Observability implies Observer Design for Switched Linear Systems

Aneel Tanwani[*]
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign, USA
tanwani2@illinois.edu

Hyungbo Shim[†]
ASRI, School of Electrical Engineering
Seoul National University, Korea
hshim@snu.ac.kr

Daniel Liberzon[*]
Coordinated Science Laboratory
University of Illinois at Urbana-Champaign, USA
liberzon@illinois.edu

ABSTRACT

This paper presents a unified framework for observability and observer design for a class of hybrid systems. A necessary and sufficient condition is presented for observability, globally in time, when the system evolves under predetermined mode transitions. A relatively weaker characterization is given for determinability, the property that concerns with unique recovery of the state at some time rather than at all times. These conditions are then utilized in the construction of a hybrid observer that is feasible for implementation in practice. The observer, without using the derivatives of the output, generates the state estimate that converges to the actual state under persistent switching.

Keywords

Switched linear systems, observability, observer design

1. INTRODUCTION

This paper studies observability conditions and observer construction for a class of hybrid systems where the continuous dynamics are modeled as linear differential equations; the state trajectories exhibit jumps during their evolution; and discrete dynamics are represented by an exogenous switching signal. Often called *switched systems*, they are described mathematically as:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \qquad t \neq \{t_q\}, \tag{1a}$$

$$x(t_q) = E_{\sigma(t_i^-)}x(t_i^-) + F_{\sigma(t_q^-)}v_q, \tag{1b}$$

$$y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t), \tag{1c}$$

where $x \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^{d_y}$ is the output, $v_q \in \mathbb{R}^{d_v}$ and $u(t) \in \mathbb{R}^{d_u}$ are the inputs, and $u(\cdot)$ is a measurable function. The dimension of the external signals denoted by d_u, d_v, and d_y for the inputs and the output, respectively, may vary for each mode, but we just treat them constant for convenience. The switching signal $\sigma : \mathbb{R} \mapsto \mathbb{N}$ is a piecewise constant and right-continuous function that changes its value at switching times $\{t_q\}$, $q \in \mathbb{N}$. It is assumed that there are a finite number of switching times in any finite time interval, thus we rule out the Zeno phenomenon in our problem formulation. The switching mode $\sigma(t)$ and the switching times $\{t_q\}$ may be governed by a supervisory logic controller, or determined internally depending on the system state, or considered as an external input. In any case, it is assumed in this paper that the signal $\sigma(\cdot)$ (and thus, the active mode and the switching time $\{t_q\}$ as well) is known. For estimation of the switching signal $\sigma(t)$, one may be referred to, e.g., [4, 7, 14, 15].

In the past decade, the structural properties of hybrid systems have been investigated by many researchers and observability along with observer construction has been one of them. In hybrid systems, the observability can be studied from various perspectives. If we allow for the use of the differential operator in the observer, then it may be desirable to determine the continuous state of the system instantaneously from the measured output. This in turn requires each subsystem to be observable, however, the problem becomes nontrivial when the switching signal is treated as a discrete state and simultaneous recovery of the discrete and continuous state is required for observability. Some results on this problem are published in [2, 6, 14].

On the other hand, if the mode transitions are represented by a known switching signal then, even though the individual subsystems are not observable, it is still possible to recover the initial state $x(t_0)$ when the output is observed over an interval $[t_0, T)$ that involves multiple switching instants. This phenomenon is of particular interest for switched systems as the notion of instantaneous observability and observability over an interval[1] coincide for linear time invariant systems. This variant of the observability in switched systems has been studied most notably by [3, 12, 17]. The authors in [8, 9] have studied the observability problem for the systems that allow jumps in the states but they do not consider the change in the dynamics that is introduced by switching to different matrices associated with the active mode. The observer design has also received some attention

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[1]See Definition 1 for precise meaning.

in the literature [1, 4, 10], where authors have assumed that each mode in the system is in fact observable admitting a state observer, and have treated the switching as a source of perturbation effect. This approach immediately incurs the need of a common Lyapunov function for the switched error dynamics, or a fixed amount of dwell-time between switching instants, because it is intrinsically a stability problem of the error dynamics.

The approach adopted in this paper is similar to [3, 17] in the sense that we consider observability over an interval. The authors in [3] have presented a coordinate dependent sufficient condition that leads to observer construction; the work of [17] primarily addresses the question whether there exists a switching signal which makes it possible to recover $x(t_0)$ from the knowledge of the output. Whereas, in this paper, similar to our recent work in [13], the switching signal is considered to be known and fixed, so that the trajectory of the system satisfies a set of time varying linear differential equations. Then for that particular trajectory, we answer the question whether it is possible to recover $x(t_0)$ from the knowledge of the measured output. We present a necessary and sufficient condition for observability over an interval that can be verified without any coordinate transformation. Since this condition depends upon the switching times and requires computation of the state transition matrices, we also provide easily verifiable conditions that are either necessary or sufficient for the main condition. Also, with a similar tool set, the notion of determinability, which is more in the spirit of recovering the current state based on the knowledge of inputs and outputs in the past, is developed. Moreover, a hybrid observer for system (1) is designed based on the proposed necessary and sufficient condition which was not the case in [17]. Since the observers are useful for various engineering applications, their utility mainly lies in their online operation method. This thought is essentially rooted in the idea for observer construction adopted in this paper: the idea of combining the partial information available from each mode and collecting them at one instance of time to get the estimate of the state. We show that under mild assumptions, such an estimate converges to the actual state of the plant. We remark that the main contribution of this paper is to present a unified framework of observability and observer design for the most general class of linear switched systems that has not been discussed in the literature, to the best of authors' knowledge.

More emphasis will be given to the case when the individual modes of the system (1) are not observable (in the classical sense of linear time-invariant systems theory) since it is obvious that the system becomes immediately observable when the system is switched to the observable mode. In order to facilitate our understanding, let us begin with an example.

Example 1. Consider a switched system characterized by

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

with $E_i = I$, $F_i = 0$, $B_i = 0$, and $D_i = 0$ for $i \in \{1, 2\}$. It is noted that the pair (A_i, C_i) is not observable for either mode $i = 1, 2$. However, if the switching signal $\sigma(t)$ changes its value in the order of $1 \to 2 \to 1$ at times t_1 and t_2, then we can recover the state. In fact, it turns out that at least two switchings are necessary and the switching sequence should contain the subsequence of modes $\{1, 2, 1\}$. For instance, if the switching happens as $1 \to 2 \to 1$, the output y at time t_1^- (just before the first switching) and t_2 (just after the second switching) are: $y(t_1^-) = x_1(t_0)$, and $y(t_2) = [1, 0]e^{A_2\tau}x(t_0) = \cos\tau \cdot x_1(t_0) + \sin\tau \cdot x_2(t_0)$, where $x(t_0) = [x_1(t_0), x_2(t_0)]^\top$ is the initial condition and $\tau = t_2 - t_1$. Then, it is obvious that $x(t_0)$ can be recovered from two measurements $y(t_1^-)$ and $y(t_2)$ if $\tau \neq k\pi$ with $k \in \mathbb{N}$. On the other hand, any switching signal whose duration for the mode 2 is an integer multiple of π is a 'singular' input (meaning the input that destroys observability).

Notation: For a square matrix A and a subspace $\mathcal{V}$, we denote by $\langle A|\mathcal{V}\rangle$ the smallest A-invariant subspace containing $\mathcal{V}$, and by $\langle \mathcal{V}|A\rangle$ the largest A-invariant subspace contained in $\mathcal{V}$. (See Property 7 in the Appendix for their computation.) For a possibly non-invertible matrix A, the subspace $A^{-1}\mathcal{V} := \{x : Ax \in \mathcal{V}\}$ and $A^{-\top}\mathcal{V} := (A^\top)^{-1}\mathcal{V}$, where $A^\top$ is the transpose of A. Similarly, it is understood that $A_2^{-1}A_1^{-1}\mathcal{V} = A_2^{-1}(A_1^{-1}\mathcal{V})$. Note that $A^{-1}\ker C = \ker(CA)$ for a matrix C. For convenience, we denote the products of matrices A_i as $\prod_{i=j}^{k} A_i := A_j A_{j+1} \cdots A_k$ when $j < k$, and $\prod_{i=j}^{k} A_i := A_j A_{j-1} \cdots A_k$ when $j > k$.

2. GEOMETRIC CONDITIONS FOR OBSERVABILITY

To make precise the notions of observability and determinability considered in this paper, let us introduce the formal definitions.

Definition 1. Let $(\sigma^i, u^i, v^i, y^i)$, for $i = 1, 2$, be any set of signals over an interval[2] $[t_0, T^+)$, and let x^i denote the resulting state trajectory that solves (1). We say that the system (1) is $[t_0, T^+)$-*observable* if the equality $(\sigma^1, u^1, v^1, y^1) = (\sigma^2, u^2, v^2, y^2)$ implies that $x^1(t_0) = x^2(t_0)$. Similarly, the system (1) is said to be $[t_0, T^+)$-*determinable* if the equality $(\sigma^1, u^1, v^1, y^1) = (\sigma^2, u^2, v^2, y^2)$ implies that $x^1(T) = x^2(T)$.

