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Output-input stability and feedback stabilization of multivariable nonlinear control systems

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Abstract— **We study the recently introduced notion of output-input stability, which is a robust variant of the minimum-phase property for general smooth nonlinear control systems. This paper develops the theory of output-input stability in the multi-input, multi-output setting. We show that output-input stability is a combination of two system properties, one related to detectability and the other to left-invertibility. For systems affine in controls, we derive a necessary and sufficient condition for output-input stability, which relies on a global version of the nonlinear structure algorithm. This condition leads naturally to a globally asymptotically stabilizing state feedback strategy for affine output-input stable systems.**

I. Introduction

For systems with inputs, two properties of interest are asymptotic stability under zero inputs and bounded state response to bounded inputs. It is well known that for linear time-invariant systems the first property implies the second one, but for nonlinear systems this is not the case. The notion of *input-to-state stability* (ISS) introduced in [15] captures both of the above properties. Its definition requires the state of the system to be bounded by a suitable function of the input, modulo a decaying term depending on initial conditions. This guarantees that bounded inputs produce bounded states and inputs converging (or equal) to zero produce states converging to zero.

Dual concepts of detectability result if one considers systems with outputs. For linear systems, one of the equivalent ways to define detectability is to demand that the state converge to zero along every trajectory for which the output is identically zero. The notion of *output-to-state stability* (OSS) introduced in [17] is a robust version of the detectability property for nonlinear systems and a dual of ISS. Its definition requires the state of the system to be bounded by a suitable function of the output plus a decaying term depending on initial conditions. This ensures that the state is bounded if the output is bounded and converges to zero if the output converges to zero.

The present line of work is concerned with the *minimum-phase* property of systems with both inputs and outputs. A linear system is minimum-phase if whenever the output is identically zero, both the state and the input must converge to zero; in the frequency domain, this is characterized by stability of system zeros. Byrnes and Isidori [2] provided an important and natural extension of the minimum-phase property to nonlinear systems (affine in controls). According to their definition, a system is minimum-phase if its *zero dynamics*—the internal dynamics of the system under the action of an input that holds the output constantly at zero—are asymptotically stable.

The above remarks suggest that to complete the picture, one should have a robust version of the minimum-phase property, which should ask the state and the input to be bounded when the output is bounded and to become small when the output is small. Such a concept was proposed in the recent paper [9] under the name of *output-input stability*. Its definition requires the state and the input of the system to be bounded by a suitable function of the output and derivatives of the output, modulo a decaying term depending on initial conditions. The resulting property is in general stronger than the minimum-phase property[1] defined in [2]. Output-input stability can be investigated with the help of the tools that have been developed over the years to study ISS, OSS, and related notions. As discussed in [9], the concept of output-input stability finds applications in feedback stabilization, adaptive control, and other areas.

The results of [9] provide a fairly complete theory of output-input stable single-input, single-output (SISO) nonlinear control systems. In this paper we continue to study the output-input stability property for multi-input, multi-output (MIMO) systems. Our goal is to investigate a connection between output-input stability and structural properties of control systems which have been studied in the context of system inversion. In particular, we show the relevance of the nonlinear structure algorithm in establishing output-input stability. Our main result is that under a global regularity assumption, this algorithm yields an equivalent characterization of output-input stability for systems affine in controls. As an application, we demonstrate that every square affine output-input stable system covered by this result can be globally asymptotically stabilized by state feedback. After providing necessary definitions in Section II, establishing preliminary results in Section III, and reviewing the nonlinear structure algorithm in Section IV, we present our main result for affine systems in Section V and then address the feedback stabilization problem in Section VI. For proofs, see [8].

[1] Strictly speaking, this statement only makes sense for systems affine in controls, because otherwise the minimum-phase property is not defined. For example, the scalar system $\dot{y} = 1 + y^2 + u^2$ is output-input stable (since $|u| \leq \sqrt{\dot{y}}$) but not minimum-phase (no input can hold the output at 0).

