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Stability of Interconnected Switched Systems and Supervisory Control of Time-Varying Plants

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***Abstract*— We discuss stability of a loop consisting of two asynchronous switched systems, in which the first switched system influences the input and the switching signal of the second switched system and the second switched system affects the first switched system's jump map. We show that when the first switched system has a small dwell-time and is switching slowly in the spirit of average dwell-time switching, all the states of the closed loop are bounded. We show how this result relates to supervisory adaptive control of time-varying plants. When the uncertain plant takes the form of a switched system with an unknown switching signal, we show that all the states of the closed-loop control system are guaranteed to be bounded provided that the plant's switching signal varies slowly enough.**

I. Introduction

We study stability of a loop consisting of two switched systems in which the first switched system's jump map is affected by the second switched system and the second switched system's switching is constrained by the first switched system. Assuming that the subsystems of both switched systems are (zero-input) exponentially stable, we want to study stability of the closed loop.

Stability of certain interconnected switched systems/hybrid systems has been studied in [1], [2], [3]. In these works, the connection between the two switched systems is the usual input-output connection and as such, those results are not easily applicable to the type of interconnected switched systems considered in this paper. We provide here new tools for analyzing such interconnected switched systems.

The type of interconnected switched systems in this paper is motivated by supervisory adaptive control of uncertain plants with time-varying parameters (see [4] for discussion on advantages of supervisory adaptive control). In supervisory adaptive control, there are multiple controllers and the active controller at every time is selected by a supervisory unit based on some switching logic (for background on supervisory adaptive control, see, *e.g* [5, Chapter 6]). If the plant's parameter is a constant, we have only one switched system in the closed loop (which comprises the multi-controllers and the supervisory unit). However, if the plant is a switched system itself, we will then have two switched systems in the loop, giving rise to interconnected switched systems.

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The paper's organization is as follows. In Section II, we define notations and symbols. In Section III, we describe interconnected switched systems. In Section IV, we provide a stability result for interconnected switched systems. In Section V, we show how interconnected switched systems arise from supervisory adaptive control of switched plants and how to apply the stability result for interconnected switched systems to study stability of the closed-loop control system. Section VI concludes and discusses future work.

II. Notations

Denote by $(\cdot)_{t_0,t}$, $t \geqslant t_0 \geqslant 0$, the *segmentation operator* such that $(f)_{t_0,t}(\tau) := \begin{cases} f(\tau) & \tau \in [t_0, t) \\ 0 & \text{else.} \end{cases}$ For a vector v, denote by $|\cdot|$ the Euclidean norm and by $\|\cdot\|$ the induced matrix-norm. Denote by $\|(\cdot)\|_{\mathcal{D}}$ the norm such that $\|x\|_{\mathcal{D}} := \sup_{t\in\mathcal{D}} |x(t)|$. For $\lambda > 0$, define the $e^{\lambda t}$*-weighted* $\mathcal{L}_2$ *norm* such that $\|(x)_{t_0,t}\|_{2,\lambda} := \left(\int_{t_0}^{t} e^{-\lambda(t-\tau)}|x(\tau)|^2 d\tau\right)^{\frac{1}{2}}$ $t \geqslant t_0$. Denote by $\|(x)_{t_0,*}\|_{2,\lambda}$ the function of t obtained when we let t be a variable in $\|(x)_{t_0,t}\|_{2,\lambda}$. We refer to $\|\cdot\|_{2,\lambda}$ as the $\mathcal{L}_{2,\lambda}$ norm (the 2 refers to the 2-norm of x and λ refers to the exponentially decaying rate). These norms are popular in functional analysis of input/output properties of systems (see, *e.g.*, [6, Chapter 3] or [7, Chapter 5]).

A *switching signal* s is a piecewise constant right continuous function. The discontinuities of s are called switches or *switching times*. Denote by $N_s(T, t_0)$ the number of switches in the interval $[t_0, T)$. For a switching signal s and a time t, denote by t_s the latest switching time of s before the time t. By convention $t_s = 0$ if t is less than or equal the first switching time of s.

III. Interconnected switched systems

Consider two switched systems. The first switched system, denoted by Γ_s, is of the following form:

$$\Gamma_s : \begin{cases} \dot{x} &= A_s x + v, \\ \dot{\xi} &= -\lambda\xi + |x|^2, \\ y &= x, \end{cases} \tag{1}$$

where $s : [0,\infty) \to \mathcal{P}$ is the switching signal, $\mathcal{P}$ is the index set, (x,ξ) is the state, and v is a bounded disturbance, $\lambda, \gamma > 0$. Assume that $\xi(0) = 0$. The second switched system, denoted by Π_σ, has the same index set $\mathcal{P}$:

$$\dot{z} = E_\sigma z + B_\sigma u \tag{2}$$

where u is the input. Without loss of generality, let $z(0) = 0$.