Since the initial state $x(t_0)$ and the inputs (u, v) uniquely determine $x(t)$ on $[t_0, T^+)$ through equation (1), observability is achieved if and only if the state trajectory $x(t)$, for each $t \in [t_0, T^+)$, is uniquely determined by the inputs and the output. Obviously, observability implies determinability by forward integration of (1), but the converse is not true due to the possibility of non-invertible matrices E_σ. In case there are no jumps in the state trajectory, or the jump maps are invertible, then observability and determinability are equivalent. The notion of determinability has also been called reconstructability in [12].

Proposition 1. For a fixed switching signal σ, the system (1) is $[t_0, T^+)$-observable (or, determinable) if, and only if, zero inputs and zero output on the interval $[t_0, T^+)$ imply that $x(t_0) = 0$ (or, $x(T) = 0$).

[2] The notation $[t_0, T^+)$ is used to denote the interval $[t_0, T + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small. In fact, because of the right continuity of the switching signal, the output $y(T)$ belongs to the next mode when T is the switching instant. Then, the point-wise measurement $y(T)$ is insufficient to contain the information for the new mode, and thus, it is imperative to consider the output signal over the interval $[t_0, T + \varepsilon)$ with $\varepsilon > 0$. This definition implicitly implies that the observability property does not change for sufficiently small ε (which is true, and becomes clear shortly).

PROOF. Since the zero solution with the zero inputs yields the zero output, the necessity follows from the fact that $x(t_0)$ (or, $x(T)$) is uniquely determined from the inputs and the outputs. For the sufficiency, suppose that the system (1) is not $[t_0, T^+)$-observable (or determinable); that is, there exist two different states $x^1(t_0)$ and $x^2(t_0)$ (or, $x^1(T)$ and $x^2(T)$) that yield the same output y under the same inputs (u, v). Let $\tilde{x}(t) := x^1(t) - x^2(t)$, where $x^i(t)$, $i = 1, 2$, is the solution of (1) with the initial condition $x^i(t_0)$. Then, by linearity, it follows that $\dot{\tilde{x}} = A_\sigma \tilde{x}$, $\tilde{x}(t_q) = E_\sigma \tilde{x}(t_q^-)$, and $C_\sigma \tilde{x} = C_\sigma x^1 - C_\sigma x^2 = y - y = 0$, but $\tilde{x}(t_0) = x^1(t_0) - x^2(t_0) \neq 0$ (or, $\tilde{x}(T) = x^1(T) - x^2(T) \neq 0$). Hence, zero inputs and zero output do not imply $x(t_0) = 0$ (or, $x(T) = 0$), and the sufficiency holds. $\square$

Because of Propostion 1, we are motivated to introduce the following homogeneous switched ODE, which has been obtained by setting the inputs (u, v) equal to zero in (1).

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad y(t) = C_{\sigma(t)} x(t), \quad t \in [t_{q-1}, t_q) \tag{2a}$$

$$x(t_q) = E_{\sigma(t)} x(t_q^-). \tag{2b}$$

If this homogenous system is observable (or, determinable), then $y \equiv 0$ implies that $x(t_0) = 0$ (or, $x(T) = 0$) and in terms of description of system (1), it means that zero inputs and zero output give $x(t_0) = 0$ (or, $x(T) = 0$); hence, (1) is observable/determinable because of Proposition 1. On the other hand, if the system (1) is observable/determinable, then it is still observable/determinable with zero inputs, which is described as system (2). Thus, the observability/determinability of systems (1) and (2) are equivalent.

Before going further, let us rename the switching sequence for convenience. For system (1), when the switching signal $\sigma(t)$ takes the mode sequence $\{q_1, q_2, q_3, \cdots\}$, we rename them as increasing integers $\{1, 2, 3, \cdots\}$, which is ever increasing even though the same mode is revisited; for convenience, this sequence is indexed by q and not $\sigma(t)$. Moreover, it is often the case that the mode of the system changes without the state jump (1b), or the state jumps without switching to another mode. In the former case, we can simply take $E_q = I$, and in the latter case, we increase the mode index by one and take $A_q = A_{q+1}$ and so on. In this way various situations fit into the description of (1) with increasing mode sequence. The switching time t_q is the instant when transition from mode q to mode $q + 1$ takes place.

2.1 Necessary and Sufficient Conditions for Observability

In this section, we present a characterization of the unobservable subspace for the system (2) with a fixed switching signal. Towards this end, let $\mathcal{N}_q^m$ $(m \geq q)$ denote the set of states at $t = t_{q-1}$ for system (2) that generate identically zero output over $[t_{q-1}, t_{m-1}^+)$. Then, it is easily seen that $\mathcal{N}_q^m$ is actually a subspace due to linearity of (2), and we call $\mathcal{N}_q^m$ the *unobservable subspace for* $[t_{q-1}, t_{m-1}^+)$. It is observed that the system (2) is an LTI system between two consecutive switching times, so that its unobservable subspace on the interval $[t_{q-1}, t_q)$ is simply given by the largest A_q-invariant subspace contained in $\ker C_q$, i.e., $\langle \ker C_q | A_q \rangle = \ker G_q$ where $G_q := \mathrm{col}(C_q, C_q A_q, \cdots, C_q A_q^{n-1})$. So it is clear that $\mathcal{N}_q^q = \ker G_q$. Now, when the measured output is available over the interval $[t_{q-1}, t_{m-1}^+)$ that includes switchings at $t_q, t_{q+1}, \ldots, t_{m-1}$, more information about the state is obtained in general so that $\mathcal{N}_q^m$ gets smaller as the difference $m - q$ gets larger, and we claim that the subspace $\mathcal{N}_q^m$, in that case, is given by

$$\mathcal{N}_q^m = \ker G_q \cap \left(\bigcap_{i=q+1}^{m} \prod_{j=q}^{i-1} e^{-A_j \tau_j} E_j^{-1} \ker G_i \right) \tag{3a}$$

$$= \ker G_q \cap \left(\bigcap_{i=q+1}^{m} \ker \left(G_i \prod_{l=i-1}^{q} E_l e^{A_l \tau_l} \right) \right) \tag{3b}$$

where $\tau_j = t_j - t_{j-1}$. The following theorem presents a necessary and sufficient condition for observability of the system (1) while proving the claim in the process.

THEOREM 1. For the system (2) with a switching signal $\sigma_{[t_0, t_{m-1}^+)}$, the unobservable subspace for $[t_0, t_{m-1}^+)$ at t_0 is given by $\mathcal{N}_1^m$ of (3). Therefore, the system (1) is $[t_0, t_{m-1}^+)$-observable if, and only if,

$$\mathcal{N}_1^m = \{0\}. \tag{4}$$

In case the interval under consideration is not finite and the switching is persistent, observability of system (1) is determined by whether there exists an $m \in \mathbb{N}$ such that (4) holds.

REMARK 1. From (3), it is not difficult to arrive at the following recursive formula for $\mathcal{N}_1^m$:

$$\begin{aligned} \mathcal{N}_m^m &= \ker G_m, \\ \mathcal{N}_q^m &= \ker G_q \cap e^{-A_q \tau_q} E_q^{-1} \mathcal{N}_{q+1}^m, \quad 1 \leq q \leq m - 1. \end{aligned} \tag{5}$$

PROOF OF THEOREM 1. *Sufficiency.* Using the result of Proposition 1, it suffices to show that the identically zero output of (2) can only be produced by $x(t_0) = 0$. Assume that $y \equiv 0$ on $[t_0, t_{m-1}^+)$. Then, it is immediate that $x(t_{m-1}) \in \mathcal{N}_m^m = \ker G_m$. We next apply the inductive argument to show that $x(t_{q-1}) \in \mathcal{N}_q^m$ for $1 \leq q \leq m - 1$. Suppose that $x(t_q) \in \mathcal{N}_{q+1}^m$, then $x(t_{q-1}) \in e^{-A_q \tau_q} E_q^{-1} \mathcal{N}_{q+1}^m$ since $x(t)$ is the solution of (2). Zero output on the interval $[t_{q-1}, t_q)$ implies that $x(t_{q-1}) \in \ker G_q$. Therefore,

$$x(t_{q-1}) \in \ker G_q \cap e^{-A_q \tau_q} E_q^{-1} \mathcal{N}_{q+1}^m.$$

From (5), it follows that $x(t_{q-1}) \in \mathcal{N}_q^m$. In particular, $x(t_0) \in \mathcal{N}_1^m = \{0\}$, so $x(t) \equiv 0$, $t \in [t_0, t_{m-1}^+)$.

