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Hybrid feedback stabilization of systems with quantized signals☆

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Abstract

This paper is concerned with global asymptotic stabilization of continuous-time systems subject to quantization. A hybrid control strategy originating in earlier work (Brockett and Liberzon, IEEE Trans. Automat. Control 45 (2000) 1279) relies on the possibility of making discrete on-line adjustments of quantizer parameters. We explore this method here for general nonlinear systems with general types of quantizers affecting the state of the system, the measured output, or the control input. The analysis involves merging tools from Lyapunov stability, hybrid systems, and input-to-state stability.

Keywords: Quantization; Hybrid control; Lyapunov function; Input-to-state stability

1. Introduction

In the classical feedback control setting, the output of the process is assumed to be passed directly to the controller, which generates the control input and in turn passes it directly back to the process. In practice, however, this paradigm often needs to be re-examined because the interface between the controller and the process features some additional information-processing devices. These considerations arise, for example, in networked control systems; see the articles in Bushnell (2001) and the references therein.

One important aspect to take into account in such situations is signal quantization. We think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking on a finite set of values. Quantization may affect the process output (this happens, for example, when the output measurements to be used for feedback are obtained by using a digital camera, stored in the memory of a digital computer, or transmitted over a digital communication channel). It may also affect the control input (examples include the standard PWM amplifier and the manual transmission on a car).

We assume that the given system evolves in continuous time. In the presence of quantization, the state space (or the input space) of the system is divided into a finite number of *quantization regions*, each corresponding to a fixed value of the quantizer. At the time of passage from one quantization region to another, the dynamics of the closed-loop system change abruptly. Therefore, systems with quantization can be naturally viewed as *hybrid* systems, i.e., systems described by a coupling between continuous and discrete dynamics.

There are two well-studied phenomena which account for changes in the system's behavior caused by quantization. The first one is saturation: if the signal is outside the range of the quantizer, then the quantization error is large, and the control law designed for the ideal case of no quantization leads to instability. The second one is deterioration of performance near the equilibrium: as the difference between the current and the desired values of the state becomes small, higher precision is required, and so in the presence of quantization errors asymptotic convergence is impossible. These phenomena manifest themselves in the existence of two nested invariant regions such that all trajectories of the quantized system starting in the bigger region approach the smaller one, while no further convergence guarantees can be given.

A standard assumption made in the literature is that parameters of the quantizer are fixed in advance and cannot be changed by the control designer; see, among many sources, (Chou, Chen, & Horng, 1996; Delchamps, 1990;

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Raisch, 1995; Feng & Loparo, 1997; Sur & Paden, 1998; Lunze, Nixdorf, & Schröder, 1999). There has been some research concerned with the question of how the choice of quantization parameters affects the behavior of the system (Wong & Brockett, 1999; Åström & Bernhardsson, 1999; Liberzon & Brockett, 2000; Elia & Mitter, 2001; Ishii & Francis, 2002). In this paper, building on the earlier work reported in (Brockett & Liberzon, 2000; Liberzon, 2000), we adopt the approach that it is possible to vary some parameters of the quantizer in real time, on the basis of collected data. For example, if a quantizer is used to represent a camera, this corresponds to zooming in and out, i.e., varying the focal length, while the number of pixels of course remains fixed. This approach fits into the framework of *control with limited information*: the state of the system is not completely known, but it is only known which one of a fixed number of quantization regions contains the current state at each instant of time. The quantizer can be thought of as a coder that generates an encoded signal taking values in a given finite set. By changing the size and relative position of the quantization regions—i.e., by modifying the coding mechanism—we can learn more about the behavior of the system, without violating the restriction on the type of information that can be communicated to the controller. This will help us overcome the two difficulties described above.

The quantization parameters will be updated at discrete instants of time (these *switching events* will be determined by the values of a suitable Lyapunov function). This results in a *hybrid quantized feedback control policy*. There are several reasons for adopting a hybrid control approach rather than varying the quantization parameters continuously. First, in specific situations there may be some constraints on how many values these parameters are allowed to take and how frequently they can be adjusted. Thus a discrete adjustment policy is more natural and easier to implement than a continuous one. Secondly, the analysis of hybrid systems obtained in this way appears to be more tractable than that of systems resulting from continuous parameter tuning. In fact, we will see that invariant regions defined by level sets of a Lyapunov function provide a simple and effective tool for studying the behavior of the closed-loop system. This also implies that precise computation of the switching times is not essential, which makes our hybrid control policies robust with respect to certain types of time delays (such as those associated with periodic sampling).

The recent paper (Brockett & Liberzon, 2000) thoroughly investigates the hybrid control methodology outlined above in the context of the feedback stabilization problem for linear control systems with output (or state) quantization. It is shown there that if a linear system can be stabilized by a linear feedback law, then it can also be *globally asymptotically stabilized* by a hybrid quantized feedback control policy. The control strategy is usually composed of two stages. The first, "zooming-out" stage consists in increasing the range of the quantizer until the state of the system can be adequately measured; at this stage, the system is open-loop. The second, "zooming-in" stage involves applying feedback and at the same time decreasing the quantization error in such a way as to drive the state to the origin. The developments of Brockett and Liberzon (2000) were restricted to quantizers that give rise to rectilinear quantization regions.

The present work generalizes the contributions of Brockett and Liberzon (2000) in three different directions. First, we consider more general types of quantizers, with quantization regions having *arbitrary shapes* as in Lunze et al. (1999). This extension is useful in several situations. For example, in the context of vision-based feedback control mentioned earlier, the image plane of the camera is divided into rectilinear regions, but the shapes of the quantization regions in the state space which result from computing inverse images of these rectangles can be rather complicated. The so-called *Voronoi tessellations* suggest that, at least in two dimensions, it may be beneficial to use hexagonal quantization regions rather than more familiar rectangular ones (Du, Faber, & Gunzburger, 1999). We will demonstrate that the principal findings of Brockett and Liberzon (2000) are still valid in this more general setting.

Another goal of this paper is to address the quantized feedback stabilization problem for *nonlinear systems*. It can be shown via a linearization argument that by using the approach of Brockett and Liberzon (2000) one can obtain local asymptotic stability for a nonlinear system, provided that the corresponding linearized system is stabilizable (Hu, Feng, & Michel, 1999). Here we are concerned with achieving global stability [1] results. We will show that the techniques developed in Brockett and Liberzon (2000) can be extended in a natural way to those nonlinear systems that are *input-to-state stabilizable with respect to measurement disturbances*. We thus reveal an interesting interplay between the problem of quantized feedback stabilization, the theory of hybrid systems, and topics of current interest in nonlinear control design. A preliminary investigation of these questions has been reported in Liberzon (2000), but only for state quantizers with rectilinear quantization regions.

