

## Quantization, Time Delays, and Nonlinear Stabilization

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**Abstract**—The purpose of this note is to demonstrate that a unified study of quantization and delay effects in nonlinear control systems is possible by merging the quantized feedback control methodology recently developed by the author and the small-gain approach to the analysis of functional differential equations with disturbances proposed earlier by Teel. We prove that under the action of a robustly stabilizing feedback controller in the presence of quantization and time delays satisfying suitable conditions, solutions of the closed-loop system starting in a given region remain bounded and eventually enter a smaller region. We present several versions of this result and show how it enables global asymptotic stabilization via a dynamic quantization strategy.

**Index Terms**—Input-to-state stability, nonlinear system, quantized control, small-gain condition, time delays.

### I. INTRODUCTION

To be applicable in realistic situations, control theory must take into account communication constraints between the plant and the controller, such as those arising in networked control systems (see, e.g., [2] and the references therein). Two most common phenomena relevant in this context are quantization and time delays. It is also important to be able to handle nonlinear dynamics. To the best of the author's knowledge, the present note is a first step towards addressing these three aspects in a unified and systematic way.

It is well known that a feedback law which globally asymptotically stabilizes a given system in the absence of quantization will in general fail to provide global asymptotic stability of the closed-loop system that arises in the presence of a quantizer with a finite number of values. One reason for this is saturation: If the quantized signal is outside the range of the quantizer, then the quantization error is large, and the control law designed for the ideal case of no quantization may lead to instability. Another reason is deterioration of performance near the equilibrium: As the difference between the current and the desired values of the state becomes small, higher precision is required, and so in the presence of quantization errors asymptotic convergence is typically lost. Due to these phenomena, instead of global asymptotic stability it is more reasonable to expect that solutions starting in a given region remain bounded and approach a smaller region. In [15], the author has developed a quantized feedback control methodology for nonlinear systems based on results of this type, under a suitable robust stabilization assumption imposed on the controller (see also [16] for further discussion and many references to prior work on quantized control). As we will see, this robustness of a stabilizing controller with respect to quantization errors (which is automatic in the linear case) plays a central role in the nonlinear results.

The effect of a sufficiently small time delay on stability of a linear system can be studied by standard perturbation techniques based on Rouché's theorem (see, e.g., [6] and [8]). When the delay is large but

known, it can in principle be attenuated by propagating the state forward from the measurement time. However, the case we are interested in is when the delay is not necessarily small and its value is not available to the controller. There is, of course, a rich literature on asymptotic stability of time-delay systems [7], [6], but these results are not very suitable here because they involve strong assumptions on the system or on the delay, and asymptotic stability will in any case be destroyed by quantization. On the other hand, there are very few results on boundedness and ultimate boundedness for nonlinear control systems with (possibly large) delays. Notable exceptions are [27], [29], [30], and this work is heavily based on [27]. In that paper, Teel uses a small-gain approach to analyze the behavior of nonlinear feedback loops in the presence of time delays and external disturbances (see also ([6, Ch. 8] and [12] for other applications of the small-gain approach to stability analysis of time-delay systems). Our main observation is that the findings of [27] are compatible with our recent results on quantization; by identifying external disturbances with quantization errors, we are able to naturally combine the two lines of work.

For concreteness, we consider the setting where there is a processing delay in collecting state measurements from the sensors and/or computing the control signal, followed by quantization of the control signal before it is transmitted to the actuators. Other scenarios can be handled similarly, as explained at the end of the note. In what follows, we will repeat some of the developments from [27] in order to make this note self-contained and also because we will need some estimates not explicitly written down in [27]. Assuming that an upper bound on the initial state is available, our main result (Theorem 1 in Section III) establishes an upper bound and a smaller ultimate bound on resulting closed-loop trajectories. Section IV contains some interpretations and modifications of this result, and explains how global asymptotic stability can be recovered by using the dynamic quantization method of [3] and [15].

We take the delay to be fixed for simplicity; however, everything can be readily generalized to the case of a time-varying bounded delay, by simply replacing the value of the delay with its upper bound in all subsequent arguments. Time-varying bounded delays are useful, in particular, for modeling sampled-data control systems; see, e.g., [28], [5], and the references therein. The developments of [28] are especially relevant here because they also directly rely on the approach of [27]. Thus, the results of this note can be used to study systems with sampled-data quantized feedback, such as those considered in [11], [21], and [20].

