Proceedings of the 38th Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Lie-algebraic conditions for exponential stability of switched systems

Andrei A. Agrachev
Steklov Math. Inst., Moscow, Russia
and S.I.S.S.A.–I.S.A.S., Trieste, Italy
agrachev@sissa.it

Daniel Liberzon*
Dept. of Electrical Engineering
Yale University, New Haven, CT
liberzon@sysc.eng.yale.edu

Abstract

It has recently been shown that a family of exponentially stable linear systems whose matrices generate a solvable Lie algebra possesses a quadratic common Lyapunov function, which implies that the corresponding switched linear system is exponentially stable for arbitrary switching. In this paper we prove that the same properties hold under the weaker condition that the Lie algebra generated by given matrices can be decomposed into a sum of a solvable ideal and a subalgebra with a compact Lie group. The corresponding local stability result for nonlinear switched systems is also established. Moreover, we demonstrate that if a Lie algebra fails to satisfy the above condition, then it can be generated by a family of stable matrices such that the corresponding switched linear system is not stable. Relevant facts from the theory of Lie algebras are collected at the end of the paper for easy reference.

1 Introduction

A switched system can be described by a family of continuous-time subsystems and a rule that orchestrates the switching between them. Such systems arise, for example, when different controllers are being placed in the feedback loop with a given process, or when a given process exhibits a switching behavior caused by abrupt changes of the environment. For a discussion of various issues related to switched systems, see the recent survey article [6].

To define more precisely what we mean by a switched system, consider a family $\{f_p : p \in \mathcal{P}\}$ of sufficiently regular functions from $\mathbb{R}^n$ to $\mathbb{R}^n$, parameterized by some index set $\mathcal{P}$. Let $\sigma : [0, \infty) \to \mathcal{P}$ be a piecewise constant function of time, called a *switching signal*. A *switched system* is then given by the following system of differential equations in $\mathbb{R}^n$:

$$\dot{x} = f_\sigma(x). \tag{1}$$

Note that infinitely fast switching (chattering), which calls for a concept of generalized solution, is not considered in this paper. In the particular case when all the individual subsystems are linear (i.e., $f_p(x) = A_p x$ where $A_p \in \mathbb{R}^{n\times n}$ for each $p \in \mathcal{P}$), we obtain a *switched linear system*

$$\dot{x} = A_\sigma x. \tag{2}$$

This paper is concerned with the following problem: find conditions on the individual subsystems which guarantee that the switched system is asymptotically stable for an arbitrary switching signal σ. In fact, a somewhat stronger property is desirable, namely, exponential stability that is uniform over the set of all switching signals. Clearly, all the individual subsystems must be asymptotically stable, and we will assume this to be the case throughout the paper. Note that it is not hard to construct examples where instability can be achieved by switching between asymptotically stable systems, so one needs to determine what additional requirements must be imposed.

Commutation relations among the individual subsystems play an important role in the context of the problem posed above. This can be illustrated with the help of the following example. Consider the switched linear system (2), take $\mathcal{P}$ to be a finite set, and suppose that the matrices A_p commute pairwise: $A_pA_q = A_qA_p$ for all $p, q \in \mathcal{P}$. Then it is easy to show directly that the switched linear system is exponentially stable, uniformly over all switching signals. Alternatively, one can construct a quadratic common Lyapunov function for the family of linear systems

$$\dot{x} = A_p x, \qquad p \in \mathcal{P} \tag{3}$$

as shown in [9], which is well known to lead to the same conclusion.

In this paper we undertake a systematic study of the connection between the behavior of the switched system and the commutation relations among the individual subsystems. In the case of the switched linear system (2), a useful object that reveals the nature of these commutation relations is the Lie algebra $\mathfrak{g} := \{A_p : p \in \mathcal{P}\}_{LA}$ generated by the matrices A_p, $p \in \mathcal{P}$ (with respect to the standard Lie bracket $[A_p, A_q] := A_pA_q - A_qA_p$). The observation that the structure of this Lie algebra is relevant to stability of

*Supported by ARO DAAH04-95-1-0114, NSF ECS 9634146, and AFOSR F49620-97-1-0108.

(2) goes back to the paper by Gurvits [4]. That paper studied the discrete-time counterpart of (2)

$$x(k+1) = A_{\sigma(k)}x(k) \tag{4}$$

where σ is a function from nonnegative integers to a finite index set $\mathcal{P}$ and $A_p = e^{L_p}$, $p \in \mathcal{P}$ for some matrices L_p. Gurvits conjectured that if the Lie algebra $\{L_p : p \in \mathcal{P}\}_{LA}$ is nilpotent (which means that Lie brackets of sufficiently high order equal zero), then the system (4) is asymptotically stable for any switching signal σ. He was able to prove this conjecture for the particular case when $\mathcal{P} = \{1, 2\}$ and the third-order Lie brackets vanish: $[L_1, [L_1, L_2]] = [L_2, [L_1, L_2]] = 0$.

