output derivatives (4). As control $u(t)$ does not directly affect $y^{(k)}(t)$, for $k = 1, \dots, r-1$, their values at t_0 are determined by the initial state. Substituting

$$u(t) = -\frac{L_f^r h(x(t))}{L_g L_f^{r-1} h(x(t))} + \frac{1}{L_g L_f^{r-1} h(x(t))} y_d^{(r)}(t) \tag{8}$$

in (4c) gives $y^{(r)}(t) = y_d^{(r)}(t)$. Using (7), repeated integration yields $y(t) = y_d(t)$.

We can now easily check for the switch-singular pairs among Γ_p, Γ_q with relative degrees r_p, r_q respectively, where $p, q \in \mathcal{P}$.

Lemma 3. *For SISO switched systems, (x_0, y) is a switch-singular pair of two subsystems Γ_p and Γ_q if and only if $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ and*

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa - 1)} \end{pmatrix}(t_0) = \begin{pmatrix} h_\kappa(x_0) \\ \vdots \\ L_{f_\kappa}^{r_\kappa - 1} h_\kappa(x_0) \end{pmatrix}, \quad \kappa = p, q. \tag{9}$$

The example below illustrates the use of these concepts.

Example 2. Consider a SISO switched system with two modes

$$\Gamma_p := \begin{cases} \dot{x} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ x_1 x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ x_2 \end{pmatrix} u, & \mathbb{M} = \mathbb{R}^3 \\ y = x_1 \end{cases}$$

$$\Gamma_q := \begin{cases} \dot{x} = \begin{pmatrix} x_2 \\ x_2 x_3 \\ -x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ x_2 \end{pmatrix} u, & \mathbb{M} = \mathbb{R}^3 \\ y = 2x_1. \end{cases}$$

If Γ_p is active, then $\dot{y} = x_1 + x_2$; if Γ_q is active, then $\dot{y} = 2x_2$. Both subsystems have relative degree 2 on $\mathbb{R}^3$ which can be verified by taking second derivative of the output. If there exists $x \in \mathbb{R}^3$ which forms a switch-singular pair with $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$, then the following equality must be satisfied

$$\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

which gives $x_1 = x_2 = 0$. This state constraint yields $y = \dot{y} = 0$. If we let $\overline{\mathcal{Y}}^\alpha := \left\{ y : [t_0, T) \to \mathbb{R} \mid \dot{y}_{[t_0, T)} \in \mathcal{F}^{AC} \text{ and } \binom{y(t)}{\dot{y}(t)} \neq 0 \text{ for almost all } t \in [t_0, T) \right\}$, then there exists no switch-singular pair between $x_0 \in \mathbb{R}^3$ and $y \in \overline{\mathcal{Y}}^\alpha$. Theorem 1 and Lemma 1 infer that the switched system generated by $\{\Gamma_p, \Gamma_q\}$ is strongly invertible with inverse submanifold $\mathbb{R}^3$ over $\overline{\mathcal{Y}}^\alpha$. Alternatively, if $x_0 \neq 0$ then (x_0, y) is not a switch-singular pair for any y and the switched system is strongly invertible with inverse submanifold $\mathbb{R}^3 \setminus \{0\}$ over $\mathcal{Y}^{\text{all}}$. $\square$

For general switched nonlinear systems, it is hard to check for the existence of switch-singular pairs. To see this, consider the system (3) with $m = 1$. For simplicity, assume that $\mathcal{P} = \{p, q\}$ and the subsystems Γ_p, Γ_q have equal relative degrees, i.e., $r_p = r_q =: r$. Lemma 3 states that Γ_p, Γ_q have a switch-singular pair (x_0, y) if and only if

$$\hat{y} = \mathcal{H}_p(x_0) = \mathcal{H}_q(x_0) \tag{10}$$

where $\hat{y} = (y, \dot{y}, \dots, y^{(r-1)})^T$ and $\mathcal{H}_\kappa = (h_p, L_{f_p} h_p, \dots, L_{f_p}^{r-1} h_p)^T$, $\kappa = \{p, q\}$. To see if there exist any switch-singular pairs between two subsystems, one is interested in solving $\mathcal{H}_p(x_0) = \mathcal{H}_q(x_0)$ for x_0; that is, x_0 that forms switch-singular pair actually lies in the solution space of r-nonlinear equations where each equation itself involves functions of an n-dimensional variable x_0. As it is hard to talk about the solutions of nonlinear equations in general, investigation into more constructive conditions for checking of switch-singular pairs is a topic of ongoing research. Nonetheless, in the case of SISO switched bilinear systems, the nonlinear equations in (9) become linear and the task of checking the existence of switch-singular pairs between two subsystems is comparatively easier, as illustrated below.

Example 3. Consider a switched system with SISO bilinear subsystems, having the dynamics of the form

$$\begin{aligned} \dot{x} &= A_{\sigma(t)} x + B_{\sigma(t)} x u, \\ y &= C_{\sigma(t)} x \end{aligned} \tag{11}$$

where $\sigma(t) = p \in \mathcal{P}$, $x \in \mathbb{R}^n$, $A_p, B_p \in \mathbb{R}^{n \times n}$, $C_p \in \mathbb{R}^{1 \times n}$. Also, $u(t), y(t) \in \mathbb{R}$.

If some mode $p \in \mathcal{P}$ is active over a time interval, then at any time t in that interval, the expression for the derivatives of output is

$$\begin{aligned} y(t) &= C_p x(t), \\ \dot{y}(t) &= C_p A_p x(t), \end{aligned}$$

$$\vdots$$
$$y^{(r_p-1)}(t) = C_pA_p^{r_p-1}x(t),$$
$$y^{(r_p)}(t) = C_pA_p^{r_p}x(t) + C_pA_p^{r_p-1}B_pxu(t) \tag{12}$$

where r_p denotes the relative degree of subsystem p. If we introduce the notations

$$\hat{y}_p(t) := \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r_p-1)(t)} \end{pmatrix} \quad \text{and} \quad Z_p := \begin{pmatrix} C_p \\ C_pA_p \\ \vdots \\ C_pA_p^{r_p-1} \end{pmatrix},$$

then based on the functional reproducibility criteria, an output $y_{[t_0,t_0+\varepsilon)}$ can be produced by a subsystem p if and only if $\hat{y}_p(t_0) = Z_px(t_0)$. Consequently, if two subsystems p, q can produce a given segment of output on an interval $[t_0, t_0+\varepsilon)$, then we will have

$$\begin{pmatrix} \hat{y}_p(t_0) \\ \hat{y}_q(t_0) \end{pmatrix} = \begin{pmatrix} Z_p \\ Z_q \end{pmatrix} x(t_0). \tag{13}$$