### II. NOTATION AND PRELIMINARIES

Consider the system

$$\dot{x} = f(x, u)$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}^m$  is the control, and  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a  $C^1$  function. The inputs to the system are subject to quantization. The (input) *quantizer* is a piecewise constant function  $q : \mathbb{R}^m \rightarrow \mathcal{Q}$ , where  $\mathcal{Q}$  is a finite subset of  $\mathbb{R}^m$ . Following [15] and [16], we assume that there exist real numbers  $M > \Delta > 0$  such that the following condition holds:

$$|u| \leq M \quad \Rightarrow \quad |q(u) - u| \leq \Delta. \quad (1)$$

This condition gives a bound on the quantization error when the quantizer does not saturate. We will refer to  $M$  and  $\Delta$  as the *range* and the

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error bound of the quantizer, respectively. We consider the one-parameter family of quantizers

$$q_\mu(u) := \mu q\left(\frac{u}{\mu}\right), \quad \mu > 0. \quad (2)$$

Here  $\mu$  can be viewed as a ‘‘zoom’’ variable. This parameter  $\mu$  is in general adjustable, but we take it to be fixed everywhere except Section IV-B. The range of the quantizer  $q_\mu$  is  $M\mu$  and the error bound is  $\Delta\mu$ .

Assumed given is some nominal state feedback law  $u = k(x)$ , which is  $C^1$  and satisfies  $k(0) = 0$ . We take this feedback law to be stabilizing robustly with respect to actuator errors, in the following sense (see Section IV for a discussion of this assumption and a way to relax it).

*Assumption 1:* There exists a  $C^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for some class  $\mathcal{K}_\infty$  functions<sup>1</sup>  $\alpha_1, \alpha_2, \alpha_3, \rho$  and for all  $x \in \mathbb{R}^n$  and  $v \in \mathbb{R}^m$  we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad (3)$$

and

$$|x| \geq \rho(|v|) \Rightarrow \frac{\partial V}{\partial x} f(x, k(x) + v) \leq -\alpha_3(|x|). \quad (4)$$

We let the state measurements be subject to a fixed delay  $\tau > 0$ . This delay followed by quantization gives the actual control law in the form

$$u(t) = q_\mu(k(x(t - \tau))) \quad (5)$$

and yields the closed-loop system

$$\dot{x}(t) = f(x(t), q_\mu(k(x(t - \tau)))). \quad (6)$$

We can equivalently rewrite this as

$$\dot{x}(t) = f(x(t), k(x(t)) + \theta(t) + e(t)) \quad (7)$$

where

$$\theta(t) := k(x(t - \tau)) - k(x(t)) \quad (8)$$

and

$$e(t) := q_\mu(k(x(t - \tau))) - k(x(t - \tau)). \quad (9)$$

To simplify notation, we will write  $a \vee b$  for  $\max\{a, b\}$ . Applying (4) with  $v = \theta + e$  and defining  $\gamma \in \mathcal{K}_\infty$  by  $\gamma(r) := \rho(2r)$ , we have

$$|x| \geq \gamma(|\theta| \vee |e|) \Rightarrow \frac{\partial V}{\partial x} f(x, k(x) + \theta + e) \leq -\alpha_3(|x|). \quad (10)$$

The quantity defined in (8) can be expressed as

$$\theta(t) = - \int_{t-\tau}^t \frac{\partial k}{\partial x}(x(s)) f(x(s), k(x(s - \tau)) + e(s)) ds. \quad (11)$$

<sup>1</sup>Recall that a function  $\alpha : [0, \infty) \rightarrow [0, \infty)$  is said to be of class  $\mathcal{K}$  if it is continuous, strictly increasing, and  $\alpha(0) = 0$ . If  $\alpha$  is also unbounded, then it is said to be of class  $\mathcal{K}_\infty$ . A function  $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  is said to be of class  $\mathcal{KL}$  if  $\beta(\cdot, t)$  is of class  $\mathcal{K}$  for each fixed  $t \geq 0$  and  $\beta(r, t)$  decreases to 0 as  $t \rightarrow \infty$  for each fixed  $r \geq 0$ . We will write  $\alpha \in \mathcal{K}_\infty, \beta \in \mathcal{KL}$ , etc.

Substituting this into the system (7) and noting the range of the time arguments in (11), we obtain a system with delay  $t_d := 2\tau$ . For initial data  $x : [-t_d, 0] \rightarrow \mathbb{R}^n$  which is assumed to be given, this system has a unique maximal solution<sup>2</sup>  $x(\cdot)$ . We adopt the following notation from [27]:  $|x_d(t)| := \max_{s \in [t-t_d, t]} |x(s)|$  for  $t \in [0, \infty)$ ,  $\|x_d\|_J := \sup_{t \in J} |x_d(t)|$  for a subinterval  $J$  of  $[0, \infty)$ , and  $|e_d(t)|$  and  $\|e_d\|_J$  are defined similarly. In view of (11) and the integral mean value theorem, for some  $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$  we have the bound

$$|\theta(t)| \leq \tau \gamma_1(|x_d(t)|) \vee \tau \gamma_2(|e_d(t)|)$$

(we are using the fact that  $f(0, k(0)) = 0$ , which follows from Assumption 1). Defining  $\gamma_\tau(r) := \gamma(\tau \gamma_1(r))$ ,  $\hat{\gamma}(r) := \gamma(\tau \gamma_2(r) \vee r)$ , and using (10), we have

$$|x(t)| \geq \gamma_\tau(|x_d(t)|) \vee \hat{\gamma}(|e_d(t)|) \Rightarrow \dot{V}(t) \leq -\alpha_3(|x|).$$