It was recently shown in [5] that the switched linear system (2) is exponentially stable for arbitrary switching if the Lie algebra $\mathfrak{g}$ is solvable (see Section A.3 for the definition). The proof relied on the facts that matrices in a solvable Lie algebra can be simultaneously put in the upper-triangular form (Lie's Theorem) and that a family of linear systems with stable upper-triangular matrices has a quadratic common Lyapunov function. For the result to hold, the index set $\mathcal{P}$ does not need to be finite. One can derive the corresponding result for discrete-time systems in similar fashion, thereby confirming and directly generalizing the statement conjectured by Gurvits (because every nilpotent Lie algebra is solvable).

In the present paper we continue the line of work initiated in the above references. Our main theorem is a direct generalization of the one proved in [5]. The new result states that one still has exponential stability for arbitrary switching if the Lie algebra $\mathfrak{g}$ is a semidirect sum of a solvable ideal and a subalgebra with a compact Lie group (which amounts to saying that all the matrices in this second subalgebra have purely imaginary eigenvalues). The corresponding local stability result for the nonlinear switched system (1) is also established. Being formulated in terms of the original data, such Lie-algebraic stability criteria have an important advantage over results that depend on a particular choice of coordinates, such as the one reported in [8]. Moreover, we demonstrate that the above condition is in some sense the strongest one that can be given on the Lie algebra level. Loosely speaking, we show that if a Lie algebra does not satisfy this condition, then it could be generated by a switched linear system that is not stable.

More precisely, the main contributions of the paper can be summarized as follows (see the Appendix for an overview of relevant definitions and facts from the theory of Lie algebras). Given a matrix Lie algebra $\hat{\mathfrak{g}}$ which contains the identity matrix, we are interested in the following question: Is it true that any set of stable generators for $\hat{\mathfrak{g}}$ gives rise to a switched system that is exponentially stable, uniformly over all switching signals? We discover that the above property depends only on the structure of $\hat{\mathfrak{g}}$ as a Lie algebra, and not on the choice of a particular matrix representation of $\hat{\mathfrak{g}}$. The following equivalent characterizations of this property can be given:

1. The factor algebra $\hat{\mathfrak{g}}$ mod $\mathfrak{r}$, where $\mathfrak{r}$ denotes the radical, is a compact Lie algebra.
2. The Killing form is negative semidefinite on $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$.
3. The Lie algebra $\hat{\mathfrak{g}}$ does not contain any subalgebras isomorphic to $sl(2)$.

2 Preliminaries

The switched system (1) is called (locally) *uniformly exponentially stable* (UES) if there exist positive constants M, c and μ such that for any switching signal σ the solution of (1) with $\|x(0)\| \le M$ satisfies

$$\|x(t)\| \le ce^{-\mu t}\|x(0)\| \quad \forall t \ge 0. \tag{5}$$

If there exist positive constants c and μ such that the estimate (5) holds for any switching signal σ and any initial condition $x(0)$, then the switched system is called *globally uniformly exponentially stable* (GUES). For switched linear systems the two concepts are equivalent [7].

In the context of the switched linear system (2), we will always assume that $\{A_p : p \in \mathcal{P}\}$ is a compact (with respect to the usual topology in $\mathbb{R}^{n\times n}$) set of real $n \times n$ matrices with eigenvalues in the open left half-plane. The following stability criterion was established in [5]. It will be crucial in proving our main result (Theorem 2 in the next section).

Theorem 1 *If $\mathfrak{g}$ is a solvable Lie algebra, the switched linear system (2) is GUES.*

Remark 1. The proof of this result given in [5] relies on a construction of a quadratic common Lyapunov function for the family of linear systems (3). The existence of such a function actually implies GUES of the time-varying system $\dot{x} = A_\sigma x$ with σ not necessarily piecewise constant. This observation will be used in the proof of Theorem 2.

The above condition can always be checked directly in a finite number of steps if $\mathcal{P}$ is a finite set. Alternatively, one can use the standard criterion for solvability in terms of the Killing form. Similar criteria exist for checking the conditions to be presented in the next section—see Sections A.3 and A.4 for details.

3 Sufficient conditions for stability

Let $\mathfrak{g}$ be the Lie algebra defined by $\mathfrak{g} = \{A_p : p \in \mathcal{P}\}_{LA}$ as before. Consider its Levi decomposition $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$, where $\mathfrak{r}$ is the radical and $\mathfrak{s}$ is a semisimple subalgebra (see Section A.4). We will now prove the following generalization of Theorem 1.

