On quantization and delay effects in nonlinear control systems

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Summary. The purpose of this paper is to demonstrate that a unified study of quantization and delay effects in nonlinear control systems is possible by merging the quantized feedback control methodology recently developed by the author and the small-gain approach to the analysis of functional differential equations with disturbances proposed earlier by Teel. We prove that under the action of a robustly stabilizing feedback controller in the presence of quantization and sufficiently small delays, solutions of the closed-loop system starting in a given region remain bounded and eventually enter a smaller region. We present several versions of this result and show how it enables global asymptotic stabilization via a dynamic quantization strategy.

1 Introduction

To be applicable in realistic situations, control theory must take into account communication constraints between the plant and the controller, such as those arising in networked embedded systems. Two most common phenomena relevant in this context are quantization and time delays. It is also important to be able to handle nonlinear dynamics. To the best of the author's knowledge, the present paper is a first step towards addressing these three aspects in a unified and systematic way.

It is well known that a feedback law which globally asymptotically stabilizes a given system in the absence of quantization will in general fail to provide global asymptotic stability of the closed-loop system that arises in the presence of a quantizer with a finite number of values. One reason for this is saturation: if the quantized signal is outside the range of the quantizer, then

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the quantization error is large, and the control law designed for the ideal case of no quantization may lead to instability. Another reason is deterioration of performance near the equilibrium: as the difference between the current and the desired values of the state becomes small, higher precision is required, and so in the presence of quantization errors asymptotic convergence is typically lost. Due to these phenomena, instead of global asymptotic stability it is more reasonable to expect that solutions starting in a given region remain bounded and approach a smaller region. In [11] the author has developed a quantized feedback control methodology for nonlinear systems based on results of this type, under a suitable robust stabilization assumption imposed on the controller (see also [12] for further discussion and many references to prior work on quantized control). As we will see, this robustness of a stabilizing controller with respect to quantization errors (which is automatic in the linear case) plays a central role in the nonlinear results.

The effect of a sufficiently small time delay on stability of a linear system can be studied by standard perturbation techniques based on Rouché's theorem (see, e.g., [5, 7]). When the delay is large but known, it can in principle be attenuated by propagating the state forward from the measurement time. However, the case we are interested in is when the delay is not necessarily small and its value is not available to the controller. There is of course a rich literature on asymptotic stability of time-delayed systems [6, 5], but these results are not very suitable here because they involve strong assumptions on the system or on the delay, and asymptotic stability will in any case be destroyed by quantization. On the other hand, there are very few results on boundedness and ultimate boundedness for nonlinear control systems with (possibly large) delays. Notable exceptions are [19, 21, 22], and the present work is heavily based on [19]. In that paper, Teel uses a small-gain approach to analyze the behavior of nonlinear feedback loops in the presence of time delays and external disturbances. Our main observation is that his findings are compatible with our recent results on quantization; by identifying external disturbances with quantization errors, we are able to naturally combine the two lines of work.

For concreteness, we consider the setting where there is a processing delay in collecting state measurements from the sensors and/or computing the control signal, followed by quantization of the control signal before it is transmitted to the actuators. Other scenarios can be handled similarly, as explained at the end of the paper. Assuming that an upper bound on the initial state is available, our main result (Theorem 1 in Section 3) establishes an upper bound and a smaller ultimate bound on resulting closed-loop trajectories. Section 4 contains some interpretations and modifications of this result, and explains how global asymptotic stability can be recovered by using the dynamic quantization method of [2, 11].

In what follows, we will repeat some of the developments from [19] in order to make the paper self-contained and also because we will need some estimates not explicitly written down in [19]. We remark that the results of [19] were also used in [20] for a different purpose, namely, to study the effects of sampled-data control implementation. We take the delay to be fixed for simplicity; however, everything readily generalizes to the case of a time-varying bounded delay (by simply replacing the value of the delay with its upper bound in all subsequent arguments). This latter type of delay is used for modeling sampled-data control systems (see, e.g., [4] and the references therein), thus such an extension could be useful for studying systems with sampled-data quantized feedback.

