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ISS and integral-ISS disturbance attenuation with bounded controls*

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Abstract

We consider the problem of achieving disturbance attenuation in the ISS and integral-ISS sense for nonlinear systems with bounded controls. For the ISS case we derive a "universal" formula which extends an earlier result of Lin and Sontag to systems with disturbances. For the integral-ISS case we give two constructions, one resulting in a smooth control law and the other in a switching control law. We also briefly discuss some issues related to input-to-state stability of switched and hybrid systems.

1 Introduction

As we know from Artstein's theorem [2], the existence of a smooth control Lyapunov function implies that there exists a state feedback control law, smooth away from the origin, which makes the closed-loop system globally asymptotically stable. This statement holds for general (possibly constrained) control spaces. In [14] Sontag derived a "universal" formula which leads to an explicit construction of a stabilizing feedback law for affine systems in the case of arbitrary unbounded controls. Such universal formulas have later been obtained for controls bounded in magnitude [11], positive controls [12], and controls restricted to Minkowski balls [13].

When a given system has external disturbances, a problem of interest is to find a state feedback control law that makes the closed-loop system input-to-state stable with respect to the disturbances. An appropriate notion in this context is that of ISS-control Lyapunov function, whose existence leads to explicit formulas for input-to-state stabilizing feedback laws [5, 8, 18, 20]. More recently, an integral variant of input-to-state stability (iISS) was defined and studied in [1, 16]. A notion of iISS-control Lyapunov function was introduced in [10], where it was shown that the knowledge of such a function allows one to construct a feedback law that makes the closed-loop system integral-input-to-state stable with respect to the disturbances. Compared with the ISS case, however, the resulting formula is more complicated (except when the values of the disturbances can be directly measured and used for control): the construction involved "patching" together several control laws defined on appropriate regions of the state space. An interesting source of motivation for the integral-input-to-state stabilization problem is discussed in [7].

The aforementioned formulas for control laws that achieve ISS and iISS disturbance attenuation, with the exception of the pointwise min-norm controls considered in [5], are only valid in the case of arbitrary unbounded controls. On the other hand, as we already indicated, the asymptotic stabilization problem has been successfully treated for various constrained control spaces. The purpose of this paper is to start filling in this gap by considering the problem of achieving ISS and iISS disturbance attenuation using controls with bounded magnitude. The results that we obtain are close in spirit to those reported in [10] for unbounded controls. In the ISS case we give a "universal" formula that naturally extends the result of [11], while in the iISS case we combine several control laws to obtain a globally defined state feedback law. We show that this can be done in two ways: continuously or by switching. The second method also applies to the case of unbounded controls and therefore provides an alternative to the construction given in [10]. It is inspired by the switching control laws used (for different purposes) in [6] and [21].

The layout of the paper is as follows. In Section 2 we review necessary definitions and background results. A universal formula for ISS disturbance attenuation with bounded controls is given in Section 3. The problem of iISS disturbance attenuation is dealt with in Section 4: we first treat the case when the values of the disturbances are available for control; after that we obtain a smooth control law in pure state feedback form, and then give an alternative construction using hysteresis switching. Section 5 contains concluding remarks.

2 Preliminaries

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If in addition $\alpha(r) \to \infty$ as $r \to \infty$, then it is said to be of

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class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

A positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *control Lyapunov function* (CLF) for the system

$$\dot{x} = f(x) + G(x)u, \qquad x \in \mathbb{R}^n,\ u \in \mathcal{U} \subset \mathbb{R}^m$$

if for all $x \neq 0$ we have

$$\inf_{u \in \mathcal{U}} \{\nabla V(x) f(x) + \nabla V(x) G(x) u\} < 0.$$

If such a CLF V is given and if $\mathcal{U} = \mathbb{R}^m$, then the well-known Sontag's formula derived in [14] can be applied to construct a state feedback control law that makes the closed-loop system globally asymptotically stable (with Lyapunov function V). If $\mathcal{U}$ is the closed unit ball in $\mathbb{R}^m$ with respect to the standard Euclidean norm, i.e., $\mathcal{U} = \{u \in \mathbb{R}^m : |u| \leq 1\}$, then one can use the universal formula for bounded controls derived in [11], which gives the feedback control law

$$k(x) := \begin{cases} -\dfrac{a(x) + \sqrt{(a(x))^2 + |b(x)|^4}}{|b(x)|^2\left(1 + \sqrt{1 + |b(x)|^2}\right)} b^T(x), & b(x) \neq 0 \\ 0, & b(x) = 0 \end{cases}$$

where $a(x) := \nabla V(x) f(x)$ and $b(x) := \nabla V(x) G(x)$.