Theorem 2 *If $\mathfrak{s}$ is a compact Lie algebra, the switched linear system (2) is GUES.*

PROOF. For an arbitrary $p \in \mathcal{P}$, write $A_p = r_p + s_p$ with $r_p \in \mathfrak{r}$ and $s_p \in \mathfrak{s}$. Let us show that r_p is a stable matrix. Writing

$$e^{(r_p+s_p)t} = e^{s_p t} B_p(t) \tag{6}$$

we have the following equation for $B_p(t)$:

$$\dot{B}_p(t) = e^{-s_p t} r_p e^{s_p t} B_p(t), \qquad B_p(0) = I. \tag{7}$$

To verify (7), differentiate the equality (6) with respect to t, which gives

$$(r_p + s_p)e^{(r_p+s_p)t} = s_p e^{s_p t} B_p + e^{s_p t} \dot{B}_p.$$

Using (6) again, we have

$$r_p e^{s_p t} B_p + s_p e^{s_p t} B_p = s_p e^{s_p t} B_p + e^{s_p t} \dot{B}_p$$

hence (7) holds. Define $c_p(t) := e^{-s_p t} r_p e^{s_p t}$. Clearly, $\text{spec}(c_p(t)) = \text{spec}(r_p)$ for all t. One has the standard expansion

$$c_p(t) = r_p + [s_p t, r_p] + \frac{1}{2}[s_p t, [s_p t, r_p]] + \dots$$

Since $[\mathfrak{s}, \mathfrak{r}] \subseteq \mathfrak{r}$, we see that $c_p(t) \in \mathfrak{r}$. According to Lie's Theorem, there exists a basis in which all matrices from $\mathfrak{r}$ are upper-triangular. Combining the above facts, it is not hard to check that $\text{spec}(B_p(t)) = e^{t\text{spec}(r_p)}$. Now it follows from (7) that $\text{spec}(r_p)$ lies in the open left half of the complex plane. Indeed, as $t \to \infty$ we have $e^{(r_p+s_p)t} \to 0$ because the matrix A_p is stable. Since $\mathfrak{s}$ is compact, there exists a constant $C > 0$ such that we have $|e^s x| \geq C|x|$ for all $s \in \mathfrak{s}$ and $x \in \mathbb{R}^n$, thus we cannot have $e^{s_p t} x \to 0$ for $x \neq 0$. Therefore, $B_p(t) \to 0$, and so r_p is stable.

Since $p \in \mathcal{P}$ was arbitrary, we see that all the matrices r_p, $p \in \mathcal{P}$ are stable. Theorem 1 implies that the switched linear system generated by these matrices is GUES. Moreover, the same property holds for matrices in the extended set $\bar{\mathfrak{r}} := \{\bar{A} : \exists p \in \mathcal{P} \text{ and } s \in \mathfrak{s} \text{ such that } \bar{A} = e^{-s} r_p e^s\}$. This is true because the matrices in this set are stable and belong to $\mathfrak{r}$ (the last statement follows from the same expansion as the one used earlier for $c_p(t)$). The transition matrix of the original switched linear system (2) at time t takes the form

$$\Phi(t,0) = e^{(r_{p_1}+s_{p_1})t_1} \cdots e^{(r_{p_k}+s_{p_k})t_k} = e^{s_{p_1}t_1} B_{p_1} \cdots e^{s_{p_k}t_k} B_{p_k}$$

where $t_1 + \dots + t_k = t$ and $\dot{B}_{p_i}(t) = e^{-s_{p_i}t} r_{p_i} e^{s_{p_i}t} B_{p_i}(t)$, $i = 1, \dots, k$. To simplify the notation, let $k = 2$ (in the general case one can adopt the same line of reasoning or use induction on k). We can then write

$$\begin{aligned}\Phi(t,0) &= e^{s_{p_1}t_1} e^{s_{p_2}t_2} e^{-s_{p_2}t_2} B_{p_1}(t_1) e^{s_{p_2}t_2} B_{p_2}(t_2) \\ &= e^{s_{p_1}t_1} e^{s_{p_2}t_2} \tilde{B}_{p_1}(t_1) B_{p_2}(t_2)\end{aligned}$$

where $\tilde{B}_{p_1}(t) := e^{-s_{p_2}t_2} B_{p_1}(t) e^{s_{p_2}t_2}$. We have

$$\begin{aligned}\frac{d}{dt}\tilde{B}_{p_1}(t) &= e^{-s_{p_2}t_2} e^{-s_{p_1}t} r_{p_1} e^{s_{p_1}t} B_{p_1}(t) e^{s_{p_2}t_2} \\ &= e^{-s_{p_2}t_2} e^{-s_{p_1}t} r_{p_1} e^{s_{p_1}t} e^{s_{p_2}t_2} \tilde{B}_{p_1}(t)\end{aligned}$$

