Brief paper

Compensation of disturbances for MIMO systems with quantized output☆

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ABSTRACT

The paper deals with the robust output feedback discrete control of continuous-time multi input multi output (MIMO) linear plants with arbitrary relative degree under parametric uncertainties and external bounded disturbances with quantized output signal. The parallel reference model (auxiliary loop) to the plant is used for extracting information about the uncertainties acting on the plant. The proposed algorithm guarantees that the output of the plant tracks the reference output with the required accuracy.

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1. Introduction

In recent years much attention has been given to the investigation of constraints on a communication channel included in a feedback control loop. Signal quantization (via quantizer or encoder) is usually considered as a source of independent discrete random noise which additively acts on the system. In Gray and Neuhoff (1998) this assumption allows to simplify the investigation of systems with quantization. However, this assumption may be too rough if the value of the quantization step is commensurate with the range of signal variation (Baillieul, 2002).

There are results relating to minimizing errors caused by quantization in the control loop. The paper of Larson (1967) is devoted to the synthesis of an optimal control system for discrete linear plants with quantization of the input signal. From Larson (1967) we have for input signal the one-dimensional density distribution.

In Fischer (1982) the algorithm for optimal quantization is derived from the solution of an optimization problem for a closed-loop system with a linear–quadratic criterion with Gaussian noise (LQG-problem). Similar results are presented by Curry, 1969 where the optimal stabilization problem of a linear stochastic discrete plant with a quadratic cost function is considered. The papers of Aldajani and Sayed (2001), Venayagamoorthy and Zha (2007) and Zierhofer (2000) are devoted to design of adaptive quantizers where the range of signal conversion is changed automatically. In subsequent works (for example, Andrievsky, Fradkov, & Peaucelle, 2007, Gomez-Estern, Canudas de Wit, Rubio, & Fornes, 2007 and Zheng, Duni, & Rao, 2007) the use of adaptive quantizers in control systems and estimation systems is considered. The paper of Liberzon (2003) is concerned with global asymptotic stabilization of continuous-time systems subject to quantization. A hybrid control strategy (Brockett & Liberzon, 2000) relies on the possibility of making discrete on-line adjustments of quantizer parameters. Sharon and Liberzon (2012) considered the problem of achieving input-to-state stability with respect to external disturbances for control systems with quantized measurements.

The special interest of the present paper is control of a plant under parametric uncertainties and disturbances. Guo, Zhang, and Zhao (2011) study the adaptive tracking control for systems with quantized output observations and one unknown parameter. The result by Konaka, Suzuki, and Okuma (2002) deals with a control problem where the continuous plant is controlled

by an adaptive discrete logic-based controller. The paper by Zheng and Yang (2012) is concerned with the quantized output feedback stabilization problem for a class of uncertain systems with nonsmooth nonlinearities in the actuator device via sliding mode algorithm. In Hayakawa, Ishii, and Tsumura (2009) a direct adaptive control framework for nonlinear systems with input quantizers is developed. The problem of feedback quantized output control for dynamical single input single output (SISO) plants with any relative degree under parametric uncertainties and uncontrollable disturbances is studied in Furtat, Fradkov, and Liberzon (2014).

As far as known to the authors of the present paper, the problem of control of multi input multi output (MIMO) plants under parametric uncertainty and external disturbances with quantized output has not been investigated. However, there are many fields where this problem is important, for example, control of networked systems, high-tech manufacturing networks, a complex crystal grid, chemical industry etc. Therefore, the present paper generalizes the results of Furtat et al. (2014) by solving this problem for MIMO plants.

As in Furtat et al. (2014) we use the method of the parallel reference model (auxiliary loop). The motivation for using this method is as follows: calculation of the regulator parameters is easy; the algorithm has low dynamical order; the system has the same structure for a broad class of plant models. The parallel reference model algorithm was first proposed by Tsykunov (2007). The idea of this method consists in implementing an auxiliary loop with desired parameters parallel to the plant. The difference between the output of the plant and the output of the auxiliary loop gives a function which depends on disturbances. This function uses for design a control law. In Tsykunov (2012) this algorithm is generalized for discrete control. In this paper we use results by Furtat et al. (2014) and Tsykunov (2012) for the synthesis of discrete control of continuous-time linear plants under parametric uncertainties and external bounded disturbance with quantization of an output signal. It is assumed that only input and output of the plant are accessible for measurement. The proposed algorithm guarantees that the output of the plant tracks the reference output with the required accuracy.