We recall from [15] that a general system

$$\dot{x} = f(x, d) \tag{1}$$

with a locally essentially bounded disturbance input d is called *input-to-state stable* (ISS) with respect to d if for some functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for all initial states $x(0)$, and all d the following estimate holds:

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \gamma(\|d_t\|) \qquad \forall t \geq 0 \tag{2}$$

where $\|d_t\| := \operatorname{ess\,sup}\{|d(s)| : s \in [0, t]\}$. As shown in [17], a necessary and sufficient condition for ISS is the existence of an *ISS-Lyapunov function*, i.e., a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some $\mathcal{K}_\infty$ functions α and χ we have

$$\nabla V(x) f(x, d) \leq -\alpha(|x|) + \chi(|d|) \qquad \forall x, d.$$

Also, recall from [16] that the system (1) is called *integral-input-to-state stable* (iISS) with respect to d if for some functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for all initial states $x(0)$, and all d the following estimate holds:

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|d(s)|) ds \qquad \forall t \geq 0.$$

It was shown in [1] that the system is iISS if and only if it is *0-GAS* (i.e., the system $\dot{x} = f(x, 0)$ is globally asymptotically stable) and *zero-output dissipative*, i.e., there exist a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ and a class $\mathcal{K}_\infty$ function ν such that

$$\nabla V(x) f(x, d) \leq \nu(|d|) \qquad \forall x, d.$$

Another necessary and sufficient condition for iISS established in [1] is the existence of an *iISS-Lyapunov function*, i.e., a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ function χ and some positive definite continuous function α we have

$$\nabla V(x) f(x, d) \leq -\alpha(|x|) + \chi(|d|) \qquad \forall x, d.$$

Comparing this with the above characterization of ISS, where α was required to be of class $\mathcal{K}_\infty$, we see clearly that iISS is a weaker property than ISS.

In this paper we will be concerned with systems that are affine in controls and disturbances. These are systems of the form

$$\dot{x} = f(x) + G_1(x) d + G_2(x) u \tag{3}$$

where $x \in \mathbb{R}^n$, $d \in \mathbb{R}^k$, $u \in \mathcal{U} \subset \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $G_1 : \mathbb{R}^n \to \mathbb{R}^{n \times k}$ and $G_2 : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are smooth functions. Throughout the paper we will use the notation $a(x) := \nabla V(x) f(x)$, $b_1(x) := \nabla V(x) G_1(x)$, and $b_2(x) := \nabla V(x) G_2(x)$, where it will be clear from the context which function V is being used. We will say that a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is an *ISS-control Lyapunov function* (ISS-CLF) for the system (3) if there exist class $\mathcal{K}_\infty$ functions α and χ such that for all x and d we have

$$\inf_{u \in \mathcal{U}} \{a(x) + b_1(x) d + b_2(x) u\} \leq -\alpha(|x|) + \chi(|d|).$$

As is not hard to show, this is equivalent to the existence of a class $\mathcal{K}_\infty$ function ρ such that for all x and d we have

$$\begin{gathered} |x| \geq \rho(|d|) \\ \Downarrow \\ \inf_{u \in \mathcal{U}} \{a(x) + b_1(x) d + b_2(x) u\} \leq -\alpha(|x|)/2. \end{gathered} \tag{4}$$

We will say that a positive definite radially unbounded smooth function $V : \mathbb{R}^n \to \mathbb{R}$ is an *iISS-control Lyapunov function*[1] (iISS-CLF) for the system (3) if there exist a class $\mathcal{K}_\infty$ function χ and a positive definite continuous function α such that for all x and d we have

$$\inf_{u \in \mathcal{U}} \{a(x) + b_1(x) d + b_2(x) u\} \leq -\alpha(|x|) + \chi(|d|).$$

