

A note on uniform global asymptotic stability of nonlinear switched systems in triangular form

David Angeli
Dip. Sistemi e Informatica
University of Florence
50139 Firenze, Italy
angeli@dsi.unifi.it

Daniel Liberzon*
Dept. of Electrical Engineering
Yale University
New Haven, CT 06520 USA
daniel.liberzon@yale.edu

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Abstract

This note examines stability properties of systems that result from switching among globally asymptotically stable nonlinear systems in triangular form. We show by means of a counterexample that, unlike in the linear case, such a switched system might not be globally asymptotically stable, uniformly over all switching signals. We then formulate conditions that guarantee this uniform global asymptotic stability property.

1 Introduction

In this note we study systems that result from switching among a given family of asymptotically stable systems. Simple examples show that switching might have a destabilizing effect. A lot of research has recently been devoted to the problem of finding conditions under which a switched system remains asymptotically stable under arbitrary switching. Many references on this subject are listed in the survey article [7].

To formalize the notion of a switched system, suppose that we are given a family of systems

$$\dot{x} = f_p(x), \quad p \in \mathcal{P} \quad (1)$$

where $f_p, p \in \mathcal{P}$ is a family of sufficiently regular (e.g., smooth) vector fields in \mathbb{R}^n , parameterized by some index set \mathcal{P} . The *switched system* is then defined by

$$\dot{x} = f_\sigma(x) \quad (2)$$

where $\sigma : [0, \infty) \rightarrow \mathcal{P}$ is a piecewise constant function of time, called a *switching signal*. We assume that the state of (2) does not jump at the switching instants,

i.e., the solution $x(\cdot)$ is everywhere continuous. The case of infinitely fast switching (chattering), which calls for a concept of generalized solution, is not considered here. Everything that follows can also be applied to the time-varying system $\dot{x} = f_\sigma(x)$ with σ not necessarily piecewise constant.

In the particular case when all the individual subsystems are linear, we obtain a *switched linear system*

$$\dot{x} = A_\sigma x \quad (3)$$

A question that has attracted considerable attention in the literature is the following one: when is the system (3) globally exponentially stable, uniformly over the set of all switching signals σ ? In other words, under what circumstances do there exist positive constants c and λ such that the solution of (3) for every initial state $x(0)$ and every switching signal σ satisfies the estimate $|x(t)| \leq ce^{-\lambda t}|x(0)| \forall t \geq 0$? (It was shown in [2] that this is actually equivalent to the seemingly weaker property that (3) is asymptotically stable for every switching signal.) To ensure this property, it is not sufficient to require that all the individual matrices $A_p, p \in \mathcal{P}$ be Hurwitz; some additional structure on these matrices must be imposed.

It is not hard to show that the switched linear system (3) is asymptotically stable for arbitrary switching if the set of Hurwitz matrices $\{A_p : p \in \mathcal{P}\}$ is compact (with respect to the usual topology in $\mathbb{R}^{n \times n}$) and each matrix in this set takes the upper-triangular form

$$A_p = \begin{pmatrix} (A_p)_{11} & (A_p)_{12} & \dots & (A_p)_{1n} \\ 0 & (A_p)_{22} & \dots & (A_p)_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (A_p)_{nn} \end{pmatrix}$$

(this observation was made, e.g., in [10, 11, 6]). Indeed, the evolution of the last component of the state vector x of the switched linear system is described by $\dot{x}_n = (A_\sigma)_{nn}x_n$. Since $\{A_p : p \in \mathcal{P}\}$ is a compact set of Hurwitz matrices, there exists a number $\epsilon > 0$

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such that $(A_p)_{nn} \leq -\epsilon$ for each $p \in \mathcal{P}$. This implies that $x_n(t)$ decays to zero exponentially. Next, consider $\dot{x}_{n-1} = (A_\sigma)_{n-1,n-1}x_{n-1} + (A_\sigma)_{n-1,n}x_n$. By virtue of the same argument as before, this equation can be viewed as an exponentially stable scalar linear system, forced by an exponentially decaying input $x_n(t)$. Therefore, $x_{n-1}(t)$ also decays to zero exponentially. Continuing this analysis, one proves that the switched system is exponentially stable, uniformly over all switching signals. Alternatively, the same result can be deduced from the fact that in the present case all the systems in the family $\dot{x} = A_p x$, $p \in \mathcal{P}$ share a common quadratic Lyapunov function $V(x) = x^T D x$, where D is a suitably chosen diagonal matrix (see [6] for details).

The goal of this paper is to investigate to what extent the above reasoning applies to nonlinear switched systems. Throughout the paper we make the following assumptions regarding the family of systems (1):

A1 The origin is a globally asymptotically stable equilibrium for all the systems in the family (1).