The two switched systems interact in the following way. For the first switched system, at a switching time τ of the switching signal s, the state satisfies

$$|x(\tau)|^2 \leqslant c_1|x(\tau^-)|^2 + c_2|z(\tau^-)|^2, \tag{3}$$

$$\xi(\tau) \leqslant c_1\xi(\tau^-) + c_2\|(z)_{0,t}\|^2_{2,\lambda} \tag{4}$$

for some $c_1, c_2 > 0$. We assume that there is no state jump at switching times for the second switched systems *i.e.* $z(\tau_i^-) = z(\tau_i^+)$ for all switching times τ_i of the switching signal σ. For the second switched system, the $e^{\lambda t}$-weighted $\mathcal{L}_2$ norm of the input u is bounded in terms of the state ξ of Γ_s:

$$\|(u)_{t_0,t}\|^2_{2,\lambda} \leqslant f_u(\xi(t)) \quad \forall t \geqslant t_0 \geqslant 0 \tag{5}$$

for some increasing function $f_u : [0,\infty) \to [0,\infty)$. The switching signal σ satisfies the following inequality:

$$N_s(t,t_0) \leqslant N_0(\xi(t)) + \frac{t-t_0}{\tau_a} \quad \forall t \geqslant t_0, \tag{6}$$

where $\tau_a > 0$ and N_0 is an increasing function. When N_0 is bounded, the bound is known as chatter bound and the number τ_a is known as *average dwell-time* [8]. However, here N_0 is a function of $\xi(t)$ (and hence, is not bounded a *priori*). With slight abuse of terminology, we also call τ_a average dwell-time in a more general sense than in [8].

IV. STABILITY

Assume that A_p, E_p are Hurwitz for all $p \in \mathcal{P}$. If the switching signal s is a constant signal, then (3) and (4) do not come into effect and Γ_s is a non-switched stable linear system. Therefore, $|x(t)|$ will be bounded for all t. Then ξ and the chatter bound N_0 are also bounded. Then s is an average dwell-time switching signal with a bounded chatter bound. The input is bounded in view of (5). Using the stability result with average dwell-time switching [8], we conclude that z is bounded if τ_a is large enough, regardless of N_0 (see also [9]). However, the situation is more complicated when s is not a constant signal and the stability results [10], [8], [9] are not applicable here.

We assume that N_0 in (6) has the following form:

$$N_0(\xi(t)) := c_3 + c_4 \ln(d + \bar{c}\xi(t)) \tag{7}$$

for some numbers $c_3, c_4, d, \bar{c} > 0$. We also assume that the function f in (5) has the following form

$$f_u(\xi(t)) := c_5(d + \bar{c}\xi(t)) \tag{8}$$

for some $c_5 > 0$. The reason for these forms is that they stem from the analysis of supervisory adaptive control later.

Let $-\lambda_0/2 < 0$ be the largest real part of the eigenvalues of E_p for all $p \in \mathcal{P}$ and $-\hat{\lambda}/2 < 0$ be the largest real part of the eigenvalues of A_p for all $p \in \mathcal{P}$. We will present a series of lemmas that characterize various input-to-state like properties for Π_σ and Γ_s. Proofs are omitted due to space limitation.

A. The switched system Γ_s

The lemma below quantifies how the state of Γ_s is bounded in terms of the states of the closed loop at the latest switching time and the disturbance v.

Lemma 1: For every $\lambda < \hat{\lambda}$, for all $t \geqslant 0$,

$$|x(t)|^2 \leqslant X(t_s)e^{-\hat{\lambda}(t-t_s)} + \hat{\gamma}\|(v)_{t_s,t}\|^2_{2,\hat{\lambda}} \tag{9}$$

$$\xi(t) \leqslant f_1(t_s)e^{-\lambda(t-t_s)} + \tfrac{\hat{\gamma}}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|^2_{2,\lambda} \tag{10}$$

where $X(t_s) := a_0(c_1|x(t_s^-)|^2 + c_2|z(t_s)|^2)$, $f_1(t_s) := c_1a_0\xi(t_s^-) + c_2a_0\|(z)_{0,t_s}\|^2_{2,\lambda} + \frac{c_1a_0}{\hat{\lambda}-\lambda}|x(t_s^-)|^2 + \frac{c_2a_0}{\hat{\lambda}-\lambda}|z(t_s)|^2$ for some constants $a_0, \hat{\gamma} > 0$.

B. The switched system Π_σ

The lemma below characterizes the state z between any interval $[t_0, t)$. Proof is omitted due to space limitation.

Lemma 2: For every $\lambda < \lambda_0$,

$$|z(t)|^2 \leqslant \gamma_1\eta^\kappa(t)e^{\lambda_\circ - \lambda\kappa}|z(t_0)|^2 + \gamma_2\eta^{\kappa+1}(t) \ \forall t \geqslant t_0 \tag{11}$$

for some constants $\gamma_1, \gamma_2 > 0, \kappa > 1$ and $\eta(t) = d + \bar{c}\xi(t)$.

The following lemma characterizes z in terms of all of the states of the closed loop at the latest switching time and the disturbance v. Proof is omitted due to space limitation.