For the remainder of the paper, we will take the control space $\mathcal{U}$ to be the closed unit ball in $\mathbb{R}^m$. We are interested in constructing input-to-state stabilizing

[1]This is the first of the two types of iISS-control Lyapunov functions considered in [10].

and integral-input-to-state stabilizing feedback control laws for the system (3), i.e., control laws that make the closed-loop system ISS and iISS, respectively. We remark that the control laws to be used in this paper lead to closed-loop systems that are in general not smooth but just continuous at the origin (and smooth everywhere else). All the results cited above are valid for this class of systems; see [10, 20] for details.

3 ISS disturbance attenuation

We will need the following technical lemma.

Lemma 1 *A function V is an ISS-CLF for the system (3) if and only if for all x we have*

$$\inf_{u\in\mathcal{U}}\{a(x)+|b_1(x)|\rho^{-1}(|x|)+b_2(x)u\}\leq-\alpha(|x|)/2$$

where $\rho\in\mathcal{K}_\infty$ is the same function as in (4).

This lemma can be easily established by a worst-case disturbance argument (cf. [8, Lemma 2.1]), and the proof is omitted. We will assume that V satisfies the following variant of the small control property: for each $\epsilon>0$ there exists a $\delta>0$ such that whenever $|x|<\delta$ there exists some u with $|u|<\epsilon$ for which

$$a(x)+|b_1(x)|\rho^{-1}(|x|)+b_2(x)u\leq-\alpha(|x|)/2. \qquad (5)$$

Note that there is no loss of generality in restricting the function α in (5) to be the same as in the definition of ISS-CLF, because we can always decrease this function in the neighborhood of 0 if necessary.

Proposition 2 *If the system (3) admits an ISS-control Lyapunov function V satisfying the small control property* (5), *then there exists an input-to-state stabilizing feedback law $u=k(x)$, taking values in the unit ball, which is smooth when $x\neq0$ and continuous everywhere.*

PROOF. Define the function $\omega(x) := a(x)+|b_1(x)|\rho^{-1}(|x|)$. Since this function is merely continuous and not necessarily smooth, we need to have another function, $\bar\omega$, which is smooth away from 0, continuous at 0, and satisfies $\omega(x)\leq\bar\omega(x)\leq\omega(x)+\alpha(|x|)/4$ for all x. Such a function can be constructed using standard smooth approximation techniques (cf. [4, Lemma 4.9]). Now define the feedback law

$$k(x):=\begin{cases}-\dfrac{\bar\omega(x)+\sqrt{(\bar\omega(x))^2+|b_2(x)|^4}}{|b_2(x)|^2\left(1+\sqrt{1+|b_2(x)|^2}\right)}b_2^T(x), & b_2(x)\neq0\\ 0, & b_2(x)=0\end{cases}$$

It follows from the results of [11] that this control law is smooth when $x\neq0$, continuous everywhere, and takes values in the unit ball. Moreover, we have

$$a(x)+|b_1(x)|\rho^{-1}(|x|)+b_2(x)k(x)<0 \qquad \forall x\neq0. \quad (6)$$

Along the solutions of the closed-loop system we have

$$\dot V=a(x)+b_1(x)d+b_2(x)k(x). \qquad (7)$$

Combining (6) and (7), we obtain $\dot V<0$ whenever $x\neq0$ and $|x|\geq\rho(|d|)$. This implies that V is an ISS-Lyapunov function for the closed-loop system (see Remark 2.4 in [17]), and the desired ISS property follows. □

4 iISS disturbance attenuation

Let us first consider the simpler case when the values of the disturbances can be used for control, i.e., when the feedback law can take the form $u=k(x,d)$. This situation is not altogether meaningless, for example, it arises in applications to control of uncertain nonlinear systems discussed in [7]. We will assume that a given iISS-CLF V satisfies the following variant of the small control property: for each $\epsilon>0$ there exists a $\delta>0$ such that whenever $|x|,|d|<\delta$ there exists some u with $|u|<\epsilon$ for which

$$a(x)+b_1(x)d-\chi(|d|)+b_2(x)u\leq-\alpha(|x|). \qquad (8)$$

Again, note that we can always decrease α in a neighborhood of 0 if necessary, so there is no loss of generality in restricting it to be the same function as in the definition of iISS-CLF.