2 Notation and preliminaries

Consider the system

$$\dot{x} = f(x, u)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a C^1 function. The inputs to the system are subject to quantization. The (input) *quantizer* is a piecewise constant function $q : \mathbb{R}^m \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^m$. Following [11, 12], we assume that there exist real numbers $M > \Delta > 0$ such that the following condition holds:

$$|u| \le M \quad \Rightarrow \quad |q(u) - u| \le \Delta. \tag{1}$$

This condition gives a bound on the quantization error when the quantizer does not saturate. We will refer to M and Δ as the *range* and the *error bound* of the quantizer, respectively. We consider the one-parameter family of quantizers

$$q_\mu(u) := \mu q\Big(\frac{u}{\mu}\Big), \qquad \mu > 0. \tag{2}$$

Here μ can be viewed as a "zoom" variable. This parameter is in general adjustable, but in this section we take μ to be fixed. The range of the quantizer q_μ is $M\mu$ and the error bound is $\Delta\mu$.

Assumed given is some nominal state feedback law $u = k(x)$, which is C^1 and satisfies $k(0) = 0$. We take this feedback law to be stabilizing robustly with respect to actuator errors, in the following sense (see Section 4 for a discussion of this assumption and a way to relax it).

Assumption 1 There exists a C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions[2] $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ we have

[2] Recall that a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \ge 0$. We will write $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, etc.

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{3}$$

and

$$|x| \ge \rho(|v|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x)+v) \le -\alpha_3(|x|). \tag{4}$$

We let the state measurements be subject to a fixed delay $\tau > 0$. This delay followed by quantization give the actual control law in the form

$$u(t) = q_\mu(k(x(t-\tau))) \tag{5}$$

and yields the closed-loop system

$$\dot{x}(t) = f(x(t), q_\mu(k(x(t-\tau)))). \tag{6}$$

We can equivalently rewrite this as

$$\dot{x}(t) = f\big(x(t), k(x(t)) + \theta(t) + e(t)\big) \tag{7}$$

where

$$\theta(t) := k(x(t-\tau)) - k(x(t)) \tag{8}$$

and

$$e(t) := q_\mu(k(x(t-\tau))) - k(x(t-\tau)). \tag{9}$$

To simplify notation, we will write $a \vee b$ for $\max\{a, b\}$. Applying (4) with $v = \theta + e$ and defining $\gamma \in \mathcal{K}_\infty$ by $\gamma(r) := \rho(2r)$, we have

$$|x| \ge \gamma(|\theta| \vee |e|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x)+\theta+e) \le -\alpha_3(|x|). \tag{10}$$

The quantity defined in (8) can be expressed as

$$\theta(t) = -\int_{t-\tau}^{t} k'(x(s)) f(x(s), k(x(s-\tau)) + e(s)) ds. \tag{11}$$

Substituting this into the system (7), we obtain a system with delay $t_d := 2\tau$. For initial data $x : [-t_d, 0] \to \mathbb{R}^n$ which is assumed to be given, this system has a unique maximal solution[3] $x(\cdot)$. We adopt the following notation from [19]: $|x_d(t)| := \max_{s\in[t-t_d,t]} |x(s)|$ for $t \in [0,\infty)$, $\|x_d\|_J := \sup_{t\in J} |x_d(t)|$ for a subinterval J of $[0,\infty)$, and $|e_d(t)|$ and $\|e_d\|_J$ are defined similarly. In view of (11), for some $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ we have

$$|\theta(t)| \le \tau\gamma_1(|x_d(t)|) \vee \tau\gamma_2(|e_d(t)|)$$

(we are using the fact that $f(0, k(0)) = 0$, which follows from Assumption 1). Defining $\gamma_\tau(r) := \gamma(\tau\gamma_1(r))$, $\hat{\gamma}(r) := \gamma(\tau\gamma_2(r) \vee r)$, and using (10), we have

[3] It is not hard to see that discontinuities of the control (5) do not affect the existence of Carathéodory solutions; cf. the last paragraph of Section 2.6 in [6].

$$|x(t)| \geq \gamma_\tau(|x_d(t)|) \vee \hat{\gamma}(|e_d(t)|) \quad \Rightarrow \quad \dot{V}(t) \leq -\alpha_3(|x|).$$

