Stability Analysis of Hybrid Systems Via Small-Gain Theorems

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Abstract. We present a general approach to analyzing stability of hybrid systems, based on input-to-state stability (ISS) and small-gain theorems. We demonstrate that the ISS small-gain analysis framework is very naturally applicable in the context of hybrid systems. Novel Lyapunov-based and LaSalle-based small-gain theorems for hybrid systems are presented. The reader does not need to be familiar with ISS or small-gain theorems to be able to follow the paper.

1 Introduction

The small-gain theorem is a classical tool for analyzing input-output stability of feedback systems; see, e.g., [1]. More recently, small-gain tools have been used extensively to study feedback interconnections of nonlinear state-space systems in the presence of disturbances; see, e.g., [2]. Hybrid systems can be naturally viewed as feedback interconnections of simpler subsystems. For example, every hybrid system can be regarded as a feedback interconnection of its continuous and discrete dynamics. This makes small-gain theorems a very natural tool to use for studying internal and external stability of hybrid systems. However, we are not aware of any systematic application of this idea in the literature.

The purpose of this paper is to bring the small-gain analysis method to the attention of the hybrid systems community. We review, in a tutorial fashion, the concept of input-to-state stability (ISS) introduced by Sontag [3] and a nonlinear small-gain theorem from [2] based on this concept. The ISS small-gain theorem states that a feedback interconnection of two ISS systems is ISS if an appropriate composition of their respective ISS gain functions is smaller than the identity function. Since a proof of this theorem can be based entirely on time-domain analysis of system signals, the result is valid for general dynamical systems, thus

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providing an “off-the-shelf” method for verifying stability of hybrid systems. We also discuss Lyapunov-based tools for checking the hypotheses of this theorem.

As an alternative to time-domain proofs, Lyapunov function constructions for interconnected systems under small-gain conditions were studied for continuous-time systems in [4] and for discrete-time systems in [5]. It is well known that having a Lyapunov function provides additional insight into the behavior of a stable system and is important for tasks such as perturbation analysis and estimating the region of attraction. In this paper, we present a novel construction of a Lyapunov function for a class of hybrid systems satisfying the conditions of the ISS small-gain theorem. We also describe another approach, based on constructing a “weak” (non-strictly decreasing) Lyapunov function and applying the LaSalle invariance principle for hybrid systems from [6]. While the basic idea of the small-gain stability analysis for hybrid systems was announced and initially examined by the authors in [7], the Lyapunov function constructions reported here are new and represent the main technical contribution of this work.

In the companion paper [7], we illustrate the power of the proposed method through a detailed treatment of several specific problems in the context of hybrid control with communication constraints. As demonstrated there, the small-gain analysis provides insightful interpretations of existing results, immediately leads to generalizations, and allows a unified treatment of problems that so far have been studied separately. Due to the pervasive nature of hybrid systems in applications, we expect that the main ideas described in this paper will be useful in many other areas as well.

2 Preliminaries

In what follows, *id* denotes the identity function and $\circ$ denotes function composition. We write $a \vee b$ for $\max\{a, b\}$ and $a \wedge b$ for $\min\{a, b\}$. The class of continuously differentiable functions is denoted by C^1 (the domain will be specified separately). The gradient operator is denoted by ∇. Given some vectors $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, we often use the simplified notation (x_1, x_2) for the “stack” vector $(x_1^T, x_2^T)^T \in \mathbb{R}^{n_1+n_2}$.

2.1 Hybrid System Model

We begin by describing the model of a hybrid system to which our subsequent results will apply. This model easily fits into standard modeling frameworks for hybrid systems (see, e.g., [8, 6, 9]), and the reader can consult these references for background and further technical details. The description to be provided here is somewhat informal, but it is sufficient for presenting the results.

We label the hybrid system to be defined below as $\mathcal{H}$. The *state variables* of $\mathcal{H}$ are divided into continuous variables $x \in \mathbb{R}^n$ and discrete variables $\mu \in \mathbb{R}^k$. We note that μ actually takes values in a discrete subset of $\mathbb{R}^k$ along every trajectory of the hybrid system, but this set need not be fixed a priori and may vary with initial conditions. The *time* is continuous: $t \in [t_0, \infty)$. We also consider *external variables* $w \in \mathbb{R}^s$, viewed as disturbances.

The *state dynamics* describing the evolution of these variables with respect to time are composed of *continuous evolution* and *discrete events.* During continuous evolution (i.e., while no discrete events occur), μ is held constant and x satisfies the ordinary differential equation $\dot{x} = f(x, \mu, w)$ with $f : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^s \to \mathbb{R}^n$ locally Lipschitz. We now describe the discrete events. Given an arbitrary time t, we will denote by $x^-(t)$, or simply by x^- when the time arguments are omitted, the quantity $x(t^-) = \lim_{s \nearrow t} x(s)$, and similarly for the other state variables. Consider a *guard map* $G : \mathbb{R}^{n+k} \to \mathbb{R}^p$ (where p is a positive integer) and a *reset map* $R : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$. The discrete events are defined as follows: whenever $G\big(x^-, \mu^-\big) \geq 0$ (component-wise), we let $(x, \mu) = R(x^-, \mu^-) = \big(R_x(x^-, \mu^-), R_\mu(x^-, \mu^-)\big)$. By construction, all signals are right-continuous.

Some remarks on the above relations are in order. In many situations, the continuous state does not jump at the event times: $R_x(x, \mu) \equiv x$. The guard map often depends on time and/or auxiliary clock variables, which we do not explicitly model here (they can be incorporated into x). We want inequality rather than equality in the reset triggering condition because for a discrete event to occur, we might need several conditions which do not become valid simultaneously (e.g., some relation between x and μ holds *and* a clock has reached a certain value). Of course, equality conditions are easily described by pairs of inequalities. Note that we allow the disturbances w to affect the discrete events only indirectly, through the continuous state x. This assumption will simplify the Lyapunov-based conditions in Sections 4 and 5; it is typically reasonable in the context of hybrid control design (see [7, 10]).