In this paper the following notations are used: R and N are the sets of real and natural numbers respectively, $p = d/dt$, $A = (a_{ij})$ is a matrix with elements a_{ij}, $\deg(L(p))$ is the degree of the linear operator $L(p)$, $\dim x$ is the dimension of the vector x, $\det(A)$ is the determinant of the square matrix A, λ is a complex variable, $\mathrm{diag}\{\cdot\}$ is a diagonal matrix, $\lambda_{\min}(\cdot)$ $(\lambda_{\max}(\cdot))$ is minimal (maximal) eigenvalue of a matrix, $\sup f(t)$ is the least upper bound of the function $f(t)$ with respect to t, I_v is $v \times v$ identity matrix, A^T is transpose of the matrix A.

2. Problem statement

Consider a plant equation

$$\dot{x}(t) = Ax(t) + Bu(t) + D\vartheta(t), \qquad y(t) = Lx(t),$$
$$x(0) = x_0, \tag{1}$$

where $x(t) \in R^n$ is a state vector, $y(t) \in R^v$ is an output being quantized, $u(t) \in R^v$ is an input, $\vartheta(t) \in R^l$ is an uncontrollable bounded disturbance, $A \in R^{n \times n}$, $B \in R^{n \times v}$, $D \in R^{n \times l}$, $L \in R^{v \times n}$ are matrices with unknown elements, x_0 is an unknown initial condition.

For simplicity, we assume that $\dim y = \dim u$.

According to Ioannou and Sun (1996, p. 38), transform Eq. (1) to the standard input–output form

$$Q(p)y(t) = R(p)u(t) + f(t), \tag{2}$$

where $Q(p) = W(p)I_v$, $W(p) = \det(pI_n - A)$, $R(p) = W(p)L(pI_n - A)^{-1}B = (R_{ij}(p))$, $f(t) = W(p)L(pI_n - A)^{-1}D\vartheta(t)$, $W(p)$ and

$R_{ij}(p)$, $i = 1, 2, \dots, v$, $j = 1, 2, \dots, v$ are linear differential operators with unknown coefficients, $\deg W(p) = n$, $\deg R_{ij}(p) = m_{ij}$, $m_{ii} \geq m_{ij}$ for $i \neq j$.

We are interested in the situation where only quantized measurements $q_i(y_i(t))$, $i = 1, 2, \dots, v$ of the output are available (Liberzon, 2003).

Assume that there exists a positive real number $\bar{\delta}$ such that if $|y_i(t)| \leq \bar{y}_i$ then

$$|q_i(y_i(t)) - y_i(t)| \leq \bar{\delta}, \quad i = 1, 2, \dots, v. \tag{3}$$

Inequality (3) defines an upper bound on the quantizer error. Let smooth bounded reference signals $y_{mi}(t)$, $i = 1, 2, \dots, v$ and a sequence of sampling times t_k, $k \in N$ be given. The problem is to design a discrete control law such that the following conditions hold:

$$|y_i(t_k) - y_{mi}(t_k)| \leq \delta \quad \text{for } i = 1, 2, \dots, v \text{ and } t_k \geq T, \tag{4}$$

where t_k is a sampling time, $t_{k-1} \leq t < t_k$, $\delta > 0$ is a prespecified required accuracy, $T > 0$ is a transient time. Moreover, the step $h = t_k - t_{k-1}$ is constant.

Assumptions. 1. Coefficients of operators $W(p)$ and $R_{ij}(p)$ belong to a known compact set $\varXi$.
2. Plant (1) is minimum phase, i.e. the trivial solution of equation $R(p)u(t) = 0$ is stable.
3. Only signals $q_i(y_i(t_k))$, $y_{mi}(t_k)$ and $u_i(t_k)$, $i = 1, 2, \dots, v$ are available for measurement in a control system.

Note that Assumption 1 is needed for choosing controller parameters such that goal (4) is ensured. This statement will be described in Section 3 and Appendix in more detail. Assumption 3 imposes the restriction on plant measurements, i.e. only quantized output is available. In Section 4 we will consider an example for model (2).