Lemma 3: For every $\lambda < \bar{\lambda}$,

$$|z(t)|^2 \leqslant g(t)e^{-\lambda(t-t_s)} + \gamma_3\nu^{\kappa+1}(t) \tag{12}$$

$$\|(z)_{0,t}\|^2_{2,\lambda} \leqslant \|(z)_{0,t_s}\|^2_{2,\lambda}e^{-\lambda(t-t_s)} + U(t), \ \forall t \geqslant 0 \tag{13}$$

where $U(t) := \frac{1}{\bar{\lambda}-\lambda}g(t)e^{-\lambda(t-t_s)} + \gamma_3\|(\nu^{\frac{\kappa+1}{2}})_{t_s,t}\|^2_{2,\lambda}$, $\bar{\lambda} :=$ $\min\{\lambda_0 - \lambda\kappa, (\kappa+1)\lambda\}$, $\nu(t) := d + \frac{\bar{c}\hat{\gamma}}{\hat{\lambda}-\lambda}\|(v)_{t_s,t}\|^2_{2,\lambda}$, and $g(t) := (\gamma_1 2^{\kappa-1}\bar{c}^\kappa f_1^\kappa(t_s) + \gamma_1 2^{\kappa-1}\|\nu\|^\kappa_{[t_s,t]})|z(t_s)|^2 + \gamma_2\hat{\gamma}2^\kappa\bar{c}^{\kappa+1}f_1^{\kappa+1}(t_s)$ and f_1 is as in Lemma 1.

C. Lyapunov-like function

Let $W(t) := c_1a_0\xi(t) + c_2a_0\|(z)_{0,t}\|^2_{2,\lambda} + \frac{c_1a_0}{\hat{\lambda}-\lambda}|x(t)|^2 + \frac{c_2a_0}{\hat{\lambda}-\lambda}|z(t)|^2$.

Lemma 4: Suppose $\exists \bar{v}$ such that $|v(t)| \leqslant \bar{v}\ \forall t$. We have

$$W(t) \leqslant (\alpha_1 W^\kappa(t_s^-) + \alpha_2)W(t_s^-)e^{-\lambda(t-t_s)} + \alpha_3 \ \forall t \geqslant 0 \tag{14}$$

for some $\alpha_1, \alpha_2, \alpha_3 > 0$. By convention, $W(0^-) = W(0)$.

D. Quantifying slow switching by curves

The variable W satisfies inequality of the following form

$$W(t) \leqslant \rho(W(t_s^-))W(t_s^-)e^{-\lambda(t-t_s)} + \alpha_3$$

for all $t_s > 0$ and $W(t) \leqslant W_0e^{-\lambda t} + \alpha_3\ \forall t \in [0, t_1)$, where ρ is a non decreasing function and $W_0 = \rho(W(0))W(0)$. Define the function

$$\begin{aligned} h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) &:= \rho^{\overline{N}}(M)W_0e^{-(\lambda - \ln\rho(M)/\bar{\tau})\delta_d} \\ &+ \alpha_3 + \alpha_3\rho^{\overline{N}-1}(M)\frac{1}{1 - e^{-(\lambda - \ln\rho(M)/\bar{\tau})\delta_d}} - M. \end{aligned} \tag{15}$$

This h function indeed stems from stability analysis of W. The inequality $h_\rho(M, \overline{N}, \bar{\tau}, W_0) \leqslant 0$ helps quantify the relationship among the initial value of $W(t)$ (via W_0) and a chatter bound, an average dwell-time, and a dwell-time of the switching signal s (via $\overline{N}$, $\bar{\tau}$, and δ_d).

1) Average dwell-time vs. chatter bound curve: Fixed a δ_d. Define the set $\mathcal{A}_{\rho,\delta_d}$ parameterized by W_0 as

$$\mathcal{A}_{\rho,\delta_d}(W_0) := \{(\overline{N}, \bar{\tau}) : \overline{N} \geqslant 1, \bar{\tau} \geqslant \delta_d > 0, \text{ and } \exists M : \tau > \ln \rho(M)/\lambda \text{ and } h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) \leqslant 0\}. \quad (16)$$

Note that for any ρ and W_0, if δ_d is large enough, we can always have $h_{\rho,\delta_d}(M, \overline{N}, \bar{\tau}, W_0) < 0$ if $\overline{N} = 1$ and $M > 2\alpha_3$.

Since h is increasing in $\overline{N}$ and decreasing in $\bar{\tau}$, in view of (16), $\exists$ a function $\bar{\tau} = \phi_{\rho,\delta_d,W_0}(\overline{N})$ that is the lower boundary of $\mathcal{A}_{\rho,\delta_d}(W_0)$ such that $\mathcal{A}_{\rho,\delta_d}(W_0) := \{(n,t) : 1 \leqslant n \leqslant N_{max}, t > \phi_{\rho,\delta_d,W_0}(n)\}$ for some N_{max} (N_{max} can be ∞). We call ϕ_{ρ,δ_d,W_0} an *average dwell-time vs. chatter bound curve*. The function ϕ_{ρ,δ_d,W_0} is not easy to characterize analytically but can be calculated numerically for given $\rho, \alpha_3, \lambda, \kappa$, δ_d, and W_0. For example, for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 0.01, W(0) = 0.01, \kappa = 1.5, \lambda = 0.1, \delta_d = 0.5$, we plot an approximation of ϕ_{ρ,δ_d,W_0} in Fig. 1.