This is equivalent to saying that

$$\begin{bmatrix} I_p \\ I_q \end{bmatrix} \hat{y}(t_0) = \begin{pmatrix} Z_p \\ Z_q \end{pmatrix} x(t_0) \tag{14}$$

where $\hat{y} := \left(y, \dot{y}, \ldots, y^{(r-1)}\right)^T$, $r := \max\{r_p, r_q\}$, and for $\kappa = \{p, q\}$, I_κ is an $r_\kappa \times r$ matrix whose ijth element is 1 if $i = j$ and 0 otherwise. Thus, the existence of switch-singular pairs in case of SISO bilinear switched systems implies that the intersection of range spaces of $\binom{I_p}{I_q}$ and $\binom{Z_p}{Z_q}$ is not empty. Since $\binom{I_p}{I_q}$ and $\binom{Z_p}{Z_q}$ are both linear operators acting on linear subspaces, the *zero vector* is always in their range space. Thus, an identically zero output always forms a switch singular pair with the kernel of $\binom{Z_p}{Z_q}$; that is, $\left(\ker\binom{Z_p}{Z_q}, 0\right)$ forms a switch-singular pair for such systems. That is the trivial case; for the nontrivial case we check if $\binom{I_p}{I_q}$ and $\binom{Z_p}{Z_q}$ have a nontrivial common range space. So, if there exists a nonzero output that forms a switch-singular pair with some state at time t, then $\hat{y}(t) \in \text{range}\binom{I_p}{I_q} \cap \text{range}\binom{Z_p}{Z_q}$, or equivalently $\text{rank}\begin{bmatrix} I_p & Z_p \\ I_q & Z_q \end{bmatrix} < \text{rank}\begin{bmatrix} I_p \\ I_q \end{bmatrix} + \text{rank}\begin{bmatrix} Z_p \\ Z_q \end{bmatrix}$.

In other words, if all the subsystems in (11) are invertible and $r := \max_{p\in\mathcal{P}} r_p < \infty$, then for all $x(t_0) := x_0 \in \mathbb{R}^n$ and $y \in \mathcal{Y}^\alpha := \{y \mid y^{(r-1)} \in \mathcal{F}^{AC} \text{ and } \hat{y}_{[t_0,t_0+\varepsilon)} \not\equiv 0, \text{ for some } \varepsilon > 0\}$, the pairs (x_0, y) are not switch-singular pairs of Γ_p, Γ_q, if and only if the following rank condition holds:

$$\text{rank}\begin{bmatrix} I_p & Z_p \\ I_q & Z_q \end{bmatrix} = \text{rank}\begin{bmatrix} I_p \\ I_q \end{bmatrix} + \text{rank}\begin{bmatrix} Z_p \\ Z_q \end{bmatrix} \tag{15}$$

for all $p \neq q$, $p, q \in \mathcal{P}$ such that $\mathcal{Y}_p \cap \mathcal{Y}_q \neq \{0\}$.

This condition is similar to the one given in Vu and Liberzon (2008, Lemma 3) for checking the existence of switch-singular pairs in switched linear systems. The common framework in both cases is the appearance of linear equations when taking the derivatives of the outputs, which makes it easier to derive the rank conditions. □

We now have a toolset to check the invertibility conditions given in Theorem 1. If these conditions are satisfied and the switched system is strongly invertible, a switched inverse system can be constructed to recover the input and switching signal σ from given output and initial state. For the switched inverse system, define the *index inversion function* $\overline{\Sigma}^{-1} : \mathbb{M}^\alpha \times \mathcal{Y}^\alpha \to \mathcal{P}$ as:

$$\overline{\Sigma}^{-1}(x_0, y) = p : y \in \mathcal{Y}_p \quad \text{and} \quad y^{(k)}(t_0) = L_{f_p}^k h_p(x_0) \tag{16}$$

where $k = 0, 1, \ldots, r_p - 1$, t_0 is the initial time of y, and $x_0 = x(t_0)$. The function $\overline{\Sigma}^{-1}$ is well-defined since p is unique by the fact that there are no switch-singular pairs. The existence of p is guaranteed because it is assumed that $y \in \mathcal{Y}^\alpha$ is an output. The dynamics of the inverse switched system Γ_σ^{-1} are:

$$\sigma(t) = \overline{\Sigma}^{-1}(z(t), y_{[t,t+\varepsilon)}),$$
$$\dot{z} = f_\sigma(z) + g_\sigma(z)\left(\frac{y^{(r_\sigma)} - L_{f_\sigma}^{r_\sigma}h_\sigma(z)}{L_{g_\sigma}L_{f_\sigma}^{r_\sigma-1}h_\sigma(z)}\right),$$
$$u(t) = \frac{y^{(r_\sigma)}(t) - L_{f_\sigma}^{r_\sigma}h_\sigma(z(t))}{L_{g_\sigma}L_{f_\sigma}^{r_\sigma-1}h_\sigma(z(t))}$$

with the initial condition $z(t_0) = x_0$. The notation $(\cdot)_\sigma$ denotes the object in the parenthesis calculated for the subsystem with index $\sigma(t)$. The initial condition $\sigma(t_0)$ determines the initial active subsystem at the initial time t_0, from which time onwards, the active subsystem indexes and the input as well as the state are determined uniquely and simultaneously.

4.2. Multiple-input multiple-output (MIMO) systems

For multiple-input multiple-output (MIMO) nonlinear systems affine in controls (1), one uses the *structure algorithm* to compute the inverse. When a system is invertible, the structure algorithm, or Singh's inversion algorithm, allows us to express the input as a function of the output, its derivatives and possibly some states.

The structure algorithm: This version of the algorithm closely follows the construction given by Di Benedetto et al. (1989), which is a slightly modified version of the algorithm by Singh (1981).