3. Algorithm of disturbances compensation

According to Furtat et al. (2014), rewrite $R_{ii}(p)$ and $W(p)$ in the form

$$R_{ii}(p) = R_{ii}^0(p) + \Delta R_{ii}(p), \qquad W(p) = W^0(p) + \Delta W(p). \tag{5}$$

Here $R_{ii}^0(\lambda)$ and $W^0(\lambda)$ are Hurwitz polynomials of degrees m_{ii} and n respectively, $\deg \Delta R_{ii}(p) \leq m_{ii}$, $\deg \Delta W(p) < n$. The polynomials $R_{ii}^0(\lambda)$ and $W^0(\lambda)$ are chosen such that $W^0(\lambda)/R_{ii}^0(\lambda) = Q_{mi}(\lambda)$, where $Q_{mi}(\lambda)$ is a Hurwitz polynomial of degree $\gamma_i = n - m_{ii}$. Taking into account (2) and (5), rewrite (2) as follows:

$$Q_{mi}(p)y_i(t) = u_i(t) + z_i(t), \quad i = 1, 2, \dots, v, \tag{6}$$

where

$$R_{ii}^0(p)z_i(t) = \Delta R_{ii}(p)u_i(t) - \Delta Q_{ii}(p)y_i(t) + R_{ii}^0(p)\tau_i(t)$$
$$- \sum_{j=1, \, i \neq j}^{v} Q_{ij}(p)y_j(t) + \sum_{j=1, \, i \neq j}^{v} R_{ij}(p)u_j(t)$$
$$+ \sum_{j=1}^{v} f_j(t),$$

$\tau_i(t)$ is an exponentially decaying function which depends on initial conditions of (1). Taking into account (6), the equation for the tracking error $e_i(t) = y_i(t) - y_{mi}(t)$ can be rewritten in the form

$$Q_{mi}(p)e_i(t) = u_i(t) + \rho_i(t), \tag{7}$$

where $\rho_i(t) = z_i(t) - Q_{mi}(p)y_{mi}(t)$. We see that the functions $\rho_i(t)$, $i = 1, 2, \dots, v$ contain a parametric uncertainty and external disturbance of plant (1). Therefore, according to Tsykunov (2007) introduce the control laws

$$u_i(t) = \alpha_i v_i(t_k), \quad i = 1, 2, \dots, v, \, t_{k-1} \leq t < t_k, \tag{8}$$

where $\alpha_i > 0$ are design parameters, $v_i(t_k)$ are new controls at time t_k.

Consider the auxiliary loop

$$Q_{mi}(p)\bar{e}_i(t) = \beta_i v_i(t_k),$$
$$p^{r-1}\bar{e}_i(t) = 0, \quad i = 1, 2, \ldots, v, \ r = 1, \ldots, \gamma_i, \tag{9}$$

where $\beta_i > 0$ are design parameters, $\bar{e}_i(t)$ are outputs of auxiliary loops (9). The auxiliary loop is a parallel reference model with desired behavior of transients. Therefore, taking into account (7)–(9), the equation for the error function $\sigma_i(t) = e_i(t) - \bar{e}_i(t)$ can be written in the form

$$Q_{mi}(p)\sigma_i(t) = \varphi_i(t), \quad i = 1, 2, \ldots, v, \tag{10}$$

where $\varphi_i(t) = (\alpha_i - \beta_i)v_i(t_k) + \rho_i(t)$ is a new disturbance function. The signals $\sigma_i(t)$, $i = 1, 2, \ldots, v$ contain information about parametric and external disturbances of plant (1).

To simplify justification of the control system let us suppose for the moment that output of plant (1) is not quantized and disturbance $\vartheta(t)$ is smooth. Taking into account (10), represent the value $\varphi_i(t_k)$ as follows:

$$\varphi_i(t_k) = q_{mi}^T \xi_i(t_k), \quad i = 1, 2, \ldots, v, \tag{11}$$

where $\bar{\varphi}_i(t_k) = (\alpha_i - \beta_i)v_i(t_k) + \bar{\rho}_i(t_k)$,

$$R_{ii}^0(p)\bar{z}_i(t_k) = \Delta R_{ii}(p)u_i(t_k) - \Delta Q_{ii}(p)q_i(y_i(t))$$
$$- \sum_{j=1, \ i \neq j}^{v} Q_{ij}(p)q_j(y_j(t)) + \sum_{j=1, \ i \neq j}^{v} R_{ij}(p)u_j(t_k)$$
$$+ \sum_{j=1}^{v} f_j(t_k) + R_{ii}^0(p)\left[\tau_i(t_k) - Q_{mi}(p)y_{mi}(t_k)\right],$$

q_{mi} is a vector composed of coefficients of $Q_{mi}(p)$, $\xi_i(t_k) = \left[\sigma_i(t_k), \ \dot{\sigma}_i(t_k), \ldots, \sigma_i^{(\gamma_i)}(t_k)\right]^T$, $\sigma_i^{(r)}(t_k)$ is the rth derivative of the signal $\sigma_i(t)$ taken at time t_k.