In light of (3), this implies

$$V(t) \geq \alpha_2 \circ \gamma_\tau(|x_d(t)|) \vee \alpha_2 \circ \hat{\gamma}(|e_d(t)|) \Rightarrow \dot{V}(t) \leq -\alpha_3(|x|)$$

hence, by the standard comparison principle (cf. [13, Ch. 4])

$$V(t) \leq \beta(V(t_0), t - t_0) \vee \alpha_2 \circ \gamma_\tau(\|x_d\|_{[t_0, \infty)}) \vee \alpha_2 \circ \hat{\gamma}(\|e_d\|_{[t_0, \infty)})$$

where  $\beta \in \mathcal{KL}$  and  $\beta(r, 0) = r$ . Using (3) again, we have

$$|x(t)| \leq \tilde{\beta}(|x(t_0)|, t - t_0) \vee \tilde{\gamma}_x(\|x_d\|_{[t_0, \infty)}) \vee \tilde{\gamma}_e(\|e_d\|_{[t_0, \infty)}) \quad (12)$$

where  $\tilde{\beta}(r, t) := \alpha_1^{-1}(\beta(\alpha_2(r), t))$ ,  $\tilde{\gamma}_x(r) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma_\tau(r)$ , and  $\tilde{\gamma}_e(r) := \alpha_1^{-1} \circ \alpha_2 \circ \hat{\gamma}(r)$ . The full expression for  $\tilde{\gamma}_x$  (which depends on the delay  $\tau$ ) is

$$\tilde{\gamma}_x(r) = \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\tau \gamma_1(r)). \quad (13)$$

Let us invoke the properties of the quantizer to upper-bound the quantization error  $e$  defined in (9). Take  $\kappa$  to be some class  $\mathcal{K}_\infty$  function with the property that

$$\kappa(r) \geq \max_{|x| \leq r} |k(x)| \quad \forall r \geq 0$$

so that we have

$$|k(x)| \leq \kappa(|x|) \quad \forall x.$$

Then (1) and (2) give

$$|x_d(t)| \leq \kappa^{-1}(M\mu) \Rightarrow |e(t)| \leq \Delta\mu. \quad (14)$$

<sup>2</sup>It is not hard to see that discontinuities of the control (5) do not affect the existence of Carathéodory solutions; cf. the last paragraph of [7, Sec. 2.6].

## III. MAIN RESULT

*Theorem 1:* Let Assumption 1 hold. Assume that the initial data satisfies

$$|x_d(t_0)| \leq E_0 \quad (15)$$

for some known  $E_0 > 0$ . Choose a (small)  $\varepsilon \geq 0$  and assume that for some  $\Lambda > 0$  we have

$$\alpha_1^{-1} \circ \alpha_2(E_0) \vee \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) < \Lambda < \kappa^{-1}(M\mu) \quad (16)$$

and

$$\tilde{\gamma}_x(r) < r \quad \forall r \in (\varepsilon, \Lambda]. \quad (17)$$

Then the solution of the closed-loop system (6) satisfies the bound

$$\|x_d\|_{[t_0, \infty)} \leq \alpha_1^{-1} \circ \alpha_2(E_0) \vee \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) \quad (18)$$

and the ultimate bound

$$\|x_d\|_{[t_0+T, \infty)} \leq \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) \quad (19)$$

for some  $T > 0$ .