Step 1: Calculate

$$\dot{y} = L_fh(x) + L_Gh(x)u = \frac{\partial h}{\partial x}[f(x) + G(x)u]$$

and write it as $\dot{y} =: a_1(x) + b_1(x)u$. Define $s_1 := \text{rank}\, b_1(x)$, which is the rank of $b_1(x)$ in some neighborhood of x_0, denoted as $N_1(x_0)$. Permute, if necessary, the components of the output so that the first s_1 rows of $b_1(x)$ are linearly dependent. Decompose y as

$$\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1(x) + \tilde{b}_1(x)u \\ \hat{a}_1(x) + \hat{b}_1(x)u \end{pmatrix}$$

where $\dot{\tilde{y}}_1$ consists of the first s_1 rows of $\dot{y}$. Since the last $m - s_1$ rows of $b_1(x)$ are linearly dependent upon the first s_1 rows, there exists a matrix $F_1(x)$ such that

$$\dot{\tilde{y}}_1 = \tilde{a}_1(x) + \tilde{b}_1(x)u,$$
$$\dot{\hat{y}}_1 = \hat{h}^1(x, \dot{\tilde{y}}_1) = \hat{a}_1(x) + F_1(x)(\dot{\tilde{y}}_1 - \tilde{a}_1(x)) \tag{17}$$

where the last equation is affine in $\dot{\tilde{y}}_1$. Finally, set $\widetilde{B}_1(x) := \tilde{b}_1(x)$.

Step $k+1$: Suppose that in steps 1 through k, $\dot{\tilde{y}}_1, \ldots, \tilde{y}_k^{(k)}, \hat{y}_k^{(k)}$ have been defined so that

$$\dot{\tilde{y}}_1 = \tilde{a}_1(x) + \tilde{b}_1(x)u,$$
$$\vdots$$
$$\begin{aligned} \tilde{y}_k^{(k)} &= \tilde{a}_k(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k-1, i \le j \le k\}) \\ &\quad + \tilde{b}_k(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k-1, i \le j \le k-1\})u, \end{aligned}$$
$$\hat{y}_k^{(k)} = \hat{h}^k(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k, i \le j \le k\})$$

where all the expressions on the right-hand side are rational functions of $\tilde{y}_i^{(j)}$. Suppose also that the matrix $\widetilde{B}_k := [\tilde{b}_1^T, \dots, \tilde{b}_k^T]^T$ (vertical stacking of the linearly independent rows obtained at each step) has full rank equal to s_k in $N_k(x_0)$. Then calculate

$$\hat{y}_k^{(k+1)} = \frac{\partial \hat{h}^k}{\partial x}[f(x) + G(x)u] + \sum_{i=1}^{k}\sum_{j=i}^{k} \frac{\partial \hat{h}^k}{\partial \tilde{y}_i^{(j)}} \tilde{y}_i^{(j+1)}$$

and write it as

$$\begin{aligned} \hat{y}_k^{(k+1)} &= a_{k+1}(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k, i \le j \le k+1\}) \\ &\quad + b_{k+1}(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k, i \le j \le k\})u. \end{aligned} \tag{18}$$

Define $B_{k+1} := [\widetilde{B}_k^T, b_{k+1}^T]^T$, and $s_{k+1} := \operatorname{rank} B_{k+1}$. Permute, if necessary, the components of $\hat{y}_k^{(k+1)}$ so that the first s_{k+1} rows of B_{k+1} are linearly independent. Decompose $\hat{y}_k^{(k+1)}$ as

$$\hat{y}_k^{(k+1)} = \begin{pmatrix} \tilde{y}_{k+1}^{(k+1)} \\ \hat{y}_{k+1}^{(k+1)} \end{pmatrix}$$

where $\tilde{y}_{k+1}^{(k+1)}$ consists of the first $(s_{k+1} - s_k)$ rows. Since the last rows of $B_{k+1}(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k, i \le j \le k\})$ are linearly dependent on the first s_{k+1} rows, we can write

$$\begin{aligned} \dot{\tilde{y}}_1 &= \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ &\ \vdots \\ \tilde{y}_{k+1}^{(k+1)} &= \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k, i \le j \le k+1\}) \\ &\quad + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k, i \le j \le k\})u, \\ \hat{y}_{k+1}^{(k+1)} &= \hat{h}^{k+1}(x, \{\tilde{y}_i^{(j)} | 1 \le i \le k+1, i \le j \le k+1\}) \end{aligned}$$

where once again everything is rational in $\tilde{y}_i^{(j)}$. Finally, set $\widetilde{B}_{k+1} := [\widetilde{B}_k^T, \tilde{b}_{k+1}^T]^T$, which has full rank equal to s_{k+1} locally.

End of Step $k+1$.

By construction, $s_1 \le s_2 \le \cdots \le m$. If for some integer α we have $s_\alpha = m$, then the algorithm terminates and the system is strongly invertible at x_0. We call α the *relative order*[6] of the system. The closed form expression for u is derived from the α-th step of the structure algorithm, which gives an invertible matrix $\widetilde{B}_\alpha := [\tilde{b}_1^T, \dots, \tilde{b}_\alpha^T]^T$ having full rank equal to m in a neighborhood $N_\alpha(x_0) =: N(x_0)$, namely,

$$u = \widetilde{B}_\alpha^{-1}\left[\begin{pmatrix} \dot{\tilde{y}}_1 \\ \vdots \\ \tilde{y}_\alpha^{(\alpha)} \end{pmatrix} - \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_\alpha \end{pmatrix}\right] =: \widetilde{B}_\alpha^{-1}[\tilde{Y}_\alpha - \tilde{A}_\alpha]. \tag{19}$$

Note that the entries of the matrix $\widetilde{B}_\alpha$ are rational functions of the derivatives of the output and there may exist an output for which the rank of $\widetilde{B}_\alpha$ drops. We denote by Y^s the values of the output and its derivatives, evaluated at a time instant t, for which the rank of $\widetilde{B}_\alpha(x, y(t))$ is less than m, while $x \in N(x_0)$. We can now formally define the sets $\mathcal{Y}^s$ and $\mathcal{Y}^\alpha$ for a subsystem as follows: $\mathcal{Y}^s := \{y : [t_0, T) \to \mathbb{R}^2 \mid y(t) \in Y^s$ for almost all $t \in [t_0, t_0 + \delta) \subseteq [t_0, T)$, where $\delta > 0$ is arbitrary$\}$, and $\mathcal{Y}^\alpha := \{y : [t_0, T) \to \mathbb{R}^2 \mid y(t) \notin Y^s$ for almost all $t \in [t_0, t_0 + \varepsilon)$ and some $\varepsilon > 0\}$. In other words, $\mathcal{Y}^s$ includes those outputs for which the matrix $\widetilde{B}_\alpha$ is not invertible and $\mathcal{Y}^\alpha$ is its complement. Hence, we work with u such that $\Gamma_{x_0}^O(u) \notin \mathcal{Y}^s$. Comparing to the SISO case, we had $\widetilde{B}_\alpha = L_g L_f^{r-1} h(x)$ which is a function of the state only and thus there exists no singular output for SISO systems. Another useful class of systems for which $\mathcal{Y}^s = \emptyset$ was discussed by Hirschorn (1979a). As was the case in SISO systems, substitution of the expression for u from (19) in (1) gives the dynamics of the inverse system. These dynamics are defined on the set $\mathbb{M}^\alpha := \{x \in \mathbb{M} \mid \operatorname{rank} \widetilde{B}_\alpha(x, y(t)) = m, y(t) \notin Y^s\}$, which is open and dense if $f(x), g(x), h(x)$ are analytic functions.