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II. Background and preliminary results

Consider the system

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= h(x)\end{aligned} \tag{1}$$

where the state x takes values in $\mathbb{R}^n$, the input u takes values in $\mathbb{R}^m$, the output y takes values in $\mathbb{R}^p$ (for some positive integers n, m, and p), and the functions f and h are smooth. In this paper we restrict admissible input (or "control") signals to be at least continuous. For every initial condition $x(0)$ and every input $u(\cdot)$, there is a solution $x(\cdot)$ of (1) defined on a maximal interval $[0, T_{\max})$, and the corresponding output $y(\cdot)$. We write $\mathcal{C}^k$ for the space of k times continuously differentiable functions $u : [0,\infty) \to \mathbb{R}^m$, where k is some nonnegative integer. Whenever the input u is in $\mathcal{C}^k$, the derivatives $\dot{y}$, $\ddot{y}$, $\dots$, $y^{(k+1)}$ exist and are continuous; they are given by

$$y^{(i)}(t) = H_i\big(x(t), u(t), \dots, u^{(i-1)}(t)\big),\ \ i = 1,\dots,k+1$$

where for $i = 0,1,\dots$ the functions $H_i : \mathbb{R}^n \times (\mathbb{R}^m)^i \to \mathbb{R}^p$ are defined recursively via $H_0 := h$ and

$$H_{i+1}(x, u_0, \dots, u_i) := \frac{\partial H_i}{\partial x} f(x, u_0) + \sum_{j=0}^{i-1} \frac{\partial H_i}{\partial u_j} u_{j+1}$$

(here the arguments of H_i are $x \in \mathbb{R}^n$ and $u_0, \dots, u_{i-1} \in \mathbb{R}^m$). Given integers $1 \le i \le j \le l$ and an $\mathbb{R}^l$-valued signal z, we will denote by $z_{i\dots j}$ the vector given by components i through j of z, i.e.,

$$z_{i\dots j} := (z_i, \dots, z_j)^T.$$

We will let $\|\cdot\|_{[a,b]}$ denote the supremum norm of a signal restricted to an interval $[a,b]$, i.e., $\|z\|_{[a,b]} := \sup\{|z(s)| : a \le s \le b\}$, where $|\cdot|$ is the standard Euclidean norm.

According to Definition 1 of [9], the system (1) is called *output-input stable* if there exist a positive integer N, a class $\mathcal{KL}$ function[2] β, and a class $\mathcal{K}_\infty$ function γ such that for every $x(0)$ and every $u \in \mathcal{C}^{N-1}$ the inequality

$$\left|\begin{pmatrix} x(t)\\ u(t)\end{pmatrix}\right| \le \beta(|x(0)|, t) + \gamma\left(\left\|\begin{pmatrix} y\\ \dot{y}\\ \vdots\\ y^{(N)}\end{pmatrix}\right\|_{[0,t]}\right) \tag{2}$$

holds for all t in the domain of the corresponding solution. (The assumption that u belongs to $\mathcal{C}^{N-1}$ is made to guarantee that $y^{(N)}$ is well defined, and can be weakened if the function H_N is independent of u_{N-1}.)

[2]Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If $\alpha \in \mathcal{K}$ is unbounded, then it is said to be of *class* $\mathcal{K}_\infty$. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \ge 0$ and $\beta(s,t)$ decreases to 0 as $t \to \infty$ for each fixed $s \ge 0$.