Proposition 3 *If the system (3) admits an iISS-control Lyapunov function V satisfying the small control property* (8), *then there exists an integral-input-to-state stabilizing control law $u=k(x,d)$, taking values in the unit ball, which is smooth when $(x,d)\neq(0,0)$ and continuous everywhere.*

PROOF. Define the function $\omega(x,d):=a(x)+b_1(x)d-2\chi(|d|)$. Take another function, $\bar\omega(x,d)$, which is smooth away from $(0,0)$, continuous at $(0,0)$, and satisfies $\omega(x,d)\leq\bar\omega(x.d)\leq\omega(x,d)+\alpha(|x|)/2+\chi(|d|)/2$ for all x and d (cf. the proof of Proposition 2). Now define the control law

$$k(x,d):=\begin{cases}-\dfrac{\bar\omega+\sqrt{\bar\omega^2+|b_2(x)|^4}}{|b_2(x)|^2\left(1+\sqrt{1+|b_2(x)|^2}\right)}b_2^T(x), & b_2(x)\neq0\\ 0, & b_2(x)=0\end{cases}$$

It follows from the results of [11] that this control law is smooth when $(x,d)\neq(0,0)$, continuous everywhere, and takes values in the unit ball. Moreover, along the solutions of the closed-loop system we have

$$\dot V<2\chi(|d|)$$

for all $x\neq0$ and all d. This implies that the closed-loop system is 0-GAS and zero-output dissipative, hence iISS. □

We now turn to the problem of constructing an integral-input-to-state stabilizing control law in the pure state feedback form $u = k(x)$. If V is an iISS-CLF for the system (3), then we have

$$a(x) + b_1(x)d \le |b_2(x)| - \alpha(|x|) + \chi(|d|) \qquad \forall x, d \quad (9)$$

(to see why, take $u = -b_2^T(x)/|b_2(x)|$). Therefore,

$$b_2(x) = 0 \Rightarrow a(x) + b_1(x)d \le -\alpha(|x|) + \chi(|d|). \quad (10)$$

Also, setting $d = 0$ in (9) we obtain

$$a(x) < |b_2(x)| \qquad \forall x \neq 0. \quad (11)$$

The last inequality implies that V is a CLF for the system $\dot{x} = f(x) + G_2(x)u$ (again, just take $u = -b_2^T(x)/|b_2(x)|$). The universal formula derived in [11] leads to the control law

$$k_0(x) := \begin{cases} -\dfrac{a(x)+\sqrt{(a(x))^2+|b_2(x)|^4}}{|b_2(x)|^2(1+\sqrt{1+|b_2(x)|^2})} b_2^T(x), & b_2(x) \neq 0 \\ 0, & b_2(x) = 0 \end{cases} \quad (12)$$

We have $|k_0(x)| < 1$, and for all $x \neq 0$

$$a(x) + b_2(x)k_0(x) = \frac{a(x)\sqrt{1+|b_2(x)|^2} - \sqrt{(a(x))^2+|b_2(x)|^4}}{1+\sqrt{1+|b_2(x)|^2}} < 0.$$

Now let us see how to construct a bounded integral-input-to-state stabilizing control law for the system (3). We could just use the "bang-bang" control law

$$k_1(x) := \begin{cases} -\dfrac{b_2^T(x)}{|b_2(x)|}, & b_2(x) \neq 0 \\ 0, & b_2(x) = 0 \end{cases} \quad (13)$$

But this control law is not continuous and may actually lead to chattering. What we propose to do to fix this problem is basically to combine the control law given by (12) for small values of $|b_1(x)|$ and $|b_2(x)|$ with that given by (13) for large values of $|b_1(x)|$ or $|b_2(x)|$. We will describe two different ways of doing this: continuously or by hysteresis switching.