Thus we see that

$$\Phi(t,0) = e^{s_{p_1}t_1} e^{s_{p_2}t_2} \cdot \bar{B}(t) \tag{8}$$

where $\bar{B}(t)$ is the transition matrix of a switched/time-varying system generated by matrices in $\bar{\mathfrak{r}}$, i.e., $\frac{d}{dt}\bar{B}(t) = \bar{A}(t)\bar{B}(t)$ with $\bar{A}(t) \in \bar{\mathfrak{r}}\ \forall t \geq 0$. The norm of the first term in the above product is bounded by compactness, while the norm of the second goes to zero exponentially by Theorem 1 (see also Remark 1), and the statement of the theorem follows. ∎

Remark 2. The fact that $\mathfrak{r}$ is the radical, implying that $\mathfrak{s}$ is semisimple, was not used in the proof. The statement of Theorem 2 remains valid for any decomposition of $\mathfrak{g}$ into the sum of a solvable ideal $\mathfrak{r}$ and a subalgebra $\mathfrak{s}$. Among all possible decompositions of this kind, the one considered above gives the strongest result. If $\mathfrak{g}$ is solvable, then $\mathfrak{s} = 0$ is of course compact, and we recover Theorem 1 as a special case.

Example 1. Suppose that the matrices A_p, $p \in \mathcal{P}$ take the form $A_p = -\lambda_p I + S_p$ where $\lambda_p > 0$ and $S_p^T = -S_p$ for all $p \in \mathcal{P}$. These are automatically stable matrices. Suppose also that $\text{span}\{A_p, p \in \mathcal{P}\} \ni I$. Then the condition of Theorem 2 is satisfied. Indeed, take $\mathfrak{r} = \{\lambda I : \lambda \in \mathbb{R}\}$ (scalar multiples of the identity matrix) and observe that the Lie algebra $\{S_p : p \in \mathcal{P}\}_{LA}$ is compact because skew-symmetric matrices have purely imaginary eigenvalues.

In [5] the GUES property was deduced from the existence of a quadratic common Lyapunov function. In the present case we found it more convenient to obtain the desired result directly. However, under the hypothesis of Theorem 2 a quadratic common Lyapunov function for the family of linear systems (3) can also be constructed, as we now show. Let $\bar{V}(x) = x^T Q x$ be a quadratic common Lyapunov function for the family of linear systems generated by matrices in $\bar{\mathfrak{r}}$ (which exists according to [5]). Define the function

$$V(x) := \int_{\mathcal{S}} \bar{V}(Sx)dS = x^T \cdot \int_{\mathcal{S}} S^T Q S dS \cdot x$$

where $\mathcal{S}$ is the Lie group corresponding to $\mathfrak{s}$ and the integral is taken with respect to the Haar measure invariant under right translation on $\mathcal{S}$ (see Section A.4). Using (8), it is straightforward to show that the derivative of V along solutions of (2) satisfies

$$\frac{d}{dt}V(x(t)) = \frac{d}{dt}\int_{\mathcal{S}} \bar{V}(S\bar{B}(t)x(0))dS = \int_{\mathcal{S}} x^T(0)\bar{B}^T(t)S^T$$
$$((S\bar{A}(t)S^{-1})^T Q + QS\bar{A}(t)S^{-1})S\bar{B}(t)x(0)dS < 0.$$

The first equality in the above formula follows from the invariance of the measure, and the last inequality holds because $S\bar{A}(t)S^{-1} \in \bar{\mathfrak{r}}$ for all $t \geq 0$ and all $S \in \mathcal{S}$.

Remark 3. It is now clear that the above results remain valid if piecewise constant switching signals are replaced by arbitrary measurable functions (cf. Remark 1).

The existence of a quadratic common Lyapunov function is needed to prove Corollary 3 below. It is also an interesting fact in its own right because, although the converse Lyapunov theorem proved in [7] implies that GUES always leads to the existence of a common Lyapunov function, in some cases it is not possible to find a quadratic one [1]. Incidentally, this clearly shows that the condition of Theorem 2 is not necessary for GUES of the switched linear system (2). Another way to see this is to note that the GUES property is robust with respect to small perturbations of the parameters of the system, whereas the above Lie-algebraic condition is not.