Necessity. Assuming that $\mathcal{N}_1^m \neq \{0\}$, we show that a nonzero initial state $x(t_0) \in \mathcal{N}_1^m$ yields the solution $x(\cdot)$ of (2) such that $y \equiv 0$, which implies unobservability. First, we show the following implication;

$$x(t_{q-1}) \in \mathcal{N}_q^m \quad \Rightarrow \quad x(t_q) \in \mathcal{N}_{q+1}^m, \qquad q < m. \tag{6}$$

Indeed, assuming that $x(t_{q-1}) \in \mathcal{N}_q^m$ with $q < m$, it follows that, $x(t_q) = E_q e^{A_q \tau_q} x(t_{q-1})$, which further gives,

$$\begin{aligned} x(t_q) &\in E_q e^{A_q \tau_q} \mathcal{N}_q^m \\ &= E_q e^{A_q \tau_q} \left(\ker G_q \cap e^{-A_q \tau_q} E_q^{-1} \mathcal{N}_{q+1}^m \right) \\ &\subseteq E_q \ker G_q \cap E_q E_q^{-1} \mathcal{N}_{q+1}^m \\ &= E_q \ker G_q \cap \mathcal{N}_{q+1}^m \cap \mathcal{R}(E_q) \subseteq \mathcal{N}_{q+1}^m \end{aligned}$$

by using (5) and Properties 2, 3, and 11 in the Appendix. Therefore, for $0 \leq q \leq m - 1$, $x(t_q) \in \mathcal{N}_{q+1}^m \subseteq \ker G_{q+1}$, and the solution $x(t) = e^{A_{q+1}(t-t_q)}x(t_q)$ for $t \in [t_q, t_{q+1})$ satisfies that $y(t) = C_{q+1}x(t) = 0$ for $t \in [t_q, t_{q+1})$ due to A_{q+1}-invariance of $\ker G_{q+1}$. $\square$

In order to test the observability of the system (2), one can compute $\mathcal{N}_1^m$ by (3) (the formula (3b) may be preferable because the computation of pre-image due to E_j^{-1} is avoided). The observability condition given in Theorem 1 is dependent upon a particular switching signal under consideration, and it is entirely possible that the system is observable for certain switching signals and unobservable for others (*cf.* Example 1). For a predetermined family of subsystems, if there is a switching signal for which (4) holds, we call it a 'regular' switching signal, whereas the term 'singular' switching signal denotes one for which (4) does not hold. Also, in practice, the computation of matrix exponent is heavy (especially for large dimensional systems) and one may resort to the following sufficient, or necessary conditions, which are independent of switching times and only take mode sequence into consideration. Hence, once the sufficient condition in Corollary 1 holds (respectively, the necessary condition in Corollary 2 is violated), then the system is observable (resp. unobservable) for any switching signal that has the same switching sequence regardless of the switching times.

COROLLARY 1. Let $\overline{\mathcal{N}}_1^m$ be an over-approximation of $\mathcal{N}_1^m$ that is defined as follows:

$$\overline{\mathcal{N}}_m^m := \ker G_m,$$
$$\overline{\mathcal{N}}_q^m := \left\langle A_q \middle| \ker G_q \cap E_q^{-1}\overline{\mathcal{N}}_{q+1}^m \right\rangle, \qquad 1 \le q \le m-1.$$

The system (1) is $[t_0, t_{m-1}^+)$-observable if $\overline{\mathcal{N}}_1^m = \{0\}$.

PROOF. Proof is completed by showing that $\mathcal{N}_q^m \subseteq \overline{\mathcal{N}}_q^m$ for $1 \le q \le m$. First, note that $\mathcal{N}_m^m = \overline{\mathcal{N}}_m^m$. Assuming that $\mathcal{N}_{q+1}^m \subseteq \overline{\mathcal{N}}_{q+1}^m$ for $1 \le q \le m-1$, we now claim that $\mathcal{N}_q^m \subseteq \overline{\mathcal{N}}_q^m$. Indeed, by Properties 3, 9, and 11 in the Appendix, and the recursion equation (5), we obtain

$$\begin{aligned}
\mathcal{N}_q^m &= \ker G_q \cap e^{-A_q\tau_q}E_q^{-1}\mathcal{N}_{q+1}^m \\
&= e^{-A_q\tau_q}\left(\ker G_q \cap E_q^{-1}\mathcal{N}_{q+1}^m\right) \\
&\subseteq \left\langle A_q \middle| \ker G_q \cap E_q^{-1}\mathcal{N}_{q+1}^m \right\rangle \\
&\subseteq \left\langle A_q \middle| \ker G_q \cap E_q^{-1}\overline{\mathcal{N}}_{q+1}^m \right\rangle = \overline{\mathcal{N}}_q^m, \quad 1 \le q \le m-1.
\end{aligned}$$

Therefore, the condition $\overline{\mathcal{N}}_1^m = \{0\}$ implies (4). $\square$

COROLLARY 2. Let $\underline{\mathcal{N}}_1^m$ be an under-approximation of $\mathcal{N}_1^m$ that is defined as follows:

$$\underline{\mathcal{N}}_m^m := \ker G_m,$$
$$\underline{\mathcal{N}}_q^m := \left\langle \ker G_q \cap E_q^{-1}\underline{\mathcal{N}}_{q+1}^m \middle| A_q \right\rangle, \qquad 1 \le q \le m-1.$$

The system (1) is $[t_0, t_{m-1}^+)$-observable only if $\underline{\mathcal{N}}_1^m = \{0\}$.

PROOF. Proof proceeds similar to Corollary 1. With $\mathcal{N}_m^m = \underline{\mathcal{N}}_m^m$, we assume that $\mathcal{N}_{q+1}^m \supseteq \underline{\mathcal{N}}_{q+1}^m$ for $1 \le q \le m-1$, and claim that $\mathcal{N}_q^m \supseteq \underline{\mathcal{N}}_q^m$. Again by Properties 3, 9, and 11 in the Appendix, and employing equation (5), we obtain

$$\begin{aligned}
\mathcal{N}_q^m &= e^{-A_q\tau_q}\left(\ker G_q \cap E_q^{-1}\mathcal{N}_{q+1}^m\right) \\
&\supseteq \left\langle \ker G_q \cap E_q^{-1}\mathcal{N}_{q+1}^m \middle| A_q \right\rangle \\
&\supseteq \left\langle \ker G_q \cap E_q^{-1}\underline{\mathcal{N}}_{q+1}^m \middle| A_q \right\rangle = \underline{\mathcal{N}}_q^m, \quad 1 \le q \le m-1.
\end{aligned}$$

The condition $\underline{\mathcal{N}}_1^m = \{0\}$ is implied by (4). $\square$

REMARK 2. By taking orthogonal complements of $\mathcal{N}_q^m$, $\overline{\mathcal{N}}_q^m$ and $\underline{\mathcal{N}}_q^m$, respectively, we get dual conditions, using Properties 5, 6, 8, and 10 in the Appendix, as follows. The system (1) is $[t_0, t_{m-1}^+)$-observable if and only if $\mathcal{P}_1^m = \mathbb{R}^n$ where

$$\mathcal{P}_1^m := (\mathcal{N}_1^m)^\perp = \mathcal{R}(G_1^\top) + \sum_{i=2}^{m}\prod_{j=1}^{i-1} e^{A_j^\top\tau_j}E_j^\top\mathcal{R}(G_j^\top).$$

Based on the above definition, one can state Corollary 1 and Corollary 2 in alternate forms. System (1) is $[t_0, t_{m-1}^+)$-observable if $\underline{\mathcal{P}}_1^m = \mathbb{R}^n$, where $\underline{\mathcal{P}}_1^m$ is computed as:

$$\underline{\mathcal{P}}_m^m = (\overline{\mathcal{N}}_m^m)^\perp = \mathcal{R}(G_m^\top)$$
$$\underline{\mathcal{P}}_q^m = (\overline{\mathcal{N}}_q^m)^\perp = \left\langle \mathcal{R}(G_q^\top) + E_q^\top\underline{\mathcal{P}}_{q+1}^m \middle| A_q^\top \right\rangle, \quad 1 \le q \le m-1.$$

Also, the system (1) is $[t_0, t_{m-1}^+)$-observable only if $\overline{\mathcal{P}}_1^m = \mathbb{R}^n$, where $\overline{\mathcal{P}}_1^m$ is defined sequentially as:

$$\overline{\mathcal{P}}_m^m = (\underline{\mathcal{N}}_m^m)^\perp = \mathcal{R}(G_m^\top)$$
$$\overline{\mathcal{P}}_q^m = (\underline{\mathcal{N}}_q^m)^\perp = \left\langle A_q^\top \middle| \mathcal{R}(G_q^\top) + E_q^\top\overline{\mathcal{P}}_{q+1}^m \right\rangle, \quad 1 \le q \le m-1.$$

