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Systems & Control Letters 54 (2005) 405–416
SYSTEMS & CONTROL LETTERS
www.elsevier.com/locate/sysconle

Common Lyapunov functions for families of commuting nonlinear systems ☆

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Received 27 May 2003; received in revised form 5 August 2004; accepted 24 September 2004
Available online 6 November 2004

Abstract

We present constructions of a local and global common Lyapunov function for a finite family of pairwise commuting globally asymptotically stable nonlinear systems. The constructions are based on an iterative procedure, which at each step invokes a converse Lyapunov theorem for one of the individual systems. Our results extend a previously available one which relies on exponential stability of the vector fields.

Keywords: Common Lyapunov functions; Switched nonlinear systems; Commutativity; Uniform asymptotic stability

1. Introduction

A family of dynamical systems and a switching signal which specifies the active system at each time give rise to a switched system. Switched systems are common in situations where system behavior can hardly be described by a single ordinary differential equation, for instance, when a physical system exhibits several modes or when there are several controllers and a switching among them. As is well known, it is possible to have unstable trajectories when switching among globally asymptotically stable (GAS) systems (see, e.g., [4,9]). It is then interesting to study uniform asymptotic stability of switched systems with respect to switching signals, which is the property that the switched system state goes to zero asymptotically regardless of what a switching sequence is [9]. If this property holds for all initial conditions, we have global uniform asymptotic stability (GUAS).

Stability of switched systems under arbitrary switching has been the subject of a number of studies, and several classes of switched systems that possess the GUAS property have been identified [1,2,8,16,17,19,7]. In particular, it is known that a switched system generated by a finite family of GAS pairwise commuting subsystems is GUAS. When the subsystems are linear, it is easy to prove this fact by manipulating matrix exponentials. When the

☆ Supported by NSF ECS-0134115 CAR, NSF ECS-0114725, and DARPA/AFOSR MURI F49620-02-1-0325 grants.

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doi:10.1016/j.sysconle.2004.09.006

subsystems are nonlinear, which is the subject of our study here, the GUAS property has been proved by using comparison functions in [11].

The converse Lyapunov theorem for switched systems asserts the existence of a common Lyapunov function when the switched system is GUAS [12]. A Lyapunov function is of theoretical interest and also useful for perturbation analysis. For a GUAS switched system generated by locally exponentially stable systems, a construction of a common Lyapunov function is studied in [15,3]. The general construction of a common Lyapunov function for GUAS switched nonlinear systems presented in [12] is a consequence of the converse Lyapunov theorem for robust stability of nonlinear systems [10]. Although these constructions can be applied to a family of pairwise commuting systems, they are too general for our setting since they do not utilize commutativity. The alternative constructions considered here involve handling the individual systems sequentially rather than simultaneously, resulting in more constructive procedures (as compared with, e.g., [3,12]). Further, one of our constructions also gives a bound on the gradient of the Lyapunov function and thus allows us to infer about stability of the switched system under perturbations (which is not possible with the approach of [12]).

For a finite family of pairwise commuting systems, we are interested in iterative procedures for constructing a common Lyapunov function, which employ Lyapunov functions of individual systems. Such a procedure was first proposed for a family of linear systems in [17] and later applied to a family of exponentially stable nonlinear systems in [18]. In this paper, we provide more general constructions of common Lyapunov functions for a finite family of pairwise commuting GAS—but not necessarily locally exponentially stable—nonlinear systems. We achieve this by basing the iterative procedures on general converse Lyapunov theorems for GAS nonlinear systems.

There are primarily two ways of constructing a converse Lyapunov function for a GAS nonlinear system. One is the integral construction due to Massera [14], the other is Kurzweil's construction [6]. Utilizing GUAS property, we describe an integral construction of a common Lyapunov function for a family of pairwise commuting GAS systems on a bounded region around the origin. This Lyapunov function is used to derive a result on stability of the corresponding switched system under perturbations. We then use Kurzweil's method to obtain a common Lyapunov function which is valid on the whole state space. The latter construction actually does not rely on GUAS of the switched system. As with non-switched systems, a smoothing procedure can be used to achieve arbitrary smoothness of the Lyapunov functions.

2. Background

2.1. Notations and definitions

Recall that a continuous function $V : D \subseteq \mathbb{R}^n \to [0,\infty)$ is *positive definite* if $V(x) = 0 \Leftrightarrow x = 0$. A continuous function $\alpha : [0,a) \to [0,\infty)$ is of *class* $\mathcal{K}$ if it is increasing and $\alpha(0) = 0$. If $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, we say that α belongs to *class* $\mathcal{K}_\infty$. A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ is of *class* $\mathcal{KL}$ if for each fixed t, $\beta(r,t)$ is of class $\mathcal{K}$ and for each fixed r, $\beta(r,s)$ is decreasing with respect to s and $\lim_{s\to\infty} \beta(r,s) = 0$.

For a nonlinear system,

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz vector field, denoted by $\phi(t,\xi)$ the solution with initial condition $x(0) = \xi \in \mathbb{R}^n$. If for each ξ, the solution is defined for all $t \in [0,\infty)$, the system is *forward complete*. If the solution is defined for all $t \in (-\infty, 0]$, the system is *backward complete*. The system (1) is *complete* if it is both backward and forward complete. The system is GAS if there exists a class $\mathcal{KL}$ function β such that $|\phi(t,\xi)| \leqslant \beta(|\xi|,t)$ $\forall \xi \in \mathbb{R}^n, \forall t \geqslant 0$, where $|\cdot|$ denotes the Euclidean norm. We denote by B_r the open ball of radius r centered at the origin, $B_r := \{x \in \mathbb{R}^n : |x| < r\}$.

Consider a family of dynamical systems

$$\dot{x}(t) = f_p(x(t)), \quad p \in \mathcal{P}, \tag{2}$$

where $f_p : \mathbb{R}^n \to \mathbb{R}^n$, $p \in \mathcal{P}$ are locally Lipschitz vector fields parameterized by a finite index set $\mathcal{P} := \{1,\ldots,m\}$ for some positive integer m. This gives rise to a *switched system*

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{3}$$

where $\sigma : [0, \infty) \to \mathscr{P}$ is a piecewise constant *switching signal*. Such a function σ has a finite number of discontinuities, which we call *switching times*, on every bounded time interval, and takes a constant value on every interval between two consecutive switching times. We denote by S the set of all admissible switching signals for the switched system (3). With the assumption of locally Lipschitz vector fields, piecewise constant switching signals, and no impulse effects, a unique solution of (3) for each $\sigma \in S$ and each initial condition exists and is continuous and piecewise differentiable. We write $\psi_\sigma(t, \xi)$ for the solution of the switched system (3) starting at ξ at time 0 for a particular switching signal $\sigma \in S$ and we write $\phi_p(t, \xi)$ for solutions of the individual nonlinear systems indexed by $p \in \mathscr{P}$. Sometimes, we will write $\phi^t_p(\xi)$ instead of $\phi_p(t, \xi)$ for the ease of writing compositions of flows. We denote by $\phi^{t_\mathscr{I}}_\mathscr{I}(\xi)$ the composition of flows over a finite index set $\mathscr{I}$,