Well-posedness (existence and uniqueness of solutions) of the hybrid system $\mathcal{H}$ is an issue; see, e.g., [8]. At the general level of the present discussion, we are going to assume it. For example, by using clocks, we can ensure that a bounded number of discrete events occurs in any bounded time interval. Then, to obtain a solution (in the sense of Carathéodory), we simply flow the continuous dynamics until either the end of their domain is reached (finite escape) or a discrete event occurs; in the latter case, we repeat from the new state, and so on. See also [11] for an interesting alternative definition of solutions of hybrid systems.

2.2 Feedback Interconnection Structure

The starting point for our results is the observation that we can view the hybrid system $\mathcal{H}$ as a feedback interconnection of its continuous and discrete parts, as shown in Figure 1(a). For simplicity, we ignore the roles of the guard map G and the continuous state reset map R_x in the diagram.

It is clear that the above decomposition is just one possible way to split the hybrid system $\mathcal{H}$ into a feedback interconnection of two subsystems. There may be many ways to do it; the best choice will depend on the structure of the problem and will be one for which the small-gain approach described below will work. Each subsystem in the decomposition can be continuous, discrete, or hybrid, and may be affected by the disturbances. This more general situation is illustrated in Figure 1(b). Here, the state variables and the external signals of $\mathcal{H}$ are split as $x = (x_1, x_2)$, $\mu = (\mu_1, \mu_2)$, $w = (w_1, w_2)$, the first subsystem $\mathcal{H}_1$

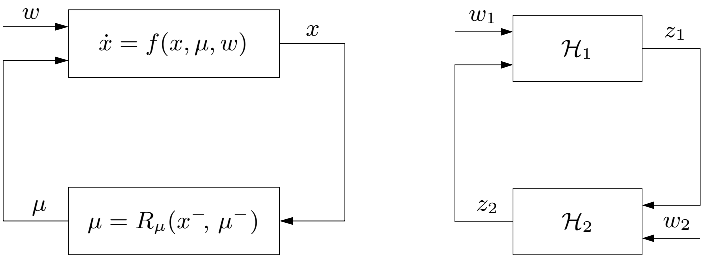

Fig. 1. Hybrid system viewed as feedback interconnection: (a) special decomposition, (b) general decomposition

has states $z_1 := (x_1, \mu_1)$ and inputs $v_1 = (z_2, w_1)$, and the second subsystem $\mathcal{H}_2$ has states $z_2 := (x_2, \mu_2)$ and inputs $v_2 = (z_1, w_2)$.

In the approach discussed here, coming up with a decomposition of the above kind is the first step in the analysis of a given hybrid system. As we pointed out, at least one such decomposition always exists. It can also happen that the hybrid system model is given from the beginning as an interconnection of several hybrid systems. Thus the structure we consider is very general and not restrictive.

2.3 Stability Definitions

A function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{K}$ (which we write as $\alpha \in \mathcal{K}$) if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of *class* $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$). A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ is decreasing to zero as $t \to \infty$ for each fixed $r \geq 0$.

We now define the stability notions of interest in this paper. Consider a hybrid system with state $z = (x, \mu)$ and input v (as a special case, it can have only continuous dynamics or only discrete events). Following [3], we say that this system is *input-to-state stable* (ISS) with respect to v if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial state $z(t_0)$ and every input $v(\cdot)$ the corresponding solution satisfies the inequality

$$|z(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma(\|v\|_{[t_0,t]}) \tag{1}$$

for all $t \geq t_0$, where $\|v\|_{[t_0,t]} := \sup\{|v(s)| : s \in [t_0, t]\}$ (except possibly on a set of measure 0). We will refer to γ as an *ISS gain function*, or just a *gain* if clear from the context. For time-invariant systems, we can take $t_0 = 0$ without loss of generality. If the inputs are split as $v = (v_1, v_2)$, then (1) is equivalent to $|z(t)| \leq \beta(|z(t_0)|, t - t_0) + \gamma_1(\|v_1\|_{[t_0,t]}) + \gamma_2(\|v_2\|_{[t_0,t]})$ for some functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$. In this case, we will call γ_1 the ISS gain from v_1 to z, and so on.

In the case of no inputs ($v \equiv 0$), the inequality (1) reduces to $|z(t)| \leq \beta(|z(t_0)|, t)$ for all $t \geq t_0$, which corresponds to the standard notion[1] of *global asymptotic stability* (GAS). In the presence of inputs, ISS captures the property

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[1] This can also be equivalently restated in the more classical ε–δ style (cf. [12]).

that bounded inputs and inputs converging to 0 produce states that are also bounded and converging to 0, respectively. We note that asymptotic stability of a *linear* system (continuous or sampled-data) can always be characterized by a class $\mathcal{KL}$ function of the form $\beta(r,t) = cre^{-\lambda t}$, $c, \lambda > 0$. Moreover, an asymptotically stable linear system is automatically ISS with respect to external inputs, with a linear ISS gain function $\gamma(r) = cr$, $c > 0$.

3 ISS Small-Gain Theorem

Consider the hybrid system $\mathcal{H}$ defined in Section 2.1, and suppose that it has been represented as a feedback interconnection of two subsystems $\mathcal{H}_1$ and $\mathcal{H}_2$ in the way described in Section 2.2 and shown in Figure 1(b). The small-gain theorem stated next reduces the problem of verifying ISS of $\mathcal{H}$ to that of verifying ISS of $\mathcal{H}_1$ and $\mathcal{H}_2$ and checking a condition that relates their respective ISS gains. The result we give is a special case of the small-gain theorem from [2]. That paper treats continuous systems, but since the statement and the proof given there involve only properties of system signals, the fact that the dynamics are hybrid in our case does not change the validity of the result. We note that the small-gain theorem presented in [2] is much more general in that it treats partial measurements (input-to-output-stability, in conjunction with detectability) and deals with practical stability notions. Many other versions are also possible, e.g., we can replace the sup norm used in (1) by an L_p norm [13].