A2 For each $p \in \mathcal{P}$, the vector field f_p takes the upper-triangular form

$$f_p(x) = \begin{pmatrix} f_{p1}(x_1, x_2, \dots, x_n) \\ f_{p2}(x_2, \dots, x_n) \\ \dots \\ f_{pn}(x_n) \end{pmatrix}$$

A3 The index set \mathcal{P} is a compact subset of some topological space.

A4 The map $p \mapsto f_p(x)$ is continuous for each $x \in \mathbb{R}^n$.

(The last two assumptions are trivially satisfied if \mathcal{P} is a finite set.)

If the linearization matrices $\frac{\partial f_p(x)}{\partial x}(0)$ are all Hurwitz and $\frac{\partial f_p}{\partial x}(x)$ depends continuously on p , then the linearized systems have a common quadratic Lyapunov function by virtue of the result mentioned above. It follows that in this case the original nonlinear switched system (2) is locally exponentially stable, uniformly over σ (this is a fairly straightforward consequence of Lyapunov's 1st method; cf. [6, Corollary 5]). However, in this paper we are concerned primarily with global stability properties of the switched system (2). Working with the nonlinear vector fields directly, one gains an advantage even if only local stability conditions are sought, because the linearization matrices may fail to be Hurwitz.

In Section 2 we show, by means of a counterexample, that for nonlinear systems the triangular structure alone does not guarantee global asymptotic stability of the switched system, uniform over the set of all switching signals. Thus, in order to ensure this uniform global

asymptotic stability property, one must impose some additional conditions. One set of such conditions is proposed in Section 3. They ensure, loosely speaking, that each component of the state vector stays small if the subsequent components are small. We address the question of how these conditions can be verified, and give a construction of a common Lyapunov function. We also discuss a generalization to block-triangular systems. Section 4 contains some concluding remarks.

2 A counterexample

We first recall some standard terminology and necessary definitions. A function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is said to be of class \mathcal{K} if it is continuous, strictly increasing, and $\alpha(0) = 0$. If in addition $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$, then it is said to be of class \mathcal{K}_∞ . A function $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is said to be of class \mathcal{KL} if $\beta(\cdot, t)$ is of class \mathcal{K} for each fixed $t \geq 0$ and $\beta(r, t)$ decreases to 0 as $t \rightarrow \infty$ for each fixed $r \geq 0$.

We will say that the switched system (2) is *uniformly globally asymptotically stable* (with respect to σ taking values in \mathcal{P}) if for each pair of positive numbers ϵ, R there exist positive numbers δ and T such that for every switching signal σ the solution of (2) exists globally and satisfies $|x(t)| \leq \epsilon \forall t \geq 0$ whenever $|x(0)| \leq \delta$ (uniform stability) and $|x(t)| \leq \epsilon \forall t \geq T$ whenever $|x(0)| \leq R$ (uniform attraction). This is equivalent to the existence of a class \mathcal{KL} function β with the property that for every initial state $x(0)$ and every switching signal σ the solution of (2) satisfies

$$|x(t)| \leq \beta(|x(0)|, t) \quad \forall t \geq 0$$

(see, e.g., Proposition 2.5 in [8]). An equivalent characterization of the uniform global asymptotic stability property is the existence of a *common Lyapunov function* for all the systems in the family (1), i.e., a positive definite radially unbounded smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that the inequality

$$\nabla V(x) f_p(x) \leq -\alpha(|x|) \quad \forall p \in \mathcal{P}$$

holds for some positive definite function α ; we can actually assume, without loss of generality, that $\alpha \in \mathcal{K}_\infty$ [8].

One might be tempted to conjecture, on the basis of the result reviewed in the Introduction, that Assumptions A1–A4 guarantee uniform global asymptotic stability of the switched system (2). We now provide a counterexample showing that this is not true. Take $\mathcal{P} = \{1, 2\}$, and consider the following vector fields:

$$f_1(x) = \begin{pmatrix} -x_1 + 2 \sin^2(x_1) x_1^2 x_2 \\ -x_2 \end{pmatrix}$$

$$f_2(x) = \begin{pmatrix} -x_1 + 2 \cos^2(x_1) x_1^2 x_2 \\ -x_2 \end{pmatrix}$$

To see that the system $\dot{x} = f_1(x)$ is globally asymptotically stable, take an arbitrary initial condition $(x_1(0), x_2(0))^T$. We have $x_2(t) = x_2(0)e^{-t}$. As for x_1 , it is not hard to see that $|x_1(t)| \leq E \forall t \geq 0$, where E is the smallest integer multiple of π that is greater than or equal to $|x_1(0)|$. Therefore, there exists a time T such that $|2x_1^2(t)x_2(t)| \leq |x_1(t)|/2 \forall t \geq T$, which implies that $x_1(t) \rightarrow 0$. We conclude that the system is globally attractive; its stability in the sense of Lyapunov easily follows from the same estimates. Global asymptotic stability of the system $\dot{x} = f_2(x)$ is established by a similar argument.