Example 4. As an illustration of the structure algorithm, let us revisit the system defined in Example 1. Step 1 of the algorithm yields $\dot{y} = \begin{pmatrix} \dot{\tilde{y}}_1 \\ \dot{\hat{y}}_1 \end{pmatrix} = \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ x_3 & 0 \end{pmatrix} u$. Using $F_1(x) = x_3/x_1$, we get $\dot{\hat{y}}_1 = \dot{y}_2 = (x_3/x_1)\dot{y}_1$. In Step 2, after differentiating $\dot{\hat{y}}_1 = \dot{y}_2$, we get the following set of equations:

$$\begin{aligned} \dot{\tilde{y}}_1 &= \dot{y}_1 = x_1 u_1 \\ \ddot{\tilde{y}}_2 &= \ddot{y}_2 = \frac{x_3 \ddot{y}_1 - \dot{y}_1 \dot{y}_2 + \dot{y}_1 u_2}{x_1} \Rightarrow \widetilde{B}_2 = \begin{pmatrix} x_1 & 0 \\ 0 & \dot{y}_1/x_1 \end{pmatrix}. \end{aligned}$$

So, $\widetilde{B}_2$ has rank 2, the number of inputs. Hence, $\alpha = 2$; $\mathbb{M}^\alpha = \{x \in \mathbb{R}^3 \mid x_1 \ne 0\}$; $Y^s = \{z \in \mathbb{R}^2 \mid z_1 = 0\}$, and $\mathcal{Y}^s = \{y : [t_0, T) \to \mathbb{R}^2 \mid \dot{y}(t) \in Y^s$ for almost all $t \in [t_0, t_0 + \delta) \subseteq [t_0, T)$, where $\delta > 0$ is arbitrary$\}$. □

Remark 5. Unlike in the SISO case, we need some differentiability assumptions on the input signals to characterize the input space for MIMO systems. In the structure algorithm, Step 1 gives $\dot{\tilde{y}}_1$ that has already u on the right-hand side and the α-th step of the algorithm involves $\{\tilde{y}_i^{(j)} | 1 \le i \le \alpha - 1, i \le j \le \alpha\}$. Thus $\tilde{y}_i^{(\alpha-1)}$ must be absolutely continuous so that $\tilde{y}_i^{(\alpha)}$ exists almost everywhere. For the input space, it means that $u^{(\alpha-1)}$ must be Lebesgue measurable and locally essentially bounded. These constraints characterize the input space $\mathcal{U}$ for MIMO case and $\mathcal{Y}$ is the corresponding set of outputs. From the structure algorithm, we deduce that the system is invertible on $\mathcal{Y}^\alpha = \mathcal{Y} \setminus \mathcal{Y}^s$. □

Based on the structure algorithm, we now study the conditions for functional reproducibility of multivariable nonlinear systems. Using the notation derived in the structure algorithm, denote by Z the vector

$$Z\left(x, \dot{\tilde{y}}_1, \dots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\right) := \begin{pmatrix} h(x) \\ \hat{h}^1(x, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{h}^{\alpha-1}\left(x, \dot{\tilde{y}}_1, \dots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\right) \end{pmatrix}$$

and let

$$\hat{y} := \begin{pmatrix} y \\ \dot{\hat{y}}_1 \\ \vdots \\ \hat{y}_{\alpha-1}^{(\alpha-1)} \end{pmatrix}, \qquad \hat{y}_d := \begin{pmatrix} y_d \\ \dot{\hat{y}}_{d_1} \\ \vdots \\ \hat{y}_{d_{\alpha-1}}^{(\alpha-1)} \end{pmatrix}. \tag{20}$$

So Z is basically a concatenation of the expressions appearing at each step of Singh's structure algorithm which get differentiated and $\hat{y}$ is the concatenation of the corresponding expressions on the left-hand side so that

$$Z\left(x, \dot{\tilde{y}}_1, \dots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\right) - \hat{y} = 0.$$

The following result is along the same line as Lemma 2 and has appeared in Singh (1982, Theorem 1). The proof is given in the Appendix and is developed differently than Singh (1982).

[6] The term was coined by Hirschorn (1979a) and is weaker than the notion of vector relative degree. Parallel to the terminology used in linear system theory, Nijmeijer and Schumacher (1985) show that α is the *highest order of zeros at infinity*.

Lemma 4. *If the system given by (1), with $x(t_0) = x_0$, has a relative order $\alpha < \infty$, then there exists a control input u such that $\Gamma^O_{x_0}(u) = y_d(\cdot)$ if and only if*

$$\hat{y}_d(t_0) = Z\left(x_0, \dot{\tilde{y}}_{d_1}(t_0), \dots, \tilde{y}^{(k)}_{d_k}(t_0)\right) \quad \forall k = 0, 1, \dots, \alpha - 1. \tag{21}$$

Another version of this result in terms of jet spaces is given by Respondek (1990). Similarly to the SISO case, the idea is that the portion of the output which is not directly affected by u is determined initially by the value of state variables; and the input u, for which $\Gamma^O_{x_0}(u) = y_d(\cdot)$, is given by (19) with y replaced by y_d in that formula.