2.2 Necessary and Sufficient Conditions for Determinability

In order to study determinability of the system (1) and arrive at a result parallel to Theorem 1, our first goal is to develop an object similar to $\mathcal{N}_q^m$. So, for system (2) with a given switching signal, let $\mathcal{Q}_q^m$ be the set of states that can be reached at time $t = t_{m-1}$ while producing the zero output on the interval $[t_{q-1}, t_{m-1}^+)$. We call $\mathcal{Q}_q^m$ the *undeterminable subspace for* $[t_{q-1}, t_{m-1}^+)$. Then, it can be shown, similarly to the proof of Theorem 1, that $\mathcal{Q}_q^m$ is computed as:

$$\begin{aligned}
\mathcal{Q}_q^m = \ker G_m \cap E_{m-1}\ker(G_{m-1}) \cap& \\
\left(\bigcap_{i=q}^{m-2}\prod_{l=m-1}^{i+1} E_l e^{A_l\tau_l}E_i\ker G_i\right)&,
\end{aligned} \tag{7}$$

with $\mathcal{Q}_q^q = \ker G_q$. In the above equation, the subspace $(\Pi_{l=m-1}^{i+1}E_l e^{A_l\tau_l}E_i\ker G_i)$ indicates the set of states at time $t = t_{m-1}$ obtained by propagating the unobservable state of the mode i, that is active during the interval $[t_{i-1}, t_i)$, under the dynamics of system (2). Intersection of these subspaces with $\ker G_m$ shows that $\mathcal{Q}_q^m$ is the set of states that cannot be determined from the zero output at time $t = t_{m-1}$. Then, the determinability can be characterized as in the following theorem (which is given without proof).

THEOREM 2. For the system (2) and a given switching signal $\sigma_{[t_0, t_{m-1}^+)}$, the undeterminable subspace for $[t_0, t_{m-1}^+)$ at t_{m-1} is given by $\mathcal{Q}_1^m$ of (7). Therefore, the system (1) is $[t_0, t_{m-1}^+)$-determinable if and only if

$$\mathcal{Q}_1^m = \{0\}. \tag{8}$$

The condition (8) is equivalent to (4) when all E_q matrices, $q = 1, \ldots, m-1$, are invertible because of the relation

$$\mathcal{Q}_1^m = \prod_{l=m-1}^{1} E_l e^{A_l\tau_l}\mathcal{N}_1^m.$$

On the other hand, if any of the jump maps E_q is a zero matrix, then (8) holds regardless of (4) (which makes sense because we can immediately determine that $x(t_{m-1}) = 0$ in this case).

A recursive expression for $\mathcal{Q}_1^m$ is again possible as

$$\mathcal{Q}_1^1 = \ker G_1$$
$$\mathcal{Q}_1^q = \ker G_q \cap E_{q-1} e^{A_{q-1}\tau_{q-1}} \mathcal{Q}_1^{q-1}, \qquad 2 \le q \le m.$$

An important observation is that the sequence $\{\mathcal{Q}_1^q\}_{q=1}^m$ is moving forward in time and the next element of the sequence is obtained when the system switches to another mode. This is a major difference in computation of $\{\mathcal{Q}_1^q\}$ when comparing it with $\{\mathcal{N}_q^m\}$, as the computation of the latter requires the knowledge of mode sequence and switching times from the future. This makes the computation of $\mathcal{Q}_1^q$ more feasible for online implementation.

Corollary 3. The system (1) is $[t_0, t_{m-1}^+)$-determinable if $\overline{\mathcal{Q}}_1^m = \{0\}$, where $\overline{\mathcal{Q}}_1^m$ is computed by

$$\overline{\mathcal{Q}}_1^m := \ker G_1$$
$$\overline{\mathcal{Q}}_1^q := E_{q-1} \left\langle A_{q-1} | \overline{\mathcal{Q}}_1^{q-1} \right\rangle \cap \ker G_q, \quad 2 \le q \le m.$$

Corollary 4. The system (1) is $[t_0, t_{m-1}^+)$-determinable only if $\underline{\mathcal{Q}}_1^m = \{0\}$, where $\underline{\mathcal{Q}}_1^m$ is computed by

$$\underline{\mathcal{Q}}_1^1 := \ker G_1$$
$$\underline{\mathcal{Q}}_1^q := E_{q-1} \left\langle \underline{\mathcal{Q}}_1^{q-1} | A_{q-1} \right\rangle \cap \ker G_q, \quad 2 \le q \le m.$$

The above corollaries are proved by showing that $\underline{\mathcal{Q}}_1^q \subseteq \mathcal{Q}_1^q \subseteq \overline{\mathcal{Q}}_1^q$. It is noted again that the computation of sequential subspaces in Corollary 3 and Corollary 4 proceeds forward in time.

Remark 3. An alternative dual characterization of determinability is possible by inspecting whether the complete state information is available while going forward in time. This is achieved in terms of the subspace $\mathcal{M}_q^m$, obtained by taking the orthogonal complement of $\mathcal{Q}_q^m$. Using Properties 5, 6, 8, and 10 in the Appendix, the following expression follows from (7):

$$\mathcal{M}_q^m := (\mathcal{Q}_q^m)^\perp = \sum_{i=q}^{m-2} \prod_{l=m-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top} \mathcal{R}(G_i^\top) + E_{m-1}^{-\top} \mathcal{R}(G_{m-1}^\top) + \mathcal{R}(G_m^\top). \tag{9}$$

In other words, $\mathcal{M}_q^m$ is the set of states at time instant $t = t_{m-1}$ that can be identified, modulo the unobservable subspace at t_{m-1}, from the information of $y(\cdot)$ over the interval $[t_{q-1}, t_{m-1}^+)$. Therefore, the dual statement for determinability is that the system (1) is $[t_0, t_{m-1}^+)$-determinable if and only if

$$\mathcal{M}_1^m = \mathbb{R}^n. \tag{10}$$

It is noted that a recursive expression for $\mathcal{M}_1^m$ is given by

$$\mathcal{M}_1^1 = \mathcal{R}(G_1^\top)$$
$$\mathcal{M}_1^q = E_{q-1}^{-\top} e^{-A_{q-1}^\top \tau_{q-1}} \mathcal{M}_1^{q-1} + \mathcal{R}(G_q^\top), \quad 2 \le q \le m,$$

and the dual statements of Corollaries 3 and 4, that are independent of switching times, are given as follows: system (1) is $[t_0, t_{m-1}^+)$-determinable if $\underline{\mathcal{M}}_1^m = \mathbb{R}^n$, where

$$\underline{\mathcal{M}}_1^1 := (\overline{\mathcal{Q}}_1^m)^\perp = \mathcal{R}(G_1^\top),$$
$$\underline{\mathcal{M}}_1^q := (\overline{\mathcal{Q}}_1^q)^\perp = E_{q-1}^{-\top} \left\langle \underline{\mathcal{M}}_1^{q-1} | A_{q-1}^\top \right\rangle + \mathcal{R}(G_q^\top), \ 2 \le q \le m.$$

Similarly, system (1) is $[t_0, t_{m-1}^+)$-determinable only if $\overline{\mathcal{M}}_1^m = \mathbb{R}^n$, where $\overline{\mathcal{M}}_1^m$ is computed as follows:

$$\overline{\mathcal{M}}_1^1 := (\underline{\mathcal{Q}}_1^m)^\perp = \mathcal{R}(G_1^\top),$$
$$\overline{\mathcal{M}}_1^q := (\underline{\mathcal{Q}}_1^q)^\perp = E_{q-1}^{-\top} \left\langle A_{q-1}^\top | \overline{\mathcal{M}}_1^{q-1} \right\rangle + \mathcal{R}(G_q^\top), \ 2 \le q \le m.$$

3. OBSERVER DESIGN

In engineering practice, an observer is designed to provide an estimate of the actual state value at current time. In this regard, determinability (weaker than observability according to Definition 1) is a suitable notion. Based on the conditions obtained for determinability in the previous section, an asymptotic observer is designed for the system (1) in this section. By asymptotic observer, we mean that the estimate $\hat{x}(t)$ converges to the plant state $x(t)$ as $t \to \infty$, and in order to achieve this convergence, we introduce the following assumptions.

Assumption 1.

1. The switching is persistent in the sense that there exists a $D > 0$ such that a switch occurs at least once in every time interval of length D; that is,
$$t_q - t_{q-1} < D, \qquad \forall\, q \in \mathbb{N}. \tag{11}$$
2. The system is persistently determinable in the sense that there exists an $N \in \mathbb{N}$ such that
$$\dim \mathcal{M}_{q-N}^q = n, \qquad \forall\, q \ge N+1. \tag{12}$$
(The integer N is interpreted as the minimal number of switches required to gain determinability.)
3. $\|A_q\|$ is uniformly bounded for all $q \in \mathbb{N}$ (which is always the case when A_q belongs to a finite set).