Theorem 1. *Suppose that:*
1. $\mathcal{H}_1$ is ISS with respect to $v_1 = (z_2, w_1)$, with gain γ_1 from z_2 to z_1, i.e.,

$$|z_1(t)| \le \beta_1(|z_1(t_0)|, t - t_0) + \gamma_1(\|z_2\|_{[t_0,t]}) + \bar{\gamma}_1(\|w_1\|_{[t_0,t]})$$

for some $\beta_1 \in \mathcal{KL}$, $\gamma_1, \bar{\gamma}_1 \in \mathcal{K}_\infty$.
2. $\mathcal{H}_2$ is ISS with respect to $v_2 = (z_1, w_2)$, with gain γ_2 from z_1 to z_2, i.e.,

$$|z_2(t)| \le \beta_2(|z_2(t_0)|, t - t_0) + \gamma_2(\|z_1\|_{[t_0,t]}) + \bar{\gamma}_2(\|w_2\|_{[t_0,t]})$$

for some $\beta_2 \in \mathcal{KL}$, $\gamma_2, \bar{\gamma}_2 \in \mathcal{K}_\infty$.
3. There exists a function $\rho \in \mathcal{K}_\infty$ such that[2]

$$(id + \rho) \circ \gamma_1 \circ (id + \rho) \circ \gamma_2(r) \le r \qquad \forall\, r \ge 0. \tag{2}$$

Then $\mathcal{H}$ is ISS with respect to the input $w = (w_1, w_2)$.

Three special cases are worth mentioning explicitly. First, in the case of no external signals ($w_1 = w_2 \equiv 0$), we conclude that $\mathcal{H}$ is GAS. Second, when the two ISS gain functions are linear: $\gamma_i(r) = c_i r$, $i = 1, 2$, the small-gain condition (2) reduces to the simple one $c_1 c_2 < 1$. Third, the theorem covers the case

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[2] If one replaces $\beta + \gamma$ with $\beta \vee \gamma$ in the definition (1) of ISS, then the small-gain condition (2) can be simplified to $\gamma_1 \circ \gamma_2(r) < r$ for all $r > 0$.

of a cascade connection, where one of the gains is 0 and hence the small-gain condition (2) is automatically satisfied.

Sometimes one wants to concentrate only on some states of the overall system, excluding the other states from the feedback interconnection. For example, one might ignore some auxiliary variables (such as clocks) which have very simple dynamics and remain bounded for all time. Theorem 1 is still valid if z_1 and z_2 include only the states of interest for each subsystem.[3]

Small-gain theorems have been widely used for analysis of continuous-time as well as discrete-time systems with feedback interconnection structure. The discussion of Section 2.2 suggests that it is also very natural to use this idea to analyze (internal or external) stability of hybrid systems. Of course, one needs to show that the subsystems in a feedback decomposition satisfy suitable ISS properties, and calculate the ISS gains in order to check the small-gain condition (2). There exist efficient tools for doing this, as exemplified in the next section.

4 Sufficient Conditions for ISS

Consider the hybrid system $\mathcal{H}$ defined in Section 2.1, and suppose that it has been represented as a special feedback interconnection shown in Figure 1(a). The two lemmas stated below provide Lyapunov-based conditions which guarantee ISS of the continuous and discrete dynamics, respectively, and give expressions for the ISS gains. Thus they can be used for verifying the hypotheses of Theorem 1 in this particular case. The first result is well established [3]; the second one is a slightly sharpened version of Theorem 4 from the recent paper [15].

Lemma 1. *Suppose that there exists a C^1 function $V_1 : \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha_{1,x}, \alpha_{2,x}, \rho_x, \sigma$, and a continuous positive definite function $\alpha_{3,x} : [0,\infty) \to [0,\infty)$ satisfying*

$$\alpha_{1,x}(|x|) \le V_1(x) \le \alpha_{2,x}(|x|) \tag{3}$$

and

$$V_1(x) \ge \rho_x(|\mu|) \vee \sigma(|w|) \quad \Rightarrow \quad \nabla V_1(x) f(x,\mu,w) \le -\alpha_{3,x}(V_1(x)). \tag{4}$$

Then the x-subsystem is ISS with respect to (μ, w), with gain $\gamma_x := \alpha_{1,x}^{-1} \circ \rho_x$ from μ to x.

The condition (3) simply says that V_1 is positive definite and radially unbounded. We can take $\alpha_{3,x}$ to be of class $\mathcal{K}_\infty$ with no loss of generality [3]. The condition (4) can be equivalently rewritten as $\nabla V_1(x) f(x,\mu,w) \le -\alpha_{4,x}(V_1(x)) + \chi_x(|\mu|)$ for some $\alpha_{4,x}, \chi_x \in \mathcal{K}_\infty$. However, using the latter condition instead of (4) in the lemma would in general lead to a more conservative ISS gain. We also note that Lemma 1 can be easily generalized by allowing V_1 to depend on t as well as on x, leaving the bounds in (3) unchanged, and adding the time derivative of V_1 in (4); we will work with a Lyapunov function of this kind in Theorem 2 below.

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[3] This amounts to modifying the hypotheses by replacing ISS with a suitable input-to-output stability notion (cf. [2, 14]) and requiring that the ISS gain from the "hidden" states in each subsystem to the states of interest in the other subsystem be 0.

Lemma 2. *Suppose that there exists a C^1 function $V_2 : \mathbb{R}^k \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha_{1,\mu}, \alpha_{2,\mu}, \rho_\mu$, and a continuous positive definite function $\alpha_{3,\mu} : [0,\infty) \to [0,\infty)$ satisfying*

$$\alpha_{1,\mu}(|\mu|) \le V_2(\mu) \le \alpha_{2,\mu}(|\mu|) \tag{5}$$

such that we have

$$V_2(\mu) \ge \rho_\mu(|x|) \quad \Rightarrow \quad V_2(R_\mu(x,\mu)) - V_2(\mu) \le -\alpha_{3,\mu}(V_2(\mu)) \tag{6}$$

and

$$V_2(\mu) \le \rho_\mu(r) \text{ and } |x| \le r \quad \Rightarrow \quad V_2(R_\mu(x,\mu)) \le \rho_\mu(r). \tag{7}$$

Suppose also that for each $t > t_0$ such that $V_2(\mu(s)) \ge \rho_\mu(\|x\|_{[t_0,s]})$ for all $s \in [t_0,t)$, the number $N(t,t_0)$ of discrete events in the interval $[t_0,t]$ satisfies

$$N(t,t_0) \ge \eta(t-t_0) \tag{8}$$

where $\eta : [0,\infty) \to [0,\infty)$ is an increasing function. Then the μ-subsystem is ISS with respect to x, with gain $\gamma_\mu := \alpha_{1,\mu}^{-1} \circ \rho_\mu$.