We conclude this section with a local stability result for the nonlinear switched system (1). Let $f_p : D \to \mathbb{R}^n$ be continuously differentiable with $f_p(0) = 0$ for each $p \in \mathcal{P}$, where D is a neighborhood of the origin in $\mathbb{R}^n$. Consider the linearization matrices

$$F_p := \frac{\partial f_p}{\partial x}(0), \qquad p \in \mathcal{P}.$$

Assume that the matrices F_p are stable, that $\mathcal{P}$ is a compact subset of some topological space, and that $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p for each $x \in D$. Consider the Lie algebra $\tilde{\mathfrak{g}} := \{F_p : p \in \mathcal{P}\}_{LA}$ and its Levi decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{r}} \oplus \tilde{\mathfrak{s}}$. The following statement is a relatively straightforward consequence of Lyapunov's first method; see also [5, Corollary 5].

Corollary 3 *If $\tilde{\mathfrak{s}}$ is a compact Lie algebra, the switched system (1) is UES.*

4 A converse result

We find it useful to introduce a possibly larger Lie algebra $\hat{\mathfrak{g}}$ by adding to $\mathfrak{g}$ the scalar multiples of the identity matrix if necessary. In other words, define $\hat{\mathfrak{g}} := \{I, A_p : p \in \mathcal{P}\}_{LA}$. The Levi decomposition of $\hat{\mathfrak{g}}$ is given by $\hat{\mathfrak{g}} = \hat{\mathfrak{r}} \oplus \mathfrak{s}$ with $\hat{\mathfrak{r}} \supseteq \mathfrak{r}$ (because the subspace $\mathbb{R}I$ belongs to the radical of $\hat{\mathfrak{g}}$). Thus $\hat{\mathfrak{g}}$ satisfies the hypothesis of Theorem 2 if and only if $\mathfrak{g}$ does. Our goal in this section is to show that if this hypothesis is not satisfied, then $\hat{\mathfrak{g}}$ can be generated by a family of stable matrices (which might in principle be different from $\{A_p : p \in \mathcal{P}\}$) with the property that the corresponding switched linear system is not stable. Such a statement could in some sense be interpreted as a converse of Theorem 2. It would imply that by working just with $\hat{\mathfrak{g}}$ it is not possible to obtain a stronger result than the one given in the previous section.

In fact, we will prove a somewhat stronger statement, by showing that a desired set of generators can always be chosen in such a way that it contains the same number of elements as the original set that was used to generate $\hat{\mathfrak{g}}$. More precisely, let $\{A_1, A_2, \dots, A_m\}$ be any finite set of stable generators for $\hat{\mathfrak{g}}$ (if the index set $\mathcal{P}$ is infinite, a suitable finite subset can always be extracted from it). We then have the following result.

Theorem 4 *If $\mathfrak{s}$ is not a compact Lie algebra, there exists a set of m stable generators for $\hat{\mathfrak{g}}$ such that the corresponding switched linear system is not UES.*

Proof. It follows from basic properties of solutions to differential inclusions that if a family of matrices gives rise to a UES switched linear system, then all convex linear combinations of these matrices are stable (this fact is easily seen to be true from the converse Lyapunov theorems of [7, 1], although in [7] it was actually used to prove the result). To prove the theorem, we will first find a pair of stable matrices $B_1, B_2 \in \hat{\mathfrak{g}}$ with an unstable convex combination, and then use them to construct a desired set of generators.

Since $\mathfrak{s}$ is not compact, it contains a subalgebra isomorphic to $sl(2,\mathbb{R})$ which can be constructed as shown in Section A.5. Therefore, our task is to find a pair of matrices in a matrix representation of $sl(2,\mathbb{R})$ with an unstable convex combination. Since any matrix representation of $sl(2,\mathbb{R})$ is a direct sum of irreducible ones, there is no loss of generality in considering only irreducible representations. Their complete classification in all dimensions (up to equivalence induced by linear coordinate transformations) is available. In particular, it is known that any irreducible representation of $sl(2,\mathbb{R})$ contains two matrices of the following form:

$$\tilde{B}_1 = \begin{pmatrix} 0 & \mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mu_r \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

(cf. Section A.2). The matrix $\tilde{B}_1$ has positive entries $\mu_1, \dots, \mu_r$ immediately above the main diagonal and zeros elsewhere, and the matrix $\tilde{B}_2$ has ones immediately below the main diagonal and zeros elsewhere. It is not hard to check that the nonnegative matrix $\tilde{B} := (\tilde{B}_1 + \tilde{B}_2)/2$ is irreducible[1], and as such satisfies the assumptions of the Perron-Frobenius Theorem (see, e.g., [2, Chapter XIII]). According to that theorem, $\tilde{B}$ has a positive eigenvalue. Then for a small enough $\epsilon > 0$ the matrix $B := \tilde{B} - \epsilon I$ also has a positive eigenvalue. We have $B = (\tilde{B}_1 - \epsilon I + \tilde{B}_2 - \epsilon I)/2$. This implies that a desired pair of matrices in $\hat{\mathfrak{g}}$ can be defined by $B_1 := \tilde{B}_1 - \epsilon I$ and $B_2 := \tilde{B}_2 - \epsilon I$. Indeed, these matrices are stable, but their average is not.

