STABILIZING A NONLINEAR SYSTEM WITH LIMITED INFORMATION

(EXTENDED ABSTRACT)

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In recent years, extensive research activity has been devoted to the question of how much information a feedback controller really needs in order to stabilize a given system. Questions of this kind are motivated by applications where communication capacity is limited (e.g., a large number of systems sharing the same network cable or wireless medium, microsystems with a large number of sensors and actuators on a single chip) as well as situations where security considerations compel one to transmit as little information as possible. Among the many references on this subject, the ones particularly close in spirit to the present work are [1, 2, 3, 4, 5, 6, 7, 8].

All results developed in the aforementioned papers are limited to linear systems. The work reported here is a first step towards understanding information-based control aspects for nonlinear systems. Specifically, we extend the result and the control scheme described in [8] to nonlinear systems, characterizing the amount of information sufficient for global asymptotic stabilization. An underlying assumption is the existence of a feedback law which stabilizes the system in the case of perfect information and, moreover, provides robustness with respect to measurement errors in the sense of *input-to-state stability* (ISS) as defined in [9]. This assumption is quite restrictive in general, although some results on designing such control laws are available; see [10, 11, 12, 13]. We also note that this assumption can be relaxed in several ways, as will be discussed elsewhere.

The set-up considered in this paper is as follows. The system to be stabilized is

$$\dot{x} = f(x, u) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control variable, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz function satisfying $f(0, 0) = 0$. Control inputs considered in this paper are piecewise Lipschitz continuous. The term "limited information feedback" refers to the following scenario:

SAMPLING. Measurements are to be received by the controller at discrete times 0, τ, 2τ, $\dots$, where $\tau > 0$ is a fixed *sampling period*.

ENCODING. At each of the above sampling times, the measurement received by the controller must be a number in the set $\{0, 1, \dots, N\}$, where N is a fixed positive integer.

Thus the data available to the controller is a stream of integers

$$q_0(x(0)), \quad q_1(x(\tau)), \quad q_2(x(2\tau)), \quad \dots$$

where $q_k(\cdot) : \mathbb{R}^n \to \{0, 1, \dots, N\}$ is, for each k, some *encoding function*. For different values of k we can use different encoding functions. As we will see, it is natural to use the previous values $q_i(x(i\tau))$, $i = 0, \dots, k-1$ to define the function $q_k(\cdot)$. We assume that the controller knows the initial encoding function $q_0(\cdot)$ as well as the rule that defines $q_k(\cdot)$ on the basis of the previously received encoded measurements, so that for each k the function q_k is known to the controller. In other words, there is a communication protocol satisfying the above constraints upon which the process and the controller agree in advance.

We find it convenient to use the infinity norm $\|x\|_\infty := \max\{|x_i| : 1 \le i \le n\}$ on $\mathbb{R}^n$. We let $B^n_\infty(x_0, r)$ denote a ball with respect to this norm with radius r and center x_0, i.e., the hypercube centered at x_0 with edges $2r$:

$$B^n_\infty(x_0, r) := \{x \in \mathbb{R}^n : \|x - x_0\|_\infty \le r\}.$$

Our first goal is to obtain an upper bound on the size of the state. We do this by "zooming out", i.e., expanding the support of the encoding function, fast enough to dominate

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the growth of the state for the uncontrolled system (no feedback is applied at this stage). The following assumption is needed to execute this task.

ASSUMPTION 1. *The unforced system*

$$\dot{x} = f(x, 0) \tag{2}$$

is forward complete. This means that for every initial state $x(0)$ the solution of (2), which we denote by $\xi(x(0), \cdot)$, is defined for all $t \geq 0$.