In light of (3), this implies

$$V(t) \geq \alpha_2 \circ \gamma_\tau(|x_d(t)|) \vee \alpha_2 \circ \hat{\gamma}(|e_d(t)|) \quad \Rightarrow \quad \dot{V}(t) \leq -\alpha_3(|x|)$$

hence by the standard comparison principle (cf. [9, Chapter 4])

$$V(t) \leq \beta(V(t_0), t - t_0) \vee \alpha_2 \circ \gamma_\tau\Big(\|x_d\|_{[t_0,\infty)}\Big) \vee \alpha_2 \circ \hat{\gamma}\Big(\|e_d\|_{[t_0,\infty)}\Big)$$

where $\beta \in \mathcal{KL}$ and $\beta(r, 0) = r$. Using (3) again, we have

$$|x(t)| \leq \tilde{\beta}(|x(t_0)|, t - t_0) \vee \tilde{\gamma}_x\Big(\|x_d\|_{[t_0,\infty)}\Big) \vee \tilde{\gamma}_e\Big(\|e_d\|_{[t_0,\infty)}\Big) \qquad (12)$$

where $\tilde{\beta}(r, t) := \alpha_1^{-1}(\beta(\alpha_2(r), t))$, $\tilde{\gamma}_x(r) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma_\tau(r)$, and $\tilde{\gamma}_e(r) := \alpha_1^{-1} \circ \alpha_2 \circ \hat{\gamma}(r)$. The full expression for $\tilde{\gamma}_x$ (which depends on the delay τ) is

$$\tilde{\gamma}_x(r) = \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\tau\gamma_1(r)). \qquad (13)$$

Let us invoke the properties of the quantizer to upper-bound the quantization error e defined in (9). Take κ to be some class $\mathcal{K}_\infty$ function with the property that

$$\kappa(r) \geq \max_{|x| \leq r} |k(x)| \qquad \forall\, r \geq 0$$

so that we have

$$|k(x)| \leq \kappa(|x|) \qquad \forall\, x.$$

Then (1) and (2) give

$$|x_d(t)| \leq \kappa^{-1}(M\mu) \quad \Rightarrow \quad |e(t)| \leq \Delta\mu. \qquad (14)$$

3 Main result

Theorem 1 *Let Assumption 1 hold. Assume that the initial data satisfies*

$$|x_d(t_0)| \leq E_0 \qquad (15)$$

for some known $E_0 > 0$. Choose a (small) $\varepsilon > 0$ and assume that for some $\Lambda > 0$ we have

$$\alpha_1^{-1} \circ \alpha_2(E_0) \vee \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) < \Lambda < \kappa^{-1}(M\mu). \qquad (16)$$

Assume that the delay τ is small enough so that

$$\tilde{\gamma}_x(r) < r \qquad \forall\, r \in (\varepsilon, \Lambda]. \qquad (17)$$

Then the solution of the closed-loop system (6) *satisfies the bound*

$$\|x_d\|_{[t_0,\infty)} \le \alpha_1^{-1} \circ \alpha_2(E_0) \vee \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) \tag{18}$$

and the ultimate bound

$$\|x_d\|_{[t_0+T,\infty)} \le \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) \tag{19}$$

for some $T > 0$*.*

Proof. As in [19], the main idea behind the proof is a small-gain argument combining (12) with the bound

$$|x_d(t)| \le |x_d(t_0)| \cdot \frac{(1 - \mathrm{sgn}(t - t_d - t_0))}{2} \vee \|x\|_{[t_0,\infty)}$$

which gives

$$\|x_d\|_{[t_0,\infty)} \le |x_d(t_0)| \vee \tilde{\beta}(|x(t_0)|, 0) \vee \tilde{\gamma}_x\Big(\|x_d\|_{[t_0,\infty)}\Big) \vee \tilde{\gamma}_e\Big(\|e_d\|_{[t_0,\infty)}\Big).$$

We know that $\tilde{\beta}(r, 0) = \alpha_1^{-1} \circ \alpha_2(r) \ge r$. The condition (15) and the first inequality in (16) imply that there exists some maximal interval $[t_0, \bar{t})$ on which $|x_d(t)| < \Lambda$. On this interval, using (14), (16), (17), and causality, we have the (slightly conservative) bound

$$\begin{aligned}\|x_d\|_{[t_0,\bar{t})} &\le \tilde{\beta}(|x_d(t_0)|, 0) \vee \tilde{\gamma}_x\Big(\|x_d\|_{[t_0,\bar{t})}\Big) \vee \tilde{\gamma}_e\Big(\|e_d\|_{[t_0,\bar{t})}\Big) \\ &\le \alpha_1^{-1} \circ \alpha_2(E_0) \vee \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) < \Lambda\end{aligned}$$