It is perhaps best to interpret output-input stability as a combination of two separate properties of the system. The first one is expressed by the inequality

$$|x(t)| \le \beta(|x(0)|, t) + \gamma\left(\left\|\begin{pmatrix} y\\ \dot{y}\\ \vdots\\ y^{(N)}\end{pmatrix}\right\|_{[0,t]}\right) \tag{3}$$

and corresponds to detectability (OSS) with respect to the output and its derivatives, uniform over inputs. Following [9], we will say that the system (1) is *weakly uniformly 0-detectable of order* N if the inequality (3) holds, or just *weakly uniformly 0-detectable* when an order is not specified. The results of [7], [17] imply that the system (1) is weakly uniformly 0-detectable of order N if there exists a continuously differentiable, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ and class $\mathcal{K}_\infty$ functions α, χ such that

$$\frac{\partial V}{\partial x} f(x, u_0) \le -\alpha(|x|) + \chi\left(\left|\begin{pmatrix} H_0(x)\\ \vdots\\ H_N(x, u_0, \dots, u_{N-1})\end{pmatrix}\right|\right) \tag{4}$$

for all $x, u_0, \dots, u_{N-1}$. As explained in [9], the class of weakly uniformly 0-detectable systems includes all affine systems in global normal form with ISS inverse dynamics.

The second ingredient of the output-input stability property is described by the inequality

$$|u(t)| \le \beta(|x(0)|, t) + \gamma\left(\left\|\begin{pmatrix} y\\ \dot{y}\\ \vdots\\ y^{(N)}\end{pmatrix}\right\|_{[0,t]}\right) \tag{5}$$

which says that the input should become small if the output and its derivatives are small. Loosely speaking, this suggests that the system has a stable left inverse in the input-output sense. Unlike uniform detectability, this property does not seem to admit a Lyapunov-like characterization. In the SISO case it is closely related to the existence of a relative degree; see [9, Theorem 1]. In general, however, this second property needs to be understood better, which is precisely the goal of the present paper. In the next section we formulate and study a useful property which, in combination with (3), yields (5).

III. Input-bounding property

Let us say that the system (1) has the *input-bounding property* if there exist a positive integer k^* and two class $\mathcal{K}_\infty$ functions ρ_1 and ρ_2 such that we have

$$|u_0| \le \rho_1(|x|) + \rho_2\left(\left|\begin{pmatrix} H_1(x, u_0)\\ \vdots\\ H_{k^*}(x, u_0, \dots, u_{k^*-1})\end{pmatrix}\right|\right) \tag{6}$$

for all $x, u_0, \dots, u_{k^*-1}$. Defined in this way, the input-bounding property represents a functional relation between the input and state variables, but one can recast this property in terms of trajectories of the system. We point out that the input-bounding property resembles in its appearance the notion of relative degree as defined in [9] but is actually much less restrictive, especially for MIMO systems. The next result reveals the connection between output-input stability, weak uniform 0-detectability, and the input-bounding property.

Proposition 1 *The system* (1) *is output-input stable if and only if it is weakly uniformly 0-detectable and has the input-bounding property.*

Proposition 1 explains the importance of the input-bounding property. As we show next, a natural way of checking this property for systems affine in controls is provided by a global variant of the nonlinear structure algorithm.

IV. NONLINEAR STRUCTURE ALGORITHM

From now on, we restrict attention to the case when $m \le p$ and the system (1) is affine in controls, i.e., takes the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) \end{aligned} \tag{7}$$

Its dynamics can also be written in more detail as

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i.$$

We assume that $f(0) = 0$ and $h(0) = 0$ (although the second assumption is only made for convenience and can be removed). All functions are assumed to have the smoothness required for all relevant derivatives to exist. Dimensions and arguments of vectors and matrices will be omitted when clear from the context.

The construction described below is based on Singh's algorithm for nonlinear system inversion [14]; this is a generalization of Hirschorn's nonlinear structure algorithm [3], which in turn is an extension of Silverman's linear structure algorithm [12], [13]. This algorithm can be used to generate a left inverse system driven by the output y and its derivatives. It corresponds to the zero dynamics algorithm for an extended system with respect to the output $y - h(x)$, and the dynamics of the left inverse reduces to the zero dynamics of the original system when driven by $y \equiv 0$; see [6]. (The differential-geometric interpretation reveals the intrinsic, coordinate-independent nature of the algorithm.) This algorithm is also closely related to the dynamic extension algorithm used to solve the dynamic state feedback input-output decoupling problem (see [10, Sections 8.2 and 11.3] for details). We now present its global version[3] suitable for our purposes (cf. [5, Section 11.5]).