For $\alpha \geq 0$, define $A_1(\alpha) := B_1 + \alpha A_1$ and $A_2(\alpha) := B_2 + \alpha A_2$. If α is small enough, then $A_1(\alpha)$ and $A_2(\alpha)$ are stable matrices, while $(A_1(\alpha) + A_2(\alpha))/2$ is unstable. Thus the matrices $A_1(\alpha), A_2(\alpha), A_3, \dots, A_m$ yield a switched system that is not UES. Moreover, it is not hard to show that for α small enough these matrices generate $\hat{\mathfrak{g}}$. Indeed, consider a basis for $\hat{\mathfrak{g}}$ formed by $A_1, \dots, A_m$, and their suitable Lie brackets. Replacing

[1] A matrix is called *irreducible* if it has no proper invariant subspaces spanned by coordinate vectors.

A_1 and A_2 in these expressions by $A_1(\alpha)$ and $A_2(\alpha)$ and writing the coordinates of the resulting elements relative to the above basis, we obtain a square matrix $\Delta(\alpha)$. Its determinant is a polynomial in α which tends to ∞ as $\alpha \to \infty$, and therefore is not identically zero. Thus $\Delta(\alpha)$ is nondegenerate for all but finitely many values of α; in particular, we will have a basis for $\hat{\mathfrak{g}}$ if we take α sufficiently small. This completes the proof. ■

The above result reveals the following important fact: the property of $\hat{\mathfrak{g}}$ which is being investigated here, namely, GUES of any switched system whose associated Lie algebra is $\hat{\mathfrak{g}}$, depends only on the structure of $\hat{\mathfrak{g}}$ (i.e., on the commutation relations between its matrices) and is independent of the choice of a particular representation.

It is also interesting to notice that a Lie algebra $\hat{\mathfrak{g}}$ containing the scalar multiples of I always has a set of stable generators such that the corresponding switched linear system is GUES. Indeed, $-I$ can be used as one of the generators, and then we can subtract λI from arbitrarily chosen other generators, where $\lambda > 0$ is large enough, so that the corresponding linear systems all share the common Lyapunov function $V(x) = x^T x$.

A Basic facts about Lie algebras

In this appendix we give an informal overview of basic properties of Lie algebras. Only those facts that play a role in the developments of the previous sections are discussed. The reader is referred to [10] and other standard references for more details.

A.1 Lie algebras and their representations

A *Lie algebra* $\mathfrak{g}$ is a finite-dimensional vector space equipped with a *Lie bracket*, i.e., a bilinear, skew-symmetric map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. Any Lie algebra $\mathfrak{g}$ can be identified with a tangent space at the identity of a Lie group $\mathcal{G}$ (an analytic manifold with a group structure). If $\mathfrak{g}$ is a matrix Lie algebra, then the elements of $\mathcal{G}$ are given by products of the exponentials of the matrices from $\mathfrak{g}$. For example, if $\mathfrak{g}$ is the Lie algebra $gl(n,\mathbb{R})$ of all real $n \times n$ matrices with the standard Lie bracket $[A,B] = AB - BA$, then the corresponding Lie group is given by the invertible matrices. Given an abstract Lie algebra $\mathfrak{g}$, one can consider its (matrix) representations. A *representation* of $\mathfrak{g}$ on an n-dimensional vector space V is a homomorphism (i.e., a linear map that preserves the Lie bracket) $\phi : \mathfrak{g} \to gl(V)$. It assigns to each element $g \in \mathfrak{g}$ a linear operator $\phi(g)$ on V, which can be described by an $n \times n$ matrix. A representation ϕ is called *irreducible* if V contains no nontrivial subspaces invariant under the action of all $\phi(g)$, $g \in \mathfrak{g}$. A particularly useful representation is the *adjoint* one, denoted by 'ad'. The vector space V in this case is $\mathfrak{g}$ itself, and for $g \in \mathfrak{g}$ the operator ad$g$ is defined by $\mathrm{ad}g(a) := [g,a]$, $a \in \mathfrak{g}$.