*Proof:* As in [27], the main idea behind the proof is a small-gain argument combining (12) with the bound

$$|x_d(t)| \leq |x_d(t_0)| \cdot \frac{(1 - \text{sgn}(t - t_d - t_0))}{2} \vee \|x\|_{[t_0, \infty)}$$

which gives

$$\|x_d\|_{[t_0, \infty)} \leq |x_d(t_0)| \vee \tilde{\beta}(|x_d(t_0)|, 0) \vee \tilde{\gamma}_x(\|x_d\|_{[t_0, \infty)}) \vee \tilde{\gamma}_e(\|e_d\|_{[t_0, \infty)}).$$

We know that  $\tilde{\beta}(r, 0) = \alpha_1^{-1} \circ \alpha_2(r) \geq r$ . The condition (15) and the first inequality in (16) imply that there exists some maximal interval  $[t_0, \bar{t})$  on which  $|x_d(t)| < \Lambda$ . On this interval, using (14), (16), (17), and causality, we have the (slightly conservative) bound

$$\begin{aligned} & \|x_d\|_{[t_0, \bar{t})} \\ & \leq \tilde{\beta}(|x_d(t_0)|, 0) \vee \tilde{\gamma}_x(\|x_d\|_{[t_0, \bar{t})}) \vee \tilde{\gamma}_e(\|e_d\|_{[t_0, \bar{t})}) \\ & \leq \alpha_1^{-1} \circ \alpha_2(E_0) \vee \varepsilon \vee \tilde{\gamma}_e(\Delta\mu) < \Lambda. \end{aligned}$$

Thus, actually  $\bar{t} = \infty$  and (18) is established.

Next, denote the right-hand side of (18) by  $E$  and pick a  $\rho > 0$  such that

$$\tilde{\beta}(E, \rho) \leq \varepsilon. \quad (20)$$

Using (12), we have

$$\begin{aligned} \|x_d\|_{[t_0+t_d+\rho, \infty)} & \leq \|x\|_{[t_0+\rho, \infty)} \\ & \leq \tilde{\beta}(E_0, \rho) \vee \tilde{\gamma}_x(\|x_d\|_{[t_0, \infty)}) \vee \tilde{\gamma}_e(\Delta\mu) \\ & \leq \varepsilon \vee \tilde{\gamma}_x(E) \vee \tilde{\gamma}_e(\Delta\mu). \end{aligned}$$

From this, using (12) again but with  $t_0 + t_d + \rho$  in place of  $t_0$ , we obtain

$$\begin{aligned} \|x_d\|_{[t_0+2(t_d+\rho), \infty)} & \leq \|x\|_{[t_0+t_d+2\rho, \infty)} \\ & \leq \tilde{\beta}(|x(t_0 + t_d + \rho)|, \rho) \\ & \quad \vee \tilde{\gamma}_x(\|x_d\|_{[t_0+t_d+\rho, \infty)}) \vee \tilde{\gamma}_e(\Delta\mu) \\ & \leq \tilde{\beta}(\varepsilon \vee \tilde{\gamma}_e(\Delta\mu), \rho) \vee \tilde{\beta}(\tilde{\gamma}_x(E), \rho) \\ & \quad \vee \tilde{\gamma}_x(\varepsilon \vee \tilde{\gamma}_e(\Delta\mu)) \vee \tilde{\gamma}_x^2(E) \vee \tilde{\gamma}_e(\Delta\mu) \\ & \leq \varepsilon \vee \tilde{\gamma}_x^2(E) \vee \tilde{\gamma}_e(\Delta\mu) \end{aligned}$$

in view of (16), (17), and (20), and the continuity of  $\tilde{\gamma}_x$  at  $\varepsilon$ . There exists a positive integer  $n$  such that  $\tilde{\gamma}_x^n(E) \leq \varepsilon \vee \tilde{\gamma}_e(\Delta\mu)$ . Repeating the previous calculation, for  $T := n(t_d + \rho)$  we have the bound (19) and the proof is complete.  $\square$

The following simple example illustrates the theorem. We also refer the reader to [9] for an interesting application of our result in the context of teleoperation.

*Example 1:* Consider the scalar system  $\dot{x} = x^2 - x + u$  and the feedback law  $u = -x^2$ . The closed-loop system in the presence of actuator errors is  $\dot{x} = -x + v$ . Assumption 1 can be verified using  $V(x) := x^2/2$ . We have  $\alpha_1(r) = \alpha_2(r) := r^2/2$ , and a simple calculation shows that for an arbitrary  $c > 1$  we can take  $\rho(r) := cr$  and  $\alpha_3(r) := (c-1)r^2/c$ . This implies  $\gamma(r) = \rho(2r) = 2cr$ . Formula (11) gives

$$\theta(t) = \int_{t-\tau}^t 2x(s)(x^2(s) - x(s) - x^2(s-\tau) + e(s)) ds$$

and it is easy to establish the bound

$$\begin{aligned} |\theta(t)| & \leq \tau(4|x_d(t)|^3 + 3|x_d(t)|^2 + |e_d(t)|^2) \\ & \leq \tau(8|x_d(t)|^3 + 6|x_d(t)|^2) \vee 2\tau|e_d(t)|^2. \end{aligned}$$