Example 5. Consider the system given in Example 1. The vector $\hat{y}$ is the portion of the output that gets differentiated and therefore,

$$\hat{y} = \begin{pmatrix} y_1 \\ y_2 \\ \dot{y}_2 \end{pmatrix} \Rightarrow \hat{y}_d = \begin{pmatrix} y_{d_1} \\ y_{d_2} \\ \dot{y}_{d_2} \end{pmatrix}.$$

The vector $Z(x, y_{d_1}, y_2, \dot{y}_{d_1})$ is given by

$$Z(x, y_{d_1}, y_2, \dot{y}_{d_1}) = \begin{pmatrix} x_1 \\ x_2 \\ \dot{y}_{d_1}(x_3/x_1) \end{pmatrix}.$$

Using Lemma 4 and calculations in Example 4, if we have $\hat{y}_d(t_0) = Z\left(x_0, y_{d_1}(t_0), y_2(t_0), \dot{y}_{d_1}(t_0)\right)$ then the control which produces y_d as an output, on a small interval, is given by

$$u_1 = \frac{\dot{y}_{d_1}}{x_1}$$
$$u_2 = \frac{x_1\ddot{y}_{d_2} - x_3\ddot{y}_{d_1} + \dot{y}_{d_1}\dot{y}_{d_2}}{\dot{y}_{d_1}}.$$

If $y_d(\cdot) \in \mathcal{Y}^\alpha$ for all times and the corresponding state trajectory $x(\cdot) \in \mathbb{M}^\alpha$, then the system can produce $y_d(\cdot)$ as an output over an arbitrary time interval. □

Lemma 4 gives the following condition for the verification of switch-singular pairs.

Lemma 5. *For MIMO switched systems, (x_0, y) is a switch-singular pair of two subsystems Γ_p, Γ_q if and only if $y \in \mathcal{Y}_p \cap \mathcal{Y}_q$ and*

$$\begin{pmatrix} y \\ \dot{\hat{y}}_1 \\ \vdots \\ \hat{y}^{\alpha_\kappa - 1}_{(\alpha_\kappa - 1)} \end{pmatrix} = \begin{pmatrix} h_\kappa(x_0) \\ \hat{h}^1_\kappa(x_0, \dot{\tilde{y}}_1) \\ \vdots \\ \hat{h}^{\alpha_\kappa - 1}_\kappa(x_0, \dot{\tilde{y}}_1, \dots, \tilde{y}^{(\alpha_\kappa - 1)}_{\alpha_\kappa - 1}) \end{pmatrix} \tag{22}$$

for $\kappa = p, q$, and α_κ denotes the relative order of subsystem Γ_κ.

The procedure for constructing the inverse from this point onwards is exactly the same as discussed earlier for the SISO case with u given by (19) instead of (5).

Remark 6. The results in this section can also be extended to include the case when there are state jumps at switching times. Denote by $\psi_{p,q} : \mathbb{M} \to \mathbb{M}$ the *reset map* when switching from subsystem p to subsystem q, $p, q \in \mathcal{P}$. Thus far, we have considered the case of identity reset maps only, where $\psi_{p,q}(x) = x \, \forall p, q \in \mathcal{P}$, $\forall x \in \mathbb{M}$. For nonidentity reset maps, Definition 3 is modified to "(x_0, y) is a switch-singular pair of the two subsystems Γ_p, Γ_q if there exist u_1, u_2 and $\varepsilon > 0$ such that $\Gamma^O_{p,x_0}(u_{1[t_0,t_0+\varepsilon)}) = \Gamma^O_{q,\psi_{p,q}(x_0)}(u_{2[t_0,t_0+\varepsilon)}) = y_{[t_0,t_0+\varepsilon)}$ or $\Gamma^O_{p,\psi_{q,p}(x_0)}(u_{1[t_0,t_0+\varepsilon)}) = \Gamma^O_{q,x_0}(u_{2[t_0,t_0+\varepsilon)}) = y_{[t_0,t_0+\varepsilon)}$". Essentially, this means that the output is indistinguishable between the two subsystems, taking into account the effect of state jumps. In case of SISO systems, instead of Eq. (9), we check for switch-singular pairs using

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa - 1)} \end{pmatrix}(t_0) = \begin{pmatrix} h_\kappa(\psi_{p,\kappa}(x_0)) \\ \vdots \\ L^{r_\kappa - 1}_{f_\kappa} h_\kappa(\psi_{p,\kappa}(x_0)) \end{pmatrix}, \tag{23}$$

or

$$\begin{pmatrix} y \\ \vdots \\ y^{(r_\kappa - 1)} \end{pmatrix}(t_0) = \begin{pmatrix} h_\kappa(\psi_{q,\kappa}(x_0)) \\ \vdots \\ L^{r_\kappa - 1}_{f_\kappa} h_\kappa(\psi_{q,\kappa}(x_0)) \end{pmatrix} \tag{24}$$

where $\kappa = p, q \in \mathcal{P}$ and $\psi_{p,p}(x_0) = \psi_{q,q}(x_0) = x_0$, $\forall p, q \in \mathcal{P}$. Eq. (22) would also be modified similarly when dealing with MIMO systems. The statement of Theorem 1, in either case, remains unchanged.

Another generalization is to include switching mechanisms, such as *switching surfaces*. Denote by $S_{p,q}$ the switching surface for subsystem p, where the switched system jumps to subsystem q. Then we only need to check for the switch-singularity of $x_0 \in S_{p,q}$ and $x_0 \in S_{q,p}$ instead of $x_0 \in \mathbb{M}$ for the two subsystems Γ_p, Γ_q. □

5. Output generation

In the previous section, we considered the question of left invertibility where the objective was to recover (σ, u) uniquely for all y in some output set $\mathcal{Y}^\alpha$. In this section, we address a different problem which concerns with finding (σ, u) (that may not be unique) such that $\mathrm{H}_{x_0}(\sigma, u) = y_d$ for a given function y_d and a state x_0. For the invertibility problem, we found conditions on the subsystems and the output set $\mathcal{Y}$ so that the map H_{x_0} is injective. Here, we are given one particular (x_0, y_d) and wish to find its preimage under the map H_{x_0}. For the switched system (3), denote by $\mathrm{H}^{-1}_{x_0}(y_d)$ the preimage of a function y_d,

$$\mathrm{H}^{-1}_{x_0}(y_d) := \{(\sigma, u) : \mathrm{H}_{x_0}(\sigma, u) = y_d\}. \tag{25}$$

If y_d is not in the image set of H_{x_0} then by convention $\mathrm{H}^{-1}_{x_0} = \emptyset$. When $\mathrm{H}^{-1}_{x_0}(y_d)$ is a singleton, the system is invertible at x_0. We want to find conditions and an algorithm to generate $\mathrm{H}^{-1}_{x_0}(y_d)$ when $\mathrm{H}^{-1}_{x_0}(y_d)$ is a finite set.