We disregard the time consumed for computation by assuming that the data processor is fairly fast compared to the plant process. The computation time, however, needs to be considered in real-time application if the plant itself is fast.

The observer we propose is a hybrid dynamical system of the form

$$\dot{\hat{x}}(t) = A_q \hat{x}(t) + B_q u(t), \qquad t \ne t_q, \tag{13a}$$
$$\hat{x}(t_q) = E_q(\hat{x}(t_q^-) - \xi_q(t_q^-)) + F_q v_q, \tag{13b}$$
$$\xi_q(t_q^-) = \begin{cases} \mathcal{L}_q(y_{[t_{q-N-1}, t_q)}, u_{[t_{q-N-1}, t_q)}, v_{[q-N, q-1]}), & q > N, \\ 0, & 1 \le q \le N, \end{cases} \tag{13c}$$

with an arbitrary initial condition $\hat{x}(t_0) \in \mathbb{R}^n$. It is seen that the observer consists of a system copy and an estimate update law by some operator $\mathcal{L}_q$. So the goal is to design the operator $\mathcal{L}_q$ such that $\hat{x}(t) \to x(t)$. It will turn out that the operator $\mathcal{L}_q$ includes dynamic observers for partial states at each mode, and some inversion algorithm logic. The design parameters of the operator $\mathcal{L}_q$ are formulated in Theorem 3; but before stating that result, we give construction of the operator $\mathcal{L}_q$ and in the process, set up the machinery required to develop the statement of Theorem 3.

With $\tilde{x} := \hat{x} - x$, the error dynamics are described by,

$$\dot{\tilde{x}}(t) = A_q \tilde{x}(t), \qquad t \neq t_q, \tag{14a}$$

$$\tilde{x}(t_q) = E_q(\tilde{x}(t_q^-) - \xi_q(t_q^-)). \tag{14b}$$

The output error can now be defined as $\tilde{y}(t) := C_q \hat{x}(t) + D_q u(t) - y(t) = C_q \tilde{x}(t)$.

Based on the description of error dynamics, we design partial observers for each mode q using the idea similar to Kalman observability decomposition [5]. Choose a matrix Z^q such that its columns are an orthonormal basis of $\mathcal{R}(G_q^\top)$, so that $\mathcal{R}(Z^q) = \mathcal{R}(G_q^\top)$. Further, choose a matrix W^q such that its columns are an orthonormal basis of $\ker G_q$. From the construction, there are matrices $S_q \in \mathbb{R}^{r_q \times r_q}$ and $R_q \in \mathbb{R}^{d_y \times r_q}$, where $r_q = \operatorname{rank} G_q$, such that $Z^{q\top} A_q = S_q Z^{q\top}$ and $C_q = R_q Z^{q\top}$, and that the pair (S_q, R_q) is observable. Let $z^q := Z^{q\top}\tilde{x}$ and $w^q := W^{q\top}\tilde{x}$, so that z^q (resp. w^q) denotes the observable (resp. unobservable) states of mode q. Thus, for the interval $[t_{q-1}, t_q)$, we obtain,

$$\dot{z}^q = Z^{q\top} A_q \tilde{x} = S_q z^q, \quad \tilde{y} = C_q \tilde{x} = R_q z^q, \tag{15a}$$

$$z^q(t_{q-1}) = Z^{q\top}\tilde{x}(t_{q-1}). \tag{15b}$$

Since z^q is observable over the interval $[t_{q-1}, t_q)$, a standard Luenberger observer, whose role is to estimate $z^q(t_q^-)$ at the end of the interval, is designed as:

$$\dot{\hat{z}}^q = S_q \hat{z}^q + L_q(\tilde{y} - R_q \hat{z}^q), \quad t \in [t_{q-1}, t_q), \tag{16a}$$

$$\hat{z}^q(t_{q-1}) = 0, \tag{16b}$$

where L_q is a matrix such that $(S_q - L_q R_q)$ is Hurwitz. Note that we have fixed the initial condition of the estimator to be zero for each interval.

Next, with $j > i$, define the state-flow matrix

$$\Psi_i^j(\tau_{\{i+1,j\}}) := e^{A_j \tau_j} E_{j-1} e^{A_{j-1}\tau_{j-1}} E_{j-2} \cdots e^{A_{i+1}\tau_{i+1}} E_i, \tag{17}$$

and for convenience $\Psi_q^q := I$. We now define a matrix $\Theta_i^q(\tau_{\{i+1,q\}})$ whose columns form the basis of the subspace $\mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp$; that is,

$$\mathcal{R}(\Theta_i^q(\tau_{\{i+1,q\}})) = \mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp, \quad i = q-N, \cdots, q.$$

where we denote the vector $[\tau_{i+1}, \cdots, \tau_j]$ simply by $\tau_{\{i+1,j\}}$ which, for succinct presentation and by appropriate use of superscripts and subscripts, is often dropped when used as an argument. As a convention, we take Θ_i^q to be a null matrix whenever $\mathcal{R}(\Psi_i^q(\tau_{\{i+1,q\}})W^i)^\perp = \{0\}$.

Using the determinability of the system, that is, Assumption 1.2, it will be shown later in the proof of Theorem 3 (equation (27)) that the matrix

$$\Theta_q := [\Theta_q^q \vdots \cdots \vdots \Theta_{q-N}^q] \tag{18}$$

has rank n. Equivalently, $\Theta_q^\top$ has n independent columns and is left-invertible, so that $(\Theta_q^\top)^\dagger = (\Theta_q \Theta_q^\top)^{-1}\Theta_q$, where $\dagger$ denotes the left-pseudo-inverse. Introduce the notation

$$\xi_{\{q-N,q-1\}}^- := \operatorname{col}(\xi_{q-N}(t_{q-N}^-), \ldots, \xi_{q-1}(t_{q-1}^-)),$$

and let $\Omega_q(z^q(t_q^-), z^{q-1}(t_{q-1}^-), \ldots, z^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-)$ denote the matrix

$$\begin{bmatrix} \Theta_q^{q\top}\Psi_q^q Z^q z^q(t_q^-) \\ \vdots \\ \Theta_{q-N}^{q\top}\left(\Psi_{q-N}^q Z^{q-N} z^{q-N}(t_{q-N}^-) - \sum_{l=q-N}^{q-1}\Psi_l^q \xi_l(t_l^-)\right) \end{bmatrix}.$$

We then define $\xi_q(t_q^-)$ in (13c) as:

$$\begin{aligned} \xi_q(t_q^-) &:= (\Theta_q^\top)^\dagger \Omega_q(\hat{z}^q(t_q^-), \ldots, \hat{z}^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-) \\ &=: \Xi_q(\hat{z}^q(t_q^-), \hat{z}^{q-1}(t_{q-1}^-), \ldots, \hat{z}^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-). \end{aligned} \tag{19}$$

Finally, as the last piece of notation, we define the matrices M_j^q, $j = q-N, \cdots, q$, as follows:

$$[M_q^q, M_{q-1}^q, \cdots, M_{q-N}^q] := E_q \begin{bmatrix} \Theta_q^{q\top} \\ \vdots \\ \Theta_{q-N}^{q\top} \end{bmatrix}^\dagger \times \operatorname{blockdiag}\left(\Theta_q^{q\top}\Psi_q^q, \Theta_{q-1}^{q\top}\Psi_{q-1}^q, \cdots, \Theta_{q-N}^{q\top}\Psi_{q-N}^q\right). \tag{20}$$

Each M_j^q, $j = q-N, \cdots, q$, is an n by n matrix whose argument is $\tau_{\{q-N+1,q\}}$, while the argument of both Θ_j^q and Ψ_j^q is $\tau_{\{j+1,q\}}$ for $j = q-N, \cdots, q-1$ (note that $\Psi_q^q = I$ and that Θ_q^q is a constant matrix).

Based on these definitions, the statement of the following theorem shows that, with suitably chosen values of L_j, the computation of $\hat{z}^q(t_q^-)$ from (16) and $\xi_q(t_q^-)$ from (19) leads to converging state estimates using (13).