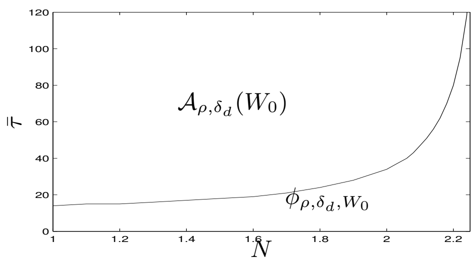

Fig. 1. Average dwell-time vs. chatter bound curve

2) Average dwell-time vs. dwell-time curve: Fixed a $\overline{N}$. Define the set $\mathcal{B}_{\rho,\overline{N}}$ parameterized by W_0 as

$$\mathcal{B}_{\rho,\overline{N}}(W_0) := \{(\bar{\tau}, \delta_d) : \bar{\tau} \geqslant \delta_d > 0, \text{ and } \exists M : \tau > \ln \rho(M)/\lambda \text{ and } h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) \leqslant 0\}. \quad (17)$$

Since h is decreasing in δ_d and also decreasing in $\bar{\tau}$, in view of (17), there exists a function $\bar{\tau} = \psi_{\rho,\overline{N},W_0}(\delta_d)$ that is the lower boundary of $\mathcal{B}_{\rho,\overline{N}}(W_0)$ such that $\mathcal{B}_{\rho,\overline{N}}(W_0) := \{(t,d) : \delta_d^{min} \leqslant d \leqslant \delta_d^{max}, t > \psi_{\rho,\overline{N},W_0}(d)\}$ for some $\delta_d^{max} > \delta_d^{min}$. We call $\psi_{\rho,\overline{N},W_0}$ an *average dwell-time vs. dwell-time curve*. The function $\psi_{\rho,\overline{N},W_0}$ is not easy to characterize analytically but can be calculated numerically for given $\rho, \alpha_3, \lambda, \kappa$, $\overline{N}$, and W_0. For example, for $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 0.01, W(0) = 0.01, \kappa = 1.5, \lambda = 0.1, \overline{N} = 2$, we plot an approximation of $\psi_{\rho,\overline{N},W_0}$ in Fig. 2.

When $\rho(M) \leqslant \bar{\rho}\ \forall M$ for some $\bar{\rho}$, then for every $W_0 \geqslant 0$, $\delta_d > 0$, $\overline{N} \geqslant 1$, and $\tau \geqslant \ln \rho/\lambda$, we can always choose M large enough so that $h_\rho(M, \overline{N}, \bar{\tau}, \delta_d, W_0) < 0$. Therefore, $\mathcal{A}_{\rho,\delta_d}(W_0) = \{(\overline{N}, \bar{\tau}) : \overline{N} \geqslant 1, \bar{\tau} > \ln \bar{\rho}/\lambda\}$, which does not depends on W_0 and δ_d and also, $\mathcal{B}_{\rho,\overline{N}}(W_0) = \{(t,d) : d > 0, t > \ln \bar{\rho}/\lambda, t \geqslant d\}$, which does not depend on $\overline{N}$ and W_0. Then both the sets $\mathcal{A}$ and $\mathcal{B}$ can be characterized by a single number $\ln \bar{\rho}/\lambda$, which is the lower bound on average dwell-time for stability of W (as reported in [8], [9]; see also [10]) (in that case, the two curves are horizonal).

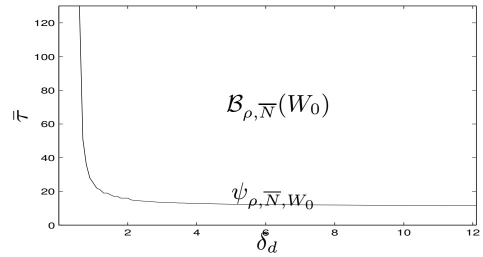

Fig. 2. Average dwell-time vs. dwell-time curve

E. Stability

Let $\rho(M) := \alpha_1 M^\kappa + \alpha_2$ where $\alpha_1, \alpha_2, \kappa$ are the constants as in Lemma 4. Suppose that the initial state is bounded by $X_0 : |x(0)|^2 \leqslant X_0$. Let $W_0 := \rho(\frac{c_1 a_0}{\bar{\lambda} - \lambda} X_0) \frac{c_1 a_0}{\bar{\lambda} - \lambda} X_0$ where c_1, a_0 are as in the definition of $W(t)$. Let ϕ_{ρ,δ_d,W_0} be the average dwell-time vs. chatter bound curve and $\psi_{\rho,\overline{N},W_0}$ be the average dwell-time vs. dwell-time curve defined as in subsection IV-D.

Theorem *1: The interconnected switched system described in Section III has all the states bounded for all $|x(0)|^2 \leqslant X_0$ and for every switching signal s having a dwell-time δ_d, a chatter bound $\overline{N}$, and an average dwell-time $\bar{\tau}$ such that $\bar{\tau} \geqslant \phi_{\rho,\delta_d,W_0}(\overline{N})$ and $\bar{\tau} \geqslant \psi_{\rho,\overline{N},W_0}(\delta_d)$.*

Proof: Proof is omitted due to space limitation. ■

Remark 1: The switching signal s is characterized by both a dwell-time δ_d, an average dwell-time $\bar{\tau}$, $\bar{\tau} > \delta_d$, and a chatter bound $\overline{N}$. For the variable W having property (14), it is not possible to guarantee stability using only average dell-time (we need $\delta_d > 0$). If using dwell-time alone ($\overline{N} = 1$), a stability result will be more restrictive (the dwell-time will be greater than δ_d whereas average dwell-time switching still allows switching intervals as small as δ_d).