4.1 Smooth control

Assume that a given iISS-CLF V satisfies the following small control property [14]: for each $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x| < \delta$ there exists some u with $|u| < \epsilon$ for which

$$a(x) + b_2(x)u < 0. \quad (14)$$

Proposition 4 *If the system (3) admits an iISS-control Lyapunov function V satisfying the small control property (14), then there exists an integral-input-to-state stabilizing feedback law $u = k(x)$, taking values in the unit ball, which is smooth when $x \neq 0$ and continuous everywhere.*

PROOF. Define the set $D_0 := \{x \in \mathbb{R}^n : b_2(x) = 0\}$. Let D_1 be a neighborhood of $D_0 \setminus \{0\}$ in $\mathbb{R}^n$ (empty if $D_0 = \{0\}$) such that for each $x \in D_1$ there exists some $x_0 \in D_0 \setminus \{0\}$ with $|a(x) - a(x_0)| < \alpha(|x_0|)$ and $|b_1(x) - b_1(x_0)| < 1$ (here α is the same as in (10)). Then for each $x \in D_1$ and each d we have (picking an appropriate x_0):

$$a(x)+b_1(x)d = a(x_0)+b_1(x_0)d+(a(x)-a(x_0))+(b_1(x) - b_1(x_0))d < -\alpha(|x_0|)+\chi(|d|)+\alpha(|x_0|)+|d| = \hat{\chi}(|d|)$$

where $\hat{\chi}(r) := \chi(r) + r$.

Let D_2 be a neighborhood of 0 in $\mathbb{R}^n$ such that $|b_1(x)| < 1$ for all $x \in D_2$. Note that $D_1 \cup D_2$ is a neighborhood of D_0 in $\mathbb{R}^n$. Let $\varphi(x) : \mathbb{R}^n \to [0, 1]$ be a smooth function such that $\varphi(x) = 0$ on some open subset of $D_1 \cup D_2$ containing D_0, and $\varphi(x) = 1$ if $x \notin D_1 \cup D_2$. Such a "bump" function is well known to exist (see, e.g., [3, Lemma 3.1.2]). Consider the feedback law

$$k(x) := k_0(x) + \varphi(x)(k_1(x) - k_0(x)).$$

Observe that for all x we have $b_2(x)k_0(x) \le 0$ and

$$b_2(x)(k_1(x) - k_0(x)) = \frac{a(x) - |b_2(x)|}{1+\sqrt{1+|b_2(x)|^2}} + \frac{\sqrt{(a(x))^2+|b_2(x)|^4} - |b_2(x)|\sqrt{1+|b_2(x)|^2}}{1+\sqrt{1+|b_2(x)|^2}} \le 0$$

because of (11), hence $b_2(x)k(x) \le 0$ for all x. For all (x, d) with $x \in D_1$ we have

$$a(x) + b_1(x)d + b_2(x)k(x) < \hat{\chi}(|d|) + b_2(x)k(x) \le \hat{\chi}(|d|).$$

For all $(x, d) \neq (0, 0)$ with $x \in D_2$ we have

$$a(x)+b_1(x)d+b_2(x)k(x) \le a(x)+|d|+b_2(x)k_0(x) < |d|.$$

Finally, for all (x, d) with $x \notin D_1 \cup D_2$ we have

$$a(x)+b_1(x)d+b_2(x)k(x) = a(x)+b_1(x)d-|b_2(x)| < \chi(|d|)$$

by (9). Putting the above inequalities together, we obtain

$$a(x) + b_1(x)d + b_2(x)k(x) < \nu(|d|) \qquad \forall (x, d) \neq (0, 0)$$

where $\nu(r) := \max\{\hat{\chi}(r), r, \chi(r)\}$. This implies that the closed-loop system is 0-GAS and zero-output dissipative, hence iISS. Finally, it is easy to see that k is smooth on $\mathbb{R}^n \setminus \{0\}$ and continuous everywhere (because k_0 has these properties by [14], k_1 is smooth away from D_0, and $\varphi(x) = 0$ on a neighborhood of D_0). □