Theorem 3. For system (1), consider the hybrid observer in (13) with the operator $\mathcal{L}_q$ computed through observer (16) and the map Ξ in (19). Suppose that Assumption 1 holds. At each switching instant $t = t_q$, $q > N$, introduce the positive constants $\lambda_j^q := \|M_j^q(\tau_{\{q-N+1,q\}})\|$, and α_j, γ_j such that $\|Z^j e^{(S_j - L_j R_j)\tau_j} Z^{j\top}\| \leq \alpha_j e^{-\gamma_j \tau_j}$. If the gains L_j are chosen so that,

$$\lambda_j^q \alpha_j e^{-\gamma_j \tau_j} \leq c < \frac{1}{N+1}, \tag{21}$$

for each $j = q-N, \cdots, q$, and a constant c, then

$$\lim_{t\to\infty} |\hat{x}(t) - x(t)| = 0. \tag{22}$$

Based on the construction of the operator $\mathcal{L}_q$ ($q > N$), and the result in Theorem 3, the implementation of our observer can be summarized as follows:

- At each time instant $t = t_q$,
  - compute the constants λ_j^q, $j = q-N, \cdots, q$, using the knowledge of $\tau_{\{q-N+1,q\}}$, A_j, E_j, and Θ_q,
  - compute the observer gain L_j, using the matrices S_j, R_j, and Z^j such that (21) holds,
  - run the individual observer (16) for j-th mode with the stored data y and u to obtain $\hat{z}^j(t_j^-)$, $j = q-N, \cdots, q$.
- Find $\xi_q(t_q^-)$ by (19), use it in (13), and repeat.

Proof of Theorem 3. Using (14), it follows from Assumptions 1.1 and 1.3 that the estimation error $\tilde{x}(t)$ for the interval $[t_q, t_{q+1}^-)$ is bounded by

$$|\tilde{x}(t)| = |e^{A_{q+1}(t-t_q)}\tilde{x}(t_q)| \leq e^{L(t-t_q)}|\tilde{x}(t_q)|$$

with a constant L such that $\|A_q\| \le L$, and thus,

$$|\tilde{x}(t)| \le e^{LD}|\tilde{x}(t_q)|.$$

Therefore, if $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$, then we achieve that

$$\lim_{t\to\infty} |\tilde{x}(t)| = 0. \tag{23}$$

Remainder of the proof shows that $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$ under the conditions stated in the theorem statement.

Note that, $\tilde{x}(t_q^-)$ can be written as,

$$\tilde{x}(t_q^-) = \begin{bmatrix} Z^{q\top} \\ W^{q\top} \end{bmatrix}^{-1} \begin{bmatrix} z^q(t_q^-) \\ w^q(t_q^-) \end{bmatrix} = Z^q z^q(t_q^-) + W^q w^q(t_q^-). \tag{24}$$

The matrix $\Psi_i^j(\tau_{\{i+1,j\}})$, defined in (17), transports $\tilde{x}(t_i^-)$ to $\tilde{x}(t_j^-)$ along (14) by

$$\tilde{x}(t_j^-) = \Psi_i^j(\tau_{\{i+1,j\}})\tilde{x}(t_i^-) - \sum_{l=i}^{j-1} \Psi_l^j(\tau_{\{l+1,j\}})\xi_l(t_l^-). \tag{25}$$

We now have the following series of equivalent expressions for $\tilde{x}(t_q^-)$:

$$\begin{aligned}
\tilde{x}(t_q^-) &= Z^q z^q(t_q^-) + W^q w^q(t_q^-) \\
&= \Psi_{q-1}^q Z^{q-1} z^{q-1}(t_{q-1}^-) + \Psi_{q-1}^q W^{q-1} w^{q-1}(t_{q-1}^-) \\
&\qquad - \Psi_{q-1}^q \xi_{q-1}(t_{q-1}^-) \\
&= \Psi_{q-2}^q Z^{q-2} z^{q-2}(t_{q-2}^-) + \Psi_{q-2}^q W^{q-2} w^{q-2}(t_{q-2}^-) \\
&\quad - \Psi_{q-2}^q \xi_{q-2}(t_{q-2}^-) - \Psi_{q-1}^q \xi_{q-1}(t_{q-1}^-) \\
&\ \ \vdots \\
&= \Psi_{q-N}^q Z^{q-N} z^{q-N}(t_{q-N}^-) \\
&+ \Psi_{q-N}^q W^{q-N} w^{q-N}(t_{q-N}^-) - \sum_{l=q-N}^{q-1} \Psi_l^q \xi_l(t_l^-).
\end{aligned} \tag{26}$$

To appreciate the implication of this equivalence, we first note that for each $q - N \le i \le q$, the term $\Psi_i^q Z^i z^i(t_i^-)$ transports the observable information of the i-th mode from the interval $[t_{i-1}, t_i)$ to the time instant t_q^-. This observable information is corrupted by the unknown term $w^i(t_i^-)$, but since the information is being accumulated at t_q^- from modes $i = q - N, \cdots, q$, the idea is to combine the partial information from each mode to recover $\tilde{x}(t_q^-)$. This is where we use the notion of determinability. By Properties 1, 5, and 6 in the Appendix, and the fact that $\mathcal{R}(W^i)^\perp = (\ker G_i)^\perp = \mathcal{R}(G_i^\top)$ and $e^{-A_q^\top \tau_q}\mathcal{R}(G_q^\top) = \mathcal{R}(G_q^\top)$, it follows under Assumption 1.2 that

$$\begin{aligned}
&\mathcal{R}(W^q)^\perp + \mathcal{R}(\Psi_{q-1}^q W^{q-1})^\perp + \cdots + \mathcal{R}(\Psi_{q-N}^q W^{q-N})^\perp \\
&= e^{-A_q^\top \tau_q}\Big(\mathcal{R}(G_q^\top) + E_{q-1}^{-\top}\mathcal{R}(G_{q-1}^\top) + \\
&\qquad \sum_{i=q-N}^{q-2} \Pi_{l=q-1}^{i+1} E_l^{-\top} e^{-A_l^\top \tau_l} E_i^{-\top}\mathcal{R}(G_i^\top)\Big) \\
&= e^{-A_q^\top \tau_q}\mathcal{M}_{q-N}^q = \mathbb{R}^n.
\end{aligned} \tag{27}$$

Thus, the matrix Θ_q defined in (18) has rank n, so that for each equality in (26), that is $i = q - N, \cdots, q$, we have the relation

$$\Theta_i^{q\top}\tilde{x}(t_q^-) = \Theta_i^{q\top}\left(\Psi_i^q Z^i z^i(t_i^-) - \sum_{l=i}^{q-1} \Psi_l^q \xi_l(t_l^-)\right).$$

Employing left-invertibility of $\Theta_q^\top$ to get,

$$\begin{aligned}
\tilde{x}(t_q^-) &= (\Theta_q^\top)^\dagger \Omega_q(z^q(t_q^-), \ldots, z^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-) \\
&= \Xi_q(z^q(t_q^-), \ldots, z^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-).
\end{aligned} \tag{28}$$

It is seen from (28) that, if we can estimate $z^i(t_i^-)$, $i = q - N, \ldots, q$, without error, then by (28) the plant state $x(t_q^-)$ is exactly recovered because $x(t_q^-) = \hat{x}(t_q^-) - \tilde{x}(t_q^-)$, and both entities on the right side of the equation are known. However, since this is not the case, we set $\xi_q(t_q^-)$ to be an estimate of $\tilde{x}(t_q^-)$ as described in (19).

Due to the linearity of Ω_q in z^i's and ξ_i's, it is noted that,

$$\begin{aligned}
\tilde{x}(t_q) &= E_q(\tilde{x}(t_q^-) - \xi_q(t_q^-)) && \text{(29a)} \\
&= E_q\Big(\Xi_q(z^q(t_q^-), \ldots, z^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-) \\
&\qquad - \Xi_q(\hat{z}^q(t_q^-), \ldots, \hat{z}^{q-N}(t_{q-N}^-), \xi_{\{q-N,q-1\}}^-)\Big) && \text{(29b)} \\
&= -E_q(\Theta_q^\top)^\dagger \Omega_q(\tilde{z}^q(t_q^-), \ldots, \tilde{z}^{q-N}(t_{q-N}^-), 0) && \text{(29c)}
\end{aligned}$$

where $\tilde{z} := \hat{z} - z$. It follows from (15) and (16) that

$$\tilde{z}^i(t_{i-1}) = \hat{z}^i(t_{i-1}) - z^i(t_{i-1}) = 0 - Z^{i\top}\tilde{x}(t_{i-1}).$$

and that

$$\tilde{z}^i(t_i^-) = e^{(S_i - L_i R_i)\tau_i}\tilde{z}^i(t_{i-1}) = -e^{(S_i - L_i R_i)\tau_i} Z^{i\top}\tilde{x}(t_{i-1})$$

Plugging this expression in (29), and using the definition of M_j^q, $j = q - N, \ldots, q$, from (20), we get

$$\tilde{x}(t_q) = \sum_{j=q-N}^{q} M_j^q(\tau_{\{q-N+1,q\}}) Z^j e^{(S_j - L_j R_j)\tau_j} Z^{j\top}\tilde{x}(t_{j-1}).$$

In order to bound the norm of $\tilde{x}(t_q)$, consider the constants $\alpha_j, \gamma_j, \lambda_j^q > 0$ defined in the theorem statement to get,

$$|\tilde{x}(t_q)| \le \sum_{j=q-N}^{q} \lambda_j^q \alpha_j e^{-\gamma_j \tau_j}|\tilde{x}(t_{j-1})|. \tag{30}$$

The statement of the following lemma, proof of which appears in the appendix, aids us in the completion of the proof.