$$\phi^{t_\mathscr{I}}_\mathscr{I}(\xi) := \phi^{t_1}_{k_1} \circ \cdots \circ \phi^{t_m}_{k_m}(\xi), \tag{4}$$

where $k_1, \ldots, k_m$ are elements of $\mathscr{I}$ in the increasing order and $t_1, \ldots, t_m$ are the corresponding times. When we write $t_\mathscr{I} \geqslant 0$ (resp. > 0, < 0, $\leqslant 0$, $= 0$), it is equivalent to writing $t_i \geqslant 0\ \forall i \in \mathscr{I}$ (resp. > 0, < 0, $\leqslant 0$, $= 0$). We write $s_\mathscr{I} = (t + \tau)_\mathscr{I}$ meaning that $s_i = t_i + \tau_i\ \forall i \in \mathscr{I}$. Accordingly, $t_\mathscr{I} \geqslant \tau_\mathscr{I}$ (resp. $>, <, \leqslant, =$) means $t_i \geqslant \tau_i\ \forall i \in \mathscr{I}$ (resp. $>, <, \leqslant, =$).

We now briefly review some stability concepts for switched systems (see, e.g., [7]). The switched system (3) is GUAS if there exists a class $\mathscr{KL}$ function β such that for every switching signal $\sigma \in S$, we have

$$|\psi_\sigma(t, \xi)| \leqslant \beta(|\xi|, t) \quad \forall\ \xi \in \mathbb{R}^n,\ t \geqslant 0. \tag{5}$$

It is easy to see that a necessary condition for GUAS is that all individual systems of (2) are GAS. A sufficient (as well as necessary) condition for GUAS is the existence of a *common Lyapunov function* for the family of systems (2), which is a positive definite, continuously differentiable function $V : \mathbb{R}^n \to [0, \infty)$ such that

$$L_{f_p} V(\xi) := \nabla V(\xi) f_p(\xi) \leqslant -W(\xi) \quad \forall\ \xi \in \mathbb{R}^n,\ p \in \mathscr{P}, \tag{6}$$

where ∇V is the gradient (row vector) of V and W is some positive definite function. A weaker version only requires V to be continuous, positive definite, locally Lipschitz away from zero and satisfying

$$D_{f_p} V(\xi) := \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[V(\phi_p(\varepsilon, \xi)) - V(\xi)] \leqslant -W(\xi) \quad \forall \xi \in \mathbb{R}^n \backslash \{0\},\ p \in \mathscr{P}, \tag{7}$$

for some positive definite W, where $D_{f_p} V(\xi)$ is the directional derivative of V along the solution of $\dot{x}(t) = f_p(x(t))$. When V is continuously differentiable, $D_{f_p} V$ becomes $L_{f_p} V$.

We say that the family of systems (2) is *pairwise commuting* if

$$\phi^t_i \circ \phi^s_j(\xi) = \phi^s_j \circ \phi^t_i(\xi) \quad \forall i, j \in \mathscr{P},\ \xi \in \mathbb{R}^n,\ t, s \geqslant 0. \tag{8}$$

If the systems are complete, it is easy to see that the property (8) is automatically extended to $t, s \in \mathbb{R}$. If the vector fields are continuously differentiable, pairwise commutativity is equivalent to the Lie bracket of every two vector fields being zero.

2.2. Converse Lyapunov theorems for nonlinear systems

In this section, we quickly review converse Lyapunov theorems for non-switched nonlinear systems (see, e.g., [5]) on which our subsequent results will be built. Basically, there are two constructions of converse Lyapunov functions for GAS nonlinear systems. One is due to Massera [14], the other is due to Kurzweil [6]. Massera's construction, sometimes referred to as the *integral construction*, seeks a converse Lyapunov function of the form

$$V(\xi) = \int_0^\infty G(|\phi(t, \xi)|)\, \mathrm{d}t \tag{9}$$

with $G : [0, \infty) \to [0, \infty)$ being chosen by the following lemma, known as *Massera's lemma*.

Lemma 1. *Let $g : [0, \infty) \to [0, \infty)$ be a continuous and decreasing function with $g(t) \to 0$ as $t \to \infty$. Let $h : [0, \infty) \to (0, \infty)$ be a continuous and nondecreasing function. There exists a function $G : [0, \infty) \to [0, \infty)$ such that*:

- *G and its derivative G' are class $\mathscr{K}$ functions defined on $[0, \infty)$.*

There exist positive real numbers c_1 and c_2 such that for all continuous functions $u : \mathbb{R} \to [0, \infty)$ satisfying $0 \leqslant u(t) \leqslant g(t) \quad \forall t \geqslant 0$, we have

$$\int_0^\infty G(u(t))\,\mathrm{d}t < c_1;$$
$$\int_0^\infty G'(u(t))h(t)\,\mathrm{d}t < c_2.$$

If the nonlinear system (1) is asymptotically stable, one can show that with G provided by Massera's lemma with suitable g and h, the integral (9) is well-defined and satisfies other conditions for V being a Lyapunov function on some bounded region around the origin. Note that if the system is exponentially stable, the function G in (9) can be taken as a quadratic one: $G(z) = z^2$. The following statement summarizes the converse Lyapunov theorem based on this construction.

Theorem 1. *Suppose that the nonlinear system* (1) *is asymptotically stable on B_r, $r < \infty$. Suppose that the Jacobian matrix $[\partial f/\partial x]$ is bounded on B_r. Then there is a constant $r_0 \in (0, r)$, and a continuously differentiable function $V : B_{r_0} \to [0, \infty)$ such that*

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|),$$
$$L_f V(\xi) \leqslant -\alpha_3(|\xi|),$$
$$|\nabla V(\xi)| \leqslant \alpha_4(|\xi|)$$

for all $\xi \in B_{r_0}$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are class $\mathscr{K}$ functions on $[0, r_0)$.

The second construction, due to Kurzweil, comprises two primary steps. The first step creates a function $g : \mathbb{R}^n \to [0, \infty)$ as

$$g(\xi) := \inf_{t \leqslant 0} \{|\phi(t, \xi)|\}$$

which is non-increasing along solutions of the nonlinear system (1) for all initial states in forward time:

$$g(\phi(t, \xi)) \leqslant g(\xi) \quad \forall t \geqslant 0, \ \xi \in \mathbb{R}^n. \tag{10}$$

The second step modifies the function g obtained in step 1 as

$$V(\xi) := \sup_{t \geqslant 0} \{g(\phi(t, \xi))k(t)\} \tag{11}$$

so that the resulting function is strictly decreasing along solutions of the nonlinear system except at the origin:

$$V(\phi(t, \xi)) < V(\xi) \quad \forall \xi \in \mathbb{R}^n \backslash \{0\}, \quad t \geqslant 0, \tag{12}$$

where $k : [0, \infty) \to \mathbb{R}$ is a strictly increasing, smooth function satisfying the following properties:

K1. $c_1 \leqslant k(t) \leqslant c_2$ for all $t \geqslant 0$ and some $0 < c_1 < c_2 < \infty$.