[3]It is straightforward to obtain local counterparts of our results, which would utilize the more commonly used local constructions to characterize an appropriately defined local variant of output-input stability.

STEP 1. We have

$$\dot{y} = \tilde{h}_1(x) + \tilde{J}_1(x)u \tag{8}$$

where $\tilde{h}_1(x) := \dfrac{\partial h}{\partial x}(x)f(x)$ and $\tilde{J}_1(x) := \dfrac{\partial h}{\partial x}(x)G(x)$. Assume that the matrix $\tilde{J}_1(x)$ has a constant rank r_1 and a fixed set of r_1 rows (empty if $r_1 = 0$) that are linearly independent for all x. Applying a permutation if necessary, we take these rows to be the first r_1 rows of $\tilde{J}_1(x)$. Partitioning all vectors in the formula (8) accordingly, we write $\dot{y}_{1\dots r_1} = h_1(x) + J_1(x)u$ and

$$\dot{y}_{r_1+1\dots p} = \hat{h}_1(x) + \hat{J}_1(x)u \tag{9}$$

where $h_1(x)$ and $\hat{h}_1(x)$ are given by the first r_1 and the last $p - r_1$ components of the vector $\tilde{h}_1(x)$, respectively, $J_1(x)$ is a matrix of full row rank, and $\hat{J}_1(x) \equiv F_1(x)J_1(x)$ for some $(p - r_1) \times r_1$ matrix $F_1(x)$. Substituting this last equation into (9), we have

$$\dot{y}_{r_1+1\dots p} = \bar{h}_1(x, \dot{y}_{1\dots r_1}) \tag{10}$$

where $\bar{h}_1(x, \dot{y}_{1\dots r_1}) := \hat{h}_1(x) + \big(\dot{y}_{1\dots r_1} - h_1(x)\big)F_1(x)$.

STEP 2. Differentiating the formula (10), we obtain

$$\ddot{y}_{r_1+1\dots p} = \tilde{h}_2(x, \dot{y}_{1\dots r_1}, \ddot{y}_{1\dots r_1}) + \tilde{J}_2(x, \dot{y}_{1\dots r_1})u \tag{11}$$

where

$$\tilde{h}_2 := \frac{\partial \bar{h}_1}{\partial x}(x, \dot{y}_{1\dots r_1})f(x) + \sum_{i=1}^{r_1} \frac{\partial \bar{h}_1}{\partial \dot{y}_i}(x, \dot{y}_{1\dots r_1})\ddot{y}_i$$

and $\tilde{J}_2(x, \dot{y}_{1\dots r_1}) := \dfrac{\partial \bar{h}_1}{\partial x}(x, \dot{y}_{1\dots r_1})G(x)$. Assume that the matrix $\begin{pmatrix} J_1(x) \\ \tilde{J}_2(x, \dot{y}_{1\dots r_1}) \end{pmatrix}$ has a constant rank r_2 and there is a fixed set of $r_2 - r_1$ rows (empty if $r_2 = r_1$) of $\tilde{J}_2(x, \dot{y}_{1\dots r_1})$ which together with the rows of $J_1(x)$ form a linearly independent set for all x and $\dot{y}_{1\dots r_1}$. Without loss of generality, we assume that these are the first $r_2 - r_1$ rows of $\tilde{J}_2(x, \dot{y}_{1\dots r_1})$. Then (11) gives

$$\ddot{y}_{r_1+1\dots r_2} = h_2(x, \dot{y}_{1\dots r_1}, \ddot{y}_{1\dots r_1}) + J_2(x, \dot{y}_{1\dots r_1})u$$