We can assume that $\alpha_{3,\mu} \in \mathcal{K}_\infty$ with no loss of generality [16]. The conditions (6) and (7) are both satisfied if we have

$$V_2(R_\mu(x,\mu)) - V_2(\mu) \le -\alpha_{4,\mu}(V_2(\mu)) + \chi_\mu(|x|) \tag{9}$$

for some $\alpha_{4,\mu}, \chi_\mu \in \mathcal{K}_\infty$. Indeed, letting $\rho_\mu(r) := \alpha_{4,\mu}^{-1}(2\chi_\mu(r))$, we see that (6) holds with $\alpha_{3,\mu} := \alpha_{4,\mu}/2$. Decreasing $\alpha_{4,\mu}$ if necessary, assume with no loss of generality that $id - \alpha_{4,\mu} \in \mathcal{K}$ (cf. [17]). We then have

$$\begin{aligned} &V_2(\mu) \le \alpha_{4,\mu}^{-1}(2\chi_\mu(r)) \text{ and } |x| \le r \quad \Rightarrow \\ &\qquad V_2(R_\mu(x,\mu)) \le \chi_\mu(|x|) + (id - \alpha_{4,\mu})\big(\alpha_{4,\mu}^{-1}(2\chi_\mu(r))\big) < \alpha_{4,\mu}^{-1}(2\chi_\mu(|x|)) \end{aligned}$$

and so (7) holds with the same ρ_μ. Moreover, (6) implies (9) and consequently (7) if the map R_μ is continuous at $(x,\mu) = (0,0)$. Still, it is useful to write two separate conditions (6) and (7) if we want the least conservative expression for the ISS gain. The former condition coupled with (8) is the main ingredient for obtaining ISS, while the latter is automatically enforced if, for example, discrete events can only decrease $V_2(\mu)$. An example of a function η that can be used in (8) is $\eta(r) = \frac{r}{\delta_a} - N_0$, where δ_a and N_0 are positive numbers (see [15]). In this case, (8) says that discrete events must happen at least every δ_a units of time on the average, modulo a finite number of events that can be "missed".

Proof of Lemma 2. Let $\bar{t} := \min\big\{t \ge t_0 : V_2(\mu(t)) \le \rho_\mu(\|x\|_{[t_0,t]})\big\} \le \infty$ (this is well defined in view of right-continuity). By virtue of (6), we have $V_2(\mu) - V_2(\mu^-) \le -\alpha_{3,\mu}(V_2(\mu^-))$ at each event time in the interval $[t_0,\bar{t})$. Therefore, there exists a function $\bar{\beta} \in \mathcal{KL}$ such that $V_2(\mu(t)) \le \bar{\beta}(V_2(\mu(t_0)), N(t,t_0))$ for all $t \in [t_0,\bar{t})$; cf. [17]. Invoking (8), we have $V_2(\mu(t)) \le \bar{\beta}\big(V_2(\mu(t_0)), \eta(t-t_0)\big)$ hence $|\mu(t)| \le \alpha_{1,\mu}^{-1}\big(\bar{\beta}\big(\alpha_{2,\mu}(|\mu(t_0)|), \eta(t-t_0)\big)\big) =: \beta_\mu(|\mu(t_0)|, t-t_0)$ for all $t \in [t_0,\bar{t})$. Next, (7) applied with $r := \|x\|_{[t_0,t]}$ at each event time guarantees that $V_2(\mu(t)) \le \rho_\mu(\|x\|_{[t_0,t]})$ hence $|\mu(t)| \le \alpha_{1,\mu}^{-1} \circ \rho_\mu(\|x\|_{[t_0,t]})$ for all $t \ge \bar{t}$. Combining the two bounds for $|\mu(t)|$ gives the desired estimate. □

5 Lyapunov-Based Small-Gain Theorems

Consider again the hybrid system $\mathcal{H}$ defined in Section 2.1 and decomposed as in Figure 1(a). Here we assume for simplicity that $R_x(x,\mu) \equiv x$ (continuous state does not jump at the event times). Theorem 1, applied to this special feedback decomposition, provides sufficient conditions for ISS. The proof of this theorem is based on trajectory analysis. Lemmas 1 and 2 can be used to check the hypotheses of Theorem 1, and involve ISS-Lyapunov functions for the two subsystems. The question naturally arises whether Theorem 1 can be formulated and proved entirely in terms of such Lyapunov functions. Such alternative formulations are available for continuous-time as well as discrete-time small-gain theorems [4, 5], but this issue has not been pursued for hybrid systems.

Here we present a preliminary result in this direction. We denote by t_k, $k = 1, 2, \ldots$ the discrete event times, which we assume to be distinct (with no significant changes, we could allow finitely many discrete events to occur simultaneously). It is also convenient to introduce a special clock variable τ, which counts the time since the most recent discrete event and is reset to 0 at the event times: $\tau(t) := t - t_k$ for $t \in [t_k, t_{k+1})$. It must be noted that the Lyapunov function V constructed in Theorem 2 below depends, besides x and μ, on this variable τ. Therefore, it can really be viewed as a Lyapunov function only if the sequence $\{t_k\}$ is independent of the initial state. Otherwise, the proof of ISS using this function is actually a trajectory-based argument (but it still represents an interesting alternative to a purely time-domain one).