K2. There is a decreasing continuous function $\tau : [0, \infty) \to (0, \infty)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} k(t) \geqslant \tau(t) \quad \forall t \geqslant 0.$$

From this point onwards, when we write k, we refer to some fixed function k satisfying the above two conditions (for example, $k(t) = (1 + 2t)/(1 + t)$ is such a function). We state Kurzweil's method in a lemma. Because we frequently refer to functions with specific properties, for a function $V : \mathbb{R}^n \to [0, \infty)$, we call the following two properties P1 and P2.

P1. Positive definite and radially unbounded. (This is equivalent to the existence of $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ such that $\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n$.)

P2. Locally Lipschitz on $\mathbb{R}^n \backslash \{0\}$ and continuous on $\mathbb{R}^n$.

Lemma 2. *Given the GAS nonlinear system* (1), *suppose that a function $g : \mathbb{R}^n \to [0, \infty)$ satisfies properties $P1$, $P2$ and inequality* (10). *Define a function V by* (11). *Then V satisfies properties $P1$, $P2$ and*

$$D_f V(\xi) \leqslant -\alpha_3(|\xi|) \quad \forall \xi \in \mathbb{R}^n \backslash \{0\},$$

where α_3 is some positive definite function on $[0, \infty)$.

The function V is continuous and locally Lipschitz but may not be continuously differentiable. However, it can be smoothed to get a smooth Lyapunov function. In particular, it can be modified to be a continuously differentiable function. Kurzweil's construction leads to the following global converse Lyapunov theorem.

Theorem 2. *Suppose that the nonlinear system* (1) *is GAS. Then there is a continuously differentiable* *function* $V : \mathbb{R}^n \to [0, \infty)$ *such that*

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|),$$
$$L_f V(\xi) \leqslant -\alpha_3(|\xi|).$$

for all $\xi \in \mathbb{R}^n$, *where* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *and* α_3 *is some positive definite function on* $[0, \infty)$.

Kurzweil's construction is defined on the whole state space and provides a global result. In contrast, the integral method works on a bounded region only. However, with the additional assumption of boundedness of the Jacobian matrix of the vector field, the latter construction gives a bound on the gradient of the converse Lyapunov function and hence, it is possible to infer about stability of the system under perturbations.

2.3. Available results on common Lyapunov functions for pairwise commuting systems

A method for constructing a common Lyapunov function for a finite family of pairwise commuting *linear* systems was first proposed in [17]. For a family of pairwise commuting Hurwitz matrices $\{A_p,\ p \in \mathcal{P}\}$, a common quadratic Lyapunov function $V(\xi) = \xi^{\mathrm{T}} P \xi,\ P > 0$ such that $A_p^{\mathrm{T}} P + P A_p < 0 \quad \forall p \in \mathcal{P}$ could be obtained as follows:

$$A_1^{\mathrm{T}} P_1 + P_1 A_1 = -P_0,$$
$$A_p^{\mathrm{T}} P_p + P_p A_p = -P_{p-1}, \quad 2 \leqslant p \leqslant m,$$
$$P := P_m,$$

where P_0 is some positive definite matrix. For a family of pairwise commuting exponentially stable *nonlinear* systems, i.e., nonlinear systems with solutions satisfying

$$|\phi_p(t, \xi)| \leqslant c|\xi| \mathrm{e}^{-\lambda t} \quad \forall\, t \geqslant 0,\ \xi \in B_{cr},\ p \in \mathcal{P} \tag{13}$$

for some $r, c, \lambda > 0$, a common Lyapunov function is constructed as follows [18]:

$$V_1(\xi) := \int_0^T |\phi_1(t, \xi)|^2\, \mathrm{d}t,$$
$$V_p(\xi) := \int_0^T V_{p-1}(\phi_p(t, \xi))\, \mathrm{d}t, \quad 2 \leqslant p \leqslant m,$$
$$V(\xi) := V_m(\xi),$$

where $T > T^*$ and T^* is some appropriately chosen constant. With the additional assumption of uniform boundedness of the Jacobian matrices of the vector fields, i.e., if there exists $L < \infty$ such that

$$\left\| \frac{\partial f_p(x)}{\partial x}(\xi) \right\| < L \quad \forall \xi \in B_{cr},\ p \in \mathcal{P}, \tag{14}$$

where $\|\cdot\|$ is the induced matrix 2-norm, one has the following theorem.

Theorem 3 (*Shim et al.* [18]). *Consider the family* (2) *of pairwise commuting systems. Suppose that they are exponentially stable as in* (13) *and have property* (14). *There is a continuously differentiable function* $V : B_{r/c^{m-1}} \to [0, \infty)$ *satisfying the following inequalities*:

$$a_1|\xi|^2 \leqslant V(\xi) \leqslant a_2|\xi|^2,$$
$$L_{f_p} V(\xi) \leqslant -a_3|\xi|^2 \quad \forall p \in \mathcal{P},$$
$$|\nabla V(\xi)| \leqslant a_4|\xi|$$

for all $\xi \in B_{r/c^{m-1}}$, *where* a_1, a_2, a_3, a_4 *are positive constants*.

The theorem is stated locally. A finite T such that $T > T^*$ ensures the existence of $a_4 > 0$. If $T = \infty$, the first two inequalities are still valid but the third one on $|\nabla V(\xi)|$ no longer holds. If $r = \infty$, the result holds globally.

We see that the above construction is based on the special case of Massera's construction with G being quadratic. The next section provides general results on common Lyapunov functions for a family of pairwise commuting GAS systems, following both Massera's construction and Kurzweil's construction.

3. Main results

3.1. Local common Lyapunov function

We need the following extension of Massera's lemma for multivariable functions.

Lemma 3. *Let* $g : [0, \infty) \to [0, \infty)$ *be a continuous and decreasing function with* $g(t) \to 0$ *as* $t \to \infty$. *Let* $h : [0, \infty) \to (0, \infty)$ *be a continuous and* *nondecreasing function. Then there exists a differentiable function* $G : [0, \infty) \to [0, \infty)$ *such that*

- *G and its derivative* G' *are class* $\mathcal{K}$ *functions on* $[0, \infty)$.
  For every positive integer l, there exist positive real numbers c_1 *and* c_2 *such that for all continuous function* $u : \mathbb{R}^l \to [0, \infty)$ *satisfying*

$$0 \leqslant u(t_1, \ldots, t_l) \leqslant g(t_1 + \cdots + t_l) \quad \forall t_i \geqslant 0,\ 1 \leqslant i \leqslant l,$$

we have

$$\int_0^\infty \cdots \int_0^\infty G(u(s_1, \ldots, s_l))\, \mathrm{d}s_1 \ldots \mathrm{d}s_l < c_1$$

and

$$\int_0^\infty \cdots \int_0^\infty G'(u(s_1, \ldots, s_l)) \times h(s_1 + \cdots + s_l)\, \mathrm{d}s_1 \ldots \mathrm{d}s_l < c_2.$$

Proof. The proof proceeds along the lines of the proof of Massera's lemma and is included in the appendix. □