V. Supervisory adaptive control

Consider time-varying uncertain plants of the form:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u + v, \\ y &= C(t)x + w, \end{aligned} \quad (18)$$

where the matrices $A(t), B(t), C(t)$ are not known *a priori* and v, w are disturbance and measurement noise, respectively. Assuming that the open loop is unstable, the objective is to stabilize the plant, making $|x(t)| \to 0$ as $t \to \infty$. Various robust adaptive control schemes for time-varying plants have been proposed (see, *e.g.*, [11], [12], [13], [14], [15] and the references therein). These works, more or less, follow the following strategy: use a continuously parameterized controller in parallel with an online parameter estimation scheme. For parameter estimation to work, it is often assumed that the unknown parameter belongs to a known convex set [12], [13] or the sign of high frequency gain is known [13], [11], [15] or some closed-loop signals are rich enough [14], [15]. A notably different approach is [16], in which the author approximates a desired control input directly using alternation of probing and control.

We will approach the problem using the *supervisory adaptive control* framework [17], [18] (see also [19])). The supervisory adaptive control technique has been shown to robustly stabilize uncertain linear plants with constant unknown parameters [17], [18], [20]. If the range of the plant's time-varying parameter is small such that the plant can be approximated by a system with a constant unknown parameter with small time-varying unmodeled dynamics, then the robustness result [20] can be applied for such time-varying plants. However, if the plant's variation is large such that in order to keep unmodeled dynamics small, the plant must be approximated by a switched system with unmodeled dynamics, then supervisory adaptive control of such large-variation time-varying plants remains an open problem (see also a related problem of identification and control of time-varying systems using multiple models [21]).

A. Supervisory adaptive control

Let Ω be the uncertainty set, *i.e.* $(A(t), B(t), C(t)) \in \Omega\ \forall t$. Assume that Ω is compact. We divide Ω into a finite number of non-overlapping subsets such that $\bigcup_{i \in \mathcal{P}} \Omega_i = \Omega$, where $\mathcal{P} = \{1, \dots, m\}$. How to divide and what the number of subsets is are interesting research questions of their own and are not pursued here (see [22]). We approximate the time-varying system (18) by a switched system with perturbation in the following way: construct a piecewise constant signal $s : [0, \infty) \to \mathcal{P}$ as $s(t) := \{i : (A(t), B(t), C(t)) \in \Omega_i\}$, and for every subset Ω_i, $i \in \mathcal{P}$, pick a nominal value $(A_i, B_i, C_i) \in \Omega_i$. Assume that (A_i, B_i) are controllable and (A_i, C_i) are detectable $\forall i \in \mathcal{P}$. We rewrite the plant (18) as

$$\begin{aligned} \dot{x} &= A_{s(t)}x + B_{s(t)}u + \delta_A(t)x + \delta_B(t)u + v, \\ y &= C_{s(t)}x + \delta_C(t)x + w, \end{aligned} \tag{19}$$

where $\delta_A(t) := A(t) - A_{s(t)}$, $\delta_B(t) := B(t) - B_{s(t)}$, and $\delta_C(t) := C(t) - C_{s(t)}$. The plant (19) can be seen as a perturbed version of the following *switched system*

$$\begin{aligned} \dot{x} &= A_{s(t)}x + B_{s(t)}u + v, \\ y &= C_{s(t)}x + w. \end{aligned} \tag{20}$$

In this paper, we will treat the case where the uncertain plant is of the form (20) with unknown signal s and with non-switched output, $C_s(t) = C\ \forall t$. The non-switched output assumption is applicable to, for example, SISO systems $y = G(s,t)u$, in which a constant C of the form $[1\ 0\ \dots 0]$ can be obtained by using state-space realization such that $y = [1\ 0\ \dots 0]x$ while leaving $A(t)$ and $B(t)$ time-varying. We also assume that $w = 0$; for the case $w \neq 0$, see Remark 4 at the end of this section.

The architecture of supervisory adaptive control comprises : 1) a multi-estimator, 2) a family of controllers, and 3) a decision maker (supervisory unit) (see Fig. 3).

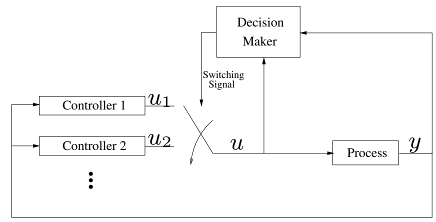

Fig. 3. supervisory adaptive control

Multi-estimator: A multi-estimator with the state $x_E = (\hat{x}_1, \dots, \hat{x}_m)$ is constructed as follows:

$$\dot{\hat{x}}_q = A_q\hat{x}_q + B_q u + L_q(C\hat{x}_q - y) \quad q \in \mathcal{P}, \tag{21}$$

where L_q are such that $A_q + L_qC$ are Hurwitz $\forall q \in \mathcal{P}$. We set $\hat{x}_q(0) = 0\ \forall q \in \mathcal{P}$. Let $\tilde{x}_q := \hat{x}_q - x$ and $\tilde{y}_q := y - C\tilde{x}_q$.
Multi-controllers: A family of *candidate feedback gains* $\{K_q\}$ is designed such that $A_q + B_qK_q$ are Hurwitz for all $q \in \mathcal{P}$. The *family of controllers* is