Lemma 1. A sequence $\{a_i\}$ satisfying

$$|a_i| \le c(|a_{i-1}| + |a_{i-2}| + \cdots + |a_{i-N-1}|), \qquad i > N,$$

with $0 \le c < 1/(N + 1)$ converges to zero: $\lim_{i\to\infty} a_i = 0$.

Applying Lemma 1 to (30), we see that $|\tilde{x}(t_q)| \to 0$ as $q \to \infty$, whence the desired result follows. □

Note that the computation of the gains requires the knowledge of switching times in order to generate converging estimates. Thus, post-processing of the switching signal is involved in computing the gains. Also, in the design of observer, we ignored the time required for computation at time instant t_q. In fact, the outcome $\xi_q(t_q^-)$ becomes available not at t_q but at $t_q + T_{comp}$ for some $T_{comp} > 0$. It is conjectured that the error caused by this time-delayed update in (13) can be suppressed by taking smaller value of c in (21) while the update is actually performed at $t_q + T_{comp}$ using another state-flow matrix. Detailed analysis on improving the quality of the observer is an ongoing work.

EXAMPLE 2. We demonstrate the working of our observer for the switched system considered in Example 1. As mentioned earlier, the system is observable with mode sequence $1 \to 2 \to 1$, and hence determinable. We assume that each mode is activated for τ seconds, so that the persistent switching signal exciting the system is:

$$\sigma(t) = \begin{cases} 1 & \text{if } t \in [2k\tau, (2k+1)\tau), \\ 2 & \text{if } t \in [(2k+1)\tau, (2k+2)\tau), \end{cases} \tag{31}$$

where $k = 0, 1, 2, \cdots$, and the underlying assumption is that $\tau \neq \kappa\pi$, for any $\kappa \in \mathbb{N}$. With this switching signal, the determinability conditions are guaranteed to hold over any time interval that involves three switches, so we pick $N = 3$. For brevity, we call $[2k\tau, (2k+1)\tau)$, the *odd* interval, and $[(2k+1)\tau, (2k+2)\tau)$, the *even* interval. With an arbitrary initial condition $\hat{x}(0)$, the observer to be implemented is:

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A_1\hat{x}(t) \\ \hat{y}(t) &= C_1\hat{x}(t) \end{aligned}\right\}, \quad t \in [2k\tau, (2k+1)\tau), \tag{32a}$$

$$\left.\begin{aligned} \dot{\hat{x}}(t) &= A_2\hat{x}(t) \\ \hat{y}(t) &= C_2\hat{x}(t) \end{aligned}\right\}, \quad t \in [(2k+1)\tau, (2k+2)\tau), \tag{32b}$$

$$\hat{x}(q\tau) = \hat{x}(q\tau^-) - \xi_q(q\tau^-), \qquad q > 3. \tag{32c}$$

In order to determine the value of $\xi_q(q\tau^-)$, we start off with the estimators for observable modes of each subsystem, denoted by z^q in (15). Note that mode 1 has one-dimensional observable subspace whereas for mode 2, the unobservable subspace is $\mathbb{R}^2$. Since mode 1 is active on every odd interval and mode 2 on every even interval, z^q for every odd q represents the partial information obtained from mode 1, and z^q for every even q is a null vector as no information is extracted from mode 2. So the one-dimensional z-observer in (16) is only implemented on odd intervals and for every odd q, the differential equation for $\hat{z}^q$ can be derived as follows:

$$G_q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{R}(G_q^\top) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, W^q = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Z^q = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

so that one may choose $S^q = 0$, and $R^q = 1$, which yields

$$\dot{\hat{z}}^q = -l_q\hat{z}^q + l_q\tilde{y}, \qquad t \in [(q-1)\tau, q\tau), \quad q\text{: odd},$$

with the initial condition $\hat{z}^q((q-1)\tau) = 0$, and $\tilde{y}$ as the difference between the measured output and the estimated output of (32). The gain l_q will be chosen later by (35).

The next step is to use the value of $\hat{z}^q(q\tau^-)$ to compute $\xi_q(q\tau^-)$, $q \geq 4$. We use the notation ξ^q to denote $\xi_q(q\tau^-)$, and let ξ_1^q be the first component of the vector ξ^q. For initialization, we pick $\xi^1 = \xi^2 = \xi^3 = \operatorname{col}(0, 0)$. The matrices appearing in the computation of ξ^q are given as follows: for every *odd* $q > 3$:

$$\Psi_{q-3}^q = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \Rightarrow \left(\Psi_{q-3}^q\mathbb{R}^2\right)^\perp = \{0\},$$

$$\Psi_{q-2}^q = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix}; \left(\Psi_{q-2}^q \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^\perp = \left\{\begin{pmatrix} \cos\tau \\ -\sin\tau \end{pmatrix}\right\},$$

$$\Psi_{q-1}^q = I_{2\times 2} \Rightarrow \left(\Psi_{q-1}^q\mathbb{R}^2\right)^\perp = \{0\},$$

$$\Psi_q^q = I_{2\times 2} \Rightarrow \left(\Psi_q^q \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^\perp = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

where the braces $\{\cdot\}$ denote the linear combination of the elements it contains. These subspaces directly lead to the expressions for Θ_j^q, $j = q-3, \ldots, q$, so that

$$\Theta_q = \begin{bmatrix} 1 & \cos\tau \\ 0 & -\sin\tau \end{bmatrix}, \quad q = 5, 7, \ldots,$$

and hence the error correction term can be computed recursively for every odd $q > 3$ by the formula:

$$\xi^q = \Theta_q^{-\top} \begin{bmatrix} z^q(t_q^-) \\ z^{q-2}(t_{q-2}^-) - \xi_1^{q-2} - [\cos\tau \ \ -\sin\tau]\xi^{q-1} \end{bmatrix}.$$

Also, it can be verified that the matrix $M_j^q = 0$ for $j = q-1, q-3$ and, for $j = q, q-2$ we get

$$M_q^q = \begin{bmatrix} 1 & 0 \\ \frac{\cos\tau}{\sin\tau} & 0 \end{bmatrix}, \quad M_{q-2}^q = \begin{bmatrix} 0 & 0 \\ -\frac{1}{\sin\tau} & 0 \end{bmatrix}. \tag{33}$$

Next, for every *even* $q > 3$, we can repeat the same calculations to get:

$$\Psi_{q-3}^q = \begin{bmatrix} \cos 2\tau & \sin 2\tau \\ -\sin 2\tau & \cos 2\tau \end{bmatrix}, \left(\Psi_{q-3}^q \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^\perp = \left\{\begin{pmatrix} \cos 2\tau \\ -\sin 2\tau \end{pmatrix}\right\},$$

$$\Psi_{q-2}^q = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \Rightarrow \left(\Psi_{q-2}^q\mathbb{R}^2\right)^\perp = \{0\},$$

$$\Psi_{q-1}^q = \begin{bmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{bmatrix} \Rightarrow \left(\Psi_{q-1}^q \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^\perp = \left\{\begin{pmatrix} \cos\tau \\ -\sin\tau \end{pmatrix}\right\},$$

$$\Psi_q^q = I_{2\times 2} \left(\Psi_q^q\mathbb{R}^2\right)^\perp = \{0\}.$$

Once again, using the the expressions for Θ_j^q, $j = q-3, \ldots, q$, based on these subspace, one gets,

$$\Theta_q = \begin{bmatrix} \cos\tau & \cos 2\tau \\ -\sin\tau & -\sin 2\tau \end{bmatrix}, \quad q = 4, 6, 8, \cdots,$$

so that

$$\xi^q = \Theta_q^{-\top} \begin{bmatrix} z^{q-1}(t_{q-1}^-) - \xi_1^{q-1} \\ z^{q-3}(t_{q-3}^-) - \xi_1^{q-3} - [\cos\tau \ \ -\sin\tau](\xi^{q-2} + \xi^{q-1}) \end{bmatrix}.$$

Again, it can be verified that the matrix $M_j^q = 0$ for $j = q, q-2$ and, for $j = q-1, q-3$ we get

$$M_q^q = \begin{bmatrix} \frac{\sin 2\tau}{\sin\tau} & 0 \\ \frac{\cos 2\tau}{\sin\tau} & 0 \end{bmatrix}, \quad M_{q-2}^q = \begin{bmatrix} -1 & 0 \\ -\frac{\cos\tau}{\sin\tau} & 0 \end{bmatrix}. \tag{34}$$

Finally, we derive the bound on gains l_q that gives converging estimates. Note that the matrix M_j^q, for each $q > 3$ and each $j = q, \cdots, q-3$, has the following induced 2-norm,

$$\lambda_j^q = \|M_j^q\| = \begin{cases} 0 & \text{if } j \text{ is even} \\ \frac{1}{|\sin\tau|} & \text{if } j \text{ is odd} \end{cases}.$$

Also, $\|Z^q e^{(S_q - l_q R_q)\tau_q} Z^{q\top}\| = e^{-l_q\tau}$ for every odd q, and null for q even. Thus, (21) is trivially satisfied when j is even,

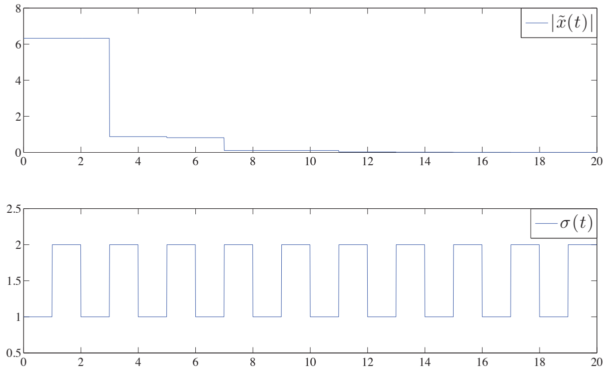

Figure 1: Switching signal and the state estimation error.

