43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

On Stability of Stochastic Switched Systems

Debasish Chatterjee and Daniel Liberzon

Abstract— **In this paper we propose a method for stability analysis of switched systems perturbed by a Wiener process. It utilizes multiple Lyapunov-like functions and is analogous to an existing result for deterministic switched systems.**

Index Terms— **switched systems, multiple Lyapunov-like functions, stochastic stability.**

I. INTRODUCTION

THERE are essentially two ways to analyze stability of deterministic switched systems; one involves construction of a common Lyapunov function and the other utilizes multiple Lyapunov functions. The former task is usually more challenging, though once a common Lyapunov function is found, the analysis is simple. The latter method, usually more amenable to applications, was first proposed in [1] and developed extensively in [2]; see also [3, Chapter 3] for a detailed discussion. We seek to extend this method to switched systems perturbed by a Wiener process.

We model each subsystem of a switched system by an Itô differential equation and make use of the stochastic differentials of Lyapunov-like functions for each subsystem along the lines of classical results on stochastic stability; see e.g. [4], [5], [6] for further details. In particular, we show that a switched system perturbed by a Wiener process is globally asymptotically stable in probability (GAS-P)—to be defined shortly—provided each subsystem is GAS-P and the sequence formed by the Lyapunov-like function corresponding to each subsystem, at the switching instants when that subsystem becomes active, is decreasing. An application of this result for the case of dwell-time switching, as well as sufficient conditions for GAS-P involving a common Lyapunov-like function, are provided.

II. PRELIMINARIES

For M_1, M_2 subsets of euclidean space, let $\mathrm{C}[M_1, M_2]$ denote the space of all continuous functions $f : M_1 \longrightarrow M_2$ and let $\mathrm{C}^2[M_1, M_2]$ denote the space of all functions $f : M_1 \longrightarrow M_2$ that are twice continuously differentiable. We say that a function $\alpha \in \mathrm{C}[\mathbb{R}_{\geqslant 0}, \mathbb{R}_{\geqslant 0}]$ is of class $\mathcal{K}$ if α is increasing with $\alpha(0) = 0$, is of class $\mathcal{K}_\infty$ if in addition $\alpha(r) \to \infty$ as $r \to \infty$, and we write $\alpha \in \mathcal{K}$ and $\alpha \in \mathcal{K}_\infty$ respectively. A function $\beta \in \mathrm{C}\big[\mathbb{R}^2_{\geqslant 0}, \mathbb{R}_{\geqslant 0}\big]$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is a function of class $\mathcal{K}$ for every fixed t and $\beta(r, t) \to 0$ as $t \to \infty$ for each fixed r, and we write $\beta \in \mathcal{KL}$.

This work was supported by NSF ECS-0134115 CAR and DARPA/AFOSR MURI F49620-02-1-0325 grants.

Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. dchatter, liberzon@control.csl.uiuc.edu

Let $\mathbf{\Omega} := (\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space and let x be a random variable on $\mathbf{\Omega}$. We will have occasion to use two inequalities (see e.g. [7] for further details): if $\phi \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}]$ is concave, then $\mathrm{E}[\phi(x)] \leqslant \phi(\mathrm{E}[x])$ provided $\mathrm{E}[\phi(x)]$ exists and is finite (*Jensen's inequality*); for $\varepsilon > 0$ and $\psi \in \mathrm{C}[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, we have $\mathrm{P}[\psi(x) \geqslant \varepsilon] \leqslant \mathrm{E}[\psi(x)]/\varepsilon$ provided $\mathrm{E}[\psi(x)]$ exists and is finite (*Chebyshev's inequality*). We assume that all expectations utilized in the analysis exist for all times $t \geqslant 0$.