Finally, in this paper we develop analogous results for systems with *input quantization*, both linear and nonlinear. In view of the examples given earlier, this expands the potential applicability of the hybrid quantized feedback control techniques. We discover that the analysis of systems with input quantization can be carried out quite similarly to the state quantization case. This analysis also yields a basis for comparing the effects of input quantization and state quantization on the performance of the system. As we will see, for nonlinear systems the case of state quantization presents a greater challenge from the viewpoint of control design.

As in Brockett and Liberzon (2000), all control laws are constructed explicitly. All vector fields and feedback laws are assumed to be sufficiently regular (e.g., smooth).

[1] Working with a given nonlinear system directly, one gains an advantage even if only local asymptotic stability is sought, because the linearization of a stabilizable nonlinear system may fail to be stabilizable.

Solutions to all differential equations are well defined, with the understanding that they are to be interpreted in the sense of Filippov (1988) if necessary (control strategies described in this paper do not rely on chattering and the analysis of the resulting closed-loop systems does not explicitly require a concept of generalized solution). With some abuse of terminology, we call a closed-loop hybrid system globally asymptotically stable if the origin is a globally asymptotically stable equilibrium of the continuous dynamics. We denote by $|\cdot|$ the standard Euclidean norm in $\mathbb{R}^n$ and by $\|\cdot\|$ the corresponding induced matrix norm in $\mathbb{R}^{n\times n}$. A function $\alpha:[0,\infty)\to[0,\infty)$ is said to be of *class* $\mathscr{K}_\infty$ if it is continuous, strictly increasing, and such that $\alpha(0)=0$ and $\alpha(r)\to\infty$ as $r\to\infty$.

2. Quantizer

Let $z\in\mathbb{R}^l$ be the variable being quantized. By a *quantizer* we mean a piecewise constant function $q:\mathbb{R}^l\to\mathscr{Q}$, where $\mathscr{Q}$ is a finite subset of $\mathbb{R}^l$. This leads to a partition of $\mathbb{R}^l$ into a finite number of *quantization regions* of the form $\{z\in\mathbb{R}^l\colon q(z)=i\}$, $i\in\mathscr{Q}$. These quantization regions are not assumed to have any particular shapes (see Fig. 1).

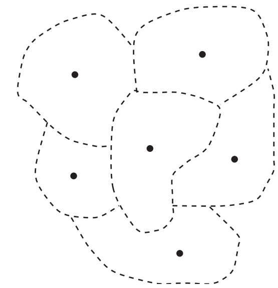

Fig. 1. Quantization regions.

When z does not belong to the union of quantization regions of finite size, the quantizer *saturates*. More precisely, we assume that there exist positive real numbers M and Δ such that the following two conditions hold:

1. If
$$|z|\leqslant M \tag{1}$$
then
$$|q(z)-z|\leqslant\Delta. \tag{2}$$
2. If
$$|z|>M$$
then
$$|q(z)|>M-\Delta.$$

Condition 1 gives a bound on the quantization error when the quantizer does not saturate. Condition 2 provides a way to detect the possibility of saturation. We will refer to M and Δ as the *range* of q and the *quantization error*, respectively. We also assume that $q(x)=0$ for x in some neighborhood of the origin, i.e., the origin lies in the interior of the set $\{x\colon q(x)=0\}$. An example of a quantizer satisfying the above requirements is provided by the quantizer with rectangular quantization regions considered in earlier works (Brockett & Liberzon, 2000; Liberzon, 2000). These conditions are also satisfied in the setting of Kofman (2001) where quantization is combined with hysteresis.

In the control strategies to be developed below, we will use quantized measurements of the form

$$q_\mu(z):=\mu q\left(\frac{z}{\mu}\right), \tag{3}$$

where $\mu>0$. The range of this quantizer is $M\mu$ and the quantization error is $\Delta\mu$. We can think of μ as the "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and quantization error, whereas decreasing μ corresponds to zooming in and obtaining a quantizer with a smaller range but also a smaller quantization error. We will update μ at discrete instants of time, so it will be the discrete state of the resulting hybrid closed-loop system. In the camera model mentioned in the Introduction, μ corresponds to the inverse of the focal length. It is possible to introduce more general, nonlinear scaling of the quantized variable, as in $v\circ q\circ v^{-1}(z)$ where v is some invertible function from $\mathbb{R}^l$ to $\mathbb{R}^l$ and $\circ$ denotes composition; however, this does not seem to introduce any significant advantages in the context of the problems studied here.

3. State quantization

To fix ideas, we treat linear systems first.

3.1. Linear systems

Consider the linear system

$$\dot{x}=Ax+Bu,\quad x\in\mathbb{R}^n,\ u\in\mathbb{R}^m. \tag{4}$$

Suppose that (4) is *stabilizable*, so that for some matrix K the eigenvalues of $A+BK$ have negative real parts. By the standard Lyapunov stability theory, there exist positive definite symmetric matrices P and Q such that

$$(A+BK)^{\mathrm{T}}P+P(A+BK)=-Q. \tag{5}$$

We will let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest eigenvalue of a symmetric matrix, respectively. The inequality

$$\lambda_{\min}(P)|x|^2\leqslant x^{\mathrm{T}}Px\leqslant\lambda_{\max}(P)|x|^2$$

will be used repeatedly below. We will assume that $B\neq 0$ and $K\neq 0$; this is no loss of generality because the case of interest is when A is not a stable matrix.

In this section we are interested in the situation where only quantized measurements $q_\mu(x)$ of the state x are available. Since the state feedback law $u=Kx$ is not implementable, consider the "certainty equivalence"[2] quantized feedback control law

$$u = Kq_\mu(x). \tag{6}$$

Assume for the moment that μ is a fixed positive number. The closed-loop system is given by

$$\dot{x} = Ax + BKq_\mu(x)$$
$$= (A+BK)x + BK\mu\left(q\left(\frac{x}{\mu}\right) - \frac{x}{\mu}\right). \tag{7}$$

The behavior of trajectories of system (7) for a fixed μ is characterized by the following result.

Lemma 1. *Fix an arbitrary $\varepsilon > 0$ and assume that M is large enough compared to Δ so that we have*

$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon), \tag{8}$$

where

$$\Theta_x := \frac{2\|PBK\|}{\lambda_{\min}(Q)} > 0.$$

Then the ellipsoids

$$\mathscr{R}_1(\mu) := \{x: x^{\mathrm{T}}Px \leqslant \lambda_{\min}(P)M^2\mu^2\} \tag{9}$$

and

$$\mathscr{R}_2(\mu) := \{x: x^{\mathrm{T}}Px \leqslant \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2\mu^2\} \tag{10}$$

are invariant regions for system (7). Moreover, all solutions of (7) that start in the ellipsoid $\mathscr{R}_1(\mu)$ enter the smaller ellipsoid $\mathscr{R}_2(\mu)$ in finite time.