Theorem 2. *Suppose that there exist positive definite, radially unbounded C^1 functions $V_1 : \mathbb{R}^n \to \mathbb{R}$ and $V_2 : \mathbb{R}^k \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions χ_1, χ_2, σ, and positive constants b_1, b_2, c, d, T such that we have*

$$V_1(x) \geq \chi_1(V_2(\mu)) \vee \sigma(|w|) \quad \Rightarrow \quad \nabla V_1(x) f(x,\mu,w) \leq -cV_1(x), \tag{10}$$

$$V_2(\mu) \geq \chi_2(V_1(x)) \quad \Rightarrow \quad V_2(R_\mu(x,\mu)) \leq e^{-d} V_2(\mu), \tag{11}$$

$$V_2(\mu) \leq e^{b_2}\chi_2(e^{b_1}V_1(x)) \quad \Rightarrow \quad V_2(R_\mu(x,\mu)) \leq \chi_2(V_1(x)), \tag{12}$$

the small-gain condition

$$e^{b_1}\chi_1(e^{b_2}\chi_2(r)) < r \qquad \forall\, r > 0 \tag{13}$$

holds, and the discrete events satisfy

$$t_{k+1} - t_k \leq T \qquad \forall\, k \geq 0. \tag{14}$$

Then there exist a locally Lipschitz function $V : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \bar{\sigma}$, a continuous positive definite function $\alpha_3 : [0,\infty) \to [0,\infty)$, and a continuous function $\alpha_4 : [0,T] \times [0,\infty) \to [0,\infty)$ satisfying $\alpha_4(\tau, r) > 0$ when $\tau r \neq 0$, such that for all $\tau \in [0,T]$ and all $(x,\mu) \in \mathbb{R}^n \times \mathbb{R}^k$ the bound

$$\alpha_1(|(x,\mu)|) \leq V(\tau,x,\mu) \leq \alpha_2(|(x,\mu)|) \tag{15}$$

holds and we have

$$\begin{aligned}&V(\tau,x,\mu) \geq \bar{\sigma}(|w|) \quad \Rightarrow \\ &\qquad \dot{V}(\tau,x,\mu) := \frac{\partial V}{\partial \tau}(\tau,x,\mu) + \frac{\partial V}{\partial x}(\tau,x,\mu) f(x,\mu,w) \leq -\alpha_3(|(x,\mu)|) \end{aligned} \tag{16}$$

for the continuous dynamics[4] *and*

$$V(0,x,R_\mu(x,\mu)) - V(\tau,x,\mu) \leq -\alpha_4(\tau,|(x,\mu)|) \tag{17}$$

for the discrete events. Consequently, $\mathcal{H}$ is ISS with respect to w.

In spirit, the hypotheses of Theorem 2 match the hypotheses of Theorem 1 and Lemmas 1 and 2, although there are some differences. We note that the condition (14) can be written as $N(t,s) \geq \frac{t-s}{T}$ for all $t > s \geq t_0$, i.e., it is a strengthened version of (8). For simplicity, we assumed in (10) and (11) that V_1 and V_2 decay at exponential rates. In the special case when the gain functions χ_1 and χ_2 are also linear, b_1 and b_2 in (12) and (13) can be set to 0. Note also that (10) only needs to hold for those states where we have continuous evolution, i.e., where $G(x,\mu) < 0$, while (11) and (12) only need to hold for those states where discrete events occur, i.e., where $G(x,\mu) \geq 0$.

Proof of Theorem 2. We have that V_1 stays constant during the discrete events while V_2 stays constant along the continuous dynamics. First, we want to construct modified functions $\overline{V}_1$ and $\overline{V}_2$ which strictly decrease during the discrete events and the continuous dynamics, respectively, while also enjoying decreasing properties similar to (10)–(12). Pick a number $L_1 \in \big(0, c \wedge (b_1/T)\big)$ and define

$$\overline{V}_1(\tau,x) := e^{L_1 \tau} V_1(x). \tag{18}$$

Using (14), we have

$$V_1(x) \leq \overline{V}_1(\tau(t),x) \leq e^{L_1 T} V_1(x) \qquad \forall\, t, x. \tag{19}$$

Similarly, pick a number $L_2 \in \big(0, (d \wedge b_2)/T\big)$ and define

$$\overline{V}_2(\tau,\mu) := e^{-L_2 \tau} V_2(\mu) \tag{20}$$

to obtain

$$e^{-L_2 T} V_2(\mu) \leq \overline{V}_2(\tau(t),\mu) \leq V_2(\mu) \qquad \forall\, t, \mu. \tag{21}$$

Define $\bar{\chi}_1(r) := e^{L_1 T}\chi_1(e^{L_2 T} r)$ and $\bar{\sigma}(r) := e^{L_1 T}\sigma(r)$. Combining (10), (18), (19), and (21), we have for the continuous dynamics

$$\begin{aligned}&\overline{V}_1(\tau,x) \geq \bar{\chi}_1(\overline{V}_2(\tau,\mu)) \vee \bar{\sigma}(|w|) \quad \Rightarrow \\ &\qquad \frac{\partial \overline{V}_1}{\partial \tau}(\tau,x) + \frac{\partial \overline{V}_1}{\partial x}(\tau,x) f(x,\mu,w) \leq -(c - L_1)\overline{V}_1(\tau,x) \end{aligned} \tag{22}$$

and for the discrete events

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[4] We will define V as a maximum of two C^1 functions, hence the gradient $\partial V/\partial x$ is in general not defined at the points where these two functions are equal. However, the derivative of $V(x(\cdot))$ with respect to time exists everywhere and is continuous almost everywhere along each trajectory. This is sufficient for establishing ISS; cf. [4].

$$\overline{V}_1(0,x) = e^{-L_1\tau}\overline{V}_1(\tau,x). \tag{23}$$

Similarly, the evolution of $\overline{V}_2$ satisfies

$$\frac{\partial \overline{V}_2}{\partial \tau}(\tau,\mu) = -L_2\overline{V}_2(\tau,\mu), \tag{24}$$

$$\overline{V}_2(\tau,\mu) \geq \chi_2(\overline{V}_1(\tau,x)) \quad \Rightarrow \quad \overline{V}_2(0,R_\mu(x,\mu)) \leq e^{-(d-L_2T)}\overline{V}_2(\tau,\mu), \tag{25}$$

$$\overline{V}_2(\tau,\mu) \leq \chi_2(\overline{V}_1(\tau,x)) \quad \Rightarrow \quad \overline{V}_2(0,R_\mu(x,\mu)) \leq \chi_2(\overline{V}_1(\tau,x)). \tag{26}$$