We define a family of systems

$$\mathrm{d}x = f_p(x)\mathrm{d}t + G_p(x)\mathrm{d}w, \qquad p \in \mathcal{P}, \tag{1}$$

where $x \in \mathbb{R}^n$, w is an m-dimensional normalized Wiener process defined on the probability space $\mathbf{\Omega}$, $\mathrm{d}x$ is a stochastic differential of x, $\mathcal{P}$ is an index set, $f_p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $G_p : \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times m}$ are sufficiently well-behaved to ensure existence and uniqueness of the corresponding solution process (see e.g. [8] for precise conditions), and $f_p(0) = 0$, $G_p(0) = 0$ for every $p \in \mathcal{P}$. To define a switched system for the family, we consider a piecewise constant function (continuous from the right by convention) $\sigma : \mathbb{R}_{\geqslant 0} \longrightarrow \mathcal{P}$, called the *switching signal*, which specifies at every time t the index $\sigma(t) = p \in \mathcal{P}$ of the active subsystem. The switched system for this family generated by σ is

$$\mathrm{d}x = f_\sigma(x)\mathrm{d}t + G_\sigma(x)\mathrm{d}w, \quad x(0) = x_0 \neq 0, \quad t \geqslant 0. \tag{2}$$

We assume that there is no jump in the state x at the switching instants, and that there is a finite number of switches on every bounded interval of time. The above definitions of f_p and G_p indicate that the solution process of (2) is trivial if $x_0 = 0$, so we exclude this case. We denote the switching instants by $t_i,\ i = 1, 2, \ldots,\ t_0 := 0$, and the sequence $\{t_i\}_{i\geqslant 0}$ is strictly increasing. The *infinitesimal generator* for each system from the family (1) acting on a function $V \in \mathrm{C}^2[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ is defined to be $\mathscr{L}_p V(x) := V_x(x) f_p(x) + \frac{1}{2}\operatorname{tr}\big(V_{xx}(x) G_p(x) G_p^{\mathrm{T}}(x)\big)$, where tr denotes the trace of a square matrix.

We adopt to the context of the stochastic switched system (2) the following notion of stochastic stability, defined in [5].

Definition 2.1: The stochastic switched system (2) is *globally asymptotically stable in probability* (GAS-P) for a given switching signal σ if for every $\eta \in]0, 1[$, there exists a function $\beta \in \mathcal{KL}$ such that the estimate

$$\mathrm{P}[|x(t)| < \beta(|x_0|, t)] \geqslant 1 - \eta, \qquad t \geqslant 0$$

holds true along all solutions of (2).

We need the following Lemma ([5, Theorem 3.3]) for our main result.

Lemma 2.2: Consider the system having index p in the family (1) and let there exist functions $V_p \in \mathrm{C}^2[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $W \in \mathcal{K}$, such that for all $x \in \mathbb{R}^n$, we have

$$\alpha_1(|x|) \leqslant V_p(x) \leqslant \alpha_2(|x|) \tag{3}$$

and

$$\mathscr{L}_p V_p(x) \leqslant -W(|x|). \tag{4}$$

Then the system is GAS-P.

There are two easy generalizations of Lemma 2.2 to switched systems which we list below. The first—Proposition 2.3—utilizes a common Lyapunov-like function and the second—Proposition 2.4—involves multiple Lyapunov-like functions satisfying a matching condition at the switching instants.

Proposition 2.3: Consider the stochastic switched system (2). Let there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V \in \mathrm{C}^2[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$, $W \in \mathcal{K}$, such that

(a) for all $x \in \mathbb{R}^n$, (3) is satisfied;
(b) for all $x \in \mathbb{R}^n$ and every $p \in \mathcal{P}$, $\mathscr{L}_p V(x) \leqslant -W(|x|)$.

Then the system (2) is GAS-P.

Proposition 2.4: Consider the stochastic switched system (2). Let there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $V_p \in \mathrm{C}^2[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ for $p \in \mathcal{P}$, $W \in \mathcal{K}$, such that

(a) for all $x \in \mathbb{R}^n$, (3) is satisfied;
(b) for all $x \in \mathbb{R}^n$ and for every $p \in \mathcal{P}$, (4) is satisfied;
(c) for every switching instant t_i, the equality $V_{\sigma(t_i)}(x(t_{i+1})) = V_{\sigma(t_{i+1})}(x(t_{i+1}))$ is satisfied.

Then the system (2) is GAS-P.