Assume for a moment that the vector $\xi_i(t_k)$ were available for measurement, then the controls could be defined by

$$v_i(t_k) = -\beta_i^{-1}q_{mi}^T \xi_i(t_k), \quad i = 1, 2, \ldots, v, \ t_{k-1} \leq t \leq t_k. \tag{12}$$

It follows from (11) and (12) that

$$v_i(t_k) = -\beta_i^{-1}\varphi_i(t_k), \quad i = 1, 2, \ldots, v, \ t_{k-1} \leq t \leq t_k. \tag{13}$$

Solving the system consisting of Eqs. (13) and $\varphi_i(t_k) = (\alpha_i - \beta_i)v_i(t_k) + \rho_i(t_k)$ with respect to $v_i(t_k)$, we get

$$v_i(t_k) = -\alpha_i^{-1}\rho_i(t_k), \quad i = 1, 2, \ldots, v, \ t_{k-1} \leq t \leq t_k. \tag{14}$$

Substituting (8) and (14) to (7), we obtain

$$Q_{mi}(p)e_i(t) = \rho_i(t) - \rho_i(t_k), \quad i = 1, 2, \ldots, v.$$

However, according to the problem statement the derivatives of the functions $\sigma_i(t)$ are not available for measurement and disturbance $\vartheta(t)$ is nonsmooth. Moreover, we need to design the control system where a quantized output of the plant is available. To overcome these difficulties, we replace derivatives with the finite differences. To this end, rewrite (11) in the form

$$\tilde{\varphi}_i(t_k) = q_{mi}^T \tilde{\xi}_i(t_k), \quad i = 1, 2, \ldots, v, \tag{15}$$

where $\tilde{\xi}_i(t_k) = \left[\tilde{\sigma}_i(t_k), \ \tilde{\sigma}_{1i}(t_k), \ldots, \tilde{\sigma}_{\gamma_i i}(t_k)\right]^T$, $\tilde{\sigma}_i(t_k) = q_i(y_i(t_k)) - y_{mi}(t_k) - \bar{e}_i(t_k)$. The vector $\tilde{\xi}_i(t_k)$ is obtained from the observers

$$\tilde{\xi}_i(t_k) = G_i\tilde{\xi}_i(t_{k-1}) + b_i\tilde{\sigma}_i(t_k), \quad i = 1, 2, \ldots, v. \tag{16}$$

Here $G_i = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ -h^{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ -h^{-\gamma_i} & \cdots & -h^{-1} & 0 \end{bmatrix}$, $b_i = \begin{bmatrix} 1 \\ h^{-1} \\ \vdots \\ h^{-\gamma_i} \end{bmatrix}$. Eq. (16) is written by using the right hand differences

$$\tilde{\sigma}_i(t_k) = \tilde{\sigma}_i(t_k),$$
$$\tilde{\sigma}_{1i}(t_k) = h^{-1}\left(\tilde{\sigma}_i(t_k) - \tilde{\sigma}_i(t_{k-1})\right),$$
$$\vdots \tag{17}$$
$$\tilde{\sigma}_{\gamma i}(t_k) = h^{-1}\left(\tilde{\sigma}_{\gamma-1, i}(t_k) - \tilde{\sigma}_{\gamma-1, i}(t_{k-1})\right).$$

Taking into account (15) and (16), rewrite controls $v_i(t_k)$ as

$$v_i(t_k) = -\beta_i^{-1}q_{mi}^T\tilde{\xi}_i(t_k), \quad i = 1, 2, \ldots, v, \ t_{k-1} \leq t < t_k. \tag{18}$$

Unlike (13), the estimates $\tilde{\xi}_i(t_k)$ of the signal $\xi_i(t_k)$ are used in (18). For implementing the auxiliary loop (9) in discrete form, first, transform (9) to a state space form

$$\dot{\bar{\varepsilon}}_i(t) = A_{mi}\bar{\varepsilon}_i(t) + \beta_i B_{mi}v_i(t_k),$$
$$\bar{e}_i(t) = L_i\bar{\varepsilon}_i(t), \quad i = 1, 2, \ldots, v, \tag{19}$$

where $\bar{\varepsilon}_i(t) \in R^{\gamma_i}$ is a state vector, the matrices A_{mi}, B_{mi} and $L_i = [1 \ 0 \ \ldots \ 0]$ are obtained according to transformation from (9) to (19). Transform Eqs. (19) to the discrete form

$$\bar{\varepsilon}_i(t_{k+1}) = \bar{A}_i\bar{\varepsilon}_i(t_k) + \beta_i\bar{B}_iv_i(t_k),$$
$$\bar{e}_i(t_k) = L_i\bar{\varepsilon}_i(t_k), \quad i = 1, 2, \ldots, v, \tag{20}$$

where $\bar{A}_i = e^{A_{mi}h}$, $\bar{B}_i = \int_t^{t+h} e^{A_{mi}(t-s)}B_{mi}ds$.