Thus actually $\bar{t} = \infty$ and (18) is established. Next, denote the right-hand side of (18) by E and pick a $\rho > 0$ such that

$$\tilde{\beta}(E, \rho) \le \varepsilon. \tag{20}$$

Using (12), we have

$$\begin{aligned}\|x_d\|_{[t_0+t_d+\rho,\infty)} \le \|x\|_{[t_0+\rho,\infty)} &\le \tilde{\beta}(E_0, \rho) \vee \tilde{\gamma}_x\Big(\|x_d\|_{[t_0,\infty)}\Big) \vee \tilde{\gamma}_e(\Delta\mu) \\ &\le \varepsilon \vee \tilde{\gamma}_x(E) \vee \tilde{\gamma}_e(\Delta\mu)\end{aligned}$$

From this, using (12) again but with $t_0 + t_d + \rho$ in place of t_0, we obtain

$$\begin{aligned}&\|x_d\|_{[t_0+2(t_d+\rho),\infty)} \le \|x\|_{[t_0+t_d+2\rho,\infty)} \\ &\le \tilde{\beta}(|x(t_0 + t_d + \rho)|, \rho) \vee \tilde{\gamma}_x\Big(\|x_d\|_{[t_0+t_d+\rho,\infty)}\Big) \vee \tilde{\gamma}_e(\Delta\mu) \\ &\le \tilde{\beta}(\varepsilon \vee \tilde{\gamma}_e(\Delta\mu), \rho) \vee \tilde{\beta}(\tilde{\gamma}_x(E), \rho) \vee \tilde{\gamma}_x(\varepsilon \vee \tilde{\gamma}_e(\Delta\mu)) \vee \tilde{\gamma}_x^2(E) \vee \tilde{\gamma}_e(\Delta\mu) \\ &\le \varepsilon \vee \tilde{\gamma}_x^2(E) \vee \tilde{\gamma}_e(\Delta\mu)\end{aligned}$$

in view of (16), (17), (20), and continuity of $\tilde{\gamma}_x$ at ε. There exists a positive integer n such that $\tilde{\gamma}_x^n(E) \le \varepsilon \vee \tilde{\gamma}_e(\Delta\mu)$. Repeating the above calculation, for $T := n(t_d + \rho)$ we have the bound (19) and the proof is complete. □

4 Discussion

Hypotheses of Theorem 1

The only constraint placed on the quantizer in Theorem 1 is the hypothesis (16). It says that the range M of q should be large enough compared to the error bound Δ, so that the last term in (16) is larger than the first one (then a suitable Λ automatically exists). A very similar condition is used in [11]. One direction for future work is to extend the result to “coarse” quantizers not satisfying such hypotheses (cf. [13], [12, Section 5.3.6]).

The small-gain condition (17) is justified by the formula (13), which ensures that for every pair of numbers $\Lambda > \varepsilon > 0$ there exists a $\tau^* > 0$ such that for all $\tau \in (0, \tau^*)$ we have

$$\tilde{\gamma}_x(r) < r \qquad \forall\, r \in (\varepsilon, \Lambda].$$

The value of τ^* depends on the relative growth rate of the functions appearing on the right-hand size of (13) for small and large arguments. In particular, if both $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$ and γ_1 have finite derivatives at 0, then for small enough τ the inequality (17) holds with $\varepsilon = 0$. In this case, the effect of ε (called the *offset* in [19]) disappears, and the ultimate bound (19) depends on the quantizer's error bound only. Also note that in the case of linear systems, $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$ and γ_1 can be taken to be linear, and for small enough τ we have

$$\tilde{\gamma}_x(r) < r \qquad \forall\, r > 0. \tag{21}$$

Assumption 1 says that the feedback law k provides *input-to-state stability* (ISS) with respect to the actuator error v, in the absence of delays (see [15, 17]). This requirement is restrictive in general. However, it is shown in [15] that for globally asymptotically stabilizable systems affine in controls, such a feedback always exists. (For linear systems and linear stabilizing feedback laws, such robustness with respect to actuator errors is of course automatic.) One way to proceed without Assumption 1 is as follows. Suppose that k just globally asymptotically stabilizes the system in the absence of actuator errors, so that instead of (4) we only have

$$\frac{\partial V}{\partial x} f(x, k(x)) \le -\alpha_3(|x|).$$

By virtue of [3, Lemma 1] or [16, Lemma 3.2], there exist a class $\mathcal{K}_\infty$ function γ and a C^1 function $G : \mathbb{R}^n \to GL(m, \mathbb{R})$, i.e., $G(x)$ is an invertible $m \times m$ matrix for each x, such that for all $x \in \mathbb{R}^n$ and $\bar{\theta}, \bar{e} \in \mathbb{R}^m$ we have

$$|x| \ge \gamma(|\bar{\theta}| \,\vee\, |\bar{e}|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x) + G(x)\bar{\theta} + G(x)\bar{e}) \le -\frac{\alpha_3(|x|)}{2}. \tag{22}$$