The proofs of the above Propositions may be constructed along similar lines as [5, Theorem 3.3]. In the next section, we present our main result; it involves multiple Lyapunov-like functions but applies to more general situations compared to Proposition 2.4 by dispensing with the matching condition in hypothesis (c).

III. MAIN RESULT

Assumption 3.1: For the rest of the paper, we assume that the index set $\mathcal{P}$ is finite: $\mathcal{P} = \{1, 2, \ldots, \mathrm{N}\}$.

The following Theorem constitutes our main result, and may be viewed as a stochastic counterpart of [3, Theorem 3.1].

Theorem 3.2: Consider the stochastic switched system (2). Let there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with $\alpha_2 \circ \alpha_1^{-1}$ concave, $V_p \in \mathrm{C}^2[\mathbb{R}^n, \mathbb{R}_{\geqslant 0}]$ for $p \in \mathcal{P}$, $W, U \in \mathcal{K}$ with $U \circ \alpha_1^{-1}$ convex, such that

(i) for all $x \in \mathbb{R}^n$, (3) is satisfied;
(ii) for all $x \in \mathbb{R}^n$ and for every $p \in \mathcal{P}$, (4) is satisfied;
(iii) for every $p \in \mathcal{P}$ and every pair of switching instants (t_i, t_j), $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p$ and $\sigma(t_k) \neq p$ for $i < k < j$, the inequality

$$\mathrm{E}[V_p(x(t_j))] - \mathrm{E}[V_p(x(t_i))] \leqslant -\mathrm{E}[U(|x(t_i)|)] \tag{5}$$

is satisfied.

Then the system (2) is GAS-P.

Proof: Consider the time interval $[t_0, t_1[$. By hypotheses (ii) and (i), we have

$$\mathrm{E}\big[V_{\sigma(t_0)}(x(t_1))\big] \leqslant \mathrm{E}\big[V_{\sigma(t_0)}(x_0)\big] \leqslant \alpha_2(|x_0|). \tag{6}$$

Over the same interval, for $p \neq \sigma(t_0)$, the estimate

$$\begin{aligned}\mathrm{E}[V_p(x(t_1))] &\leqslant \mathrm{E}[\alpha_2(|x(t_1)|)] \\ &= \mathrm{E}\big[\alpha_2 \circ \alpha_1^{-1} \circ \alpha_1(|x(t_1)|)\big] \\ &\leqslant \alpha_2 \circ \alpha_1^{-1}\big(\mathrm{E}\big[V_{\sigma(t_0)}(x(t_1))\big]\big)\end{aligned}$$

holds true, where we have utilized Jensen's inequality and hypothesis (i). In the light of (6), the above inequality implies

$$\mathrm{E}[V_p(x(t_1))] \leqslant \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2(|x_0|).$$

Now consider the interval $[t_1, t_2[$. Proceeding as before, we have

$$\mathrm{E}\big[V_{\sigma(t_1)}(x(t_2))\big] \leqslant \mathrm{E}\big[V_{\sigma(t_1)}(x(t_1))\big] \leqslant \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2(|x_0|). \tag{7}$$

Over the same interval, for $p \neq \sigma(t_1)$, the estimate

$$\begin{aligned}\mathrm{E}[V_p(x(t_2))] &\leqslant \mathrm{E}[\alpha_2(|x(t_2)|)] \\ &= \mathrm{E}\big[\alpha_2 \circ \alpha_1^{-1} \circ \alpha_1(|x(t_2)|)\big] \\ &\leqslant \alpha_2 \circ \alpha_1^{-1}\big(\mathrm{E}\big[V_{\sigma(t_1)}(x(t_2))\big]\big)\end{aligned}$$

holds true. In the light of (7), the above inequality implies

$$\mathrm{E}[V_p(x(t_2))] \leqslant \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2 \circ \alpha_1^{-1} \circ \alpha_2(|x_0|).$$