It is seen that Eqs. (8), (16), (18), (20) do not contain derivatives of the functions $\sigma_i(t_k)$ which depend on the quantized output of (1) and nonsmooth function $\vartheta(t)$, because the control system is using the estimates $\tilde{\sigma}_i(t_k)$. Therefore, algorithm (8), (16), (18), (20) can be implemented. Taking into account (8), (11), (13), (15) and (18), rewrite Eq. (7) in the form

$$Q_{mi}(p)e_i(t) = q_{mi}^T\left(\xi_i(t_k) - \tilde{\xi}_i(t_k)\right), \quad i = 1, 2, \ldots, v. \tag{21}$$

Rewrite Eq. (21) in a state space form

$$\dot{\varepsilon}_i(t) = A_{mi}\varepsilon_i(t) + B_{mi}q_{mi}^T\left(\xi_i(t_k) - \tilde{\xi}_i(t_k)\right),$$
$$e_i(t) = L_i\varepsilon_i(t), \quad i = 1, 2, \ldots, v. \tag{22}$$

For the formulation of our main result introduce the following notations: $\varepsilon = \left[\varepsilon_1^T, \ \varepsilon_2^T, \ldots, \varepsilon_v^T\right]^T$, $\xi = \left[\xi_1^T, \ \xi_2^T, \ldots, \xi_v^T\right]^T$, $\tilde{\xi} = \left[\tilde{\xi}_1^T, \ \tilde{\xi}_2^T, \ldots, \tilde{\xi}_v^T\right]^T$, $B_m = \text{diag}\left\{B_iq_{m,i}^T\right\}$, $i = 1, 2, \ldots, v$, $P = P^T > 0$ are solutions of Lyapunov equation $A_m^T P + P A_m = -Q$, $Q = Q^T > 0$, $\chi = \lambda_{\min}(R)\lambda_{\max}^{-1}(P)$, $R = Q - 2\mu^{-1}PB_mB_m^TP$, $\mu > 0$, $\theta = \sup\left|\xi(t) - \tilde{\xi}(t_k)\right|^2$.

Theorem. *Let Assumptions 1–3 hold. Then there exist coefficients $\alpha_i > 0$, $\beta_i > 0$ and small enough value $h_0 > 0$ such that for any step $h \leq h_0$ objective (4) in control system (8), (16), (18), (20) is achieved for any parameters of plant (1) from the set Ξ, where in (4) the accuracy δ is calculated as follows:*

$$\delta = \sqrt{\lambda_{\min}^{-1}(P)\left(e^{-\chi T}\varepsilon^T(0)P\varepsilon(0) + \left(1 - e^{-\chi T}\right)\chi^{-1}\mu\theta\right)}. \tag{23}$$

Moreover, the following upper bound holds:

$$|q_i(y_i(t_k)) - y_{mi}(t_k)| < \delta + \bar{\delta}. \tag{24}$$

Theorem will be proved in the Appendix.

It follows from (23) that the value δ continuously depends on α_i, β_i, $\bar{\delta}$, h, θ and Ξ. Moreover, the value θ depends on $|\xi(t)|$, $\left|\tilde{\bar{\xi}}(kh)\right|$ and, according to (11) and (15), on coefficients of $W(p)$, $R_{ij}(p)$ in (2) which belong to the compact set Ξ. Therefore, the values δ and θ are bounded since the set Ξ is compact.

4. Example

Consider a plant model in the form

$$
\begin{aligned}
&\left(p^3 + q_1 p^2 + q_2 p + q_3\right) y_1(t) + \left(p^2 + q_4 p + q_5\right) y_2(t) \\
&\quad = (r_1 p + r_2) u_1(t) + r_3 u_2(t) + f_1(t), \\
&\left(p^2 + q_6 p + q_7\right) y_2(t) + (q_8 p + q_9) y_1(t) \\
&\quad = r_4 u_1(t) + r_5 u_2(t) + f_2(t).
\end{aligned}
\tag{25}
$$

The set Ξ is determined by the following inequalities: $-3 \leq q_j \leq 3$, $j = 1, \ldots, 7$, $-2 \leq q_8 \leq 2$, $-2 \leq q_9 \leq 2$, $2 \leq r_1 \leq 3$, $-5 \leq r_2 \leq 10$, $2 \leq r_3 \leq 3$, $1 \leq r_4 \leq 2$, $1 \leq r_5 \leq 2$, $|f_1(t)| \leq 30$, $|f_2(t)| \leq 30$. The analysis of the set Ξ shows that plant (25) is minimum phase, i.e. Assumption 2 holds.