Proof. Whenever inequality (1), and consequently (2), hold with $z = x/\mu$, the derivative of $x^{\mathrm{T}}Px$ along solutions of (7) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x^{\mathrm{T}}Px = -x^{\mathrm{T}}Qx + 2x^{\mathrm{T}}PBK\mu\left(q\left(\frac{x}{\mu}\right) - \frac{x}{\mu}\right)$$
$$\leqslant -\lambda_{\min}(Q)|x|^2 + 2|x|\,\|PBK\|\Delta\mu$$
$$= -|x|\lambda_{\min}(Q)(|x| - \Theta_x\Delta\mu).$$

This implies the following formula:

$$\Theta_x\Delta(1+\varepsilon)\mu \leqslant |x| \leqslant M\mu$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}x^{\mathrm{T}}Px \leqslant -|x|\lambda_{\min}(Q)\Theta_x\Delta\varepsilon\mu. \tag{11}$$

Define the balls

$$\mathscr{B}_1(\mu) := \{x: |x| \leqslant M\mu\}$$

and

$$\mathscr{B}_2(\mu) := \{x: |x| \leqslant \Theta_x\Delta(1+\varepsilon)\mu\}.$$

[2] This standard term (borrowed from adaptive control) refers to the fact that the controller treats quantized state measurements as if they were exact state values, even though they are not.

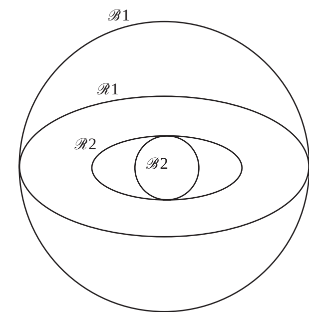

Fig. 2. The regions used in the proofs.

In view of inequality (8), we have

$$\mathscr{B}_2(\mu) \subset \mathscr{R}_2(\mu) \subset \mathscr{R}_1(\mu) \subset \mathscr{B}_1(\mu).$$

Combined with (11), this immediately implies that the ellipsoids $\mathscr{R}_1(\mu)$ and $\mathscr{R}_2(\mu)$ are both invariant. The fact that the trajectories starting in $\mathscr{R}_1(\mu)$ approach $\mathscr{R}_2(\mu)$ in finite time follows from the bound on the derivative of $x^{\mathrm{T}}Px$ given by (11). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathscr{R}_1(\mu)$ and if we let

$$T := \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)\Theta_x^2\Delta^2(1+\varepsilon)^2}{\Theta_x^2\Delta^2(1+\varepsilon)\lambda_{\min}(Q)\varepsilon} \tag{12}$$

then $x(t_0+T)$ is guaranteed to belong to $\mathscr{R}_2(\mu)$. Fig. 2 illustrates the proof (and will also be useful later). □

As we explained before, a hybrid quantized feedback control policy involves updating the value of μ at discrete instants of time. An open-loop "zooming-out" stage is followed by a closed-loop "zooming-in" stage, so that the resulting control law takes the form

$$u(t) = \begin{cases} 0, & 0 \leqslant t < t_0, \\ Kq_{\mu(t)}(x(t)), & t \geqslant t_0. \end{cases}$$

Using this idea and Lemma 1, it is possible to achieve global asymptotic stability, as we now show.

Theorem 1. *Assume that M is large enough compared to Δ so that we have*

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M > 2\Delta\max\left\{1, \frac{\|PBK\|}{\lambda_{\min}(Q)}\right\}. \tag{13}$$

Then there exists a hybrid quantized feedback control policy that makes system (4) globally asymptotically stable.

Proof. *The "zooming-out" stage*. Set u equal to 0. Let $\mu(0) = 1$. Then increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $\|\mathrm{e}^{At}\|$. For example, one can fix a positive number τ and let $\mu(t) = 1$ for $t \in [0, \tau)$, $\mu(t) = \tau\mathrm{e}^{2\|A\|\tau}$ for $t \in [\tau, 2\tau)$, $\mu(t) = 2\tau\mathrm{e}^{2\|A\|2\tau}$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \geqslant 0$

such that

$$\left|\frac{x(t)}{\mu(t)}\right| \leqslant \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - 2\Delta$$

(by (13), the right-hand side of this inequality is positive). In view of condition 1 imposed in Section 2, this implies

$$\left|q\left(\frac{x(t)}{\mu(t)}\right)\right| \leqslant \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M - \Delta$$

which is equivalent to

$$|q_\mu(x(t))| \leqslant \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M\mu(t) - \Delta\mu(t). \tag{14}$$

We can thus pick a time t_0 such that (14) holds with $t = t_0$. Therefore, in view of conditions 1 and 2 of Section 2, we have

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \leqslant \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M;$$

hence, $x(t_0)$ belongs to the ellipsoid $\mathscr{R}_1(\mu(t_0))$ given by (9). Note that this event can be detected using only the available quantized measurements.

The "zooming-in" stage. Choose an $\varepsilon > 0$ such that inequality (8) is satisfied; this is possible because of (13). We know that $x(t_0)$ belongs to $\mathscr{R}_1(\mu(t_0))$. We now apply the control law (6). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T)$, where T is given by formula (12). Then $x(t_0 + T)$ belongs to the ellipsoid $\mathscr{R}_2(\mu(t_0))$ given by (10). For $t \in [t_0 + T, t_0 + 2T)$, let

$$\mu(t) = \Omega\mu(t_0)$$

where

$$\Omega := \frac{\sqrt{\lambda_{\max}(P)}\Theta_x\Delta(1+\varepsilon)}{\sqrt{\lambda_{\min}(P)}M}.$$

We have $\Omega < 1$ by (8), hence $\mu(t_0+T) < \mu(t_0)$. It is easy to check that $\mathscr{R}_2(\mu(t_0)) = \mathscr{R}_1(\mu(t_0 + T))$. This means that we can continue the analysis for $t \geqslant t_0 + T$ as before. Namely, $x(t_0 + 2T)$ belongs to the ellipsoid $\mathscr{R}_2(\mu(t_0 + T))$ defined by (10). For $t \in [t_0 + 2T, t_0 + 3T)$, let $\mu(t) = \Omega\mu(t_0 + T)$. Repeating this procedure, we obtain the desired control policy. Indeed, we have $\mu(t) \to 0$ as $t \to \infty$, and the above analysis implies that $x(t) \to 0$ as $t \to \infty$. To show stability of the equilibrium $x = 0$ of the continuous dynamics in the sense of Lyapunov, take an arbitrary $\varepsilon > 0$. Find a positive integer k such that the ellipsoid $\mathscr{R}_1(\Omega^k)$ is contained in the ball of radius ε around the origin. Pick a $\delta > 0$ such that solutions of $\dot{x} = Ax$ with $|x(0)| \leqslant \delta$ stay in the intersection of this ε-ball with the region $\{x: q(x) = q(x/\Omega) = q(x/\Omega^{k-1}) = 0\}$ for all $t \in [0, kT]$ (recall that $q(x) = 0$ on some neighborhood of the origin). Then these solutions satisfy $|x(t)| \leqslant \varepsilon$ for all $t \geqslant 0$. □

We see from the proof of Theorem 1 that the state of the closed-loop system belongs, at equally spaced instants of time, to ellipsoids whose sizes decrease according to consecutive integer powers of Ω. Therefore, $x(t)$ converges to zero exponentially for $t \geqslant t_0$; see Brockett and Liberzon (2000) for details.