Define the function

$$\begin{aligned}\alpha(\cdot) := \max\Big\{&\alpha_2(\cdot), \big(\alpha_2 \circ \alpha_1^{-1}\big) \circ \alpha_2(\cdot), \ldots, \\ &\overbrace{\big(\alpha_2 \circ \alpha_1^{-1}\big) \circ \cdots \circ \big(\alpha_2 \circ \alpha_1^{-1}\big)}^{\mathrm{N}-1 \text{ times}} \circ \alpha_2(\cdot)\Big\}.\end{aligned}$$

Considering all possible switching sequences and keeping in mind hypothesis (iii), it is possible to show that the estimate

$$\mathrm{E}\big[V_{\sigma(t)}(x(t))\big] \leqslant \alpha(|x_0|) \qquad \forall\, t \geqslant 0 \tag{8}$$

is valid.

Clearly, there can be two possibilities:

Case 1 *Switching stops in finite time.* Due to (8), $\mathrm{E}\big[V_{\sigma(t)}(x(t))\big]$ is finite. Now, since σ eventually becomes constant at some index q (say), hypotheses (i) and (ii), together with Lemma 2.2, imply the GAS-P property of the switched system.

Case 2 *Switching continues indefinitely.* There exists at least one index $p \in \mathcal{P}$ such that the positive subsequence $\big\{\mathrm{E}\big[V_{\sigma(t_i)}(x(t_i))\big]\big\}_{\{i \geqslant 0, \sigma(t_i) = p\}}$ is infinite in length. Clearly the sequence is also monotonically decreasing by hypothesis (iii), and therefore must attain a limit, say $c \geqslant 0$. Taking limits on both sides of (5), we have

$$c - c \leqslant -\lim_{\substack{i \uparrow \infty \\ \sigma(t_i) = p}} \mathrm{E}[U(|x(t_i)|)],$$

which leads to $\lim_{i\uparrow\infty,\sigma(t_i)=p}\mathrm{E}[U(|x(t_i)|)] = 0$. By convexity of $U\circ\alpha_1^{-1}$ it follows that

$$\lim_{\substack{i\uparrow\infty\\ \sigma(t_i)=p}} \mathrm{E}[\alpha_1(|x(t_i)|)] = 0.$$

Combined with (8) and hypothesis (i), this leads to

$$\mathrm{E}[\alpha_1(|x(t)|)] \longrightarrow 0 \quad \text{as} \quad t\longrightarrow\infty. \tag{9}$$

By virtue of (8) and (9), it follows that there exists a function $\widetilde{\beta}\in\mathcal{KL}$ (the construction of such a function is a standard procedure) such that

$$\mathrm{E}[\alpha_1(|x(t)|)] \leqslant \widetilde{\beta}(|x_0|,t) \qquad \forall\, t\geqslant 0. \tag{10}$$

For an arbitrary $\eta\in\,]0,1[$, consider a class $\mathcal{KL}$ function $\overline{\beta}$ such that $\overline{\beta}(r,s) > \widetilde{\beta}(r,s)/\eta$ for all positive r and nonnegative s. Utilizing Chebyshev's inequality and (10) for each $t\geqslant 0$, we obtain

$$\mathrm{P}\big[\alpha_1(|x(t)|)\geqslant\overline{\beta}(|x_0|,t)\big] \leqslant \frac{\mathrm{E}[\alpha_1(|x(t)|)]}{\overline{\beta}(|x_0|,t)} < \eta.$$

Defining $\beta(r,s) := \alpha_1^{-1}\circ\overline{\beta}(r,s)$, we see that for each $t\geqslant 0$ the estimate

$$\mathrm{P}[|x(t)| < \beta(|x_0|,t)] \geqslant 1-\eta$$

is valid, which proves that (2) is GAS-P. ■

Theorem 3.2 requires the function $\alpha_2\circ\alpha_1^{-1}$ to be concave; this holds, for instance, in the case of purely quadratic or quartic functions α_1 and α_2. For example, consider the linear version of (2):

$$\mathrm{d}x = A_\sigma x\mathrm{d}t + B_\sigma x\mathrm{d}w, \qquad x(0)=x_0\neq 0, \quad t\geqslant 0,$$