In practice model (25) can describe a distillation column (Skogestad, Morari, & Doyle, 1988), where y_1 is a top composition, y_2 is a bottom composition, u_1 is a reflux, u_2 is a boil up. Parametric uncertainties and external disturbances depend on physical and chemical parameters of a distillation process and a feed composition (a mixture of a light and heavy component into a distillate product). For digital sensor the measurements are quantized. Therefore, the problem is to design the discrete algorithm under parametric uncertainties, external disturbance and quantized measurements.

The reference signal $y_{mi}(t)$ is chosen as follows:

$$
y_{m1}(t) = 0.2 + 0.7 \sin 1.7t, \qquad y_{m2}(t) = 0.1 + 0.7 \cos 1.1t.
$$

Choose the equations of auxiliary loop (9) in the form

$$
\begin{aligned}
&\left(p^2 + 5p + 6\right) \bar{e}_i(t) = \beta_i v(t_k), \\
&\bar{e}_i(0) = \dot{\bar{e}}_i(0) = 0, \; i = 1, 2.
\end{aligned}
\tag{26}
$$

Eqs. (26) with step $h = 0.01$ have the form

$$
\bar{\varepsilon}_i(t_{k+1}) = \begin{pmatrix} 1 & 0.01 \\ -0.06 & 0.1 \end{pmatrix} \bar{\varepsilon}_i(t_k) + \begin{pmatrix} -5 \cdot 10^{-5} \\ 0.01 \end{pmatrix} \beta_i v_i(t_k),
\tag{27}
$$

$$
\bar{e}_i(t_k) = [1\ 0]\bar{\varepsilon}_i(t_k), \qquad \bar{\varepsilon}_i(0) = [0\ 0]^T, \quad i = 1, 2.
$$

Introduce the observer Eqs. (16) as follows:

$$
\tilde{\bar{\xi}}_i(t_k) = \begin{pmatrix} 0 & 0 & 0 \\ -10^2 & 0 & 0 \\ -10^4 & -10^2 & 0 \end{pmatrix} \tilde{\bar{\xi}}_i(t_{k-1}) + \begin{pmatrix} 1 \\ 10^2 \\ 10^4 \end{pmatrix} \sigma_i(t_k),
\tag{28}
$$

$$
\tilde{\bar{\xi}}_i(0) = [0\ 0\ 0]^T, \quad i = 1, 2.
$$

Let $\alpha_i = 1$ and $\beta_i = 1$, $i = 1, 2$. According to (8) and (18), rewrite the control as

$$
u_i(t) = v_i(t_k), \qquad v_i(t_k) = -[6\ 5\ 1]\tilde{\bar{\xi}}_i(t_k), \quad i = 1, 2.
\tag{29}
$$

Let quantization interval be equal to 0.1. In Fig. 1 the transients of the tracking errors $e_i(t)$, $i = 1, 2$ are presented by the following parameters in (25): $q_i = 3$, $j = 1, \ldots, 5$, $q_6 = -3$, $q_7 = 3$, $q_8 = q_9 = 2$, $r_1 = 2$, $r_2 = 8$, $r_3 = 3$, $r_4 = 2$, $r_5 = 2$, $f_1(t) = 3 + 27 \sin t$, $f_2(t) = 10 + 20 \cos 1.2t$, $y_1(0) = \dot{y}_1(0) = \ddot{y}_1(0) = 0.9$, $y_2(0) = \dot{y}_2(0) = 0.7$.

Let quantization interval be equal to 0.01. In Fig. 2 the simulation results of the tracking errors $e_i(t)$, $i = 1, 2$ are given by the same parameters in (25).

It follows from Fig. 1 that parametric uncertainties and external disturbances are compensated by control system (27)–(29) with the required accuracy $\delta = 0.3$ achieved after 0.02 s. It follows
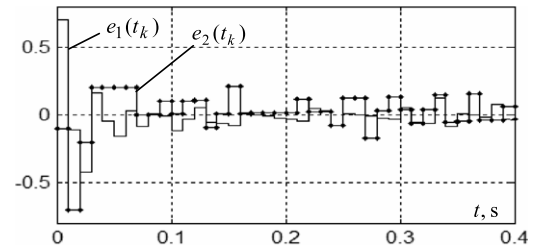