Consider the family (2) of pairwise commuting asymptotically stable systems on a ball B_r. There exist a class $\mathcal{KL}$ function β and a positive number r_0 such that for every nonempty subset $\mathcal{Q}$ of $\mathcal{P}$, we have

$$|\phi_{\mathcal{Q}}^{t_{\mathcal{Q}}}(\xi)| \leqslant \beta(|\xi|, t_1 + \cdots + t_q) \quad \forall \xi \in B_{r_0}, t_{\mathcal{Q}} \geqslant 0, q = \mathrm{card}(\mathcal{Q}). \tag{15}$$

If the individual systems are GAS, the foregoing inequality holds for all $\xi \in \mathbb{R}^n$. This is the uniform asymptotic stability property of a switched system generated by a family of pairwise commuting asymptotically stable systems [11]; details are in the proof below. Suppose that the vector fields are continuously differentiable on B_r. Since the ball B_r is compact and $\mathcal{P}$ is a finite index set, there exists a positive number L such that

$$\left\| \frac{\partial f_p(x)}{\partial x}(\xi) \right\| < L \quad \forall p \in \mathcal{P},\ \xi \in B_r. \tag{16}$$

For $\xi \in B_{r_0}$, construct a function V as follows:

$$V_1(\xi) := \int_0^\infty G(|\phi_1(t, \xi)|)\, \mathrm{d}t, \tag{17}$$

$$V_p(\xi) := \int_0^\infty V_{p-1}(\phi_p(t, \xi))\, \mathrm{d}t, \quad 2 \leqslant p \leqslant m, \tag{18}$$

$$V(\xi) := V_m(\xi) \tag{19}$$

for some function G satisfying Lemma 3 with $g(t) = \beta(r, t), h(t) = \exp(Lt)$. We have the following theorem.

Theorem 4. *Consider the family* (2) *of pairwise commuting asymptotically stable systems on* B_r. *Suppose that for each* $p \in \mathcal{P}$, *the vector field* f_p *is continuously differentiable on* B_r. *There is a constant* $r_0 \in (0, r)$, *such that the continuously differentiable function V constructed in* (17)–(19) *satisfies the following inequalities*:

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|), \tag{20}$$

$$L_{f_p} V(\xi) \leqslant -\alpha_3(|\xi|) \quad \forall p \in \mathcal{P}, \tag{21}$$

$$|\nabla V(\xi)| \leqslant \alpha_4(|\xi|) \tag{22}$$

for all $\xi \in B_{r_0}$, *where* $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ *are class* $\mathcal{K}$ *functions on* $[0, r_0)$.

Proof. By the asymptotic stability assumption, there are class $\mathcal{KL}$ functions $\beta_p,\ p \in \mathcal{P}$ such that for each $p \in \mathcal{P}$, we have

$$|\phi_p(t, \xi)| \leqslant \beta_p(|\xi|, t) \quad \forall \xi \in B_r,\ t \geqslant 0. \tag{23}$$

Let $\tilde{\alpha}_p(s) := \beta_p(s, 0),\ p \in \mathcal{P}$; they are class $\mathcal{K}$ functions on $[0, \infty)$. Define $\tilde{\alpha}(s) := \max\{\tilde{\alpha}_p(s),\ p \in \mathcal{P}\}$. Let $r_0^* = \min\{r, \tilde{\alpha}^{-1}(r), \ldots, \tilde{\alpha}^{-1} \circ \cdots \circ \tilde{\alpha}^{-1}(r)\}$ where $\tilde{\alpha}^{-1} \circ \cdots \circ \tilde{\alpha}^{-1}$ is the composition of $\tilde{\alpha}^{-1}$ with itself l times, $1 \leqslant l \leqslant m$. For every nonempty $\mathcal{Q} \subseteq \mathcal{P}$, since $|\phi_{\mathcal{Q}}^{t_{\mathcal{Q}}}(\xi)| \leqslant \tilde{\alpha}_1 \circ \cdots \circ \tilde{\alpha}_q(|\xi|),\ q = \mathrm{card}(\mathcal{Q})$, it is guaranteed that $|\phi_{\mathcal{Q}}^{t_{\mathcal{Q}}}(\xi)| < r,\ \forall t_{\mathcal{Q}} \geqslant 0,\ \forall \xi \in B_{r_0}$ if $0 < r_0 < r_0^*$. It is easy to prove that for arbitrary $\beta_1, \beta_2 \in \mathcal{KL}$, there is a $\beta \in \mathcal{KL}$ such that $\beta_1(\beta_2(r, s), t) \leqslant \beta(r, s + t)\ \forall r, s, t \geqslant 0$ (see [11, Lemma 2.2]). It follows that for every nonempty $\mathcal{Q} \subseteq \mathcal{P}$, there exists a $\beta_{\mathcal{Q}} \in \mathcal{KL}$ such that

$$|\phi_{\mathcal{Q}}^{t_{\mathcal{Q}}}(\xi)| \leqslant \beta_{\mathcal{Q}}(|\xi|, t_1 + \cdots + t_q) \quad \forall \xi \in B_{r_0}, \forall t_{\mathcal{Q}} \geqslant 0,\ q = \mathrm{card}(\mathcal{Q}).$$

Let $\beta(r,t) := \max\{\beta_{\mathscr{Q}}(r,t),\ \mathscr{Q} \subseteq \mathscr{P},\ \mathscr{Q} \neq \emptyset\}$. We then have the inequality (15) for every nonempty $\mathscr{Q} \subseteq \mathscr{P}$. For a nonempty $\mathscr{Q} \subseteq \mathscr{P}$, let

$$V_{\mathscr{Q}}(\xi) := \int_0^\infty \cdots \int_0^\infty G(|\phi_{\mathscr{Q}}^{t_{\mathscr{Q}}}(\xi)|)\,\mathrm{d}t_1 \ldots \mathrm{d}t_q,\quad q = \mathrm{card}(\mathscr{Q}). \tag{24}$$

As can be seen from (17), (18) and (24), $V_{\mathscr{Q}}(\xi)$ is obtained by iteratively following (17), (18) on the index set $\mathscr{Q}$ starting with $\phi_j(t,\xi)$ for some $j \in \mathscr{Q}$. We have

$$V_{\mathscr{Q}}(\xi) \leqslant \int_0^\infty \cdots \int_0^\infty G(\beta(|\xi|, t_1 + \cdots + t_q))\,\mathrm{d}t_1 \ldots \mathrm{d}t_q =: \alpha_{\mathscr{Q}}(|\xi|) \quad \forall \xi \in B_r,\ q = \mathrm{card}(\mathscr{Q}) \tag{25}$$

by virtue of (15). The function $\alpha_{\mathscr{Q}}$ is well-defined, in view of Lemma 3 applied with $u = |\phi_{\mathscr{Q}}^{t_{\mathscr{Q}}}|$. It is clear that $\alpha_{\mathscr{Q}} \in \mathscr{K}$.