Thus, we can let  $\gamma_1(r) := 8r^3 + 6r^2$  and  $\gamma_2(r) := 2r^2$ . Next, we have  $\tilde{\gamma}_x(r) = \gamma(\tau\gamma_1(r)) = 16c\tau r^3 + 12c\tau r^2$  and  $\tilde{\gamma}_e(r) = \hat{\gamma}(r) = 4c\tau r^2 \vee 2cr$ . Finally, we can define  $\kappa(r) := r^2$ . We are now ready to apply Theorem 1. For simplicity, we let  $\mu := 1$ . We also take  $\varepsilon := 0$ . Condition (16) becomes  $E_0 \vee 4c\tau\Delta^2 \vee 2c\Delta < \Lambda < \sqrt{M}$ . The small-gain condition (17) is equivalent to  $\tau < (16c\Lambda^2 + 12c\Lambda)^{-1}$ . The bound (18) and the ultimate bound (19) take the form  $\|x_d\|_{[t_0, \infty)} \leq E_0 \vee 4c\tau\Delta^2 \vee 2c\Delta$  and  $\|x_d\|_{[t_0+T, \infty)} \leq 4c\tau\Delta^2 \vee 2c\Delta$ , respectively. One specific scenario covered by this result is as follows. Suppose that  $\Delta = 0.05$ ,  $M = 5$ , and  $E_0 = 2$  are given. Selecting  $\Lambda := 2.1$ , we can verify that for every delay  $\tau \leq 0.01$  the state remains in the interval  $[-2, 2]$  and asymptotically approaches the interval  $[-0.1, 0.1]$ .  $\square$

## IV. DISCUSSION

## A. Hypotheses of Theorem 1

The only constraint placed on the quantizer in Theorem 1 is the hypothesis (16). It says that the range  $M$  of  $q$  should be large enough compared to the error bound  $\Delta$ , so that the last term in (16) is larger than the first one (then a suitable  $\Lambda$  automatically exists). A very similar condition is used in [15]. One direction for future work is to extend the result to ‘‘coarse’’ quantizers not satisfying such hypotheses (cf. [17], [16, Sec. 5.3.6]).

The small-gain condition (17) is justified by the formula (13), which ensures that for every pair of numbers  $\Lambda > \varepsilon > 0$  there exists a  $\tau^* > 0$  such that (17) holds for all  $\tau \in (0, \tau^*)$ . The value of  $\tau^*$  depends on

the relative growth rate of the functions appearing on the right-hand side of (13) for small and large arguments. In particular, if both  $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$  and  $\gamma_1$  have finite derivatives at 0, then for small enough  $\tau$  the inequality (17) holds with  $\varepsilon = 0$ . In this case, the effect of  $\varepsilon$  (called the *offset* in [27]) disappears, and the ultimate bound (19) depends on the quantizer's error bound only; this was the case in Example 1. Also note that in the case of linear systems,  $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$  and  $\gamma_1$  can be taken to be linear, and for small enough  $\tau$  we have

$$\tilde{\gamma}_x(r) < r \quad \forall r > 0. \quad (21)$$

We emphasize that in general, the condition (17) does not necessarily force the delay to be small. As we see from Example 1, if  $\Lambda$  is small then  $\tau$  might be quite large. In view of (16), there is a trade-off between the amount of admissible delay, the quantizer parameters, and the initial state bound.

Assumption 1 says that the feedback law  $k$  provides *input-to-state stability* (ISS) with respect to the actuator error  $v$ , in the absence of delays (see [23] and [25]). This requirement is restrictive in general. However, it is shown in [23] that for globally asymptotically stabilizable systems affine in controls, such a feedback always exists. (For linear systems and linear stabilizing feedback laws, such robustness with respect to actuator errors is of course automatic.) One way to proceed without Assumption 1 is as follows. Suppose that  $k$  just globally asymptotically stabilizes the system in the absence of actuator errors, so that instead of (4) we only have

$$\frac{\partial V}{\partial x} f(x, k(x)) \leq -\alpha_3(|x|).$$

By virtue of [4, Lemma 1] or [24, Lemma 3.2], there exist a class  $\mathcal{K}_\infty$  function  $\gamma$  and a  $C^1$  function  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$  with  $G(x)$  invertible for each  $x$ , such that for all  $x \in \mathbb{R}^n$  and  $\bar{\theta}, \bar{e} \in \mathbb{R}^m$  we have

$$|x| \geq \gamma(|\bar{\theta}| \vee |\bar{e}|) \Rightarrow \frac{\partial V}{\partial x} f(x, k(x)) + G(x)\bar{\theta} + G(x)\bar{e} \leq -\frac{\alpha_3(|x|)}{2}. \quad (22)$$