The condition (13) implies $\bar{\chi}_1 \circ \chi_2(r) < r$ for all $r > 0$, which is equivalent to $\chi_2(r) < \bar{\chi}_1^{-1}(r)$ for all $r > 0$. As in [4], pick a C^1, class $\mathcal{K}_\infty$ function ρ with

$$\rho'(r) > 0 \qquad \forall\, r > 0 \tag{27}$$

such that

$$\chi_2(r) < \rho(r) < \bar{\chi}_1^{-1}(r) \qquad \forall\, r > 0. \tag{28}$$

We are now ready to define a (time-varying) candidate ISS-Lyapunov function for the closed-loop system $\mathcal{H}$ as

$$V(\tau,x,\mu) := \begin{cases} \rho(\overline{V}_1(\tau,x)) & \text{if } \rho(\overline{V}_1(\tau,x)) \geq \overline{V}_2(\tau,\mu) \\ \overline{V}_2(\tau,\mu) & \text{if } \rho(\overline{V}_1(\tau,x)) < \overline{V}_2(\tau,\mu) \end{cases} \tag{29}$$

We claim that it satisfies (15)–(17). To prove this, pick arbitrary $\tau \in [0,T]$ and $(x,\mu) \neq (0,0)$. Let us first consider the case when $V(\tau,x,\mu) \geq \bar{\sigma}(|w|)$. We further distinguish between the following two cases.

Case 1: $\rho(\overline{V}_1(\tau,x)) \geq \overline{V}_2(\tau,\mu)$, so that $V(\tau,x,\mu) = \rho(\overline{V}_1(\tau,x))$. If $\rho(\overline{V}_1(\tau,x)) > \overline{V}_2(\tau,\mu)$, then we have, using (22), (27), (28), and positive definiteness of V_1 and V_2, that $x \neq 0$ and

$$\begin{aligned} \dot{V}(\tau,x,\mu) &= \rho'(\overline{V}_1(\tau,x))\left(\frac{\partial \overline{V}_1}{\partial \tau}(\tau,x) + \frac{\partial \overline{V}_1}{\partial x}(\tau,x)f(x,\mu,w)\right) \\ &\leq -\rho'(\overline{V}_1(\tau,x))(c-L_1)\overline{V}_1(\tau,x) < 0 \end{aligned}$$

If $\rho(\overline{V}_1(\tau,x)) = \overline{V}_2(\tau,\mu)$, then by positive definiteness of V_1 and V_2 both x and μ are nonzero and, invoking also (24), we have

$$\begin{aligned} \dot{V}(\tau,x,\mu) &= \rho'(\overline{V}_1(\tau,x))\left(\frac{\partial \overline{V}_1}{\partial \tau}(\tau,x) + \frac{\partial \overline{V}_1}{\partial x}(\tau,x)f(x,\mu,w)\right) \vee \frac{\partial \overline{V}_2}{\partial \tau}(\tau,\mu) \\ &\leq -\rho'(\overline{V}_1(\tau,x))(c-L_1)\overline{V}_1(\tau,x) \;\vee\; -L_2\overline{V}_2(\tau,\mu) < 0 \end{aligned}$$

Turning to the discrete events, we have three possible cases. If $\rho(\overline{V}_1(0,x)) \geq \overline{V}_2(0,R_\mu(x,\mu))$, then from (23) we have $V(0,x,R_\mu(x,\mu)) = \rho(\overline{V}_1(0,x)) = \rho(e^{-L_1\tau}\overline{V}_1(\tau,x)) \leq \rho(\overline{V}_1(\tau,x)) = V(\tau,x,\mu)$, and the inequality is strict if

$\tau > 0$. If $\rho(\overline{V}_1(0,x)) < \overline{V}_2(0, R_\mu(x,\mu))$ and $\overline{V}_2(\tau,\mu) \geq \chi_2(\overline{V}_1(\tau,x))$, then (25) gives $V(0,x,R_\mu(x,\mu)) = \overline{V}_2(0,x,R_\mu(x,\mu)) < \overline{V}_2(\tau,\mu) \leq \rho(\overline{V}_1(\tau,x)) = V(\tau,x,\mu)$. Finally, if $\rho(\overline{V}_1(0,x)) < \overline{V}_2(0,R_\mu(x,\mu))$ and $\overline{V}_2(\tau,\mu) \leq \chi_2(\overline{V}_1(\tau,x))$, then using (26) we obtain $V(0,x,R_\mu(x,\mu)) = \overline{V}_2(0,x,R_\mu(x,\mu)) \leq \chi_2(\overline{V}_1(\tau,x)) < \rho(\overline{V}_1(\tau,x)) = V(\tau,x,\mu)$.

Case 2: $\rho(\overline{V}_1(\tau,x)) < \overline{V}_2(\tau,\mu)$, so that $V(\tau,x,\mu) = \overline{V}_2(\tau,\mu)$. Using (24) and positive definiteness of V_2, we have $\mu \neq 0$ and $\dot{V}(\tau,x,\mu) = \frac{\partial \overline{V}_2}{\partial \tau}(\tau,\mu) = -L_2\overline{V}_2(\tau,\mu) < 0$. As for the discrete events, (25) and (28) imply that $\overline{V}_2(0,R_\mu(x,\mu)) < \overline{V}_2(\tau,\mu)$. If $\overline{V}_2(0,R_\mu(x,\mu)) > \rho(\overline{V}_1(0,x))$, then we have $V(0,x,R_\mu(x,\mu)) = \overline{V}_2(0,R_\mu(x,\mu)) < \overline{V}_2(\tau,\mu) = V(\tau,x,\mu)$. On the other hand, if $\overline{V}_2(0,R_\mu(x,\mu)) \leq \rho(\overline{V}_1(0,x))$, then by virtue of (23) we have $V(0,x,R_\mu(x,\mu)) = \rho(\overline{V}_1(0,x)) \leq \rho(\overline{V}_1(\tau,x)) < \overline{V}_2(\tau,\mu) = V(\tau,x,\mu)$.