4.2 Switching control

Let V be an iISS-CLF for the system (3) which satisfies the small control property (14). Let the sets D_0, D_1 and D_2 and the control laws k_0 and k_1 be as defined on the previous page. Let D_1' be a closed neighborhood of D_0 in $\mathbb{R}^n$ (empty if $D_0 = \{0\}$) such that for each $x \in D_1'$ there exists some $x_0 \in D_0 \setminus \{0\}$ with $|a(x) - a(x_0)| \leq \alpha(|x_0|)/2$ and $|b_1(x) - b_1(x_0)| \leq 1/2$. Let D_2' be a closed neighborhood of 0 in $\mathbb{R}^n$ such that $|b_1(x)| \leq 1/2$ for all $x \in D_2'$. Denote by S_0 the union of D_1 and D_2, and denote by S_1 the complement of $D_1' \cup D_2'$ in $\mathbb{R}^n$. The union of these two open overlapping regions S_0 and S_1 is the entire $\mathbb{R}^n$.

We can now define the switching control law $u = k_\sigma(x)$, where $\sigma : [0, \infty) \to \{0, 1\}$ is a piecewise constant *switching signal*, as follows. Let $\sigma(0) = 0$ if $x(0) \in S_0$ and $\sigma(0) = 1$ otherwise. For each $t > 0$, let $\sigma(t) = 0$ if $\sigma(t^-) = 1$ but $x(t) \notin S_1$. Similarly, let $\sigma(t) = 1$ if $\sigma(t^-) = 0$ but $x(t) \notin S_0$. On the other hand, if $\sigma(t^-) = i$ and $x(t) \in S_i$, keep $\sigma(t) = i$ $(i = 0, 1)$. Repeating this procedure, we generate a piecewise constant signal σ that is continuous from the right everywhere. Since σ can change its value only after the state trajectory has passed through the intersection of S_0 and S_1, chattering is avoided. This idea is known as *hysteresis*. The resulting closed-loop system is a hybrid system, σ being its discrete state.

Proposition 5 *The switching control law $u = k_\sigma(x)$ defined above makes the closed-loop system integral-input-to-state stable. More precisely, there exist functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that for all initial states $x(0)$ and all d the state of the closed-loop system with this control law satisfies*

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|d(s)|)ds \qquad \forall t \geq 0.$$

PROOF. Using the notation introduced in the proof of Proposition 4 and the inequalities obtained there, we conclude that for all $(x, d) \neq (0, 0)$ with $x \in S_0$ we have

$$a(x) + b_1(x)d + b_2(x)k_0(x) < \nu(|d|)$$

and that for all (x, d) with $x \in S_1$ we have

$$a(x) + b_1(x)d + b_2(x)k_1(x) < \nu(|d|).$$

Let α_0 be any positive definite continuous function such that for all $r \geq 0$, all $x \in S_0$ with $|x| = r$, and all $|d| \leq r$ we have

$$\alpha_0(r) \leq -a(x) - b_1(x)d - b_2(x)k_0(x) + \nu(|d|).$$

When $x \in S_0$ and $|x| \geq |d|$, we have

$$a(x) + b_1(x)d + b_2(x)k_0(x) - \nu(|d|) \leq -\alpha_0(|x|). \quad (15)$$

Next, let ν_0 be any class $\mathcal{K}_\infty$ function such that

$$\nu_0(r) \geq a(x) + b_1(x)d + b_2(x)k_0(x) - \nu(|d|) + \alpha_0(|x|)$$

for all $r \geq 0$, all $x \in S_0$ with $|x| \leq r$, and all $|d| = r$. When $x \in S_0$ and $|x| < |d|$, we have

$$a(x) + b_1(x)d + b_2(x)k_0(x) - \nu(|d|) \leq -\alpha_0(|x|) + \nu_0(|d|).$$

Together with (15) this implies that for all (x, d) with $x \in S_0$ we have

$$a(x) + b_1(x)d + b_2(x)k_0(x) \leq -\alpha_0(|x|) + \chi_0(|d|)$$

where $\chi_0(r) := \nu(r) + \nu_0(r)$.