$$u_q = K_q\hat{x}_q \quad q \in \mathcal{P}. \tag{22}$$

The *injected system* with index $p \in \mathcal{P}$ for some k, $1 \leqslant p \leqslant m$, is formed by the combination of the multi-estimator and the controller with index p:

$$\dot{x}_{\mathbb{CE}} = \bar{A}_p x_{\mathbb{CE}} + \bar{B}(C\hat{x}_p - y), \tag{23}$$

where $\bar{A}_p$ is the square matrix of dimension $\dim(x_{\mathbb{CE}})$. The formula of $\bar{A}_p$ is omitted due to space limitation. From the fact that $A_q + B_qK_q$ and $A_q + L_qC$ are Hurwitz for all $q \in \mathcal{P}$, it follows that $\bar{A}_p$ are Hurwitz for all $p \in \mathcal{P}$.

$\exists$ a family of Lyapunov functions V_q such that $\forall p \in \mathcal{P}$,

$$a_1|x_{\mathbb{CE}}|^2 \leqslant V_p(x_{\mathbb{CE}}) \leqslant a_2|x_{\mathbb{CE}}|^2 \tag{24a}$$

$$\frac{\partial V_p(x_{\mathbb{CE}})}{\partial x}(\bar{A}_p x_{\mathbb{CE}} + \bar{B}\tilde{y}_p) \leqslant -\lambda_\circ V_p(x_{\mathbb{CE}}) + \gamma|\tilde{y}_p|^2. \tag{24b}$$

for some constants $a_1, a_2, \lambda_\circ, \gamma > 0$ (the existence of such common constants for the family of injected systems is guaranteed since we have a finite number of systems).

There exists a number $\mu \geqslant 1$ such that

$$V_q(x) \leqslant \mu V_p(x) \quad \forall x \in \mathbb{R}^n, \forall p, q \in \mathcal{P}. \tag{25}$$

We can always pick $\mu = a_2/a_1$ but there may be other smaller μ satisfying (25) (for example, $\mu = 1$ if V_p are the same for all p even though $a_2/a_1 > 1$).
Supervisory unit: The decision maker, consisting of a *monitoring signal generator* and a switching logic, produces a switching signal that indicates at every time the active controller. The monitoring signals μ_p are generated as

$$\dot{\hat{\mu}}_p = -\lambda\hat{\mu}_p + \gamma|\tilde{y}_p|^2,\ \hat{\mu}_p(0) = 0, \mu_p = \varepsilon + \hat{\mu}_p, \tag{26}$$

for some $\varepsilon > 0$, $\lambda \in (0, \lambda_\circ)$, where $\lambda_\circ$, γ as in (24b). The switching signal is produced by the *scale-independent hysteresis switching logic* [8]:

$$\sigma(t) := \begin{cases} \operatorname*{argmin}_{q \in \mathcal{P}} \mu_q(t) & \text{if } \exists\, q \in \mathcal{P} \text{ such that} \\ & (1+h)\mu_q(t) \leqslant \mu_{\sigma(t^-)}(t), \\ \sigma(t^-) & \text{else,} \end{cases} \tag{27}$$

where $h > 0$ is called a *hysteresis constant*. At a switching time τ of σ, we replace the current controller by the new controller with index $\sigma(\tau)$ and we set $x_{\mathrm{CE}}(\tau) = x_{\mathrm{CE}}(\tau-)$.

The following lemma characterizing σ is [9, Lemma 4.2] with $\bar{\mu}_p = e^{\lambda t}(\varepsilon + \mu_p(t))$ (see also [23]):

Lemma 5: For every $q \in \mathcal{P}$ and for all t_0, we have

$$N_\sigma(t, t_0) \leqslant m + \tfrac{m}{\ln(1+h)} \ln\left(\tfrac{\mu_q(t)}{\varepsilon}\right) + \tfrac{m\lambda(t-t_0)}{\ln(1+h)}, \quad (28)$$

$$\|(\tilde{y}_\sigma)_{t_0,t}\|^2_{2\lambda} \leqslant \tfrac{m(1+h)}{\gamma} \mu_q(t) \quad \forall t \geqslant t_0. \quad (29)$$

B. Design parameters

Let $-\hat{\lambda}/2 < 0$ be the maximum real part of the eigenvalues of $A_p + L_pC$ over all $p \in \mathcal{P}$. For the stability proof, the parameter $h > 0$ is chosen such that

$$\ln(1+h) \leqslant m \ln \mu, \quad (30)$$

then chose $\lambda > 0$ such that

$$(\kappa + 1)\lambda < \lambda_\circ, \ \kappa := \frac{m \ln \mu}{\ln(1+h)}, \quad (31a)$$

$$\lambda < \hat{\lambda}. \quad (31b)$$

Remark 2: For the case of plants with constant unknown parameters, we only need (31a), not the extra conditions (31b) and (30), to prove stability of the closed-loop system [8] (the condition (31a) can be rewritten as $\frac{\ln(1+h)}{\lambda m} > \frac{\ln \mu}{\lambda_\circ - \lambda}$ exactly as in [8]). We can give the conditions (30), (31a), and (31b) the following interpretation: (30) means that the switching logic must be active enough (smaller h) to cope with changing parameters in the plant; (31a) implies that the "forgetting rate" λ of the monitoring signal generator must be less than the "convergence rate" $\lambda_\circ$ of the injected systems; and (31b) can be seen as saying that the "estimation rate" $\hat{\lambda}$ of the multi-estimator must be faster than the "forgetting rate" λ of the monitoring signal generator.