Since V_1 and V_2 are positive definite and radially unbounded, there exist functions $\alpha_{1,x}, \alpha_{2,x}, \alpha_{1,\mu}, \alpha_{2,\mu} \in \mathcal{K}_\infty$ such that (3) and (5) hold. Using (19), (21), and (29), we obtain

$$\rho(\alpha_{1,x}(|x|)) \vee e^{-L_2T}\alpha_{1,\mu}(|\mu|) \leq V(\tau,x,\mu) \leq \rho\big(e^{L_1T}\alpha_{2,x}(|x|)\big) \vee \alpha_{2,\mu}(|\mu|).$$

It is now a routine exercise to construct functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ for which (15) holds. Next, observe that the condition $V(\tau,x,\mu) \geq \bar{\sigma}(|w|)$ was used, via (22), only to prove the decrease of V along the continuous dynamics but not during the discrete events. Thus (16) and (17) are established (constructing α_3 and α_4 is again a simple exercise). Finally, ISS of $\mathcal{H}$ with respect to w follows from (15)–(17) via standard arguments (cf. [3, 15]). □

Remark 1. ISS of $\mathcal{H}$ would still hold if instead of (17) we had the weaker condition $V(0,x,R_\mu(x,\mu)) \leq V(\tau,x,\mu)$, with (16) unchanged. To construct a function V with these properties, we could set $L_1 = 0$ in the above proof, i.e., work with the original function V_1 in place of $\overline{V}_1$; accordingly, we could set $b_1 = 0$, and also the linearity of the right-hand side of (10) in V_1 would not be important. On the other hand, the stronger condition (17) makes the Lyapunov function V more useful for quantifying the effect of the discrete events. In particular, if we impose a dwell-time constraint $t_{k+1} - t_k \geq \varepsilon > 0$ for all $k \geq 0$, then a uniform decrease condition of the form $V - V^- \leq -\bar{\alpha}_4(V^-)$, with $\bar{\alpha}_4$ continuous positive definite, holds for all discrete events, yielding the stronger property of ISS with respect to a "hybrid time domain" in which the continuous time t and the discrete event index k play essentially equivalent roles (see [11]). □

As an alternative to constructing a Lyapunov function strictly decreasing along solutions, we can work with a *weak* Lyapunov function non-strictly decreasing along solutions and apply a LaSalle invariance principle for hybrid systems, such as the one proved in [6] (see also [18] for recent generalizations and improvements). As can be seen from the proof of the result given next, such an approach is perhaps simpler and more natural in the situation at hand, and the relevant hypotheses more closely match those of Theorem 1 and Lemmas 1 and 2.

However, the result has inherent limitations characteristic of LaSalle theorems; in particular, it is restricted to disturbance-free, time-invariant dynamics.

Consider the same hybrid system $\mathcal{H}$ as in Theorem 2, but assume that there are no disturbances, i.e., the continuous dynamics are described by $\dot{x} = f(x, \mu)$. We assume as before that the resulting discrete event times are distinct (the extension to the case when a finite number of discrete events can occur simultaneously is straightforward). We also assume that the behavior of $\mathcal{H}$ is continuous with respect to initial conditions, in the sense defined and characterized in [6].

Theorem 3. *Suppose that there exist positive definite, radially unbounded* C^1 *functions* $V_1 : \mathbb{R}^n \to \mathbb{R}$ *and* $V_2 : \mathbb{R}^k \to \mathbb{R}$*, class* $\mathcal{K}_\infty$ *functions* χ_1, χ_2*, and continuous positive definite functions* $\alpha_1, \alpha_2 : [0, \infty) \to [0, \infty)$ *such that we have*

$$V_1(x) \geq \chi_1(V_2(\mu)) \quad \Rightarrow \quad \nabla V_1(x) f(x, \mu) \leq -\alpha_1(V_1(x)), \tag{30}$$

$$V_2(\mu) \geq \chi_2(V_1(x)) \quad \Rightarrow \quad V_2(R_\mu(x, \mu)) - V_2(\mu) \leq -\alpha_2(V_2(\mu)), \tag{31}$$

$$V_2(\mu) \leq \chi_2(V_1(x)) \quad \Rightarrow \quad V_2(R_\mu(x, \mu)) \leq \chi_2(V_1(x)), \tag{32}$$

the small-gain condition

$$\chi_1 \circ \chi_2(r) < r \qquad \forall\, r > 0 \tag{33}$$

holds, and for each $t > t_0$ *such that* $V_2(\mu(s)) \geq \chi_2(V_1(x(s)))$ *for all* $s \in [t_0, t)$*, the number* $N(t, t_0)$ *of discrete events in the interval* $[t_0, t]$ *satisfies* (8) *for some increasing function* $\eta : [0, \infty) \to [0, \infty)$*. Then there exists a positive definite, radially unbounded, locally Lipschitz function* $V : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ *such that for all* $(x, \mu) \in \mathbb{R}^n \times \mathbb{R}^k$ *we have*

$$\dot{V}(x, \mu) := \frac{\partial V}{\partial x}(x, \mu) f(x, \mu) \leq 0 \tag{34}$$

for the continuous dynamics,[5]

$$V(x, R_\mu(x, \mu)) \leq V(x, \mu) \tag{35}$$

for the discrete events, and there is no forward invariant set except for the origin inside the set $S_1 \cup S_2$*, where* $S_1 := \{(x, \mu) : \dot{V}(x, \mu) = 0,\, G(x, \mu) < 0\}$ *and* $S_2 := \{(x, \mu) : V(x, R_\mu(x, \mu)) = V(x, \mu),\, G(x, \mu) \geq 0\}$*. Consequently,* $\mathcal{H}$ *is GAS.*

As in Theorem 2, the condition (30) only needs to hold for those states where we have continuous evolution, i.e., where $G(x, \mu) < 0$, while (31) and (32) only need to hold for those states where discrete events occur, i.e., where $G(x, \mu) \geq 0$.

Proof of Theorem 3. The condition (33) is equivalent to $\chi_2(r) < \chi_1^{-1}(r)$ for all $r > 0$. As in [6], pick a C^1, class $\mathcal{K}_\infty$ function ρ satisfying (27) and

$$\chi_2(r) < \rho(r) < \chi_1^{-1}(r) \qquad \forall\, r > 0. \tag{36}$$

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[5] See footnote 4.

Define a candidate weak Lyapunov function for $\mathcal{H}$ as

$$V(x,\mu) := \begin{cases} \rho(V_1(x)) & \text{if } \rho(V_1(x)) \geq V_2(\mu) \\ V_2(\mu) & \text{if } \rho(V_1(x)) < V_2(\mu) \end{cases}$$

This function is positive definite and radially unbounded by construction. We now prove that it satisfies (34) and (35). We consider two cases, similarly to the proof of Theorem 2.