From the construction, the explicit formula for $V_p(\xi)$ is

$$V_p(\xi) = \int_0^\infty \cdots \int_0^\infty G(|\phi_{\mathscr{I}_p}^{t_{\mathscr{I}_p}}(\xi)|)\,\mathrm{d}t_1 \ldots \mathrm{d}t_p = V_{\mathscr{I}_p}(\xi) \quad \forall \xi \in B_r,\ p \in \mathscr{P}, \tag{26}$$

where $\mathscr{I}_p := \{1, \ldots, p\}$. We then have

$$V_p(\xi) \leqslant \alpha_{2,p}(|\xi|) \quad \forall \xi \in B_r,\ p \in \mathscr{P} \tag{27}$$

for some $\alpha_{2,p} \in \mathscr{K}$ by virtue of (25). For all $p \in \mathscr{P}$, V_p is positive definite since $G \in \mathscr{K}$. The composition of flows $\phi_{\mathscr{Q}}^{t_{\mathscr{Q}}}(\xi)$ for every nonempty $\mathscr{Q} \subseteq \mathscr{P}$ is continuous since the flow $\phi_p(t,\xi)$ is continuous for all $p \in \mathscr{P}$. By commutativity, for each $p \in \mathscr{P}$, we have

$$V(\xi) = \int_0^\infty \overline{V}_p(\phi_p(t,\xi))\,\mathrm{d}t \quad \forall \xi \in B_r, \tag{28}$$

where

$$\overline{V}_p(\xi) := V_{\mathscr{P}_p}(\xi), \quad \mathscr{P}_p := \mathscr{P}\backslash\{p\}. \tag{29}$$

Eq. (28) is justified by Fubini's theorem (see, e.g., [13]); we can change the order of the integrals since for every nonempty $\mathscr{Q} \subseteq \mathscr{P}$, $V_{\mathscr{Q}}(\xi)$ is well-defined and $G(|\phi_{\mathscr{Q}}^{t_{\mathscr{Q}}}(\xi)|)$ is continuous. By boundedness of the Jacobian matrices of the vector fields as in (16), it is easy to show (see, e.g., [5, Excercise 3.17]) that for each $p \in \mathscr{P}$, we have

$$|\phi_p(t,\xi)| \geqslant |\xi| \exp(-Lt) \quad \forall t \geqslant 0, \tag{30}$$

$$\left\| \frac{\partial \phi_p(t,\xi)}{\partial \xi} \right\| \leqslant \exp(Lt) \quad \forall t \geqslant 0 \tag{31}$$

for all $\xi \in B_{r_0}$. From (30), for every nonempty subset $\mathscr{Q} \subseteq \mathscr{P}$, we have

$$|\phi_{\mathscr{Q}}^{t_{\mathscr{Q}}}(\xi)| \geqslant |\xi| \exp(-L(t_1 + \cdots + t_q)) \quad \forall t_{\mathscr{Q}} \geqslant 0,\ q = \mathrm{card}(\mathscr{Q}). \tag{32}$$

The inequality (32) together with (24) yields

$$V_{\mathscr{Q}}(\xi) \geqslant \int_0^{\mathrm{T}} \cdots \int_0^{\mathrm{T}} G\left(\frac{1}{2}|\xi|\right) \mathrm{d}t_1 \ldots \mathrm{d}t_q = T^q G\left(\frac{1}{2}|\xi|\right) =: \alpha_{1,q}(|\xi|), \tag{33}$$

where $T = \ln 2/(qL),\ q = \mathrm{card}(\mathscr{Q})$. Since $G \in \mathscr{K}$, the function $\alpha_{1,p}$ is of class $\mathscr{K}$. Combining (27) and (33) with $\mathscr{Q} = \mathscr{I}_p$ yields the inequality (20) for V, in which $p = m$; $\alpha_1 := \alpha_{1,m},\ \alpha_2 := \alpha_{2,m}$.

For each $p \in \mathscr{P}$, the derivative of V along $\phi_p(t,\xi)$ will be

$$L_{f_p} V(\xi) = \overline{V}_p(\phi_p(t,\xi))|_0^\infty = -\overline{V}_p(\xi) \quad \forall \xi \in B_{r_0}$$

since $\lim_{t\to\infty} \phi_p(t,\xi) = 0$ and $\overline{V}(0) = 0$. There is a class $\mathscr{K}$ function $\overline{\alpha}_p$ such that $\overline{V}_p(\xi) \geqslant \overline{\alpha}_p(|\xi|)\ \forall \xi \in B_r$ by using (32), (24) and (33) with $\mathscr{Q} = \mathscr{P}_p$. If we define

$$\alpha_3(s) := \min_{p \in \mathscr{P}} \overline{\alpha}_p(s), \quad s \in [0, r),$$

then $\alpha_3 \in \mathscr{K}$ and the inequality (21) follows.

By the chain rule, the gradient of V is

$$\frac{\partial V(\xi)}{\partial \xi} = \int_0^\infty \cdots \int_0^\infty G'(|\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)|) \frac{\phi_1^{\mathrm{T}}}{|\phi_1|} \times [\phi_1'(t_1,\xi)] \ldots [\phi_{m-1}'(t_{m-1},\xi)] \times \frac{\partial \phi_m}{\partial \xi}(t_m,\xi)\,\mathrm{d}t_1 \ldots \mathrm{d}t_m, \tag{34}$$

where $[\phi_i'(t,\xi)]$ denotes the partial derivative with respect to ξ of the solution $\phi_i(t,\xi)$ evaluated at $\xi = \phi_{\mathscr{Q}_i}^{t_{\mathscr{Q}_i}}(\xi),\ \mathscr{Q}_i = \{i+1, \ldots, m\}$ for $1 \leqslant i \leqslant m-1$.

Taking the norm of both sides of (34) and using (31), we have the inequality (22):

$$\left|\frac{\partial V(\xi)}{\partial \xi}\right| \leqslant \int_0^\infty \cdots \int_0^\infty G'(|\phi^{t_\mathscr{I}}_\mathscr{I}(\xi)|) \times e^{Lt_1} \ldots e^{Lt_m}\, dt_1 \ldots d_{t_m} = \int_0^\infty \cdots \int_0^\infty G'(|\phi^{t_\mathscr{I}}_\mathscr{I}(\xi)|) \times h(t_1 + \cdots + t_m)\, dt_1 \ldots d_{t_m} =: \alpha_4(|\xi|)$$

for all $\xi \in B_{r_0}$, where $h(t) = \exp(Lt)$. The function α_4 exists by the choice of G as in Lemma 3 with $g(t) = \beta(r, t)$, $h(t) = \exp(Lt)$, and $u(t_1, \ldots, t_m) = |\phi^{t_\mathscr{P}}_\mathscr{P}(\xi)|$. □

We stated the theorem for a family of pairwise commuting *asymptotically stable* systems. If individual systems are GAS, clearly they satisfy the conditions of the theorem. Further, if the Jacobian matrices of the vector fields are uniformly bounded on the whole state space $\mathbb{R}^n$, then there is no restriction on how large r_0 is, provided it is a finite number. Note also that the boundedness of the Jacobian matrices helps establish the bound (22) on the gradient of *V*. If we do not have the inequality (16), we still have the constructed *V* satisfying conditions (20) and (21) on a bounded region B_{r_0}, which imply local asymptotic stability of the corresponding switched system. The existence of a function $\alpha_1 \in \mathscr{K}$ in (20) can be concluded from the positive definiteness of *V*, without relying on (33).