C. Stability

We now show how the interconnected switched system in Section III arises in the supervisory adaptive control context. There are two switched systems in the closed loop:

1) The first one arises from the error dynamics. Then from (21) and (20), since s is constant in $[t_s, t)$, we have $\dot{\tilde{x}}_{s(t_s)}(t) = (A_{s(t_s)} + L_{s(t_s)}C)\tilde{x}_{s(t_s)}(t) + v(t)$. The foregoing equation is rewritten as a switched system

$$\dot{\zeta} = \mathbf{A}_s \zeta + v \quad (32)$$

where $\zeta(t) = \tilde{x}_{s(t_s)}$ and $\mathbf{A}_p = A_p + L_pC$; $\mathbf{A}_p$ are Hurwitz $\forall p \in \mathcal{P}$. We augment ζ by the variable $\xi(t) = \|(x_{t_s})_{0,t}\|^2_{2,\lambda}$ to arrive at the first switched system:

$$\begin{aligned}\dot{\zeta} &= \mathbf{A}_s\zeta + v\\ \dot{\xi} &= -\lambda\xi + |\zeta|^2\end{aligned}. \quad (33)$$

2) The second switched system is the switched injected system from (23):

$$\dot{z} = \mathbf{E}_\sigma z + \bar{B}u_1 \quad (34)$$

where $z(t) := x_{\mathrm{CE}}(t)$, $\mathbf{E}_p := \bar{A}_p$, and $u_1 := (C\hat{x}_p - Cx) = -\tilde{y}_p$. $\mathbf{E}_p$ are Hurwitz $\forall p \in \mathcal{P}$.

These two switched systems interact as follows:

1) Since $\tilde{x}_p(t) + \hat{x}_p(t) = \tilde{x}_q(t) + \hat{x}_q(t) = x(t)\ \forall t, \forall p, q \in \mathcal{P}$, we have $|\tilde{x}_p(t)|^2 \leqslant 2|\tilde{x}_q(t)|^2 + 2|\hat{x}_p - \hat{x}_q|^2$. Then

$$|\tilde{x}_p(t)|^2 \leqslant 2|\tilde{x}_q(t)|^2 + 4|x_{\mathrm{CE}}(t)|^2\ \forall t \quad (35)$$

in view of $x_{\mathrm{CE}} = (\hat{x}_1, \ldots, \hat{x}_q)^T$. Therefore, in view of $\zeta(t) = \tilde{x}_{s(t)}(t)$, we have that at every switching time t_i of the switching signal s,

$$|\zeta(t_i)|^2 \leqslant 2|\zeta(t_i^-)|^2 + 4|z(t_i)|^2. \quad (36)$$

Also from (35), $\|(\tilde{x}_p)_{0,t}\|^2_{2,\lambda} \leqslant 2\|(\tilde{x}_q)_{0,t}\|^2_{2,\lambda} + 4\|(x_{\mathrm{CE}})_{0,t}\|^2_{2,\lambda}$ so

$$\xi(t_s) \leqslant 2\xi(t_s^-) + 4\|(z)_{0,t}\|^2_{2,\lambda}. \quad (37)$$

2) We have $\tilde{y}_p(t) = C\tilde{x}_p(t)\ \forall p \in \mathcal{P}, t \geqslant 0$, so

$$|\tilde{y}_p| \leqslant \gamma_C |\tilde{x}_p|, \quad (38)$$

where $\gamma_C := \|C\|$. From (28), we have

$$N_\sigma(t, t_0) \leqslant N_0(\xi(t)) + \frac{t - t_0}{\tau_a} \quad (39)$$

where $\tau_a = \frac{\ln(1+h)}{m\lambda}$ and $N_0(\xi(t)) := m + \frac{m}{\ln(1+h)}(\ln(\varepsilon + \gamma\gamma_C^2\xi(t)) - \ln\varepsilon)$ in view of $\mu_{s(t)}(t) = \varepsilon + \gamma\|(\tilde{y}_{s(t)})_{0,t}\|^2_{2,\lambda}$, equation (38), and the definition of $\xi(t)$. From (29), (38), and the definition of u_1 in (34), we have

$$\|(u_1)_{0,t}\|^2_{2,\lambda} \leqslant \gamma_1(\varepsilon + \gamma\gamma_C^2\xi(t)),\ \gamma_1 := \tfrac{m(1+h)}{\gamma}. \quad (40)$$

Equations (33), (34), (36), (37), (39), and (40) describe an interconnected switched system in the framework presented in Section III. We then use the stability result in the previous section to conclude about stability of the closed-loop adaptive control system.

For the interconnected switched system with equations (33), (34), (36), (37), (39) and (40), let $\rho(M) := \alpha_1 M^\kappa + \alpha_2$ where $\alpha_1, \alpha_2, \kappa$ are the constants as in Lemma 4, $W_0 := \rho(\frac{c_1 a_0}{\hat{\lambda}-\lambda}X_0)\frac{c_1 a_0}{\hat{\lambda}-\lambda}X_0$ for some $X_0 > 0$. Let ϕ_{ρ,δ_d,W_0} be the average dwell-time vs chatter bound curve and $\psi_{\rho,\overline{N},W_0}$ be the average dwell-time vs. dwell-time curve defined as in subsection IV-D.