Case 1: $\rho(V_1(x)) \geq V_2(\mu)$, so that $V(x,\mu) = \rho(V_1(x))$. If $\rho(V_1(x)) > V_2(\mu)$, then we have, using (27), (30), (36), and positive definiteness of V_1 and V_2, that $x \neq 0$ and

$$\dot{V}(x,\mu) = \rho'(V_1(x))\frac{\partial V_1}{\partial x}(x)f(x,\mu) \leq -\rho'(V_1(x))\alpha_1(V_1(x)) < 0.$$

If $\rho(V_1(x)) = V_2(\mu)$ then, since V_2 stays constant along the continuous dynamics, we have $\dot{V}(x,\mu) \leq -\rho'(V_1(x))\alpha_1(V_1(x)) \vee 0 \leq 0$. We know that the discrete events do not change the value of $\rho(V_1(x))$. If $V_2(\mu) \geq \chi_2(V_1(x))$, then using (31) we have $V_2(x, R_\mu(x,\mu)) \leq V_2(\mu) \leq \rho(V_1(x))$. If $V_2(\mu) \leq \chi_2(V_1(x))$, then with the help of (32) we obtain $V_2(x, R_\mu(x,\mu)) \leq \chi_2(V_1(x)) \leq \rho(V_1(x))$. In either case we have $V_2(R_\mu(x,\mu)) \leq \rho(V_1(x))$, hence $V(x, R_\mu(x,\mu)) = \rho(V_1(x)) = V(x,\mu)$.

Case 2: $\rho(V_1(x)) < V_2(\mu)$, so that $V(x,\mu) = V_2(\mu)$. For the continuous dynamics, we have $\dot{V}(x,\mu) = 0$. As for the discrete events, (31) and (36) imply that $V_2(R_\mu(x,\mu)) < V_2(\mu)$. If $V_2(R_\mu(x,\mu)) > \rho(V_1(x))$, then $V(x, R_\mu(x,\mu)) = V_2(R_\mu(x,\mu)) < V_2(\mu) = V(x,\mu)$. If $V_2(R_\mu(x,\mu)) \leq \rho(V_1(x))$, then we have $V(x, R_\mu(x,\mu)) = \rho(V_1(x)) < V_2(\mu) = V(x,\mu)$.

The properties (34) and (35) are therefore established. Next, we turn to the claim about the absence of a nonzero invariant set inside $S_1 \cup S_2$. The previous analysis implies that we have $S_1 \subseteq \tilde{S}_1$ and $S_2 \subseteq \tilde{S}_2$, where $\tilde{S}_1 := \{(x,\mu) : \rho(V_1(x)) \leq V_2(\mu),\ G(x,\mu) < 0\}$ and $\tilde{S}_2 := \{(x,\mu) : \rho(V_1(x)) \geq V_2(\mu),\ G(x,\mu) \geq 0\}$. Hence it is enough to prove the claim for $\tilde{S}_1 \cup \tilde{S}_2$. By (36) and the hypotheses placed on the discrete events, no subset of either $\tilde{S}_1$ or $\tilde{S}_2$ can be invariant. Indeed, while the state is in $\tilde{S}_1$, (8) holds and so a discrete event must eventually occur, which means that the state must leave $\tilde{S}_1$. On the other hand, since consecutive discrete events are assumed to be separated by positive intervals of continuous evolution, $\tilde{S}_2$ is not invariant. It remains to show that discrete events cannot take the state from $\tilde{S}_2 \setminus \{(0,0)\}$ to $\tilde{S}_1$. Consider an arbitrary $(x,\mu) \in \tilde{S}_2 \setminus \{(0,0)\}$. If $V_2(\mu) \geq \chi_2(V_1(x))$, then from (31) we have $V_2(x, R_\mu(x,\mu)) < V_2(\mu) \leq \rho(V_1(x))$. If $V_2(\mu) \leq \chi_2(V_1(x))$, then from (32) we have $V_2(x, R_\mu(x,\mu)) \leq \chi_2(V_1(x)) < \rho(V_1(x))$. We conclude that $(x, R_\mu(x,\mu))$ cannot be in $\tilde{S}_1$, which establishes the claim.

Stability in the sense of Lyapunov and boundedness of all solutions follow from (34), (35), and the fact that V is positive definite and radially unbounded. Since $\mathcal{H}$ is non-blocking and deterministic by construction, the invariance principle for hybrid systems from [6] applies. To conclude GAS, we need to rule out the existence of an invariant set other than the origin inside the set on which V does not strictly decrease. But this latter set is $S_1 \cup S_2$, and we are done. □

We see that although the function V in Theorem 3 is a weak Lyapunov function, it has the right properties for applying the LaSalle invariance principle and concluding GAS. However, for other purposes (such as, for example, analyzing stability under perturbations of the right-hand side) it is still desirable to have a strictly decreasing Lyapunov function. One may try to construct such a Lyapunov function by modifying V (e.g., see results of this kind for continuous systems under appropriate "detectability" conditions in [19] and "observability" conditions in [20]).

6 Conclusions and Future Work

The main purpose of this paper was to bring the small-gain analysis method to the attention of the hybrid systems community. We argued that general hybrid systems can be viewed as feedback interconnections of simpler subsystems, and thus the small-gain analysis framework is very naturally applicable to them. While the small gain theorem based on time-domain analysis provides an "off-the-shelf" tool for studying stability of hybrid systems, Lyapunov function constructions are also of interest and were addressed in this paper. For a class of hybrid systems satisfying the conditions of the small-gain theorem, we described a construction of a Lyapunov function and another construction of a weak Lyapunov function, each of which can be used to establish stability.

Further research is needed for improving Lyapunov function constructions of Section 5, which are currently not quite satisfactory. First, Theorem 2 falls short of recovering the result of Theorem 1. Second, both Theorem 2 and Theorem 3 are restricted to the special feedback interconnection shown in Figure 1(a). Another direction for future work is to systematically exploit the proposed method in application-motivated contexts. As demonstrated in the companion paper [7] (see also [13] and the subsequent work [21]), quantized control and networked control systems represent very promising application areas, but we expect the small-gain analysis to be useful for hybrid systems arising in many other areas as well.

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