Set the control u equal to 0. Let $\mu_0 := 1$. Pick a sequence $\mu_1, \mu_2, \dots$ that increases fast enough to dominate the rate of growth of $\|x(t)\|_\infty$ at the times $\tau, 2\tau, \dots$; for example, define $\mu_1 := 2\max_{\|x(0)\|_\infty \leq \tau,\, t\in[0,\tau]} \|\xi(x(0), t)\|_\infty$, $\mu_2 := 2\max_{\|x(0)\|_\infty \leq 2\tau,\, t\in[0,2\tau]} \|\xi(x(0), t)\|_\infty$, and so on. This construction guarantees the existence of an integer $k_0 \geq 0$ such that $\|x(k_0\tau)\|_\infty \leq \mu_{k_0}$. For $k = 0, 1, \dots$, define the encoding function q_k by the formula

$$q_k(x) := \begin{cases} 1 & \text{if } x \in B^n_\infty(0, \mu_k) \\ 0 & \text{otherwise} \end{cases}$$

Then we can take k_0 to be the smallest k for which we have $q_k(x(k\tau)) = 1$. We have thus obtained the bound

$$\|x(k_0\tau)\|_\infty \leq E_0 := \mu_{k_0} \tag{3}$$

using the encoded state measurements with $N = 1$. (Such binary encoding can be realized by a quantizer taking 3^n values; cf. [2].)

The inequality (3) means that the state of the system at time $t = k_0\tau$ lies in $B^n_\infty(0, E_0)$. In other words, $\hat{x}(k_0\tau) := 0$ can be viewed as an estimate of $x(k_0\tau)$ with estimation error of infinity norm at most E_0. Our goal now is to generate state estimates with estimation errors approaching 0 as $t \to \infty$, while using these estimates to compute the feedback law.

ASSUMPTION 2. *The system* (1) *admits a locally Lipschitz feedback law $u = k(x)$ which satisfies $k(0) = 0$ and renders the closed-loop system input-to-state stable (ISS) with respect to measurement errors.* Written in terms of the infinity norm and for piecewise continuous inputs (which is sufficient for our purposes), this condition means that there exist functions[1] $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every initial condition $x(t_0)$ and every piecewise continuous signal e the corresponding solution of the system

$$\dot{x} = f(x, k(x + e)) \tag{4}$$

satisfies

$$\begin{aligned} \|x(t)\|_\infty \leq\ & \beta(\|x(t_0)\|_\infty, t - t_0) \\ & + \gamma\big(\sup\nolimits_{s\in[t_0,t]} \|e(s)\|_\infty\big) \qquad \forall\, t \geq t_0. \end{aligned} \tag{5}$$

Take κ to be some class $\mathcal{K}_\infty$ function with the property that $\kappa(r) \geq \max_{\|x\|_\infty \leq r} \|k(x)\|_\infty$ for all $r \geq 0$. Then

$$\|k(x)\|_\infty \leq \kappa(\|x\|_\infty) \qquad \forall\, x. \tag{6}$$

Let L be the Lipschitz constant for f on the region

$$\{(x, u) : \|x\|_\infty \leq D,\ \|u\|_\infty \leq \kappa(D)\} \tag{7}$$

where

$$D := \beta(E_0, 0) + \gamma\big(\sqrt[n]{N} E_0\big) + \sqrt[n]{N} E_0. \tag{8}$$

Define

$$\Lambda := e^{L\tau} \geq 1. \tag{9}$$

For $t \in [k_0\tau, k_0\tau + \tau)$, let $u(t) = 0$. At time $t = k_0\tau + \tau$, consider the box $B^n_\infty(0, \Lambda E_0)$.

ASSUMPTION 3. *The number $\sqrt[n]{N}$ is an odd integer.* This assumption is made mostly for notational convenience. If $\sqrt[n]{N}$ is not an integer, we can work with some $N' \leq N$ such that $\sqrt[n]{N'}$ is an integer. The reason for taking this integer to be odd is to ensure that the control strategy described below preserves the equilibrium at the origin. By making slight modifications, we can also achieve this property when the above integer is even.