The fact that the scaling of μ is performed at $t = t_0 + T, t_0 + 2T, \ldots$ is not crucial: since the ellipsoids considered in the proof are invariant regions for the closed-loop system, we could instead take another sequence of switching times $t_1, t_2, \ldots$ satisfying $t_i - t_{i-1} \geqslant T$, $i \geqslant 1$. However, doing this in an arbitrary fashion would sacrifice the exponential rate of decay. The constant T is usually referred to as the *dwell time*.

At the "zooming-in" stage described in the proof of Theorem 1, the switching strategy is *time-based*, in the sense that the values of the discrete state μ are changed at precomputed times at which the continuous state x is known to belong to a certain region. An alternative approach would be to employ *event-based* switching, i.e., use the quantized measurements to determine when x enters the desired region; this is done in Liberzon (2000). An event-based switching strategy relies more on feedback measurements and less on off-line computations than a time-based one; it is therefore likely to be more robust with respect to modeling errors, although the expressions for the switching times are somewhat more straightforward in the case of time-based switching.

In the preceding, μ takes on a countable set of values which is not assumed to be fixed in advance. In some situations μ may be restricted to take values in a given countable set S. It is not difficult to see that the proposed method, suitably modified, still works in this case, provided that the set S has the following properties:

1. S contains a sequence $\mu_{11}, \mu_{21}, \ldots$ that increases to ∞.
2. Each μ_{i1} from this sequence belongs to a sequence $\mu_{i1}, \mu_{i2}, \ldots$ in S that decreases to 0 and is such that we have $\Omega \leqslant \mu_{i,j+1}/\mu_{ij}$ for each j.

If the set of possible values for μ is finite rather than countable, we can only obtain practical stability and not global asymptotic stability; cf. Brockett and Liberzon (2000) and Liberzon (2000).

3.2. Nonlinear systems

Consider the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m. \tag{15}$$

We take all vector fields and control laws to be sufficiently regular (e.g., smooth). It is natural to assume that there exists a state feedback law $u = k(x)$ that makes the closed-loop system globally asymptotically stable. Actually, we need to assume that k satisfies the following stronger condition (which will be examined in more detail later): there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ such that for some class $\mathscr{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x, e \in \mathbb{R}^n$ we have

$$\alpha_1(|x|) \leqslant V(x) \leqslant \alpha_2(|x|) \tag{16}$$

and

$$|x| \geqslant \rho(|e|) \Rightarrow \nabla V(x) f(x, k(x+e)) \leqslant -\alpha_3(|x|). \tag{17}$$

According to the results of Sontag (1989) and Sontag and Wang (1995), this is equivalent to saying that the perturbed closed-loop system

$$\dot{x} = f(x, k(x+e)) \tag{18}$$

is *input-to-state stable* (ISS) with respect to the measurement disturbance input e.

Since only quantized measurements of the state are available, we again consider the "certainty equivalence" quantized feedback control law, which in this case is given by

$$u = k(q_\mu(x)), \tag{19}$$

where q_μ is defined by (3). For a fixed μ, the closed-loop system is

$$\dot{x} = f(x, k(q_\mu(x))) \tag{20}$$

and this takes the form (18) with

$$e = q_\mu(x) - x. \tag{21}$$

The behavior of trajectories of (20) for a fixed value of μ is characterized by the following lemma.

Lemma 2. *Assume that we have*

$$\alpha_1(M\mu) > \alpha_2 \circ \rho(\Delta\mu). \tag{22}$$

Then the sets

$$\mathcal{R}_1(\mu) := \{x\colon V(x) \leqslant \alpha_1(M\mu)\} \tag{23}$$

and

$$\mathcal{R}_2(\mu) := \{x\colon V(x) \leqslant \alpha_2 \circ \rho(\Delta\mu)\} \tag{24}$$

are invariant regions for system (20). *Moreover, all solutions of* (20) *that start in the set* $\mathcal{R}_1(\mu)$ *enter the smaller set* $\mathcal{R}_2(\mu)$ *in finite time.*

Proof. Whenever inequality (1), and consequently (2), hold with $z = x/\mu$, the quantization error e given by (21) satisfies

$$|e| = \left|\mu q\left(\frac{x}{\mu}\right) - \mu\frac{x}{\mu}\right| \leqslant \Delta\mu.$$

Combined with (17), this implies the following formula:

$$\rho(\Delta\mu) \leqslant |x| \leqslant M\mu \Rightarrow \dot{V} \leqslant -\alpha_3(|x|), \tag{25}$$

where $\dot{V}$ denotes the derivative of V along solutions of (20). Define the balls

$$\mathcal{B}_1(\mu) := \{x\colon |x| \leqslant M\mu\}$$

and

$$\mathcal{B}_2(\mu) := \{x\colon |x| \leqslant \rho(\Delta\mu)\}.$$

As before, in view of (16) and (22) we have

$$\mathcal{B}_2(\mu) \subset \mathcal{R}_2(\mu) \subset \mathcal{R}_1(\mu) \subset \mathcal{B}_1(\mu).$$

Combined with (25), this implies that the ellipsoids $\mathcal{R}_1(\mu)$ and $\mathcal{R}_2(\mu)$ are both invariant. The fact that the trajectories starting in $\mathcal{R}_1(\mu)$ approach $\mathcal{R}_2(\mu)$ in finite time follows from the bound on the derivative of V deduced from (25). Indeed, if a time t_0 is given such that $x(t_0)$ belongs to $\mathcal{R}_1(\mu)$ and if we let

$$T_\mu := \frac{\alpha_1(M\mu) - \alpha_2 \circ \rho(\Delta\mu)}{\alpha_3 \circ \rho(\Delta\mu)} \tag{26}$$

then $x(t_0 + T_\mu)$ is guaranteed to belong to $\mathcal{R}_2(\mu)$. □

The unforced system

$$\dot{x} = f(x, 0) \tag{27}$$

is called *forward complete* if for every initial state $x(0)$ the solution of (27), which we denote by $\xi(x(0), t)$, is defined for all $t \geqslant 0$. Our goal now is to show that this assumption, combined with the input-to-state stabilizability assumption stated earlier and a certain additional technical condition, allow one to extend the result expressed by Theorem 1 to the nonlinear system (15).