Now, if the switched system (2) defined by these two systems were uniformly globally asymptotically stable, the converse Lyapunov theorem of [8] would guarantee that the two systems have a common Lyapunov function V . This in turn would imply that V is a Lyapunov function for an arbitrary ‘‘convex combination’’ of these systems, given by

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2(\alpha \sin^2(x_1) + (1 - \alpha) \cos^2(x_1))x_1^2x_2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

where $0 \leq \alpha \leq 1$. In particular, for $\alpha = 1/2$ we arrive at the following system:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2x_2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

This system was discussed in [5, p. 8] in the context of adaptive control. Its solutions are given by the formulas

$$\begin{aligned}x_1(t) &= \frac{2x_1(0)}{x_1(0)x_2(0)e^{-t} + (2 - x_1(0)x_2(0))e^t} \\ x_2(t) &= x_2(0)e^{-t}\end{aligned}$$

We see from this that solutions with $x_1(0)x_2(0) = 2$ are unbounded, and solutions with $x_1(0)x_2(0) > 2$ escape to infinity in finite time. Thus the switched system is not uniformly globally asymptotically stable, even though the individual subsystems satisfy Assumptions A1–A4.

3 Stability results

Following Sontag [12], we will say that a general system of the form

$$\dot{x} = f(x, u)$$

is *input-to-state stable* (ISS) with respect to the measurable and locally essentially bounded input u if for some functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, for all initial states $x(0)$, and all u the following estimate holds:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u_t\|) \quad \forall t \geq 0$$

where $\|u_t\| := \text{ess sup}\{ |u(s)| : s \in [0, t] \}$. As established in [14], a necessary and sufficient condition for ISS is the

existence of an *ISS-Lyapunov function*, i.e., a positive definite radially unbounded smooth function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that for some \mathcal{K}_∞ functions α and χ we have

$$\nabla V(x)f(x, u) \leq -\alpha(|x|) + \chi(|u|)$$

We now use this concept to state the following result.

Theorem 1 *If for each $i = 1, \dots, n-1$ and each $p \in \mathcal{P}$ the system*

$$\dot{x}_i = f_{pi}(x_i, x_{i+1}, \dots, x_n)$$

is input-to-state stable (ISS) with respect to the input $u = (x_{i+1}, \dots, x_n)^T$, then the switched system (2) is uniformly globally asymptotically stable.

PROOF. Assumptions A1 and A2 imply that each of the scalar systems $\dot{x}_n = f_{pn}(x_n)$, $p \in \mathcal{P}$ is globally asymptotically stable. In view of Assumptions A3 and A4, it is well known and not hard to show that the scalar switched system $\dot{x}_n = f_{\sigma n}(x_n)$ is uniformly globally asymptotically stable. Equivalently, there exists a positive definite radially unbounded smooth function $V_n : \mathbb{R} \rightarrow \mathbb{R}$ and a class \mathcal{K}_∞ function α_n with the property that

$$V'_n(x_n)f_{pn}(x_n) \leq -\alpha_n(|x_n|) \quad \forall p \in \mathcal{P} \quad (4)$$

Fix an arbitrary $i \in \{1, \dots, n-1\}$. Since all the systems in the family

$$\dot{x}_i = f_{pi}(x_i, x_{i+1}, \dots, x_n), \quad p \in \mathcal{P} \quad (5)$$

are ISS with respect to the input $u = (x_{i+1}, \dots, x_n)^T$, it follows from the results of [14] that there exist functions $\varphi_p \in \mathcal{K}_\infty$, $p \in \mathcal{P}$ such that each one of the auxiliary systems

$$\dot{x}_i = f_{pi}(x_i, \varphi_p(|x_i|)d), \quad p \in \mathcal{P}$$

is uniformly globally asymptotically stable with respect to d taking values in the unit ball of \mathbb{R}^{n-i} . The same conclusion then holds for the systems

$$\dot{x}_i = f_{pi}(x_i, \varphi(|x_i|)d), \quad p \in \mathcal{P}$$

where $\varphi(r) := \min_{p \in \mathcal{P}} \{ \varphi_p(r) \}$. In other words, there exists a common ‘‘gain margin’’ $\varphi \in \mathcal{K}_\infty$. It is now a standard exercise to verify that the inequality

$$W'_i(x_i)f_{pi}(x_i, \varphi(|x_i|)d) \leq -\gamma(|x_i|) \quad \forall p \in \mathcal{P}$$

holds with $W_i(x_i) := x_i^2/2$ and

$$\gamma(r) := \min_{|x_i|=r, |d| \leq 1, p \in \mathcal{P}} \{ -W'_i(x_i)f_{pi}(x_i, \varphi(|x_i|)d) \}$$