and

$$\ddot{y}_{r_2+1\dots p} = \hat{h}_2(x, \dot{y}_{1\dots r_1}, \ddot{y}_{1\dots r_1}) + \hat{J}_2(x, \dot{y}_{1\dots r_1})u \tag{12}$$

where $h_2(x, \dot{y}_{1\dots r_1}, \ddot{y}_{1\dots r_1})$ and $\hat{h}_2(x, \dot{y}_{1\dots r_1}, \ddot{y}_{1\dots r_1})$ are given by the first $r_2 - r_1$ and the last $p - r_2$ components of the vector $\tilde{h}_2(x, \dot{y}_{1\dots r_1}, \ddot{y}_{1\dots r_1})$, respectively, $J_2(x, \dot{y}_{1\dots r_1})$ is a matrix of full row rank, and $\hat{J}_2(x, \dot{y}_{1\dots r_1}) \equiv F_2(x, \dot{y}_{1\dots r_1})J_2(x, \dot{y}_{1\dots r_1})$ for some $(p - r_2) \times (r_2 - r_1)$ matrix $F_2(x, \dot{y}_{1\dots r_1})$. Using this last equation, we can rewrite (12) as

$$\ddot{y}_{r_2+1\dots p} = \bar{h}_2(x, \dot{y}_{1,\dots,r_1}, \ddot{y}_{1,\dots,r_2})$$

where $\bar{h}_2 := \hat{h}_2 + \big(\ddot{y}_{r_1+1\dots r_2} - h_2\big)F_2$.

STEP k. Differentiating the formula

$$y^{(k-1)}_{r_{k-1}+1\dots p} = \bar{h}_{k-1}\Big(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}}\Big)$$

obtained at step $k-1$, we have

$$y^{(k)}_{r_{k-1}+1\dots p} = \tilde{h}_k\Big(x, \dot{y}_{1\dots r_1}, \dots, y^{(k)}_{1\dots r_{k-1}}\Big) + \tilde{J}_k\Big(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}}\Big)u \qquad (13)$$

where

$$\tilde{h}_k := \frac{\partial \bar{h}_{k-1}}{\partial x} f(x) + \sum_{j=1}^{k-1}\sum_{i=1}^{r_j} \frac{\partial \bar{h}_{k-1}}{\partial y_i^{(j)}} y_i^{(j+1)}$$

and

$$\tilde{J}_k := \frac{\partial \bar{h}_{k-1}}{\partial x}\Big(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}}\Big)G(x).$$

The global regularity assumption that we need to make at this general step reads as follows.

ASSUMPTION 1. The $p\times m$ matrix

$$\begin{pmatrix} J_1(x) \\ J_2(x, \dot{y}_{1\dots r_1}) \\ \vdots \\ \tilde{J}_k\Big(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}}\Big) \end{pmatrix}$$

has a constant rank r_k and there is a fixed set of $r_k - r_{k-1}$ rows (empty if $r_k = r_{k-1}$) of the matrix $\tilde{J}_k(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}})$ which together with the rows of $J_1(x)$, $\dots$, $J_{k-1}(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-2)}_{1\dots r_{k-2}})$ form a linearly independent set for all x, $\dot{y}_{1\dots r_1}$, $\dots$, $y^{(k-1)}_{1\dots r_{k-1}}$.

After a possible permutation, we take the desired rows of $\tilde{J}_k(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}})$ to be the first $r_k - r_{k-1}$ rows of this matrix. Then we use (13) to write

$$y^{(k)}_{r_{k-1}+1\dots r_k} = h_k + J_k u \qquad (14)$$

and

$$y^{(k)}_{r_k+1\dots p} = \hat{h}_k + \hat{J}_k u \qquad (15)$$

where h_k and $\hat{h}_k$ are given by the first $r_k - r_{k-1}$ and the last $p - r_k$ components of the vector $\tilde{h}_k(x, \dot{y}_{1\dots r_1}, \dots, y^{(k)}_{1\dots r_{k-1}})$, respectively, $J_k(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}})$ is a matrix of full row rank, and $\hat{J}_k \equiv F_k J_k$ for some $(p-r_k)\times(r_k - r_{k-1})$ matrix $F_k(x, \dot{y}_{1\dots r_1}, \dots, y^{(k-1)}_{1\dots r_{k-1}})$. In view of the last equation, (15) implies that