A.2 Example: $sl(2,\mathbb{R})$

The *special linear Lie algebra* $sl(2,\mathbb{R})$ consists of all real 2×2 matrices of trace 0. A canonical basis for this Lie algebra is given by the matrices

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (9)$$

They satisfy the relations $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$, and form what is sometimes called an $sl(2)$-*triple*. One can also consider other representations of $sl(2,\mathbb{R})$. It turns out that any irreducible representation of $sl(2,\mathbb{R})$ on a vector space of dimension n is equivalent (under a linear change of coordinates) to that given by

$$h \mapsto \begin{pmatrix} n-1 & \cdots & \cdots & 0 \\ \vdots & n-3 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -n+1 \end{pmatrix},$$

$$e \mapsto \begin{pmatrix} 0 & \mu_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mu_{n-1} \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

where $\mu_i = i(n-i)$, $i = 1,\ldots,n-1$. When $n = 2$, we recover the natural representation (9). Any representation of $sl(2,\mathbb{R})$ is a direct sum of irreducible ones.

A.3 Nilpotent and solvable Lie algebras

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are linear subspaces of a Lie algebra $\mathfrak{g}$, one writes $[\mathfrak{g}_1,\mathfrak{g}_2]$ for the linear space spanned by all the products $[g_1,g_2]$ with $g_1 \in \mathfrak{g}_1$ and $g_2 \in \mathfrak{g}_2$. Given a Lie algebra $\mathfrak{g}$, the sequence $\mathfrak{g}^{(k)}$ is defined inductively as follows: $\mathfrak{g}^{(1)} := \mathfrak{g}$, $\mathfrak{g}^{(k+1)} := [\mathfrak{g}^{(k)},\mathfrak{g}^{(k)}] \subset \mathfrak{g}^{(k)}$. If $\mathfrak{g}^{(k)} = 0$ for k sufficiently large, then $\mathfrak{g}$ is called *solvable*. Similarly, one defines the sequence $\mathfrak{g}^k$ by $\mathfrak{g}^1 := \mathfrak{g}$, $\mathfrak{g}^{k+1} := [\mathfrak{g},\mathfrak{g}^k] \subset \mathfrak{g}^k$, and calls $\mathfrak{g}$ *nilpotent* if $\mathfrak{g}^k = 0$ for k sufficiently large. Every nilpotent Lie algebra is solvable, but the converse is not true.

The *Killing form* on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form K given by $K(a,b) := \mathrm{tr}(\mathrm{ad}a \circ \mathrm{ad}b)$ for $a,b \in \mathfrak{g}$. *Cartan's 1st criterion* says that $\mathfrak{g}$ is solvable if and only if its Killing form vanishes identically on $[\mathfrak{g},\mathfrak{g}]$. Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field, and let ϕ be a representation of $\mathfrak{g}$ on a vector space V. *Lie's Theorem* states that there exists a basis for V with respect to which all the matrices $\phi(g)$, $g \in \mathfrak{g}$ are upper-triangular.

A.4 Semisimple and compact Lie algebras

A subalgebra $\bar{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is called an *ideal* if $[g,\bar{g}] \in \bar{\mathfrak{g}}$ for all $g \in \mathfrak{g}$ and $\bar{g} \in \bar{\mathfrak{g}}$. Any Lie algebra has a unique maximal solvable ideal $\mathfrak{r}$, the *radical*. A Lie algebra $\mathfrak{g}$ is called *semisimple* if its radical is 0. *Cartan's 2nd criterion* says that $\mathfrak{g}$ is semisimple if and only if its

Killing form is nondegenerate (meaning that if for some $g \in \mathfrak{g}$ we have $K(g, a) = 0\ \forall a \in \mathfrak{g}$, then g must be 0.)

A semisimple Lie algebra is called *compact* if its Killing form is negative definite. A general *compact Lie algebra* is a direct sum of a semisimple compact Lie algebra and a commutative Lie algebra (with zero Killing form). This terminology is justified by the facts that the tangent algebra of any compact Lie group is compact according to this definition, and that for any compact Lie algebra $\mathfrak{g}$ there exists a connected compact Lie group $\mathcal{G}$ with tangent algebra $\mathfrak{g}$. Compactness of a semisimple matrix Lie algebra $\mathfrak{g}$ amounts to the property that the eigenvalues of all matrices in $\mathfrak{g}$ lie on the imaginary axis. If $\mathcal{G}$ is a compact Lie group, one can associate to any continuous function $f : \mathcal{G} \to \mathbb{R}$ a real number $\int_{\mathcal{G}} f(G)dG$ so as to have $\int_{\mathcal{G}} 1dG = 1$ and $\int_{\mathcal{G}} f(AGB)dG = \int_{\mathcal{G}} f(G)dG$ $\forall A, B \in \mathcal{G}$ (left and right invariance). The measure dG is called the *Haar measure*.