Theorem *2: Consider the supervisory adaptive control scheme described in Section V-A. All the states of the closed-loop control system are bounded for all $|x(0)|^2 \leqslant X_0$ and for every switching signal s having a dwell-time δ_d, a chatter bound $\overline{N}$, and an average dwell-time $\bar{\tau}$ such that $\bar{\tau} \geqslant \phi_{\rho,\delta_d,W_0}(\overline{N})$ and $\bar{\tau} \geqslant \psi_{\rho,\overline{N},W_0}(\delta_d)$.*

Proof: Using Theorem 1, we have that $\tilde{x}_{s(t_s)}(t)$, $x_{\mathrm{CE}}(t)$ are bounded for all t. Then $\hat{x}_p$ is bounded for all $p \in \mathcal{P}$. Since $x(t) = \tilde{x}_{s(t_s)}(t) + \hat{x}_{s(t_s)}(t)$, we have x bounded. Since $\tilde{x}_p = x - \hat{x}_p$, we have $\tilde{x}_p$ bounded for all $p \in \mathcal{P}$. Then $\hat{\mu}_p$ are bounded for all $p \in \mathcal{P}$, and hence, μ_p are bounded. ■

Remark 3: The final state bound depends on ε in the following way. The constant $c_3 = \varepsilon - \frac{m}{\ln(1+h)} \ln\varepsilon$ and $d = \varepsilon$ where c_3 and d as in (7). The constants defined as in Section IV for the interconnected switched system arising from the control system in this section depend on ε as follows: $\gamma_1, \gamma_2 \sim \mu^{\varepsilon}\varepsilon^{-\kappa}$, $\gamma_4 \sim (\gamma_1, \gamma_2)$, $\gamma_5 \sim \gamma_1$. Then the constants $\alpha_1, \alpha_2, \alpha_3$ in (14) depend on ε as $\alpha_1 \sim \gamma_4 \sim \varepsilon^{-\kappa}$, $\alpha_2 \sim \gamma_5 + const \sim \varepsilon^{-\kappa} + const$, and $\alpha_3 \sim \gamma_3\varepsilon^{\kappa+1} + \bar{v} \sim \varepsilon + \bar{v}$. The bound on $W(t)$ as $t \to \infty$ is approximately $\alpha_3 + \alpha_3\bar{\mu}^{\overline{N}-1}\frac{1}{1-e^{-\lambda\delta_d}}$ and so, for every $\varepsilon > 0$, there exists $T > 0$ such that $W(t) < \varepsilon + \alpha_3 + \alpha_3\bar{\mu}^{\overline{N}-1}\frac{1}{1-e^{-\lambda\delta_d}}$ $\forall t \geqslant T$. If $\bar{v} = 0$ or if the disturbance is vanishing $\bar{v} \to 0$, we can make the final state bound as small as possible by reducing ε gradually to 0, possibly in a piecewise constant fashion. Note that we cannot simply choose a very small constant ε from the beginning because $\alpha_1, \alpha_2 \uparrow$ as $\varepsilon \downarrow$. However, we can reduce ε later when W as in (14) has become small enough so that $(\alpha_1 W(t_s^-)^{\kappa} + \alpha_2)W(t_s)$ is still bounded by the same bound as before ε is reduced.

Remark 4: The case of measurement noise ($w \neq 0$) can be incorporated in the analysis presented here. The variable v in (32) will become $v - L_{s(t)}w$, which is bounded if v, w are bounded. Equation (34) will have a disturbance $-\bar{B}w$. Equation (38) will be of the form $|\tilde{y}_p| \leqslant \gamma_C|\tilde{x}_p| + \delta_w$ where $\delta_w = \|w\|_\infty$. Equation (40) and formula for $N_0(\xi(t))$ needs to be modified. Some constants in Lemma 1, Lemma 2, Lemma 3, Lemma 4, and Theorem 1 will depend on w but the statements of the lemmas and the theorem are the same.

VI. Conclusions

We considered interconnected switched systems, in which the first switched system affects the input and the switching signal of the second switched system and the second switched system affects the jump map of the first switched system. We provided a stability condition for the closed loop, which says that the first switched system should have a small dwell-time and switch slowly enough on average. Unlike the case of single switched systems where there is a constant lower bound on average dwell-time for stability, we use average dwell-time vs. chatter bound curves and average dwell-time vs. dwell-time curves as lower bounds. We showed how the stability result of interconnected switched systems can be applied to analyze supervisory adaptive control of uncertain time-varying plants in which the plant is a switched system with unknown switching signal and non-switched output and there is no measurement noise.

Remaining to be investigated are supervisory adaptive control of switched plants with switched outputs and with unmodeled dynamics. In these cases, the inequality (5) no longer holds (the input u will be bounded by $\xi(t)$ as well as z). The case will require robustness study of interconnected switched systems. Further study of the average dwell-time vs. chatter bound curves will broaden understanding of the problem of characterizing slowly switching signals.

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