Theorem 2. *Assume that system* (27) *is forward complete and that we have*

$$\alpha_2^{-1} \circ \alpha_1(M\mu) > \max\{\rho(\Delta\mu), \chi(\mu) + 2\Delta\mu\} \quad \forall \mu > 0 \tag{28}$$

for some class $\mathcal{K}_\infty$ *function* χ. *Then there exists a hybrid quantized feedback control policy that makes system* (15) *globally asymptotically stable.*

Proof. As in the linear case, the control strategy is composed of two stages.

The "zooming-out" stage. Set the control equal to 0. Let $\mu(0) = 1$. Increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $|x(t)|$. For example, fix a positive number τ and let $\mu(t) = 1$ for $t \in [0, \tau)$, $\mu(t) = \chi^{-1}(2\max_{|x(0)|, t \leqslant \tau} |\xi(x(0), t)|)$ for $t \in [\tau, 2\tau)$, $\mu(t) = \chi^{-1}(2\max_{|x(0)|, t \leqslant 2\tau} |\xi(x(0), t)|)$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t \geqslant 0$ such that

$$|x(t)| \leqslant \chi(\mu(t)) < \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - 2\Delta\mu(t)$$

where the second inequality follows from (28). This implies

$$\left|\frac{x(t)}{\mu(t)}\right| < \frac{1}{\mu(t)}\alpha_2^{-1} \circ \alpha_1(M\mu(t)) - 2\Delta.$$

By virtue of condition 1 of Section 2 we have

$$\left|q\left(\frac{x(t)}{\mu(t)}\right)\right| \leqslant \frac{1}{\mu(t)}\alpha_2^{-1} \circ \alpha_1(M\mu(t)) - \Delta$$

which is equivalent to

$$|q_\mu(x(t))| \leqslant \alpha_2^{-1} \circ \alpha_1(M\mu(t)) - \Delta\mu(t). \tag{29}$$

Picking a time t_0 at which (29) holds and using conditions 1 and 2 of Section 2, we obtain

$$\left|\frac{x(t_0)}{\mu(t_0)}\right| \leqslant \frac{1}{\mu(t_0)}\alpha_2^{-1} \circ \alpha_1(M\mu(t_0))$$

hence $x(t_0)$ belongs to the set $\mathscr{R}_1(\mu(t_0))$ given by (23). This event can be detected solely on the basis of quantized measurements.

The "zooming-in" stage. We have established that $x(t_0)$ belongs to $\mathscr{R}_1(\mu(t_0))$. We will now use the control law (19). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T_{\mu(t_0)})$, where $T_{\mu(t_0)}$ is given by formula (26). Then $x(t_0 + T_{\mu(t_0)})$ will belong to the set $\mathscr{R}_2(\mu(t_0))$ given by (24). Calculate $T_{\omega(\mu(t_0))}$ using (26) again, where the function ω is defined as

$$\omega(r) := \frac{1}{M}\alpha_1^{-1} \circ \alpha_2 \circ \rho(\Delta r), \qquad r \geqslant 0.$$

For $t \in [t_0 + T_{\mu(t_0)},\ t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$, let

$$\mu(t) = \omega(\mu(t_0)).$$

We have $\omega(r) < r$ for all $r > 0$ by (28), thus $\mu(t_0 + T_{\mu(t_0)}) < \mu(t_0)$. One easily checks that $\mathscr{R}_2(\mu(t_0)) = \mathscr{R}_1(\mu(t_0 + T_{\mu(t_0)}))$. This means that we can continue the analysis and conclude that $x(t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$ belongs to $\mathscr{R}_2(\mu(t_0 + T_{\mu(t_0)}))$. We then repeat the procedure, letting $\mu = \omega(\mu(t_0 + T_{\mu(t_0)}))$ for the next time interval whose length is calculated from (26). Lyapunov stability of the equilibrium $x = 0$ of the continuous dynamics follows from the adjustment policy for μ as in the linear case. Moreover, we have $\mu(t) \to 0$ as $t \to \infty$, and the above analysis implies that $x(t) \to 0$ as $t \to \infty$. □

As in Section 3.1, we could pick a different sequence of switching times $t_1, t_2, \ldots$ as long as they satisfy $t_i - t_{i-1} \geqslant T_{\mu(t_{i-1})}$, $i \geqslant 1$. Similarly to the linear case, we could also implement event-based switching instead of using a dwell time; see Liberzon (2000).

Example. Consider the following system, which is a simplified version of the system treated in the example in Jiang, Mareels, and Hill (1999, p. 811):

$$\dot{x} = x^3 + xu, \qquad x, u \in \mathbb{R}.$$

In Jiang et al. (1999) it is shown how to construct a feedback law k such that the closed-loop system

$$\dot{x} = x^3 + xk(x + e)$$

is ISS with respect to e. It follows from the analysis of Jiang et al. (1999) that inequalities (16) and (17) hold with $V(x) = x^2/2$, $\alpha_1(r) = \alpha_2(r) = r^2/2$, $\alpha_3(r) = r^2$, and $\rho(r) = cr$ for an arbitrary $c > 1$. We have $(\alpha_2^{-1} \circ \alpha_1)(r) = r$, so (28) is valid for every $M > \Delta \max\{c, 2\}$.

In general, the requirement that the original system (15) be input-to-state stabilizable with the respect to the measurement error is quite restrictive for nonlinear systems. It is shown in Sontag (1989) that if an affine system of the form

$$\dot{x} = f(x) + G(x)u$$

is asymptotically stabilizable by a feedback law $u = k_0(x)$, then one can always find a feedback law $u = k(x)$ that makes the system

$$\dot{x} = f(x) + G(x)(k(x) + e)$$

ISS with respect to an *actuator* disturbance e. However, there might not exist a feedback law that makes the system

$$\dot{x} = f(x) + G(x)k(x + e)$$

ISS with respect to a *measurement* disturbance e, as was shown by way of counterexamples in Freeman (1995) and later in Fah (1999). Thus the problem of finding control laws that achieve ISS with respect to measurement disturbances is a nontrivial one, even for systems affine in controls (of course, for linear systems the distinction between the three notions mentioned above disappears). This problem has recently attracted considerable attention in the literature (Fah, 1999; Freeman & Kokotović, 1993; Jiang et al., 1999) and continues to be a subject of research efforts.