$$y^{(k)}_{r_k+1\dots p} = \bar{h}_k\Big(x, \dot{y}_{1\dots r_1}, \dots, y^{(k)}_{1\dots r_k}\Big)$$

where $\bar{h}_k := \hat{h}_k + \Big(y^{(k)}_{r_{k-1}+1\dots r_k} - h_k\Big)F_k$.

By construction, $r_1 \le r_2 \le \dots \le m$. If for some k^* we have $r_{k^*} = m$, then the algorithm terminates. If $r_n < m$, then such a k^* does not exist.

The following example illustrates the application of the algorithm to a system satisfying Assumption 1 at each step.

EXAMPLE 1. Consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 + x_2 u_1 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= -x_4 + x_1^2 \\ y &= (x_1, x_2)^T \end{aligned} \qquad (16)$$

The output derivatives are given by

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ x_2 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

so $r_1 = 1$. We have

$$\dot{y}_2 = x_3 + x_2 u_1 = x_3 + x_2 \dot{y}_1. \qquad (17)$$

Differentiating this equation yields

$$\ddot{y}_2 = x_3\dot{y}_1 + x_2\ddot{y}_1 + \begin{pmatrix} x_2\dot{y}_1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \qquad (18)$$

The matrix $\begin{pmatrix} 1 & 0 \\ x_2\dot{y}_1 & 1 \end{pmatrix}$ is nonsingular for all x and $\dot{y}_1$, hence $r_2 = 2$ and the algorithm terminates with $k^* = 2$. □

V. OUTPUT-INPUT STABILITY OF AFFINE SYSTEMS

Theorem 1 *Let Assumption 1 hold for each $k \ge 0$. Then the system (7) is output-input stable if and only if it is weakly uniformly 0-detectable and the algorithm of Section IV gives $r_{k^*} = m$ for some k^*.*

EXAMPLE 1 (continued). Consider again the system (16). We have $|u_1| = |\dot{y}_1|$, while from the formula (18) we obtain

$$u_2 = \ddot{y}_2 - x_3\dot{y}_1 - x_2\ddot{y}_1 - x_2\dot{y}_1^2$$

hence

$$|u_2| \le |\ddot{y}_2| + \frac{1}{2}x_3^2 + \frac{1}{2}\dot{y}_1^2 + x_2^2 + \frac{1}{2}\ddot{y}_1^2 + \frac{1}{2}\dot{y}_1^4.$$

Thus the system has the input-bounding property. It is also weakly uniformly 0-detectable of order 1, as is seen from

$$|x_3| = |\dot{y}_2 - y_2\dot{y}_1| \le |\dot{y}_2| + \frac{1}{2}y_2^2 + \frac{1}{2}\dot{y}_1^2$$

and the fact that the equation for x_4, which describes the inverse dynamics, is ISS with respect to x_1. (In view of the above calculations, it is straightforward to check that the Lyapunov-like sufficient condition for weak uniform 0-detectability, expressed by the inequality (4), applies with $V(x) := x^T x$.) Therefore, (16) is output-input stable. □

Remark 1 The above results can be used to establish output-input stability of some nonaffine systems. Note that to have the input-bounding property, we only need to be able to bound—and not necessarily solve for—the input in terms of the state and derivatives of the output. As a simple generalization, consider a system of the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)\gamma_i(u_i)$$

where the functions $\gamma_i, i = 1, \dots, m$ are bounded from below by some class $\mathcal{K}_\infty$ functions. It is easy to show that if the associated "virtual input" system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)v_i$$

is covered by the sufficiency part of Theorem 1 (i.e., if it is weakly uniformly 0-detectable and globally left-invertible), then the original nonaffine system is output-input stable. Of course, left-invertibility of the virtual input system is not necessary. One can even have more inputs than outputs; for example, the scalar system $\dot{y} = u_1^2 + u_2^4$ is clearly output-input stable. □

The next example illustrates what can happen when Assumption 1 is violated.