We require the individual subsystems to be invertible at x_0 because if this is not the case, then the set $\mathrm{H}^{-1}_{x_0}(y_d)$ may be infinite. When a square nonswitched nonlinear system is not invertible, the matrix $\widetilde{B}^{-1}_\alpha$ in (19) is not defined and the expression for u is modified to:

$$u(t) = \widetilde{B}^\dagger_\alpha[\tilde{Y}_\alpha - \tilde{A}_\alpha] + K(x, \tilde{Y}_{\alpha-1})v \tag{26}$$

where K is a matrix whose columns form a basis for the null space of $\widetilde{B}_\alpha$ and $\widetilde{B}^\dagger_\alpha := \widetilde{B}^T_\alpha(\widetilde{B}_\alpha\widetilde{B}^T_\alpha)^{-1}$ is a right pseudo-inverse of $\widetilde{B}_\alpha$. If an output is generated by some input u obtained from (26) with some initial state, then due to arbitrary choice of v, there always exist infinitely many different inputs that generate the same output with the same initial state. Hence to avoid infinite loop reasoning, we will assume that the individual subsystems Γ_p are invertible at x_0 for all $p \in \mathcal{P}$. However, we do not assume that the switched system is invertible as the subsystems may have switch-singular pairs. We will only consider the functions $y_d(\cdot)$ over finite time intervals so that there is only a finite number of switches under consideration.

We now present a switching inversion algorithm for switched systems similar to the one given by Vu and Liberzon (2008). The algorithm takes the parameters $x_0 \in \mathbb{M}$, $y_d \in \mathcal{F}^{pc}$ (defined over a finite interval) and returns the set $\mathrm{H}_{x_0}^{-1}(y_d)$. It uses the *index-matching map*[7] $\Sigma^{-1}: \mathbb{M} \times \mathcal{F}^{pc} \to 2^{\mathcal{P}}$ defined as $\Sigma^{-1}(x_0, y_d) := \{p$ such that $y_d \in \mathcal{Y}_p^\alpha$ and y_d satisfies (21)$\}$, obtained via the structure algorithm of Γ_p. The index-matching map returns the indexes of the subsystems that are capable of generating y_d starting from x_0. If the returned set is empty, no subsystem is able to generate that y_d starting from x_0. Note that the index-matching map Σ^{-1} is defined for every pair (x_0, y_d) and always returns a set, whereas the index inversion function $\overline{\Sigma}^{-1}$ in (16) is defined only for (x_0, y_d) which are not switch-singular pairs and returns an element of $\mathcal{P}$.

In the algorithm, $\Gamma_{p,x_0}^{-1,O}(y_d)$ denotes the output of the inverse subsystem Γ_p^{-1}; the symbol "$\leftarrow$" reads "assigned as", and ":=" reads "defined as". The concatenation of an element η and a set S is $\eta \oplus S := \{\eta \oplus \zeta, \zeta \in S\}$. By convention, $\eta \oplus \emptyset = \emptyset, \forall \eta$. Finally, the concatenation of two sets S and T is $S \oplus T := \{\eta \oplus \zeta, \eta \in S, \zeta \in T\}$.

begin $\mathrm{H}_{x_0}^{-1}(y_{d[t_0,T)})$
- Let $\overline{\mathcal{P}} := \{p \in \mathcal{P} : y_{d[t_0,t_0+\varepsilon)} \in \mathcal{Y}_p^\alpha$ and $x_0 \in \mathbb{M}_p^\alpha, \varepsilon > 0\}$
- Let $t^* := \min\{t \in [t_0, T) : y_{d[t,t+\varepsilon)} \notin \mathcal{Y}_p^\alpha$ for some $p \in \overline{\mathcal{P}},\ \varepsilon > 0\}$ otherwise $t^* = T$.
- Let $\mathcal{P}^* := \Sigma^{-1}(x_0,\ y_{d[t_0,t_0+\varepsilon)})$.
- **if** $\mathcal{P}^* \neq \emptyset$ **then**
  - Let $\mathcal{A} := \emptyset$
  - **foreach** $p \in \mathcal{P}^*$ **do**
    - Let $x := \Gamma_{p,x_0}^{-1}(y_{d[t_0,t^*)})$
    - **if** $x \in \mathbb{M}_p^\alpha$ *and* $y_{d[t_0,t^*)} \in \mathcal{Y}_p^\alpha$ **then**
      - Let $u := \Gamma_{p,x_0}^{-1,O}(y_{d[t_0,t^*)})$
      - $\mathcal{T} := \{t \in (t_0, t^*) : (x(t), y_d(t))$ is a switch-singular pair of $\Gamma_p,\ \Gamma_q$ for some $q \neq p\}$.
      - **if** $\mathcal{T}$ *is a finite set* **then**
        - **foreach** $\tau \in \mathcal{T}$ **do**
          - let $\xi := \Gamma_p(u)(\tau)$
          - $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,\tau)},\ u_{[t_0,\tau)}) \oplus \mathrm{H}_\xi^{-1}(y_{d[\tau,T)})\}$
      - **else if** $\mathcal{T} = \emptyset$ *and* $t^* < T$ **then**
        - let $\xi := \Gamma_p(u)(t^*)$
        - $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,t^*)},\ u) \oplus \mathrm{H}_\xi^{-1}(y_{d[t^*,T)})\}$
      - **else if** $\mathcal{T} = \emptyset$ *and* $t^* = T$ **then**
        - $\mathcal{A} \leftarrow \mathcal{A} \cup \{(\sigma_{[t_0,T)}, u)\}$
      - **else**
        - $\mathcal{A} := \emptyset$
    - **else**
      - $\mathcal{A} := \emptyset$
- **else**
  - $\mathcal{A} := \emptyset$
- **return** $\mathrm{H}_{x_0}^{-1}(y_d) := \mathcal{A}$

end

The return set $\mathcal{A}$ is always finite and, if nonempty, it contains the pairs of switching signals and inputs that generate the given y_d starting from x_0. If the return is an empty set, it means that there is no switching signal and input that generate y_d, or there is an infinite number of possible switching times. Also by our concatenation notation, if at any instant of time, the return of the procedure is an empty set, then that branch of the search will be empty because $\eta \oplus \emptyset = \emptyset$.