The technical assumption (28) also appears to be restrictive and hard to check. It depends on the relative growth of the functions α_1, α_2, and ρ. If the function $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$, where $\gamma(r) := \max\{\rho(\Delta r), \chi(r) + 2\Delta r\}$, is globally Lipschitz, then (28) is satisfied for every M greater than the Lipschitz constant. However, there is a weaker and more easily verifiable assumption which enables one to prove asymptotic stability in the case when a bound on the magnitude of the initial state is known (semiglobal asymptotic stability). To see how this works, take a positive number E_0 such that $|x(0)| \leqslant E_0$. Suppose that

$$(\alpha_1^{-1} \circ \alpha_2 \circ \rho)'(0) < \infty. \tag{30}$$

Then it is an elementary exercise to verify that for M sufficiently large we have

$$\alpha_2^{-1} \circ \alpha_1(M\mu) > \rho(\Delta\mu) \quad \forall \mu \in (0, 1]$$

and also

$$\alpha_2^{-1} \circ \alpha_1(M) \geqslant E_0.$$

Thus $x(0)$ belongs to the set $\mathscr{R}_1(1)$ defined by (23), the "zooming-out" stage is not necessary, and the "zooming-in" stage can be carried out as in the proof of Theorem 2, starting at $t_0 = 0$ and $\mu(0) = 1$. Forward completeness of the unforced system (27) is not required here.

If (30) does not hold, it is still possible to ensure that all solutions starting in a given compact set approach an arbitrary prespecified neighborhood of the origin (semiglobal practical stability). This is not difficult to show if the feedback law k is robust with respect to *small* measurement errors. All continuous stabilizing feedback laws possess such robustness, and discontinuous control laws for a large class of systems can also be shown to have this robustness property (Sontag, 1999). Therefore, in this context the assumption of ISS with respect to measurement disturbance inputs can be dropped altogether. For a detailed discussion of this topic, see Liberzon (2000).

In view of Sontag and Wang (1996, Lemma I.2), every asymptotically stabilizing feedback law is automatically input-to-state stabilizing with respect to the measurement error e locally, i.e., for sufficiently small values of x and e. This leads at once to local versions of the above results.

4. Input quantization

In this section we obtain results analogous to those given in Section 3 for systems whose input, rather than state, is quantized.

4.1. Linear systems

Consider the linear system (4). Suppose again that there exists a matrix K such that the eigenvalues of $A+BK$ have negative real parts, so that for some positive definite symmetric matrices P and Q Eq. (5) holds.

The state feedback law $u=Kx$ is not implementable because only quantized measurements $q_\mu(u)$ of the input u are available, where q_μ is defined by (3). We therefore consider the "certainty equivalence" quantized feedback control law

$$u=q_\mu(Kx). \tag{31}$$

This yields the closed-loop system

$$\dot{x}=Ax+B\mu q_\mu(x)=(A+BK)x + B\mu\left(q\left(\frac{Kx}{\mu}\right)-\frac{Kx}{\mu}\right). \tag{32}$$

The behavior of trajectories of (32) for a fixed value of μ is characterized as follows.

Lemma 3. *Fix an arbitrary $\varepsilon>0$ and assume that M is large enough compared to Δ so that we have*

$$\sqrt{\lambda_{\min}(P)}M > \sqrt{\lambda_{\max}(P)}\Theta_u\|K\|\Delta(1+\varepsilon),$$

where

$$\Theta_u := \frac{2\|PB\|}{\lambda_{\min}(Q)}.$$

Then the ellipsoids

$$\mathscr{R}_1(\mu) := \{x\colon x^{\mathrm{T}}Px \leqslant \lambda_{\min}(P)M^2\mu^2/\|K\|^2\} \tag{33}$$

and

$$\mathscr{R}_2(\mu) := \{x\colon x^{\mathrm{T}}Px \leqslant \lambda_{\max}(P)\Theta_u^2\Delta^2(1+\varepsilon)^2\mu^2\}$$

are invariant regions for system (32). Moreover, all solutions of (32) that start in the ellipsoid $\mathscr{R}_1(\mu)$ enter the smaller ellipsoid $\mathscr{R}_2(\mu)$ in finite time.

The proof is very similar to that of Lemma 1, and is omitted. An upper bound on the time to enter $\mathscr{R}_2(\mu)$ is

$$T := \frac{\lambda_{\min}(P)M^2-\lambda_{\max}(P)\Theta_u^2\|K\|^2\Delta^2(1+\varepsilon)^2}{\Theta_u^2\|K\|^2\Delta^2(1+\varepsilon)\lambda_{\min}(Q)\varepsilon}. \tag{34}$$

A hybrid quantized feedback control policy which combines the control law (31) with a switching strategy for μ is now obtained similarly to the state quantization case studied in the previous section.

Theorem 3. *Assume that M is large enough compared to Δ so that we have*

$$\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}M > 2\Delta\frac{\|PB\|\|K\|}{\lambda_{\min}(Q)}. \tag{35}$$

Then there exists a hybrid quantized feedback control policy that makes system (4) globally asymptotically stable.

Proof. At the "zooming-out" stage, set u equal to 0 and increase μ in a piecewise constant fashion as in the proof of Theorem 1. Then there will be a time $t_0\geqslant 0$ such that

$$|x(t_0)| \leqslant \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}\frac{M\mu(t_0)}{\|K\|}$$

which implies that $x(t_0)$ belongs to the ellipsoid $\mathscr{R}_1(\mu(t_0))$ given by (33). The "zooming-in" stage exactly parallels the one in the proof of Theorem 1, using formula (34). □

It is interesting to observe that in view of the inequality

$$\|PBK\| \leqslant \|PB\|\|K\|$$

condition (35) is in general more restrictive than the corresponding condition for the case of state quantization (see Theorem 1). On the other hand, the "zooming-in" stage for input quantization is more straightforward and does not require any additional assumptions. The remarks made after the proof of Theorem 1 concerning the exponential rate of convergence, robustness to time delays, and the alternative method of event-based switching carry over to the present case without any changes.

4.2. Nonlinear systems

Consider the nonlinear system (15). Assume that there exists a feedback law $u=k(x)$ that makes the closed-loop system globally asymptotically stable and, moreover, ensures that for some class $\mathscr{K}_\infty$ functions $\alpha_1,\alpha_2,\alpha_3,\rho$ there exists a smooth function $V:\mathbb{R}^n\to\mathbb{R}$ satisfying inequalities (16) and

$$|x|\geqslant\rho(|e|)\Rightarrow \nabla V(x)f(x,k(x)+e)\leqslant-\alpha_3(|x|)$$

for all $x,e\in\mathbb{R}^n$. According to the results of Sontag (1989) and Sontag and Wang (1995), this is equivalent to saying that the perturbed closed-loop system

$$\dot{x} = f(x, k(x) + e) \tag{36}$$

is ISS with respect to the actuator disturbance input e. Take κ to be some class $\mathscr{K}_\infty$ function with the property that

$$\kappa(r) \geqslant \max_{|x| \leqslant r} |k(x)| \quad \forall r \geqslant 0.$$

Then we have

$$|k(x)| \leqslant \kappa(|x|) \quad \forall x.$$

The closed-loop system with the "certainty equivalence" quantized feedback control law

$$u = q_\mu(k(x)) \tag{37}$$

becomes

$$\dot{x} = f(x, q_\mu(k(x))) \tag{38}$$

and this takes the form (36) with

$$e = q_\mu(k(x)) - x. \tag{39}$$

The behavior of trajectories of (38) for a fixed μ is characterized by the following result.