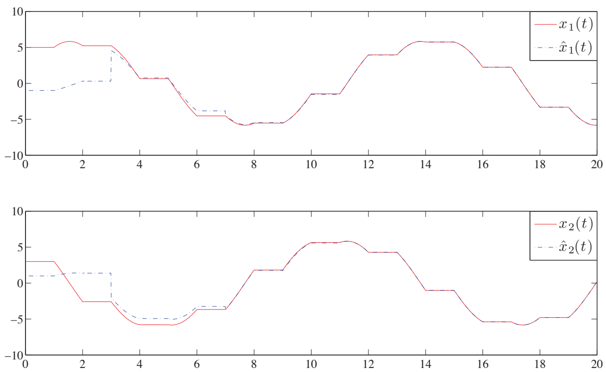

Figure 2: Converging state estimates.

and for odd values of j, the inequality

$$\frac{1}{|\sin\tau|}e^{-l_q\tau} < \frac{1}{4}$$

holds if, and only if,

$$l_q > \frac{1}{\tau}\ln\frac{4}{|\sin\tau|}\,. \tag{35}$$

Once again it can be seen that, if τ is an integer multiple of π, or even when τ approaches this point of singularity, then the gain required for convergence gets arbitrarily large. This also explains why the knowledge of switching signal is required in general to compute the observer gains.

The results of the simulation for $\tau = 1$ and $l_q = 2$ with q odd, are shown in Fig. 1 and Fig. 2. Because of the error correction term, it can be seen that there is jump discontinuity in estimation error at switching times, and the error remains constant between the switching times. This is because the subsystem 2 rotates any given initial condition in a circle of constant radius, thus not letting the error grow. If instead there were an unstable system then the error would grow in between the switching times but the error correction term would guarantee that the sequence formed by taking the value of the estimation error at switching times is indeed a decreasing sequence.

4. CONCLUSION

This paper presented conditions for observability and determinability of switched linear systems with state jumps. Based on these conditions, an observer is constructed that combines the partial information obtained from each mode at some time instant to get an estimate of the state vector. Under the assumption of persistent switching, the error analysis shows that the estimate converges to the actual state. It is noted that the transportation of the partial information in (26), even under the added unknown information, is achievable for linear systems, and may not be possible for nonlinear systems, e.g., in [11].

Several directions for future work are being pursued. For observability conditions, denseness of regular switching signals is being studied. This, in turn, leads to the question whether there exists necessary and sufficient conditions if we seek observability uniformly over all switching signals. Furthermore, the quality of observer can be investigated at various stages. The consideration of computation time may lead to a delayed update and hence an additional error term which is required to be suppressed in computing the estimate. Also, in order to avoid post-processing of the switching signal to compute the observer gains, we believe that it would be possible to pre-compute the observer gains with conditions depending on mode-sequence independently of switching times. Moreover, the construction of observers without the assumptions of maximal switching time or dwell-time switching is an interesting question that requires further investigation.

Acknowledgement

The authors appreciate the discussions with Stephan Trenn related to observability conditions.

Appendix: Proof of Lemma 1

Let $c = \alpha/(N+1)$ with $0 \le \alpha < 1$. Then it is obvious that

$$|a_i| \le \alpha \max_{i-N-1\le k\le i-1} |a_k|, \qquad i > N. \tag{36}$$

The above inequality implies that

$$\begin{aligned}
|a_{i+1}| &\le \frac{\alpha}{N+1}\sum_{k=i-N}^{i}|a_k| \le \alpha \max_{i-N\le k\le i}|a_k| \\
&\le \alpha \max\left\{|a_{i-N-1}|, \max_{i-N\le k\le i-1}|a_k|, |a_i|\right\} \\
&\le \alpha \max_{i-N-1\le k\le i-1}|a_k|.
\end{aligned}$$

By induction, this leads to

$$\max_{i\le j\le i+N}|a_j| \le \alpha \max_{i-N-1\le k\le i-1}|a_k|.$$

that is, the maximum value of the sequence $\{a_i\}$ over the length of window $N+1$ is strictly decreasing and converging to zero, which proves the desired result. □

Appendix: Some Useful Facts

Let $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}$ be any linear subspaces, A be a (not necessarily invertible) $n\times n$ matrix, and B, C be matrices of suitable dimension. For a matrix B, $\mathcal{R}(B)$ denotes the column space (range space) of B. The pre-image of $\mathcal{V}$ through A is given by $A^{-1}\mathcal{V} = \{x : Ax \in \mathcal{V}\}$. The following properties can be found in the literature such as [16], or developed with little effort.

1. $A\mathcal{R}(B) = \mathcal{R}(AB)$ and $A^{-1}\ker B = \ker(BA)$.

2. $A^{-1}A\mathcal{V} = \mathcal{V} + \ker A$, and $AA^{-1}\mathcal{V} = \mathcal{V} \cap \mathcal{R}(A)$.
3. $A^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2) = A^{-1}\mathcal{V}_1 \cap A^{-1}\mathcal{V}_2$, and $A(\mathcal{V}_1 \cap \mathcal{V}_2) \subseteq A\mathcal{V}_1 \cap A\mathcal{V}_2$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \ker A = \mathcal{V}_1 \cap \ker A + \mathcal{V}_2 \cap \ker A$, which holds, in particular, for any invertible A).
4. $A\mathcal{V}_1 + A\mathcal{V}_2 = A(\mathcal{V}_1 + \mathcal{V}_2)$, and $A^{-1}\mathcal{V}_1 + A^{-1}\mathcal{V}_2 \subseteq A^{-1}(\mathcal{V}_1 + \mathcal{V}_2)$ (with equality if and only if $(\mathcal{V}_1 + \mathcal{V}_2) \cap \mathcal{R}(A) = \mathcal{V}_1 \cap \mathcal{R}(A) + \mathcal{V}_2 \cap \mathcal{R}(A)$, which holds, in particular, for any invertible A).
5. $(\ker A)^\perp = \mathcal{R}(A^\top)$.
6. $(A^\top \mathcal{V})^\perp = A^{-1}\mathcal{V}^\perp$ and $(A^{-1}\mathcal{V})^\perp = A^\top \mathcal{V}^\perp$.
7. $\langle A|\mathcal{V}\rangle = \mathcal{V} + A\mathcal{V} + A^2\mathcal{V} + \cdots + A^{n-1}\mathcal{V}$ and $\langle \mathcal{V}|A\rangle = \mathcal{V} \cap A^{-1}\mathcal{V} \cap A^{-2}\mathcal{V} \cap \cdots \cap A^{-(n-1)}\mathcal{V}$.
8. $\langle \mathcal{V}_1 \cap \mathcal{V}_2 | A\rangle = \langle \mathcal{V}_1|A\rangle \cap \langle \mathcal{V}_2|A\rangle$ and $\langle A|\mathcal{V}_1 \cap \mathcal{V}_2\rangle \subset \langle A|\mathcal{V}_1\rangle \cap \langle A|\mathcal{V}_2\rangle$.
9. $e^{At}\mathcal{V} \subseteq \langle A|\mathcal{V}\rangle$ and $\langle \mathcal{V}|A\rangle \subseteq e^{At}\mathcal{V}$ for any t.
10. $\langle A|\mathcal{V}\rangle^\perp = \langle \mathcal{V}^\perp | A^\top \rangle$.

Now, with $G := \mathrm{col}(C, CA, \ldots, CA^{n-1})$,

11. $e^{At} \ker G = \ker G$ and $e^{A^\top t}\mathcal{R}(G^\top) = \mathcal{R}(G^\top)$ for all t.
12. $\langle \ker G | A\rangle = \ker G$ and $\langle A^\top | \mathcal{R}(G^\top)\rangle = \mathcal{R}(G^\top)$.

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