Based on the semigroup property for the trajectories of dynamical systems, the algorithm determines the switching signal and the input on a subinterval $[t_0, t)$ of $[t_0, T)$ and then concatenates these signals with the corresponding preimage on the rest of the interval $[t, T)$. If t is the first switching time after t_0, then we can find $H_{x_0}^{-1}(y_{d[t_0,t)})$ by singling out which subsystems are capable of generating $y_{d[t_0,t)}$ using the index-matching map. The obvious candidate for first switching time, denoted by t^* in the algorithm, is the time at which the output loses smoothness. Note that in the SISO case, t^* is the time at which one of the first $r - 1$ derivatives of the output lose continuity (see Section 4.1). But, it is entirely possible that we have a switching at some time instant τ and the output is still smooth (see Example 6). In this case, $(x(\tau), y_{[\tau,\tau+\varepsilon)})$ forms a switch-singular which, in SISO case, can be checked by using (9), or for the systems with reset maps, using (23) or (24). The algorithm keeps track of all the switch-singular pairs encountered along the trajectory of the motion and uses a switch at a later time to recover a "hidden switch" earlier (e.g. a switch at which the output is smooth). This makes the switching inversion algorithm a recursive procedure calling itself with different parameters within the main algorithm (e.g. the function $\mathrm{H}_{x_0}^{-1}(y_d)$ uses the returns of $\mathrm{H}_\xi^{-1}(y_{d[t^*,T)})$).

The following example should help understand this algorithm.

Example 6. Consider a SISO switched system with two modes

$$\Gamma_1: \begin{cases} \dot{x} = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, & \mathbb{M} = \mathbb{R}^2 \\ y = x_2 \end{cases}$$

$$\Gamma_2: \begin{cases} \dot{x} = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} + \begin{pmatrix} e^{x_2} \\ e^{x_2} \end{pmatrix} u, & \mathbb{M} = \mathbb{R}^2 \\ y = x_1. \end{cases}$$

We wish to reconstruct the switching signal $\sigma(\cdot)$ and the input $u(\cdot)$ which will generate the following output:

$$y_d(t) = \begin{cases} \cos t & \text{if } t \in [0, t^*) \\ 2\cos t & \text{if } t \in [t^*, T) \end{cases}$$

where $t^* = \pi$ and $T = 4.5$, with the given initial state $x_0 = (0, 1)^T$.

In this example, $(x_0, y_{[t_0,t_0+\varepsilon)})$ form a switch-singular pair if, for some $c \in \mathbb{R}$, $x_0 = \binom{c}{c}$ and $y(t_0) = c$.

We now use the above switching inversion algorithm to find (σ, u) such that $\Gamma_{x_0,\sigma}^O(u) = y_d$. We have $\overline{\mathcal{P}} = \{1, 2\}$ and $\mathcal{P}^* := \Sigma^{-1}(x_0, y_{d[0,t^*)}) = \{1\}$ by using the index-matching map with given x_0 and $y_d(0) = 1$. The inverse of Γ_1 on $[0, t^*)$ is

$$\Gamma_1^{-1}: \begin{cases} \dot{z} = \begin{pmatrix} z_1 z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \dot{y}_d, & \mathbb{M}_1^\alpha = \mathbb{R}^2 \\ u(t) = -z_2 + \dot{y}_d \end{cases}$$

with $z(0) = x_0$, which then gives

$$\begin{aligned} z(t) &= \begin{pmatrix} 0 \\ \cos t \end{pmatrix} =: x(t) \qquad t \in [0, t^*). \\ u(t) &= -\cos t - \sin t \end{aligned} \tag{27}$$

We want to find $\mathcal{T} := \{t \leq t^* : (x(t), y_{d[t,t^*)})$ is a switch-singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving

$\cos t = x_1(t) = 0, \quad t \in (0, t^*)$.

This equation has a solution $t = \pi/2 =: t_1 < t^*$, and hence $\mathcal{T} = \{t_1\}$, a finite set. We have $\xi = x(t_1) = (0, 0)^T$ and we repeat the procedure for the initial state ξ and the output $y_{d[t_1,T)}$ with $\mathcal{P}^* := \Sigma^{-1}(\xi, y_{d[t_1,t^*)}) = \{1, 2\}$. We analyze these two cases:

Case 1. $p = 1$. This implies t_1 is not a switching time, i.e., $\sigma(t) = 1$ for $t \in [t_0, t^*)$ and $u(t), x(t)$ are given by (27) for $0 \leq t < t^*$, which gives $\xi = x(t^*) = (0, -1)^T$. At t^*, Γ_2 must be active, but then $y(t^*) = x_1(t^*) = 0 \neq -2 = y_d(t^*)$, thus the index-matching map returns an empty set, $\Sigma^{-1}(\xi, y_{d[t^*,T)}) = \emptyset$.

[7] The set $2^{\mathcal{P}}$ denotes the set of all subsets of the set $\mathcal{P}$.

Case 2. $p = 2$, which means that t_1 is a switching instant. So we work with the inverse system of Γ_2,

$$\Gamma_2^{-1}: \begin{cases} \dot{z} = \begin{pmatrix} 0 \\ z_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \dot{y}_d, \qquad \mathbb{M}_1^\alpha = \mathbb{R}^2 \\ u(t) = \mathrm{e}^{-z_2} \dot{y}_d \end{cases}$$

with initial state $z(t_1) = \xi$, which gives

$$\begin{aligned} z(t) &= \begin{pmatrix} \cos t \\ \cos t + \sin t - 1 \end{pmatrix} =: x(t) \qquad t \geq t_1. \\ u(t) &= -\mathrm{e}^{(1-\cos t - \sin t)} \sin t \end{aligned}$$

We find $\mathcal{T} = \{t_1 < t \leq t^* : (x(t), y_{d[t,t^*)})$ is a switch-singular pair of $\Gamma_1, \Gamma_2\}$, which is equivalent to solving for

$$\cos t = \cos t + \sin t - 1, \qquad \frac{\pi}{2} = t_1 < t \leq t^* = \pi.$$

It is easy to see that this equation has no solution and thus there exist no switch-singular pairs in the interval (t_1, t^*). So, we let $\xi = x(t^*) = (-1, -2)^T$ and repeat the procedure with ξ and $y_{d[t^*,T)}$, which gives the unique solution $\sigma_{[t^*,T)} = 1$ and $u_{[t^*,T)} = -2(\cos t + \sin t)$.