Theorem 4 enables one to infer about stability of a switched system generated by a family of pairwise commuting asymptotically stable systems under perturbations, similarly to well-known results for non-switched nonlinear systems (see, e.g., [5, Lemma 9.3]). Consider a family of pairwise commuting asymptotically stable systems in the presence of perturbations,

$$\dot{x}(t) = f_p(x(t)) + \tilde{f}_p(x(t)), \quad p \in \mathscr{P}, \tag{35}$$

where $\tilde{f}_p : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function on B_r for each $p \in \mathscr{P}$. The corresponding switched system is

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) + \tilde{f}_{\sigma(t)}(x(t)).$$

For non-vanishing perturbations, the origin is no longer a common equilibrium. However, if the perturbation is small in some sense and the initial state is close enough to the origin, the trajectory of the perturbed switched system is ultimately bounded for arbitrary switching.

Corollary 1. *Consider the perturbed switched system* (35) *on* B_r. *Assume that the family of non-perturbed systems* (2) *satisfies the hypotheses of Theorem* 4. *There are constants* $\bar{r} > 0$, $\bar{\delta} > 0$ *such that if the perturbation terms* $\tilde{f}_p(x)$ *satisfy*

$$|\tilde{f}_p(\xi)| \leqslant \delta \quad \forall \xi \in B_r, \; p \in \mathscr{P}$$

for some $\delta \in (0, \bar{\delta})$, *then there exist* $M > 0$ *and* $\beta \in \mathscr{KL}$ *such that for every initial state* $\xi \in B_{\bar{r}}$, *and every switching signal* $\sigma \in S$, *the solution* $\psi_\sigma(t, \xi)$ *of the perturbed switched system satisfies*

$$|\psi_\sigma(t, \xi)| \leqslant \beta(|\xi|, t) \quad 0 \leqslant t \leqslant T$$

and

$$|\psi_\sigma(t, \xi)| \leqslant M \quad \forall t > T.$$

for some finite $T > 0$.

Proof. By Theorem 4, there exist a positive constant $r_0 < r$ and a function *V* satisfying (20)–(22). Let $\bar{\delta} := \theta\alpha_3(\alpha_2^{-1}(\alpha_1(r_0)))/\alpha_4(r_0)$ for some positive constant $\theta < 1$, $\bar{r} := \alpha_2^{-1}(\alpha_1(r_0))$, and $M := \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(\delta\alpha_4(r_0)/\theta)))$. The proof is similar to the proof for non-switched systems but the common Lyapunov function *V* is used in place of a single Lyapunov function. Due to space limitation, details are omitted (cf. [5, Lemma 9.3]). □

3.2. Global common Lyapunov function

In this section, we construct a global common Lyapunov function for the family of pairwise commuting GAS systems (2), following Kurzweil's construction. In the previous section, we only required individual systems to be forward complete and utilized GUAS property of the corresponding switched system. Here, we do not invoke GUAS property but assume that the individual systems are complete.[1] Hence, our construction yields a Lyapunov-based proof of the fact

[1] Note that completeness was not assumed in Theorem 2 because it can always be achieved by time rescaling [5, p. 669]; however, we cannot apply this technique to the system (2) because it does not preserve commutativity.

that a switched system generated by a finite family of pairwise commuting GAS complete subsystems is GUAS.

Define a function $g : \mathbb{R}^n \to [0, \infty)$ as

$$g(\xi) := \inf\{|\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)|,\ t_{\mathscr{P}} \leqslant 0\}. \tag{36}$$

Thus, the function g is the infimum of the solutions $\psi_\sigma(t, \xi)$ of the switched system (3) running backward in time for all possible switching signals $\sigma \in S$. Construct a function V as follows:

$$V_1(\xi) := \sup_{t \geqslant 0}\{g(\phi_1(t, \xi))k(t)\}, \tag{37}$$

$$V_p(\xi) := \sup_{t \geqslant 0}\{V_{p-1}(\phi_p(t, \xi))k(t)\}, \quad 2 \leqslant p \leqslant m, \tag{38}$$

$$V(\xi) := V_m(\xi), \tag{39}$$

where k is some function satisfying conditions K1 and K2 as in Section 2.2. We have the following theorem.

Theorem 5. *Consider the family* (2) *of pairwise commuting GAS complete systems. The function V constructed in* (37)–(39) *is continuous on* $\mathbb{R}^n$ *and locally Lipschitz on* $\mathbb{R}^n \backslash \{0\}$ *and satisfies the following inequalities*:

$$\alpha_1(|\xi|) \leqslant V(\xi) \leqslant \alpha_2(|\xi|) \quad \forall \xi \in \mathbb{R}^n, \tag{40}$$

$$D_{f_p} V(\xi) \leqslant -\alpha_3(|\xi|) \quad \forall \xi \in \mathbb{R}^n \backslash \{0\},\ p \in \mathscr{P}, \tag{41}$$

where $\alpha_1, \alpha_2 \in \mathscr{K}_\infty$ *and* α_3 *is some positive definite function on* $[0, \infty)$.

Proof. Firstly, we prove that the function g constructed as in (36) has properties P1, P2 and satisfies the following inequality:

$$g(\phi_p(t, \xi)) \leqslant g(\xi) \quad \forall t \geqslant 0,\ \xi \in \mathbb{R}^n,\ p \in \mathscr{P}. \tag{42}$$

From the definition of g in (36), we have $g(\xi) \leqslant |\xi|$, as can be seen by taking $t_{\mathscr{P}} = 0$. By commutativity, we have $\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi) = \phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\phi_p(t, \xi)) = \phi_p^t \circ \phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi)$ with $\mathscr{P}_p$ defined as in (29), and inequality (42) is verified as

$$\begin{aligned} g(\phi_p(t, \xi)) &= \inf\{|\phi_p^{t_p+t} \circ \phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi)|,\ t_p \leqslant 0,\ t_{\mathscr{P}_p} \leqslant 0\} \\ &\leqslant \inf\{|\phi_p^{t_p} \circ \phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi)|,\ t_p \leqslant 0,\ t_{\mathscr{P}_p} \leqslant 0\} \\ &\equiv g(\xi) \forall t \geqslant 0,\ \xi \in \mathbb{R}^n,\ p \in \mathscr{P}. \end{aligned}$$

Since $|\phi_p(t, \xi)| \leqslant \beta_p(|\xi|, t) \leqslant \beta_p(|\xi|, 0) =: \alpha_p(|\xi|)$ $\forall\ t_{\mathscr{P}} \geqslant 0,\ \xi \in \mathbb{R}^n$, we have

$$|\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)| \leqslant \alpha_1 \circ \cdots \circ \alpha_m(|\xi|) =: \alpha(|\xi|) \quad \forall t_{\mathscr{P}} \geqslant 0,\ \xi \in \mathbb{R}^n.$$

Replacing ξ by $\phi_{\mathscr{P}}^{\tau_{\mathscr{P}}}(\xi)$, by commutativity property, the above inequality becomes

$$|\phi_{\mathscr{P}}^{(t+\tau)_{\mathscr{P}}}(\xi)| \leqslant \alpha(|\phi_{\mathscr{P}}^{\tau_{\mathscr{P}}}(\xi)|) \quad \forall t_{\mathscr{P}} \geqslant 0,\ \xi \in \mathbb{R}^n.$$