To use this property, let us rewrite the system (7) as

$$\dot{x}(t) = f\big(x(t), k(x(t)) + G(x(t))\bar{\theta}(t) + G(x(t))\bar{e}(t)\big)$$

where

$$\bar{\theta}(t) := G^{-1}(x(t))\theta(t)$$

and

$$\bar{e}(t) := G^{-1}(x(t))e(t).$$

In view of (11), for some $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3 \in \mathcal{K}_\infty$ and $a > 0$ we have

$$|\bar{\theta}(t)| \le \tau\bar{\gamma}_1(|x_d(t)|) \vee \tau\bar{\gamma}_2(|e_d(t)|)$$

and

$$|\bar{e}(t)| \le (\bar{\gamma}_3(|x(t)|) \vee a)|e(t)|.$$

Using (22), we have

$$|x(t)| \ge \gamma\Big(\tau\bar{\gamma}_1(|x_d(t)|) \vee \tau\bar{\gamma}_2(|e_d(t)|) \vee \bar{\gamma}_3(|x(t)|)|e(t)| \vee a|e(t)|\Big) \quad \Rightarrow \quad \dot{V}(t) \le -\frac{\alpha_3(|x|)}{2}.$$

As before, we obtain from this that

$$|x(t)| \le \bar{\beta}(|x(t_0)|, t - t_0) \vee \alpha_1^{-1} \circ \alpha_2 \circ \gamma\Big(\tau\bar{\gamma}_1(\|x_d\|_{[t_0,\infty)}) \vee \tau\bar{\gamma}_2(\|e_d\|_{[t_0,\infty)}) \vee \bar{\gamma}_3(\|x\|_{[t_0,\infty)})\|e\|_{[t_0,\infty)} \vee a\|e\|_{[t_0,\infty)}\Big) \qquad (23)$$

where $\bar{\beta} \in \mathcal{KL}$ and $\bar{\beta}(r, 0) = \alpha_1^{-1} \circ \alpha_2(r)$. Define

$$\bar{\gamma}_x(r) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma\big(\tau\bar{\gamma}_1(r) \vee \bar{\gamma}_3(r)\Delta\mu\big)$$

and

$$\bar{\gamma}_e(r) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\tau\bar{\gamma}_2(r) \vee ar).$$

Then we have a counterpart of Theorem 1, without Assumption 1 and with $\bar{\gamma}_x, \bar{\gamma}_e$ replacing $\tilde{\gamma}_x, \tilde{\gamma}_e$ everywhere in the statement. The proof is exactly the same modulo this change of notation, using (23) instead of (12). The price to pay is that the function $\bar{\gamma}_x$ depends on $\Delta\mu$ as well as on τ, so the modified small-gain condition requires not only the delay but also the error bound of the quantizer to be sufficiently small. This can be seen particularly clearly in the special case of no delay ($\tau = 0$); in this case we arrive at a result complementary to [11, Lemma 2] in that it applies to every globally asymptotically stabilizing feedback law but only when the quantizer has a sufficiently small error bound.[4]

[4] The argument we just gave actually establishes that global asymptotic stability under the zero input guarantees ISS with a given offset on a given bounded region ("semiglobal practical ISS") for sufficiently small inputs. (A related result proved in [18] is that global asymptotic stability under the zero input implies ISS for sufficiently small states and inputs.) This observation confirms the essential role that ISS plays in our developments.