An arbitrary Lie algebra $\mathfrak{g}$ can be decomposed into the semidirect sum $\mathfrak{g} = \mathfrak{r} \ominus \mathfrak{s}$, where $\mathfrak{r}$ is the radical, $\mathfrak{s}$ is a semisimple subalgebra, and $[\mathfrak{s}, \mathfrak{r}] \subseteq \mathfrak{r}$ because $\mathfrak{r}$ is an ideal. This is known as a *Levi decomposition.* To compute $\mathfrak{r}$ and $\mathfrak{s}$, switch to a basis in which the Killing form K is diagonalized. The subspace on which K is not identically zero corresponds to $\mathfrak{s} \oplus (\mathfrak{r} \bmod \mathfrak{n})$, where $\mathfrak{n}$ is the maximal nilpotent subalgebra of $\mathfrak{r}$. Construct the Killing form $\bar{K}$ for the factor algebra $\mathfrak{s} \oplus (\mathfrak{r} \bmod \mathfrak{n})$. This form will vanish identically on $(\mathfrak{r} \bmod \mathfrak{n})$ and will be nonsingular on $\mathfrak{s}$. The subalgebra $\mathfrak{s}$ identified in this way is compact if and only if $\bar{K}$ is negative definite on it. For more details on this construction and examples, see [3, pp. 256–258].

A.5 Subalgebras isomorphic to $sl(2, \mathbb{R})$

Let $\mathfrak{g}$ be a real, noncompact, semisimple Lie algebra. Our goal here is to show that $\mathfrak{g}$ has a subalgebra isomorphic to $sl(2, \mathbb{R})$. To this end, consider a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$ and $\mathfrak{p}$ is its orthogonal complement with respect to K. The Killing form K is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal commuting subalgebra of $\mathfrak{p}$. Then it is easy to check using the Jacobi identity that the operators ad$a$, $a \in \mathfrak{a}$ are commuting. These operators are also symmetric with respect to a suitable inner product on $\mathfrak{g}$ (for $a, b \in \mathfrak{g}$ this inner product is given by $-K(a, \Theta b)$, where Θ is the map sending $k + p$, with $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$, to $k - p$), hence they are simultaneously diagonalizable. Thus $\mathfrak{g}$ can be decomposed into a direct sum of subspaces invariant under ad$a$, $a \in \mathfrak{a}$, on each of which every operator ad$a$ has exactly one eigenvalue. The unique eigenvalue of ad$a$ on each of these invariant subspaces is given by a linear function λ on $\mathfrak{a}$, and accordingly the corresponding subspace is denoted by $\mathfrak{g}_\lambda$. Since $\mathfrak{p} \neq 0$ (because $\mathfrak{g}$ is not compact) and since K is positive definite on $\mathfrak{p}$, the subspace $\mathfrak{g}_0$ associated with λ being identically zero cannot be the entire $\mathfrak{g}$. Summarizing, we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda\right)$$

where Σ is a finite set of nonzero linear functions on $\mathfrak{a}$ (which are called the *roots*) and $\mathfrak{g}_\lambda = \{g \in \mathfrak{g} : \text{ad}a(g) = \lambda(a)g\ \forall a \in \mathfrak{a}\}$. Using the Jacobi identity, one can show that $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu]$ is a subspace of $\mathfrak{g}_{\lambda+\mu}$ if $\lambda+\mu \in \Sigma \cup \{0\}$, and equals 0 otherwise. This implies that the subspaces $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to K unless $\lambda+\mu = 0$ (cf. [10, p. 38]). Since K is nondegenerate on $\mathfrak{g}$, it follows that if λ is a root, then so is $-\lambda$. Moreover, the subspace $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]$ of $\mathfrak{g}_0$ has dimension 1, and λ is not identically zero on it (cf. [10, pp. 39–40]). This means that there exist some elements $e \in \mathfrak{g}_\lambda$ and $f \in \mathfrak{g}_{-\lambda}$ such that $h := [e, f] \neq 0$. It is now easy to see that, multiplying e, f and h by constants if necessary, we obtain an $sl(2)$-triple. Alternatively, we could finish the argument by noting that if $g \in \mathfrak{g}_\lambda$ for some $\lambda \in \Sigma$, then the operator ad$g$ is nilpotent (because it maps each $\mathfrak{g}_\mu$ to $\mathfrak{g}_{\mu+\lambda}$ to $\mathfrak{g}_{\mu+2\lambda}$ and eventually to 0 since Σ is a finite set), and the existence of a subalgebra isomorphic to $sl(2, \mathbb{R})$ is guaranteed by the Jacobson-Morozov Theorem.

Acknowledgment. The second author is grateful to V. Protsak for helpful discussions on Lie algebras.

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