Letting $(t + \tau)_{\mathscr{P}} = 0$, we obtain

$$|\xi| \leqslant \alpha(|\phi_{\mathscr{P}}^{\tau_{\mathscr{P}}}(\xi)|) \leqslant \alpha(g(\xi)) \quad \forall \tau_{\mathscr{P}} \leqslant 0,\ \xi \in \mathbb{R}^n.$$

We then have property P1 for g,

$$\alpha^{-1}(|\xi|) \leqslant g(\xi) \leqslant |\xi| \quad \forall \xi \in \mathbb{R}^n.$$

We now prove that g has property P2. Consider a compact set $H = \{\xi \in \mathbb{R}^n : a_1 \leqslant |\xi| \leqslant a_2\}$ where $0 < a_1 < a_2$. From the GAS property of the individual systems, we have $|\xi| \leqslant \beta_p(|\phi_p(t, \xi)|, -t)$, $\forall t < 0$, $p \in \mathscr{P}$, $\xi \in \mathbb{R}^n$. Then,

$$a_1 \leqslant |\xi| \leqslant \beta_m^{-t_1} \circ \cdots \circ \beta_1^{-t_m}(|\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)|), \quad \forall \xi \in H,\ t_{\mathscr{P}} < 0, \tag{43}$$

where $\beta_p^t(x) := \beta_p(x, t)$. Since $\beta_p^0(r) \geqslant r\ \forall r \geqslant 0$, there exists $\tau_{\mathscr{P}} \geqslant 0$ such that

$$\beta_m^{\tau_1} \circ \cdots \circ \beta_1^{\tau_m}(2a_2) = a_1. \tag{44}$$

For example, we can take some $\tau_{\mathscr{P}}$ such that $\beta_i(a_1 + \Delta i, \tau_i) = a_1 + \Delta(i-1)$ where $\Delta = (2a_2 - a_1)/m$, $1 \leqslant i \leqslant m$; note that there may be more than one $\tau_{\mathscr{P}} \leqslant 0$ such that (44) holds. For all $t_{\mathscr{P}} \leqslant -\tau_{\mathscr{P}}$,

$$a_1 \leqslant \beta_1^{\tau_1} \circ \cdots \circ \beta_m^{\tau_m}(|\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)|) \quad \forall \xi \in H$$

by virtue of (43). The foregoing inequality together with (44) yield

$$|\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)| \geqslant 2a_2 \geqslant 2|\xi| \geqslant 2g(\xi) \quad \forall \xi \in H, t_{\mathscr{P}} \leqslant -\tau_{\mathscr{P}}. \tag{45}$$

This implies that g is well-defined (infimum is achieved for some $t_i \in [-\tau_i, 0]$ and is unique for each fixed $\xi \in H$). Since f_p is locally Lipschitz for all $p \in \mathscr{P}$, $\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi)$ is Lipschitz for $\xi \in H$ for any compact time interval. Since the norm function $|\cdot|$ is locally Lipschitz on H, it follows that g is Lipschitz on H and hence, g is locally Lipschitz on $\mathbb{R}^n \backslash \{0\}$. It is clear that $g(\xi) \to 0$ as $\xi \to 0$ following (45).

Since $g(0)=0$, it follows that $g(\xi)$ is continuous everywhere. Thus, g has property P2.

The function V_1 defined by (37) is well-defined and has properties P1 and P2 by Lemma 2 with the function g as in (36). We have

$$\begin{aligned}V_1(\phi_2(t,\xi)) &= \sup_{t_1\geqslant 0}\{g(\phi_1^{t_1}\circ\phi_2^{t}(\xi))k(t_1)\}\\ &= \sup_{t_1\geqslant 0}\{g(\phi_2^{t}\circ\phi_1^{t_1}(\xi))k(t_1)\}\\ &\leqslant \sup_{t_1\geqslant 0}\{g(\phi_1^{t_1}(\xi))k(t_1)\}\equiv V_1(\xi)\\ &\quad \forall t\geqslant 0,\ \forall \xi\in\mathbb{R}^n,\end{aligned}$$

by virtue of (42). It follows that V_2 is well-defined and has properties P1 and P2 by Lemma 2. Continuing this procedure, we see that V_p is well-defined and has properties P1 and P2 for every $p\in\mathscr{P}$. It remains to prove (41). For each $p\in\mathscr{P}$, we have

$$\begin{aligned}V(\xi) &= \sup_{t_m\geqslant 0}\left\{\left\{\cdots\sup_{t_1\geqslant 0}\{g(\phi_1^{t_1}\circ\cdots\circ\phi_m^{t_m}(\xi))\right.\right.\\ &\qquad \left.\left. k(t_1)\}\cdots\right\}k(t_m)\right\},\\ &= \sup_{t_{\mathscr{P}}\geqslant 0}\{g(\phi_{\mathscr{P}}^{t_{\mathscr{P}}}(\xi))k_1(t_1)\dots k(t_m)\}\\ &= \sup_{t_{\mathscr{P}}\geqslant 0}\{\overline{V}_p(\phi_p(t_p,\xi))k(t_p)\}\quad \forall p\in\mathscr{P},\end{aligned}$$

where

$$\begin{aligned}\overline{V}_p(\xi) &:= \sup_{t_{\mathscr{P}_p}\geqslant 0}\{g(\phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi))k(t_1)\dots k(t_{m-1})\},\\ \mathscr{P}_p &= \mathscr{P}\backslash\{p\}.\end{aligned}$$

The function $\overline{V}_p(x)$ is obtained by following the procedure in (37) and (38) along the index set $\mathscr{P}_p$ starting with $\phi_j(t,x)$ for some $j\in\mathscr{P}_p$. As proved previously, $\overline{V}_p(x)$ is well-defined and has properties P1 and P2. In particular, it is continuous and locally Lipschitz on $\mathbb{R}^n\backslash\{0\}$ and there exist $\overline{\alpha}_{1,p},\overline{\alpha}_{2,p}\in\mathscr{K}_\infty$ such that

$$\overline{\alpha}_{1,p}(|\xi|)\leqslant\overline{V}_p(\xi)\leqslant\overline{\alpha}_{2,p}(|\xi|)\quad \forall\xi\in\mathbb{R}^n.$$