EXAMPLE 2. Consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 + x_2 u_2 \\ \dot{x}_3 &= u_2 \\ \dot{x}_4 &= -x_4 + x_1^2 \\ y &= (x_1, x_2)^T \end{aligned} \tag{19}$$

We have

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and the rank of the matrix on the right-hand side drops from 2 to 1 when $x_2 = 0$. It is not difficult to show that we can pick a bounded sequence of initial states along which $x_2(0)$ converges to 0, a sequence of values of $u_2(0)$ converging to ∞, and appropriately chosen sequences of values for $\dot{u}_2(0)$, $\ddot{u}_2(0), \dots$ such that the derivatives $\dot{y}_2(0), \ddot{y}_2(0), \dots$ are all kept at zero. Also, let $u_1 \equiv 0$ so that $y_1 \equiv 0$. This implies that the inequality (5) is violated for small t, hence (19) is not output-input stable. (The proof of [9, Theorem 1] contains a general argument along these lines.) □

It is instructive to note that both the system (16) considered in Example 1 and the system (19) considered in Example 2 are minimum-phase, with zero dynamics in both cases being given by $\dot{x}_4 = -x_4$. An important fact not elucidated by zero dynamics is that the minimum-phase property of (16) is "robust" (small $y, \dot{y}, \dots$ force x and u to be small) while the minimum-phase property of (19) is "fragile" (small y, $\dot{y}$, $\dots$ can correspond to arbitrarily large u).

For SISO affine systems, Assumption 1 and the existence of a k^* such that $r_{k^*} = m$ reduce to the property that the system has a uniform relative degree as defined, e.g., in [4]. In the SISO case, output-input stability actually implies the existence of a relative degree for a class of systems which includes systems affine in controls; see [9, Theorem 1]. For MIMO systems, the existence of a uniform (vector) relative degree in the sense of [4] is a sufficient but not necessary condition for the structure algorithm to terminate at a k^* satisfying $r_{k^*} = m$, and neither Assumption 1 nor the existence of a uniform relative degree is necessary for output-input stability. Note that the system considered in Example 1 does not have a uniform relative degree. The next example demonstrates that the system may still be output-input stable when Assumption 1 does not hold.

EXAMPLE 3. The system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_5 + x_4 u_2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u_2 \\ \dot{x}_5 &= u_3 \\ y &= (x_1, x_2, x_3)^T \end{aligned} \tag{20}$$

is output-input stable, as can be seen from the formulas $u_1 = \dot{y}_1$, $u_2 = \ddot{y}_3$, $u_3 = \ddot{y}_2 - \ddot{y}_3^2 - \dot{y}_3\dddot{y}_3$, $x_4 = \dot{y}_3$, and $x_5 = \dot{y}_2 - \dot{y}_3\ddot{y}_3$. However, when we try to apply the nonlinear structure algorithm, we obtain

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_5 \\ x_4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

and the matrix does not have a constant rank[4]. □

For affine systems in global normal form, the weak uniform 0-detectability property amounts to ISS of the inverse dynamics with respect to the outputs and their derivatives[5]. As is well known, ISS admits a necessary and sufficient Lyapunov-like characterization [16]. To transform the affine system (7) to global normal form, one must require completeness of appropriate vector fields, to ensure that the coordinate transformation map defined by the outputs, their derivatives, and the states of the inverse dynamics is onto (see [4, Section 9.1] for SISO systems and [5, Section 11.5] for MIMO systems). Our formulation, which avoids such completeness assumptions, is more general and applies to not necessarily affine systems. However, checking weak uniform 0-detectability in the absence of a global normal form is more difficult, because of the need to handle the states not appearing as states of the reduced inverse dynamics. These states are expressed statically in terms of the outputs, their

[4]The system (20) has trivial zero dynamics ($x \equiv 0$) but $x = 0$ is not a regular point of the zero dynamics algorithm. This problem could be corrected if we allowed greater flexibility in choosing the order of output differentiation (e.g., in the present case, differentiate y_3 before y_2).