Assumption 3 allows us to define the encoding function q_{k_0+1} as follows: divide $B^n_\infty(0, \Lambda E_0)$ into N equal hypercubic boxes, numbered from 1 to N in some specific way, and let $q_{k_0+1}(x)$ be the number of the box that contains x if $x \in B^n_\infty(0, \Lambda E_0)$, and 0 otherwise. In case x lies on the boundary of several boxes, the value $q_{k_0+1}(x)$ can be chosen arbitrarily among the candidates. If $q_{k_0+1}(x(k_0\tau + \tau)) > 0$, then the encoded measurement specifies a box with edges at most $2\Lambda E_0/\sqrt[n]{N}$ which contains $x(k_0\tau + \tau)$. Letting $\hat{x}(k_0\tau + \tau)$ be the center of this box, we obtain

$$\|\hat{x}(k_0\tau + \tau) - x(k_0\tau + \tau)\|_\infty \leq \Lambda E_0/\sqrt[n]{N}.$$

If $q_{k_0+1}(x(k_0\tau + \tau)) = 0$, we interpret this as an error and return to the "zooming-out" stage described earlier.

For $t \in [k_0\tau + \tau, k_0\tau + 2\tau)$, we apply the control law

$$u(t) = k(\hat{x}(t)) \tag{10}$$

where $\hat{x}(\cdot)$ is the solution of the "copy" of the system (1), given by

$$\dot{\hat{x}} = f(\hat{x}, u)$$

with the initial condition $\hat{x}(k_0\tau + \tau)$ specified before. At time $t = k_0\tau + 2\tau$, consider the box

$$B^n_\infty(\hat{x}(k_0\tau + 2\tau^-), \Lambda^2 E_0/\sqrt[n]{N}).$$

[1] Recall that a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. If α is also unbounded, then it is said to be of class $\mathcal{K}_\infty$. A function $\beta : [0, \infty)\times[0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \to \infty$ for each fixed $r \geq 0$.

To define the encoding function q_{k_0+2}, divide this box into N equal hypercubic boxes and let $q_{k_0+2}(x)$ be the number of the box that contains x or, if it happens that $x \notin B^n_\infty(\hat{x}(k_0\tau + 2\tau^-), \Lambda^2 E_0/\sqrt[n]{N})$, let $q_{k_0+2}(x) = 0$. If we have $q_{k_0+2}(x(k_0\tau + 2\tau)) > 0$, then the encoded measurement singles out a box with edges at most $2\Lambda^2 E_0/(\sqrt[n]{N})^2$ which contains $x(k_0\tau + 2\tau)$. Let $\hat{x}(k_0\tau + 2\tau)$ be the center of this box to obtain

$$\|\hat{x}(k_0\tau + 2\tau) - x(k_0\tau + 2\tau)\|_\infty \leq \Lambda^2 E_0/(\sqrt[n]{N})^2$$

and continue. If $q_{k_0+2}(x(k_0\tau + 2\tau)) = 0$, go back to the "zooming-out" stage.

Repeating the above procedure, we see that as long as the encoded measurements received by the controller are positive, the upper bounds on the norm of the estimation error $\|\hat{x} - x\|_\infty$ at the sampling times $k_0\tau$, $k_0\tau+\tau$, $k_0\tau+2\tau$, ... form a geometric progression with ratio $\Lambda/\sqrt[n]{N}$. The goal of forcing the estimation error to approach 0 motivates our final assumption.

ASSUMPTION 4. *We have* $\Lambda < \sqrt[n]{N}$.

In view of the definition of Λ via the formula (9), this inequality characterizes the trade-off between the amount of information provided by the encoder at each sampling time and the required sampling frequency. This relationship depends explicitly on the Lipschitz constant L which, as we will see, can be interpreted as a measure of expansiveness of the system (1).

Our main result can now be stated as follows. See [14] for further details.

Theorem 1 *Under Assumptions 1–4, the above control law globally asymptotically stabilizes the system* (1).

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