To use this property, let us rewrite the system (7) as

$$\dot{x}(t) = f(x(t), k(x(t))) + G(x(t))\bar{\theta}(t) + G(x(t))\bar{e}(t)$$

where

$$\bar{\theta}(t) := G^{-1}(x(t))\theta(t)$$

and

$$\bar{e}(t) := G^{-1}(x(t))e(t).$$

In view of (11), for some  $\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3 \in \mathcal{K}_\infty$ , and  $a > 0$  we have

$$|\bar{\theta}(t)| \leq \tau \bar{\gamma}_1(|x_d(t)|) \vee \tau \bar{\gamma}_2(|e_d(t)|)$$

and

$$|\bar{e}(t)| \leq (\bar{\gamma}_3(|x(t)|) \vee a)|e(t)|.$$

Using (22), we have

$$\begin{aligned} |x(t)| &\geq \gamma(\tau \bar{\gamma}_1(|x_d(t)|) \vee \tau \bar{\gamma}_2(|e_d(t)|) \\ &\quad \vee \bar{\gamma}_3(|x(t)|)|e(t)| \vee a|e(t)|) \\ &\Rightarrow \dot{V}(t) \leq -\frac{\alpha_3(|x|)}{2}. \end{aligned}$$

As before, we obtain from this that

$$\begin{aligned} |x(t)| &\leq \bar{\beta}(|x(t_0)|, t - t_0) \\ &\quad \vee \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\tau \bar{\gamma}_1(\|x_d\|_{[t_0, \infty)})) \\ &\quad \vee \tau \bar{\gamma}_2(\|e_d\|_{[t_0, \infty)}) \\ &\quad \vee \bar{\gamma}_3(\|x\|_{[t_0, \infty)}) \|e\|_{[t_0, \infty)} \\ &\quad \vee a \|e\|_{[t_0, \infty)} \end{aligned} \quad (23)$$

where  $\bar{\beta} \in \mathcal{KL}$  and  $\bar{\beta}(r, 0) = \alpha_1^{-1} \circ \alpha_2(r)$ . Define

$$\tilde{\gamma}_x(r) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\tau \bar{\gamma}_1(r) \vee \bar{\gamma}_3(r) \Delta \mu)$$

and

$$\tilde{\gamma}_e(r) := \alpha_1^{-1} \circ \alpha_2 \circ \gamma(\tau \bar{\gamma}_2(r) \vee ar).$$

Then, we have a counterpart of Theorem 1, without Assumption 1 and with  $\tilde{\gamma}_x, \tilde{\gamma}_e$  replacing  $\tilde{\gamma}_x, \tilde{\gamma}_e$  everywhere in the statement. The proof is exactly the same modulo this change of notation, using (23) instead of (12). The price to pay is that the function  $\tilde{\gamma}_x$  depends on  $\Delta \mu$  as well as on  $\tau$ , so the modified small-gain condition requires not only the delay but also the error bound of the quantizer to be sufficiently small. This can be seen particularly clearly in the special case of no delay ( $\tau = 0$ ); in this case we arrive at a result complementary to [15, Lemma 2] (see also [19]) in that it applies to every globally asymptotically stabilizing feedback law but only when the quantizer has a sufficiently small error bound.<sup>3</sup>

### B. Extensions

In Theorem 1, the effects of quantization manifest themselves in the need for the known initial bound  $E_0$  in (15) and in the strictly positive term  $\tilde{\gamma}_e(\Delta \mu)$  in the ultimate bound (19). If the ultimate bound is strictly smaller than the initial bound, and if the quantization ‘‘zoom’’ parameter  $\mu$  in (2) can be adjusted online, then both of these shortcomings can be removed using the method proposed in [3] and [15]. First, the state measurement at time  $t_0$  gives us the value of  $x(t_0 - \tau)$ . We assume that no control was applied for  $t \in [t_0 - \tau, t_0]$ . If an upper bound  $\tau^*$  on the delay is known and the system is forward complete under zero control, then we have an upper bound of the form (15). This is because for a forward complete system, the reachable set from a given initial condition in bounded time is bounded (see, e.g., [1]). An over-approximation of the reachable set can be used to actually compute such an upper bound (see [14] and [10] for some results on approximating reachable sets). Now, we can select a value of  $\mu$  large enough to satisfy (16) and start applying the control law (5) for  $t \geq t_0$ . Let us assume for simplicity that (21) is satisfied, so that we can take  $\varepsilon = 0$ . Applying Theorem 1, we have from (19) that

$$|x_d(t_0 + T)| \leq \tilde{\gamma}_e(\Delta \mu).$$

Next, at time  $t_0 + T$  we want to select a smaller value of  $\mu$  for which (16) holds with  $E_0$  replaced by this new bound (which depends on the previous value of  $\mu$ ). For this to be possible, we need to assume that

$$\alpha_1^{-1} \circ \alpha_2 \circ \tilde{\gamma}_e(\Delta \mu) < \kappa^{-1}(M \mu) \quad \forall \mu > 0.$$

<sup>3</sup>The argument we just gave actually establishes that global asymptotic stability under the zero input guarantees ISS with a given offset on a given bounded region (‘‘semiglobal practical ISS’’) for sufficiently small inputs. (A related result proved in [26] is that global asymptotic stability under the zero input implies ISS for sufficiently small states and inputs.) This observation confirms the essential role that ISS plays in our developments.