Lemma 4. *Assume that we have*

$$\alpha_1 \circ \kappa^{-1}(M\mu) > \alpha_2 \circ \rho(\Delta\mu).$$

Then the sets

$$\mathscr{R}_1(\mu) := \{x\colon V(x) \leqslant \alpha_1 \circ \kappa^{-1}(M\mu)\} \tag{40}$$

and

$$\mathscr{R}_2(\mu) := \{x\colon V(x) \leqslant \alpha_2 \circ \rho(\Delta\mu)\} \tag{41}$$

are invariant regions for system (38). *Moreover, all solutions of* (38) *that start in the set* $\mathscr{R}_1(\mu)$ *enter the smaller set* $\mathscr{R}_2(\mu)$ *in finite time.*

The proof parallels that of Lemma 2. An upper bound on the time to enter $\mathscr{R}_2(\mu)$ is

$$T_\mu := \frac{\alpha_1 \circ \kappa^{-1}(M\mu) - \alpha_2 \circ \rho(\Delta\mu)}{\alpha_3 \circ \rho(\Delta\mu)}. \tag{42}$$

As in Section 3.2, we will denote by $\xi(x(0), t)$ the solutions of the unforced system (27).

Theorem 4. *Assume that system* (27) *is forward complete and that we have*

$$\alpha_2^{-1} \circ \alpha_1 \circ \kappa^{-1}(M\mu) > \rho(\Delta\mu) \quad \forall \mu > 0. \tag{43}$$

Then there exists a hybrid quantized feedback control policy that makes system (15) *globally asymptotically stable.*

Proof. *The "zooming-out" stage*. Set the control to 0, and let $\mu(0) = 1$. Then increase μ in a piecewise constant fashion, fast enough to dominate the rate of growth of $|x(t)|$. For example, fix a positive number τ and let $\mu(t) = 1$ for $t \in [0, \tau)$, $\mu(t) = \rho^{-1}(2 \max_{|x(0)|, t \leqslant \tau} |\xi(x(0), t)|)/\Delta$ for $t \in [\tau, 2\tau)$, $\mu(t) = \rho^{-1}(2 \max_{|x(0)|, t \leqslant 2\tau} |\xi(x(0), t|)/\Delta$ for $t \in [2\tau, 3\tau)$, and so on. Then there will be a time $t_0 \geqslant 0$ such that

$$|x(t_0)| \leqslant \rho(\Delta\mu(t_0)) < \alpha_2^{-1} \circ \alpha_1 \circ \kappa^{-1}(M\mu(t_0))$$

hence $x(t_0)$ belongs to the set $\mathscr{R}_1(\mu(t_0))$ given by (40).

The "zooming-in" stage. For $t \geqslant t_0$ apply the control law (37). Let $\mu(t) = \mu(t_0)$ for $t \in [t_0, t_0 + T_{\mu(t_0)})$, where $T_{\mu(t_0)}$ is given by formula (42). Then $x(t_0 + T_{\mu(t_0)})$ belongs to the set $\mathscr{R}_2(\mu(t_0))$ given by (41). Use (42) again to compute $T_{\omega(\mu(t_0))}$, where ω is the function defined by

$$\omega(r) := \frac{1}{M} \kappa \circ \alpha_1^{-1} \circ \alpha_2 \circ \rho(\Delta r), \quad r \geqslant 0.$$

For $t \in [t_0 + T_{\mu(t_0)}, t_0 + T_{\mu(t_0)} + T_{\omega(\mu(t_0))})$, let

$$\mu(t) = \omega(\mu(t_0)).$$

We have $\mu(t_0 + T_{\mu(t_0)}) < \mu(t_0)$ by (43), and $\mathscr{R}_2(\mu(t_0)) = \mathscr{R}_1(\mu(t_0 + T_{\mu(t_0)}))$. The proof can now be completed exactly as the proof of Theorem 2. □

As we explained in Section 3.2, the requirement of ISS with respect to actuator errors is not as severe as that of ISS with respect to measurement errors. This means that for nonlinear systems the stabilization problem in the presence of input quantization is less challenging from the point of view of control design than the corresponding problem for state quantization. When condition (43) is not satisfied, weaker results can be obtained as described in Section 3.2.

Remark 1. It is relatively straightforward to extend the results obtained so far to systems with quantization affecting both the state and the input. For example, consider the linear system (4) with the control law

$$u = q_\mu^u(K q_\mu^x(x)),$$

where q^x is a state quantizer with range M_x and error Δ_x and q^u is an input quantizer with range M_u and error Δ_u. We are taking μ in both quantizers to be the same for simplicity, i.e., we are assuming that the two quantizers are changed synchronously. The closed-loop system can be written as

$$\dot{x} = (A + BK)x + BK\mu e_m + B\mu e_a,$$

where

$$e_m := q\left(\frac{x}{\mu}\right) - \frac{x}{\mu},$$

$$e_a := q\left(Kq\left(\frac{x}{\mu}\right)\right) - Kq\left(\frac{x}{\mu}\right).$$

It is easy to check that we have $|e_m| \leqslant \Delta_x$ and $|e_a| \leqslant \Delta_u$ provided that

$$|x| \leqslant \min\left\{M_x\mu, M_u\mu, \left(\frac{M_u}{\|K\|} - \Delta_u\right)\mu\right\}.$$

The analysis can then be carried out along the same lines as before. For nonlinear systems, we need to impose the assumption of input-to-state stabilizability with respect to both measurement and actuator errors.

5. Observer-based dynamic output feedback

We now extend some of the above results to linear systems with output feedback. The developments that follow are essentially based on the ideas from Brockett and Liberzon (2000, Section 5). Other approaches are also possible; see Delchamps (1989) and Sur and Paden (1998).

Consider the linear system

$$\dot{x} = Ax + Bu,$$
$$y = Cx, \tag{44}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. Suppose that (A, B) is a stabilizable pair and (C, A) is an observable pair. This implies that there exist a feedback matrix K and an output injection matrix L such that the eigenvalues of $A + BK$ and $A + LC$ have negative real parts. The eigenvalues of the matrix

$$\bar{A} := \begin{pmatrix} A + BK & -BK \\ 0 & A + LC \end{pmatrix}$$

then also have negative real parts, and so there exist positive definite symmetric $2n \times 2n$ matrices $\bar{P}$ and $\bar{Q}$ such that

$$\bar{A}^T \bar{P} + \bar{P}\bar{A} = -\bar{Q}.$$

In this section we are interested in the situation where only quantized measurements $q_\mu(y)$ of the output y are available, where q_μ is defined by (3). We therefore consider the following dynamic output feedback law, which is based on the standard Luenberger observer but uses $q_\mu(y)$ instead of y:

$$\dot{\hat{x}} = (A + LC)\hat{x} + Bu - Lq_\mu(y),$$
$$u = K\hat{x},$$

where $\hat{x} \in \mathbb{R}^n$. The closed-loop system takes the form

$$\dot{x} = Ax + BK\hat{x},$$
$$\dot{\hat{x}} = (A + LC)\hat{x} + BK\hat{x} - Lq_\mu(y).$$

In the coordinates given by

$$\bar{x} := \begin{pmatrix} x \\ x - \hat{x} \end{pmatrix} \in \mathbb{R}^{2n},$$

we can rewrite this system more compactly as

$$\dot{\bar{x}} = \bar{A}\bar{x} + L \begin{pmatrix} 0 \\ q_\mu(y) - y \end{pmatrix}. \tag{45}$$

For a fixed value of μ, the behavior of trajectories of system (45) is characterized by the following result.