Extensions

In Theorem 1, the effects of quantization manifest themselves in the need for the known initial bound E_0 in (15) and in the strictly positive term $\tilde{\gamma}_e(\Delta\mu)$ in the ultimate bound (19). If the ultimate bound is strictly smaller than the initial bound, and if the quantization "zoom" parameter μ in (2) can be adjusted on-line, then both of these shortcomings can be removed using the method proposed in [2, 11]. First, the state measurement at time t_0 gives us the value of $x(t_0-\tau)$. We assume that no control was applied for $t \in [t_0-\tau, t_0)$. If an upper bound τ^* on the delay is known and the system is forward complete under zero control, then we have an upper bound of the form (15). This is because for a forward complete system, the reachable set from a given initial condition in bounded time is bounded (see, e.g., [1]). An over-approximation of the reachable set can be used to actually compute such an upper bound (see [10, 8] for some results on approximating reachable sets). Now we can select a value of μ large enough to satisfy (16) and start applying the control law (5) for $t \geq t_0$. Let us assume for simplicity that (21) is satisfied, so that we can take $\varepsilon = 0$. Applying Theorem 1, we have from (19) that

$$|x_d(t_0+T)| \leq \tilde{\gamma}_e(\Delta\mu).$$

Next, at time t_0+T we want to select a smaller value of μ for which (16) holds with E_0 replaced by this new bound (which depends on the previous value of μ). For this to be possible, we need to assume that

$$\alpha_1^{-1} \circ \alpha_2 \circ \tilde{\gamma}_e(\Delta\mu) < \kappa^{-1}(M\mu) \qquad \forall\, \mu > 0.$$

Repeating this "zooming-in" procedure, we recover global asymptotic stability.

We mention an interesting small-gain interpretation of the above strategy, which was given in [14]. The closed-loop system can be viewed as a hybrid system with continuous state x and discrete state μ. After the "zooming-out" stage is completed, it can be shown that the x-subsystem is ISS with respect to μ with gain smaller than $\hat{\gamma}(\Delta\cdot)$, while the μ-subsystem is ISS with respect to x with gain $\tilde{\gamma}_e^{-1}/\Delta = \hat{\gamma}^{-1} \circ \alpha_2^{-1} \circ \alpha_1/\Delta$. The composite gain is less than identity, and asymptotic stability follows from the nonlinear small-gain theorem. One important advantage of the small-gain viewpoint (which was also used to establish Theorem 1) is that it enables an immediate incorporation of external disturbances, under suitable assumptions. We refer the reader to [14] and [19] for further details.

We combined quantization and delays as in (5) just for concreteness; other scenarios can be handled similarly. Assume, for example, that the control takes the form

$$u(t) = k(q_\mu(x(t-\tau)))$$

i.e., both the quantization and the delay[5] affect the state before the control is computed. The resulting closed-loop system can be written as

$$\dot{x}(t) = f\big(x(t), k\big(x(t) + \theta(t) + e(t)\big)\big)$$

where

$$\theta(t) := x(t - \tau) - x(t)$$

and

$$e(t) := q_\mu(x(t - \tau)) - x(t - \tau).$$

Assumption 1 needs to be modified by replacing (4) with

$$|x| \geq \rho(|v|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x + v)) \leq -\alpha_3(|x|).$$

This is the requirement of ISS with respect to *measurement* errors, which is restrictive even for systems affine in controls (see the discussion and references in [11]). For $\gamma(r) := \rho(2r)$ we have

$$|x| \geq \gamma(|\theta| \vee |e|) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, k(x + \theta + e)) \leq -\alpha_3(|x|)$$

and we can proceed as before. The ISS assumption can again be relaxed at the expense of introducing a constraint on the error bound of the quantizer (see also footnote 4 above). A bound of the form (15) for some time greater than the initial time can be obtained by "zooming out" similarly to how it is done in [11], provided that an upper bound τ^* on the delay is known and the system is forward complete under zero control. Global asymptotic stability can then be achieved by "zooming in" as before. It is also not difficult to extend the results to the case when quantization affects both the state and the input (cf. [11, Remark 1]).

5 Conclusions

The goal of this paper was to show how the effects of quantization and time delays in nonlinear control systems can be treated in a unified manner by using Lyapunov functions and small-gain arguments. We proved that under the action of an input-to-state stabilizing feedback law in the presence of both quantization and small delays, solutions of the closed-loop system starting in a given region remain bounded and eventually enter a smaller region. Based on this result, global asymptotic stabilization can be achieved by employing a dynamic quantization scheme. These findings demonstrate that the quantized control algorithms proposed in our earlier work are inherently robust to

[5] Their order is not important in this case as long as the quantizer is fixed and does not saturate.

time delays, which increases their potential usefulness for applications such as networked embedded systems.

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