Similarly, we can show that there exist a positive definite function α_1 and a class $\mathcal{K}_\infty$ function χ_1 such that for all (x, d) with $x \in S_1$ we have

$$a(x) + b_1(x)d + b_2(x)k_1(x) \leq -\alpha_1(|x|) + \chi_1(|d|).$$

Thus if we define $\bar{\alpha}(r) := \min\{\alpha_0(r), \alpha_1(r)\}$ and $\bar{\chi}(r) := \max\{\chi_0(r), \chi_1(r)\}$, then we have

$$a(x) + b_1(x)d + b_2(x)k_i(x) \leq -\bar{\alpha}(|x|) + \bar{\chi}(|d|)$$

for all (x, d) with $x \in S_i$, $i = 0, 1$.

According to our description of the switching control law, when $x \notin S_1$ we always have $u = k_0(x)$, while when $x \notin S_0$ we always have $u = k_1(x)$. This guarantees that along the solutions of the closed-loop system we have

$$\dot{V} \leq -\bar{\alpha}(|x|) + \bar{\chi}(|d|) \qquad \forall x, d.$$

Thus V is an iISS-Lyapunov function for the closed-loop system, and the desired iISS property follows (a careful examination of the argument given in [1] reveals that it goes through in the presence of switching without any changes). □

From Proposition 5 it follows, in particular, that when the above control law is used and when the disturbance d is such that $\int_0^\infty \gamma(|d(s)|)ds < \infty$, only a finite number of switches occurs and we have $x(t) \to 0$. In fact, it is demonstrated in [1] that we can take $\gamma = 2\bar{\chi}$, where $\bar{\chi}$ is the function constructed in the above proof. An advantage of the switching control law over the smooth one is that it is in some sense easier to implement (no "bump" function φ is needed). This approach may also be used in the unbounded control case treated in [10].

5 Concluding remarks

We addressed the problem of achieving ISS and iISS disturbance attenuation using bounded controls. For the ISS case we derived a "universal" formula, thereby extending an earlier result of Lin and Sontag to systems with disturbances. For the iISS case we proposed a solution that involves combining several control laws to obtain a globally defined state feedback law. We presented two ways of doing that, one resulting in an almost smooth control law (i.e., a continuous control law that is smooth away from the origin) and the other resulting in a switching (in fact, hybrid) control law.

These results complement the work reported in [10] for the case of unbounded controls. It is easy to establish converse results, in the spirit of those in [10], stating that the existence of a feedback law implies the existence of a control Lyapunov function (although some care must be taken with regard to regularity at zero, as discussed in [10]). In fact, as noted in [10], an efficient method for finding a CLF is to search for a control law of the form $u = k(x, d)$ which achieves an appropriate form of disturbance attenuation (effectively, by canceling some or all terms containing the disturbance). Our results can then be applied to construct a state feedback law $u = k(x)$ that serves the same purpose, which is usually a more difficult task. Recent results of Teel and Praly [19] are also of direct relevance. Methods similar to the ones employed in that paper lead to alternative (and actually more general) constructions, as will be discussed in a future publication.

We have exploited the fact that a sufficient condition for iISS established in [1] carries over without any changes to switched systems (more generally, to hybrid and time-varying ones), provided that it is satisfied uniformly with respect to switching signals (discrete states, time). The same observation is valid regarding the corresponding sufficient condition for ISS given in [15], [17, Lemma 2.14]. In fact, much of the existing theory of ISS and iISS can be applied to these classes of systems (we are assuming here that the switching signal takes values in a compact set). For example, it is clear that if a family of systems possesses a common ISS-Lyapunov function triple (V, α, χ), then the switched system is ISS for arbitrary switching, and similarly for iISS. This is a generalization of the well-known fact that the existence of a common Lyapunov function implies global asymptotic stability under arbitrary switching for systems without inputs (see, e.g., [9]). The same argument as in [17] leads to the converse statement that ISS uniform over the set of all switching signals implies the existence of a common ISS-Lyapunov function. Another fact worth mentioning is that, under suitable assumptions, the ISS property is preserved under switching if the intervals between switching instants are large enough (it is not hard to show that this is true provided that each of the systems being switched satisfies (2) with β of the form $cx(0)e^{-\lambda t}$).

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