Fig. 1. The transients of the tracking errors e_1 (solid line) and e_2 (dash dot line) for quantization interval 0.1.
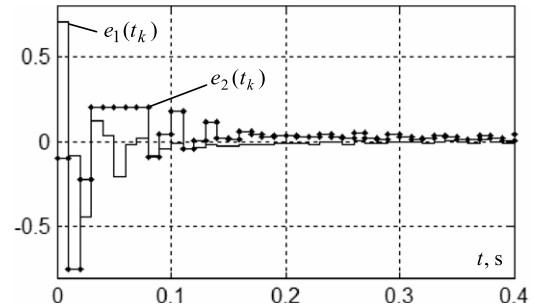

Fig. 2. The transients of the tracking errors e_1 (solid line) and e_2 (dash dot line) for quantization interval 0.01.

from Fig. 2 that the required accuracy $\delta = 0.1$ is achieved after 0.15 s. Simulation results show that the error $e_i(t)$ can be reduced by decreasing the value α_i, h, $\bar{\delta}$ and increasing the value β_i.

5. Conclusions

In this paper, we have treated the problem of robust output feedback discrete control of continuous-time linear MIMO plants under parametric uncertainties and external bounded disturbance with quantized output signal. The parallel reference model (auxiliary loop) which allows obtaining a function containing parametric uncertainties and external disturbances acting on the plant was considered. We proposed an algorithm that provides tracking by the plant output of the reference output with the required accuracy. Relationships between the tracking accuracy and the quantization error, uncertainties of the plant, and parameters of the regulator were derived.

The algorithm is proposed for the case when dimensions of the output vector are equal the input vector of the plant. Consideration of the case of different dimensions may be addressed in the future work.

Appendix

Proof of theorem. Rewrite first equation of (22) as follows:

$$
\dot{\varepsilon}(t) = A_m \varepsilon(t) + B_m \left(\xi(t) - \tilde{\bar{\xi}}(kh)\right).
\tag{30}
$$

To analyze system (30) the following lemma is needed.

Lemma. *Consider a hybrid system described by the differential and difference equation*

$$
\begin{aligned}
&\dot{x}(t) = f(x(t), u(t), \mu, t), \\
&u(t) = u(t_k) + h\varphi(x(t_k), \mu, t_k), \quad t_{k-1} \leq t < t_k,
\end{aligned}
\tag{31}
$$

where $x \in R^s$, $f(x, u, \mu, t)$ is Lipschitz continuous function in x, u, μ uniformly in t, piecewise continuous on t; $\varphi(x, \mu, t)$ is Lipschitz

continuous function in x, μ uniformly in t, piecewise continuous on t; $h = t_k - t_{k-1}$ and $\mu > 0$ are small parameters.

Consider the continuous time system for (31) in the form

$$\dot{\bar{x}}(t) = \bar{f}\left(\bar{x}(t),\ \bar{u}(t),\ t\right), \qquad \dot{\bar{u}}(t) = \bar{\varphi}\left(\bar{x}(t),\ t\right), \qquad (32)$$

where $\bar{x} \in R^s, \bar{f}\left(\bar{x},\ \bar{u},\ t\right) = f\left(\bar{x},\ \bar{u},\ 0,\ t\right), \bar{\varphi}\left(\bar{x},\ t\right) = \varphi\left(\bar{x},\ 0,\ t\right)$.

Let system (32) have a bounded closed set of attraction $\Omega = \{\bar{x},\ \bar{u}:\ F(\bar{x},\ \bar{u}) \leq C\}$, where $F(\bar{x},\ \bar{u})$ is a smooth, positive definite function in R^s. In addition let there exist some number $C_1 > 0$ such that the following condition holds:

$$\sup\left[\left[\frac{\partial F(\bar{x},\ \bar{u})}{\partial \bar{x}}\right]^T \cdot \bar{f}(\bar{x}, \bar{u},\ t):\ F(\bar{x},\ \bar{u}) = C\right] \leq -C_1. \qquad (33)$$

Then for any $\varepsilon > 0$ there exist numbers $\bar{\mu} > 0$ and $\bar{h} > 0$ such for $\mu < \bar{\mu}$ and $h < \bar{h}$ the condition $\overline{\lim}_{t\to\infty}$ dist $(\{x(t),\ u(t)\}, \Omega) < \varepsilon$ holds.

Lemma is a straightforward extension of Theorems 3.7, 3.13 in Derevitsky and Fradkov (1981).