Repeating this “zooming-in” procedure, we recover global asymptotic stability.

We mention an interesting small-gain interpretation of this strategy, which was given in [18]. The closed-loop system can be viewed as a hybrid system with continuous state  $x$  and discrete state  $\mu$ . After the “zooming-out” stage is completed, it can be shown that the  $x$ -subsystem is ISS with respect to  $\mu$  with an ISS gain function smaller than  $\hat{\gamma}(\Delta \cdot)$ , while the  $\mu$ -subsystem is ISS with respect to  $x$  with ISS gain function  $\hat{\gamma}_e^{-1}(\cdot)/\Delta = \hat{\gamma}^{-1} \circ \alpha_2^{-1} \circ \alpha_1(\cdot)/\Delta$ . The composite gain is less than identity, and asymptotic stability follows from the nonlinear small-gain theorem. One important advantage of the small-gain viewpoint (which was also used to establish Theorem 1) is that it enables an immediate incorporation of external disturbances, under suitable assumptions. We refer the reader to [18] and [27] for further details.

We combined quantization and delays as in (5) just for concreteness; other scenarios can be handled similarly. Assume, for example, that the control takes the form

$$u(t) = k(q_\mu(x(t - \tau)))$$

i.e., both the quantization and the delay<sup>4</sup> affect the state before the control is computed. The resulting closed-loop system can be written as

$$\dot{x}(t) = f(x(t), k(x(t) + \theta(t) + e(t)))$$

where

$$\theta(t) := x(t - \tau) - x(t)$$

and

$$e(t) := q_\mu(x(t - \tau)) - x(t - \tau).$$

Assumption 1 needs to be modified by replacing (4) with

$$|x| \geq \rho(|v|) \Rightarrow \frac{\partial V}{\partial x} f(x, k(x + v)) \leq -\alpha_3(|x|).$$

This is the requirement of ISS with respect to *measurement* errors, which is restrictive even for systems affine in controls (see the discussion and references in [15], as well as [22]). For  $\gamma(r) := \rho(2r)$  we have

$$|x| \geq \gamma(|\theta| \vee |e|) \Rightarrow \frac{\partial V}{\partial x} f(x, k(x + \theta + e)) \leq -\alpha_3(|x|)$$

and we can proceed as before. The ISS assumption can again be relaxed at the expense of introducing a constraint on the error bound of the quantizer (see also footnote 3). A bound of the form (15) for some time greater than the initial time can be obtained by “zooming out” similarly to how it is done in [15], provided that an upper bound  $\tau^*$  on the delay is known and the system is forward complete under zero control. Global asymptotic stability can then be achieved by “zooming in” as before. It is also not difficult to extend the results to the case when quantization affects both the state and the input (cf. [15, Rem. 1]).

## V. CONCLUSION

The goal of this note was to show how the effects of quantization and time delays in nonlinear control systems can be treated in a unified manner by using Lyapunov functions and small-gain arguments. We proved that under the action of an input-to-state stabilizing feedback law in the presence of both quantization and delays subject to suitable conditions, solutions of the closed-loop system starting in a given region remain bounded and eventually enter a smaller region. Based on this result, global asymptotic stabilization can be achieved

<sup>4</sup>Their order is not important in this case as long as the quantizer is fixed and does not saturate.

by employing a dynamic quantization scheme. These findings demonstrate, in particular, that the quantized control algorithms proposed in our earlier work are inherently robust to time delays, which increases their potential usefulness for applications such as networked control systems.

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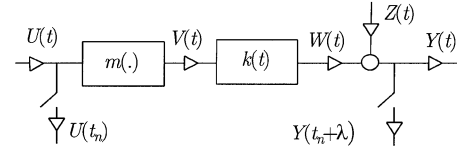


Fig. 1. Hammerstein system.

from sampled input–output signals, i.e., not from functions of time but from number sequences being a result of sampling. The distance between consecutive sampling instants may change in time, sampling can be uniform or not. We show that our kernel algorithms, one offline and two semirecursive, converge to the unknown characteristic and give convergence rates. We also examine the relation between various ways of sampling and convergence and demonstrate which methods of sampling are good and which are not. The time interval the sampling is performed on must increase to infinity sufficiently quickly.