[5]More precisely, we need ISS with respect to all possible signals that the outputs and their relevant derivatives can produce; these two notions are in general not the same (see [1]).

derivatives, and the states of the reduced inverse dynamics [6]. In Examples 1 and 3 above, this dependence was rather simple (polynomial), and the desired bound (3) could be obtained. The following example illustrates a different situation.

EXAMPLE 4. Consider the scalar system $\dot{x} = u$, $y = \arctan x$. From the equation $\dot{y} = u/(1+x^2)$ we easily deduce the input-bounding property as before. On the other hand, this system is not weakly uniformly 0-detectable. Indeed, when $u \equiv 0$, all derivatives of y are zero. Since $|y| \leq \pi/2$, it is not possible to obtain a bound of the form $|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|y\|_{[0,t]})$ for all corresponding trajectories. This system also does not admit a global normal form because the map $x \mapsto y$ is not onto. Note that the inverse of this map is given by the solution of the differential equation $dx/dy = 1 + x^2$ with initial condition $x(0) = 0$, which is not globally defined because the vector field $\tilde{g}(x) := 1 + x^2$ is not complete. □

VI. FEEDBACK STABILIZATION

As an application of the above concepts and results, we now discuss the relationship suggested in the title of this paper. Namely, we show that if the affine system (7) with the same number of inputs and outputs satisfies the necessary and sufficient condition for output-input stability provided by Theorem 1, then it can be globally asymptotically stabilized by state feedback. The main idea is that if a feedback law stabilizes the output—more precisely, if the resulting output is a solution of a globally asymptotically stable system—then weak uniform 0-detectability implies that the overall closed-loop system is automatically stabilized. The existence of an output-stabilizing feedback is, in turn, guaranteed by left-invertibility. An explicit construction of a static output-stabilizing state feedback law can be given. (It is known that a dynamic output-stabilizing state feedback can be obtained after rendering the system noninteractive; see [10, Section 8.2] or [4, Section 5.4].) An independent—and more extensive—study of an essentially equivalent static feedback stabilization scheme appears in [11].

Theorem 2 *Suppose that the system* (7) *with* $m = p$ *is weakly uniformly 0-detectable, Assumption 1 holds for each* $k \geq 0$, *and the algorithm of Section IV gives* $r_{k^*} = m$ *for some* k^*. *Then there exists a static state feedback law which makes the closed-loop system globally asymptotically stable.*

EXAMPLE 1 (revisited). For the system (16) we can let, e.g., $u_1 = -x_1$ to obtain $\dot{y}_1 = -x_1 = -y_1$. Substituting this into the equations (17) and (18) gives $\dot{y}_2 = x_3 - x_2 x_1$ and

$$\ddot{y}_2 = -x_3 x_1 + x_2 x_1 + x_2 x_1^2 + u_2.$$

The control $u_2 = x_3 x_1 - x_2 x_1^2 - x_2 - x_3$ makes y_2 satisfy the equation $\ddot{y}_2 = -\dot{y}_2 - y_2$, and the output is stabilized. □

We remark that the system (20) considered in Example 3 can be easily stabilized by static state feedback, even though it fails to satisfy Assumption 1. Indeed, first stabilize x_1 by a linear feedback law $u_1 = k_{11} x_1$, then stabilize x_3 and x_4 by $u_2 = k_{23} x_3 + k_{24} x_4$, and finally stabilize x_2 and x_5 by a linearizing feedback law $u_3 = k_3(x_2, x_3, x_4, x_5)$.

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