Let us check conditions of lemma for system (30). Consider (30) for small parameters h and $\bar{\delta}$. Let $\bar{\delta} = 0$ and $h \to 0$. From (17) it follows that $\sigma_{ij}(t_k) \to \sigma_i^{(j)}(t_k)$, $j = 0, 1, \ldots, \gamma_i$ for $h \to 0$. Hence $\tilde{\bar{\xi}}_i(t_k) \to \xi_i(t)$. Therefore, we have the system in the form

$$\dot{\bar{\varepsilon}}(t) = A_m \bar{\varepsilon}(t). \qquad (34)$$

Choose smooth, positive definite function as Lyapunov function in the form

$$F(\bar{\varepsilon}) = \bar{\varepsilon}^T(t)P\bar{\varepsilon}(t). \qquad (35)$$

Obviously, condition (33) holds for (34). Since $y_{mi}(t)$ are smooth bounded functions then $p^j y_{mi}(t)$, $j = 1, \ldots, \gamma_i$ are bonded. Function $|u(t)|$ is bounded from (1) and Assumption 2. Thus, the functions $z_i(t)$, $\rho_i(t)$ and $\varphi_i(t)$ are bounded. According to lemma there exist coefficients h_0 and $\bar{\delta}_0$ such that for $h \leq h_0$ and $\bar{\delta} \leq \bar{\delta}_0$ all functions are bounded in the closed-loop system.

We note that values of the numbers h_0 and $\bar{\delta}_0$ depend on $|\xi(t)|$ and $\left|\tilde{\bar{\xi}}(kh)\right|$ which, according to (11) and (15), depend on coefficients of $W(p)$ and $R_{ij}(p)$ in (2). In fact, it can be shown that the numbers h_0 and $\bar{\delta}_0$ depend continuously on control system parameters and elements of the compact set Ξ.

However, asymptotic stability of system solution (30) does not follow from asymptotic stability of the solution of system (34). Therefore, consider (30) for $h = h_0$ and $\bar{\delta} = \bar{\delta}_0$. According to (35), choose Lyapunov function $F(\varepsilon)$. Taking the derivative of $F(\varepsilon)$ along the trajectories of (30), we obtain

$$\dot{F}(\varepsilon) = -\varepsilon^T(t)Q\varepsilon(t) + 2\varepsilon^T(t)PB_m\left(\xi(t) - \tilde{\xi}(kh_0)\right). \qquad (36)$$

Consider the following upper bound for the second term on the right-hand side of (36) as follows:

$$2\varepsilon^T PB_m\left(\xi - \tilde{\xi}\right) \leq 2\mu^{-1}\varepsilon^T PB_m B_m^T P\varepsilon + \mu\theta. \qquad (37)$$

From (11) and (15) it follows that value θ depends on parameters h and $\bar{\delta}$. Moreover, as shown earlier, decreasing the values h and $\bar{\delta}$ the value of θ is being reduced. Taking into account bounds (39) rewrite (36) as follows:

$$\dot{F} \leq -\varepsilon^T(t)R\varepsilon(t) + \mu\theta. \qquad (38)$$

Rewrite (38) in the form

$$\dot{F}(t) \leq -\chi F(t) + \mu\theta. \qquad (39)$$

Solving inequality (39) with respect to $F(t)$, we obtain

$$F(t) \leq e^{-\chi t}F(0) + \chi^{-1}\left(1 - e^{-\chi t}\right)\mu\theta. \qquad (40)$$

It follows from (40) that $F \leq \chi^{-1}\mu\theta$ as $t \to \infty$. Taking into account (35) and (40), we have

$$|e_i(t)| \leq |\varepsilon(t)| \leq \sqrt{\lambda_{\min}^{-1}(P)\left(e^{-\chi t}F(0) + (1 - e^{-\chi t})\chi^{-1}\mu\theta\right)}. \quad (41)$$

Rewrite (41) for the sampling time t_k, $t_{k-1} \leq t < t_k$ in the form

$$|e_i(t_k)| \leq \sqrt{\lambda_{\min}^{-1}(P)\left(e^{-\chi t_k}F(0) + (1 - e^{-\chi t_k})\chi^{-1}\mu\theta\right)}. \qquad (42)$$

Taking into account (4) and (42) for time $t_k = T$ we have relation (23). Let us consider the following upper bound:

$$|q_i(y_i(t_k)) - y_{mi}(t_k)|$$
$$\leq |y_i(t_k) - y_{mi}(t_k)| + |q_i(y_i(t_k)) - y_i(t_k)|. \qquad (43)$$

It follows from (3) and (43) that relation (24) holds. Obviously, the value θ of the right hand side of (23) depends on the values α_i, β_i and h. Therefore, the value δ in (4) can be reduced by decreasing the value α_i, h and increasing the value β_i. Theorem is proved.

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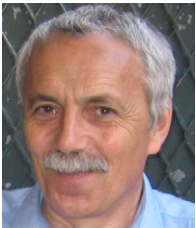
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