## Continuous-Time Hammerstein System Identification From Sampled Data

Włodzimierz Greblicki

**Abstract**—A continuous-time Hammerstein system driven by a random signal is identified from observations sampled in time. The sampling may be uniform or not. The *a-priori* information about the system is nonparametric, functional forms of both the nonlinear characteristic and the impulse response are completely unknown. Three kernel algorithms, one offline and two semirecursive are presented. Their convergence to the true characteristic of the nonlinear subsystem is shown. The distance between consecutive sampling times must not decrease too fast for the algorithms to converge.

**Index Terms**—Hammerstein system, nonparametric estimation, nonparametric identification, system identification.

### I. INTRODUCTION

In a Hammerstein system, a memoryless nonlinear subsystem is followed by a linear dynamic one. The identification of the nonlinear subsystem has been the subject of a significant number of works; see [1]–[3]. This note belongs to those in which the *a-priori* information about the system is nonparametric. Classes of all possible nonlinear characteristics and equations of the dynamic subsystem are so wide that they cannot be represented in any parametric form. Moreover, the system is driven by a random signal and the output is disturbed by a random noise.

Kernel nonparametric algorithms were applied in [4]–[8] to recover the nonlinearity in discrete-time systems. Recently, kernel algorithms have been applied to recover the nonlinear characteristics in continuous-time systems, [9], [10]. The novelty of the note is that we infer

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### II. STATEMENT OF THE PROBLEM

The identified continuous-time Hammerstein system is shown in Fig. 1. The input signal  $\{U(t); t \in (-\infty, \infty)\}$  is a stationary white random process with autocovariance function  $\sigma_U^2 \delta(t)$ , where  $\delta(t)$  is the Dirac impulse and  $\sigma_U^2 < \infty$ . The random variable  $U(t)$  has a probability density  $f$ . The first subsystem has a characteristic  $m$  which means that  $V(t) = m(U(t))$ . We assume that  $m$  is a Borel measurable function such that  $EV^2(t) < \infty$ . This is, e.g., the case for  $|m(u)| \leq c_1 + c_2|u|$  with any  $c_1, c_2 > 0$ . Thus,  $\{V(t); t \in (-\infty, \infty)\}$  is a stationary white random process with autocovariance function  $\sigma_V^2 \delta(t)$ , where  $\sigma_V^2 < \infty$ .

The unknown impulse response  $k(t)$  of the dynamic subsystem satisfies the following inequality:

$$|k(t)| \leq \alpha e^{-\beta t} \quad (1)$$

with some unknown positive  $\alpha$  and  $\beta$ , which is always the case for a stable dynamic subsystem described by a state equation. Clearly,  $W(t) = \int_{-\infty}^t k(t - \tau)V(\tau)d\tau$ . Due to the fact that  $\sigma_V^2 < \infty$  and (1),  $\{W(t); t \in (-\infty, \infty)\}$  is a stationary correlated random process such that  $\sigma_W^2 < \infty$ . Additive stationary random noise  $\{Z(t); t \in (-\infty, \infty)\}$  has zero mean, finite variance and is independent of input signal. Therefore,  $Y(t) = W(t) + Z(t)$ , where  $Y(t)$  is output of the whole system.

We estimate the nonlinear characteristic  $m$  from sampled observations, i.e., from pairs  $(U(t_1), Y(t_1 + \lambda)), (U(t_2), Y(t_2 + \lambda)), (U(t_3), Y(t_3 + \lambda)), \dots$  with  $\lambda \geq 0$  and  $0 \leq t_1 < t_2 < \dots$ . The inner signal of the system is not measured. In the note,  $\lambda$  is fixed and known.

It will be convenient to denote  $\Delta_n = \min_{i=1,2,\dots,n-1} |t_{i+1} - t_i|$ . Observe that  $\Delta_n$  is monotonous and converges to either a positive number or zero as  $n \rightarrow \infty$ . We will say that the sampling is uniform if  $t_{n+1} - t_n = \Delta$ , some  $\Delta > 0$ , for all  $n = 1, 2, \dots$ . If  $a_n/b_n$  has a nonzero limit as  $n \rightarrow \infty$ , we write  $a_n \sim b_n$ .

For the sake of simplicity,  $U$  denotes a random variable distributed like  $U(t)$ .

### III. ALGORITHMS

The basis for our identification algorithms is the fact that  $E\{Y(t + \lambda)|U(t) = u\} = k(\lambda)m(u) + \int_0^\infty k(\tau)d\tau Em(U)$ , see [9]. To make