Lemma 5. *Fix an arbitrary $\varepsilon > 0$ and assume that M is large enough compared to Δ so that we have*

$$\sqrt{\lambda_{\min}(\bar{P})}M > \sqrt{\lambda_{\max}(\bar{P})}\Theta_y \|C\|\Delta(1 + \varepsilon),$$

where

$$\Theta_y := \frac{2\|\bar{P}L\|}{\lambda_{\min}(\bar{Q})}.$$

Then the ellipsoids

$$\mathcal{R}_1(\mu) := \{\bar{x}\colon \bar{x}^{\mathrm{T}}\bar{P}\bar{x} \leqslant \lambda_{\min}(\bar{P})M^2\mu^2/\|C\|^2\} \tag{46}$$

and

$$\mathcal{R}_2(\mu) := \{\bar{x}\colon \bar{x}^{\mathrm{T}}\bar{P}\bar{x} \leqslant \lambda_{\max}(\bar{P})\Theta_y^2\Delta^2(1 + \varepsilon)^2\mu^2\}$$

are invariant regions for system (45). Moreover, all solutions of (45) that start in the ellipsoid $\mathcal{R}_1(\mu)$ enter the smaller ellipsoid $\mathcal{R}_2(\mu)$ in finite time.

The proof is similar to the proof of Lemma 1. An upper bound on the time to enter $\mathcal{R}_2(\mu)$ is

$$T := \frac{\lambda_{\min}(\bar{P})M^2 - \lambda_{\max}(\bar{P})\Theta_y^2\|C\|^2\Delta^2(1 + \varepsilon)^2}{\Theta_y^2\|C\|^2\Delta^2(1 + \varepsilon)\lambda_{\min}(\bar{Q})\varepsilon}. \tag{47}$$

A hybrid quantized feedback control policy of the next theorem combines the above dynamic output feedback law with the idea of updating the value of μ at discrete instants of time as before.

Theorem 5. *Assume that M is large enough compared to Δ so that we have*

$$\sqrt{\frac{\lambda_{\min}(\bar{P})}{\lambda_{\max}(\bar{P})}}M > \max\left\{3\Delta, 2\Delta\frac{\|\bar{P}L\|\|C\|}{\lambda_{\min}(\bar{Q})}\right\}. \tag{48}$$

Then there exists a hybrid quantized feedback control policy that makes system (44) globally asymptotically stable.

Proof. *The "zooming-out" stage*. Set u equal to 0. Increase μ in a piecewise constant fashion as before, starting from $\mu(0) = 1$, fast enough to dominate the rate of growth of $\|e^{At}\|$. Then there will be a time $t \geqslant 0$ such that

$$\left|\frac{y(t)}{\mu(t)}\right| \leqslant M - 3\Delta$$

(by (48), the right-hand side of this inequality is positive). In view of condition 1 imposed in Section 2, this implies

$$\left|q\left(\frac{y(t)}{\mu(t)}\right)\right| \leqslant M - 2\Delta$$

which is equivalent to

$$|q_\mu(y(t))| \leqslant M\mu(t) - 2\Delta\mu(t). \tag{49}$$

We can thus pick a time t_0 such that (49) holds with $t=t_0$. Fix an arbitrary $\delta > 0$ and let $\tau > 0$ be such that $\|e^{At}\| < 1+\delta$ for all $t \in [0,\tau]$. Define

$$\hat{x}(t_0) := W^{-1} \int_{t_0}^{t_0+\tau'} e^{A^T(t-t_0)} C^T \mu(t_0) q\left(\frac{y(t)}{\mu(t_0)}\right) dt, \quad (50)$$

where W denotes the (full-rank) observability Gramian $\int_0^{\tau'} e^{A^T t} C^T C e^{At}\, dt$ and τ' is the largest number in the interval $(0,\tau]$ such that

$$|q_\mu(y(t))| \leqslant M\mu(t_0) - \Delta\mu(t_0) \quad \forall t \in [t_0, t_0+\tau'].$$

In view of this and the equality

$$\int_{t_0}^{t_0+\tau'} e^{A^T(t-t_0)} C^T y(t)\, dt = Wx(t_0),$$

we have

$$|x(t_0) - \hat{x}(t_0)| \leqslant \|W^{-1}\|\tau(1+\delta)\|C\|\Delta\mu(t_0)$$

(recall that $\|C^T\| = \|C\|$). Defining $\hat{x}(t_0+\tau') := e^{A\tau'}\hat{x}(t_0)$, we obtain

$$|x(t_0+\tau') - \hat{x}(t_0+\tau')| \leqslant \|W^{-1}\|\tau(1+\delta)^2\|C\|\Delta\mu(t_0)$$

and hence

$$\begin{aligned}|\bar{x}(t_0+\tau')| &\leqslant |x(t_0+\tau')| + |x(t_0+\tau') - \hat{x}(t_0+\tau')| \\ &\leqslant |\hat{x}(t_0+\tau')| + 2|x(t_0+\tau') - \hat{x}(t_0+\tau')| \\ &\leqslant |\hat{x}(t_0+\tau')| + 2\|W^{-1}\|\tau(1+\delta)^2\|C\|\Delta\mu(t_0)\end{aligned}$$

Now, choose $\mu(t_0+\tau')$ large enough to satisfy

$$\sqrt{\frac{\lambda_{\min}(\bar{P})}{\lambda_{\max}(\bar{P})}} \frac{M\mu(t_0+\tau')}{\|C\|} \geqslant |\hat{x}(t_0+\tau')| + 2\|W^{-1}\|\tau(1+\delta)^2\|C\|\Delta\mu(t_0).$$

Then $\bar{x}(t_0+\tau')$ belongs to the ellipsoid $\mathscr{R}_1(\mu(t_0+\tau'))$ given by (46).

The "zooming-in" stage proceeds along the same lines as in the state quantization case, using the formula (47). □

The "zooming-out" stage in the above proof is somewhat more complicated than in the state quantization case. Note, however, that the integral in (50) is easy to compute (in closed form) because the function being integrated is the product of a matrix exponential and a piecewise constant function.

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