For all $p\in\mathscr{P}$, we have

$$\begin{aligned}&\overline{V}_p(\phi_p(t,\xi))\\ &\quad = \sup_{t\geqslant 0,t_{\mathscr{P}_p}\geqslant 0}\{g(\phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi)\circ\phi_p^{t}(x))k(t_1)\dots k(t_{m-1})\}\\ &\quad = \sup_{t\geqslant 0,t_{\mathscr{P}_p}\geqslant 0}\{g(\phi_p^{t}\circ\phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi))k(t_1)\dots k(t_{m-1})\}\\ &\quad \leqslant \sup_{t_{\mathscr{P}_p}\geqslant 0}\{g(\phi_{\mathscr{P}_p}^{t_{\mathscr{P}_p}}(\xi))k(t_1)\dots k(t_{m-1})\}\\ &\quad \equiv \overline{V}_p(\xi)\quad \forall t\geqslant 0,\ \xi\in\mathbb{R}^n,\end{aligned}$$

by virtue of (42). Applying Lemma 2 with $g=\overline{V}_p$, we then have that V satisfies

$$D_{f_p}V(\xi)\leqslant -\overline{\alpha}_{3,p}(|\xi|)\quad \forall\xi\in\mathbb{R}^n\backslash\{0\},\ \ p\in\mathscr{P},$$

where $\alpha_{3,p}\in\mathscr{K}$. Define

$$\alpha_3(r) := \min_{p\in\mathscr{P}}\{\overline{\alpha}_{3,p}(r)\},\quad r\in[0,\infty), \tag{46}$$

so that $\alpha_3\in\mathscr{K}_\infty$ and the inequality (41) is verified. □

Remark 1. The function V constructed in Theorem 5 is not necessarily continuously differentiable. For a single nonlinear system, smoothing a locally Lipschitz Lyapunov function is a well-known result (see, e.g., [20]). The function V constructed here can be smoothed by the smoothing procedure described in [10]. In particular, we can obtain a global continuously differentiable Lyapunov function. It is noted that we can also smooth the local Lyapunov function V constructed in Section 3.1 to get a local smooth common Lyapunov function.

4. Conclusion

We have presented iterative constructions of common Lyapunov functions for a family of pairwise commuting GAS nonlinear systems, both local and global. Based on the iterative procedure proposed in [17], our constructions relax the exponential stability assumption imposed in [18] by employing general converse Lyapunov theorems for nonlinear systems. The local construction leads to the result that for the perturbed switched system, the state is ultimately bounded for arbitrary switching if perturbations are uniformly bounded and the initial state is sufficiently small. The global construction directly implies GUAS of the switched system generated by a family of pairwise commuting GAS complete subsystems, thereby providing a Lyapunov-based proof of this fact (established in [11] by time-domain arguments).

Acknowledgements

We thank Michael Malisoff for useful comments on an earlier draft.

Appendix A. Proof of Lemma 3

Proof. There exists a sequence t_n such that $g(t_n) \leqslant 1/(1+n),\ n=1,2,\dots$ since $g(t)$ is decreasing. We construct $\eta(t)$ as follows:

- $\eta(t_n)=1/n$, between t_n and t_{n+1}, $\eta(t)$ is linear, in the interval $0<t\leqslant t_1,\ \eta(t)=(t_1/t)^p$ where p is a large enough positive integer such that $\eta'(t_1^-)<\eta'(t_1^+)$.

The function η is decreasing by construction and $g(t)<\eta(t)$ for $t\geqslant t_1$. We also have $\eta(t)\to\infty$ as $t\to 0^+$. The inverse function $\eta^{-1}(t)$ is a decreasing function and $\eta^{-1}(s)\to\infty$ as $s\to 0^+$. Then for all $s_i\geqslant 0$ such that $s_1+\dots+s_l\geqslant t_1$, we have

$$\begin{aligned}\eta^{-1}(u(s_1,\dots,s_l)) &\geqslant \eta^{-1}(g(s_1+\dots+s_l))\\ &> \eta^{-1}(\eta(s_1+\dots+s_l)) = s_1+\dots+s_l.\end{aligned} \tag{47}$$

Define

$$H(s) := \frac{\exp[-\eta^{-1}(s)]}{h(\eta^{-1}(s))}, \quad s>0, \qquad \text{and} \quad H(0):=0$$

then H is of class $\mathcal{K}$. Define

$$G(r) := \int_0^r H(s)\,\mathrm{d}s$$

then G is well-defined and is also of class $\mathcal{K}$. Its derivative $G'(r)=H(r)$ is of class $\mathcal{K}$. By virtue of (47), we have

$$\begin{aligned}G'(u(s_1,\dots,s_l)) &= \frac{\exp[-\eta^{-1}(u(s_1,\dots,s_l))]}{h(\eta^{-1}(u(s_1,\dots,s_l)))}\\ &\leqslant \frac{\mathrm{e}^{-(s_1+\dots+s_l)}}{h(s_1+\dots+s_l)} \quad \forall s_1+\dots+s_l\geqslant t_1.\end{aligned} \tag{48}$$

The foregoing inequality leads to

$$\begin{aligned}&\int_{t_1}^{\infty}\cdots\int_{t_1}^{\infty} G'(u(s_1,\dots,s_l)) \times h(s_1+\dots+s_l)\,\mathrm{d}s_1\dots\mathrm{d}s_l\\ &\quad\leqslant \int_{t_1}^{\infty}\cdots\int_{t_1}^{\infty} \mathrm{e}^{-s_1}\cdots\mathrm{e}^{-s_l}\,\mathrm{d}s_1\dots\mathrm{d}s_l \leqslant 1.\end{aligned}$$

For a given index i, $1\leqslant i\leqslant l$, if $s_i\geqslant t_1$ we then have $s_1+\dots+s_l\geqslant t_1$ since $s_j\geqslant 0,\ 1\leqslant j\leqslant l$. Thus the inequality (48) holds and hence, the integral

$$\int_{0}^{\infty}\cdots\int_{0}^{\infty} G'(u(s_1,\dots,s_l)) \times h(s_1+\dots+s_l)\,\mathrm{d}s_1\dots\mathrm{d}s_l$$

is bounded by some constant c_2 (loosely speaking, the integral $\int_0^\infty\cdots\int_0^\infty$ is the sum of multiple integrals, each of which is a combination of $\int_0^{t_1}$ and $\int_{t_1}^\infty$; there are 2^l of them; $\int_0^{t_1}$ is always bounded).

For all $s_i\geqslant 0$ such that $s_1+\dots+s_l\geqslant t_1$, we have

$$\begin{aligned}G(u(s_1,\dots,s_l)) &= \int_0^{u(s_1,\dots,s_l)} \frac{\exp[-\eta^{-1}(s)]}{h(\eta^{-1}(s))}\,\mathrm{d}s\\ &< \frac{\mathrm{e}^{-(s_1+\dots+s_l)}}{h(0)}\,u(s_1,\dots,s_l)\\ &\leqslant \frac{\mathrm{e}^{-(s_1+\dots+s_l)}}{h(0)}\,g(s_1+\dots+s_l)\\ &\leqslant \frac{\mathrm{e}^{-(s_1+\dots+s_l)}}{h(0)}\end{aligned}$$

since h is non-increasing and by virtue of (47). Then,

$$\begin{aligned}&\int_{t_1}^{\infty}\cdots\int_{t_1}^{\infty} G(u(s_1,\dots,s_l))\,\mathrm{d}s_1\dots\mathrm{d}s_l\\ &\quad\leqslant \frac{1}{h(0)}\int_{t_1}^{\infty}\cdots\int_{t_1}^{\infty} \mathrm{e}^{-(s_1+\dots+s_l)}\,\mathrm{d}s_1\dots\mathrm{d}s_l < \infty.\end{aligned}$$

It follows that

$$\int_0^\infty \cdots \int_0^\infty G(u(s_1, \ldots, s_l))\, \mathrm{d}s_1 \ldots \mathrm{d}s_l$$

is bounded by some constant c_1. $\square$

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