where $A_p, B_p\in\mathbb{R}^{n\times n}$, $p\in\mathcal{P}$. If we can find functions $V_p(x) = x^{\mathrm{T}}P_p x$, $p\in\mathcal{P}$, where every P_p is a symmetric positive definite matrix that solves the linear matrix inequality $A_p^{\mathrm{T}}P_p + P_pA_p + B_pP_pB_p^{\mathrm{T}} \leqslant -Q$, for some symmetric positive definite matrix Q, then V_p serves as a Lyapunov function for the subsystem with index p satisfying (4); see [4] for further details. Considering the finiteness of the set $\mathcal{P}$, it follows that there exists $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, that are quadratic functions of $|x|$, with V_p satisfying (3).

IV. Application and Conclusion

As an application of our result, consider a switched system in which the switching signal has a *dwell-time* τ_{D}, i.e. any two switching instants are separated by at least τ_{D} units of time: $t_{i+1}-t_i \geqslant \tau_{\mathrm{D}}$ for all $i\geqslant 0$ (see e.g. [3, Section 3.2.1]). Suppose that hypothesis (i) of Theorem 3.2 is satisfied, and there exists $\lambda,\mu > 0$ such that for all $x\in\mathbb{R}^n$ and $p,q\in\mathcal{P}$, the inequalities $V_p(x) \leqslant \mu V_q(x)$ and $\mathscr{L}_pV_p(x) \leqslant -\lambda V_p(x)$ are valid. Clearly, hypothesis (ii) of Theorem 3.2 is also satisfied. At any switching instant, utilizing the construction in the proof of [4, Chapter V, Theorem 7.1], we obtain the estimate

$$\mathrm{E}\big[V_{\sigma(t_{i+1})}(x(t_{i+1}))\big] \leqslant \mu\exp(-\lambda\tau_{\mathrm{D}})\mathrm{E}\big[V_{\sigma(t_i)}(x(t_i))\big]$$

for all $i\geqslant 0$. Let τ_{D} be larger than $\frac{\ln\mu}{\lambda}$. Combining this lower bound on τ_{D} with the above inequality, it follows that the sequence $\big\{\mathrm{E}\big[V_{\sigma(t_i)}(x(t_i))\big]\big\}_{i\geqslant 0}$ is monotonically decreasing. It is now easy to show, considering the finiteness of $\mathcal{P}$, that hypothesis (iii) of Theorem 3.2 is satisfied with the function $U(r) = (1-\exp(\ln\mu - \lambda\tau_{\mathrm{D}}))\alpha_1(r)$, so that $U\circ\alpha_1^{-1}$ is convex. Therefore, the system is GAS-P by Theorem 3.2.

The material presented in this paper is a part of an ongoing investigation into the qualitative properties of more general nonautonomous stochastic switched systems, see [9] for further details. There we combine the method of analysis using multiple Lyapunov-like functions with a stochastic version of the comparison principle to reach a general framework. Applications of this framework include a generalization of the above example to the case of *average dwell time* switching—introduced in [10].

References

[1] P. Peleties and R. A. DeCarlo, "Asymptotic stability of m-switched systems using Lyapunov-like functions," in *Proceedings of the American Control Conference*, 1991, pp. 1679–1684.

[2] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[3] D. Liberzon, *Switching in Systems and Control*. Birkhäuser, 2003.

[4] R. Z. Haśminskii, *Stochastic Stability of Differential Equations*. Sijthoff & Noordhoff, 1980.

[5] M. Krstić and H. Deng, *Stabilization of Nonlinear Uncertain Systems*. Springer-Verlag, 1998.

[6] H. J. Kushner, *Stochastic Stability and Control*. Academic Press, 1967.

[7] R. B. Ash and C. A. Doléans-Dade, *Probability and Measure Theory*. Harcourt / Academic Press, 1999.

[8] B. K. Øksendal, *Stochastic Differential Equations*. Springer-Verlag, 1998.

[9] D. Chatterjee, "Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions," Master's thesis, University of Illinois at Urbana-Champaign, submitted for publication, 2004.

[10] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control*, no. 3, 1999, pp. 2655–2660.