Thus, the switching inversion algorithm returns (σ, u), where

$$(\sigma, u) = \begin{cases} (1, -\cos t - \sin t), & \text{if } 0 \leq t < t_1 \\ (2, -\mathrm{e}^{(1-\cos t - \sin t)} \sin t), & \text{if } t_1 \leq t < t^* \\ (1, -2(\cos t + \sin t)), & \text{if } t^* \leq t \leq T. \end{cases}$$

In this example, two switches are required to generate the given output. One of the switching instants is t^* as the output loses smoothness at that instant. The other switching instant is t_1 where the output does not lose smoothness. Without the concept of switch-singular pairs, one may try all the four possible combinations with t^* as the only switching instant and arrive at the false conclusion that there is no switching signal and input that generate $y_d(t)$; but instead the use of the switching inversion algorithm allows us to construct the input and switching signal. □

6. Conclusions

In this paper, we addressed the invertibility problem of switched nonlinear systems. The concepts introduced by Vu and Liberzon (2008) for linear systems were extended to nonlinear systems. We gave a necessary and sufficient condition for a switched system to be invertible, according to which the individual subsystems should be invertible and there should be no switch-singular pairs. We developed formulae for checking if (x_0, y) is a switch-singular pair of two subsystems and then gave an algorithm that finds switching signals and inputs, possibly non-unique, which generate a given output with a given initial state.

For future work, one interesting problem is to develop conditions for checking the existence of switch-singular pairs which are more constructive as it is in general not feasible to verify (22) for every output and state. Another research direction is to approach the problem geometrically and investigate characterizations equivalent to nonexistence of switch-singular pairs to obtain geometric criteria for left invertibility of switched systems.

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Appendix

Proof of Lemma 4 (*Necessity*)**.** Supposing $\exists \varepsilon > 0$ and input u defined over the interval $[t_0, t_0 + \varepsilon)$, such that $\Gamma_{x_0}^O(u(t)) = y_d(t)$, $\forall t \in [t_0, t_0 + \varepsilon)$, then

$$\begin{aligned} & y_d(t_0) = y(t_0) = h(x_0) \\ & \hat{y}_{d_1}(t_0) = \hat{y}_1(t_0) = \hat{h}^1(x, \dot{\tilde{y}}_1) = \hat{h}^1(x, \dot{\tilde{y}}_{d_1}) \\ & \vdots \\ & \hat{y}_{d_{\alpha-1}}^{(\alpha-1)}(t_0) = \hat{y}_{\alpha-1}^{(\alpha-1)}(t_0) \\ & \quad = \hat{h}^{\alpha-1}\left(x_0, \dot{\tilde{y}}_1, \ldots, \tilde{y}_1^{(\alpha-1)}, \ldots, \tilde{y}_{\alpha-1}^{(\alpha-1)}\right) \\ & \quad = \hat{h}^{\alpha-1}\left(x_0, \dot{\tilde{y}}_{d_1}, \ldots, \tilde{y}_{d_1}^{(\alpha-1)}, \ldots, \tilde{y}_{d_{\alpha-1}}^{(\alpha-1)}\right) \end{aligned}$$

and hence Eq. (21) is satisfied.

Sufficiency: If we inject $y_d(t)$ into the inverse system, then the control input produced by this inverse system is given by (19) with $\tilde{y}$ replaced by $\tilde{y}_d$, and substituting it in the α-th step of the structure algorithm

$$\begin{aligned} & \dot{\tilde{y}}_1 = \tilde{a}_1(x) + \tilde{b}_1(x)u, \\ & \vdots \\ & \tilde{y}_\alpha^{(\alpha)} = \tilde{a}_\alpha(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \alpha - 1, i \leq j \leq \alpha\}) \\ & \qquad + \tilde{b}_\alpha(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq \alpha - 1, i \leq j \leq \alpha - 1\})u, \end{aligned}$$

we get

$$\dot{\tilde{y}}_1(t) = \dot{\tilde{y}}_{d_1}(t), \quad \forall t \in [t_0, t_0 + \varepsilon). \tag{28}$$

Here $t_0 + \varepsilon$ characterizes the time instant at which the trajectory of the inverse system hits the singular point in the state space. As the system is strongly invertible at x_0, it is guaranteed that $\varepsilon > 0$.

Using hypothesis (21), we have $h(x_0) = y_d(t_0)$, and integrating (28) on both sides over the interval $[t_0, t_0 + \varepsilon)$ to get

$$\tilde{y}_1(t) = \tilde{y}_{d_1}(t), \quad \forall t \in [t_0, t_0 + \varepsilon). \tag{29}$$

Using the initial conditions characterized by (21), the desired result can now be derived by induction. Suppose Eqs. (28) and (29) are true for index k, that is

$$\begin{aligned} & \tilde{y}_k^{(k)}(t) = \tilde{y}_{d_k}^{(k)}(t) \quad \forall t \in [t_0, t_0 + \varepsilon), \\ & \tilde{y}_k(t) = \tilde{y}_{d_k}(t) \quad \forall t \in [t_0, t_0 + \varepsilon). \end{aligned}$$

Since $\tilde{y}_{k+1}^{(k+1)} = \tilde{a}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k, i \leq j \leq k+1\}) + \tilde{b}_{k+1}(x, \{\tilde{y}_i^{(j)} \mid 1 \leq i \leq k, i \leq j \leq k\})u$, substituting u from (19) as generated by the inverse system, with $\tilde{y}$ replaced by $\tilde{y}_d$, gives

$$\tilde{y}_{k+1}^{(k+1)}(t) = \tilde{y}_{d_{k+1}}^{(k+1)}(t) \quad \forall\, t \in [t_0, t_0 + \varepsilon).$$

Again using hypothesis (21) and integrating both sides, we get

$$\tilde{y}_{k+1}(t) = \tilde{y}_{d_{k+1}}(t) \quad \forall\, t \in [t_0, t_0 + \varepsilon).$$

As $y(t) = (\tilde{y}_1(t), \ldots, \tilde{y}_\alpha(t))$, we get $\Gamma_{x_0}^O(u(t)) = y_d(t), \forall t \in [t_0, t_0 + \varepsilon)$. □

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