Along the same lines as in [14] one can show that this is equivalent to the existence of a common ISS-Lyapunov

function triple (V_i, α_i, χ_i) for the systems in the family (5); in other words, there exist a positive definite radially unbounded smooth function $V_i : \mathbb{R} \rightarrow \mathbb{R}$ and class \mathcal{K}_∞ functions α_i and χ_i such that we have

$$\begin{aligned} V_i'(x_i) f_{pi}(x_i, x_{i+1}, \dots, x_n) &\leq -\alpha_i(|x_i|) \\ &+ \chi_i(|(x_{i+1}, \dots, x_n)^T|) \quad \forall p \in \mathcal{P} \end{aligned} \quad (6)$$

We now have the functions V_1, V_2, \dots, V_n which satisfy the inequalities (4) and (6) for each $i \in \{1, \dots, n-1\}$, with the functions $\alpha_1, \dots, \alpha_n$ and $\chi_1, \dots, \chi_{n-1}$ being of class \mathcal{K}_∞ . In fact, it follows from the findings of [8, 14] that each V_i is of the form $V_i(x_i) = \rho_i(x_i^2)$ for some smooth \mathcal{K}_∞ function ρ_i . Using the technique described in [13], we can modify the functions V_{n-1} and V_n if necessary, to have $\chi_{n-1} = \alpha_n/2$. Then for each $p \in \mathcal{P}$ we will have

$$\begin{aligned} &V_{n-1}'(x_{n-1}) f_{p,n-1}(x_{n-1}, x_n) + V_n'(x_n) f_{pn}(x_n) \\ &\leq -\alpha_{n-1}(|x_{n-1}|) + \chi_{n-1}(|x_n|) - \alpha_n(|x_n|) \\ &= -\alpha_{n-1}(|x_{n-1}|) - \alpha_n(|x_n|)/2 \\ &\leq -\bar{\alpha}_{n-1,n}(|(x_{n-1}, x_n)^T|) \end{aligned}$$

for a suitable choice of $\bar{\alpha}_{n-1,n} \in \mathcal{K}_\infty$. We can write this as

$$\begin{aligned} &\nabla \bar{V}_{n-1,n}(x_{n-1}, x_n) \begin{pmatrix} f_{p,n-1}(x_{n-1}, x_n) \\ f_{pn}(x_n) \end{pmatrix} \\ &\leq -\bar{\alpha}_{n-1,n}(|(x_{n-1}, x_n)^T|) \quad \forall p \in \mathcal{P} \end{aligned}$$

where $\bar{V}_{n-1,n}(x_{n-1}, x_n) := V_{n-1}(x_{n-1}) + V_n(x_n)$. Repeating this procedure, we construct a positive definite radially unbounded smooth function $\bar{V} : \mathbb{R}^n \rightarrow \mathbb{R}$ for which the inequality

$$\nabla \bar{V}(x) f_p(x) \leq -\bar{\alpha}(|x|) \quad \forall p \in \mathcal{P}$$

holds with some $\bar{\alpha} \in \mathcal{K}_\infty$. We conclude that all the systems in the family (1) share a common Lyapunov function, and the statement of the theorem follows. \square

Remark. We saw from the counterexample given in the previous section that global asymptotic stability is not necessarily preserved under switching among triangular systems. Since asymptotic stability always implies ISS locally (i.e., for sufficiently small values of the state and the input [15]), Theorem 1 can be used to show that *local* asymptotic stability can never be lost due to switching.

The assumption that $x_n \in \mathbb{R}$ was used in the above proof to deduce that the switched system $\dot{x}_n = f_{\sigma n}(x_n)$ is uniformly globally asymptotically stable. Likewise, for each $i \in \{1, \dots, n-1\}$ the assumption that $x_i \in \mathbb{R}$ was used to deduce that the family of systems $\dot{x}_i = f_{pi}(x_i, u)$, $p \in \mathcal{P}$, with $u = (x_{i+1}, \dots, x_n)^T$, has a common ISS-Lyapunov function triple (V_i, α_i, χ_i) . The last

property is equivalent to *uniform input-to-state stability* (uniform ISS) of the switched system $\dot{x}_i = f_{\sigma i}(x_i, u)$, which amounts to the existence of functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that the estimate

$$|x_i(t)| \leq \beta(|x_i(0)|, t) + \gamma(\|u_t\|) \quad \forall t \geq 0$$

holds for all switching signals σ , all initial states $x_i(0)$, and all u . Once the above properties are established, the fact that the components x_i , $i = 1, \dots, n$ of the state vector x are scalar is of no significance. The proof could in fact be carried out directly in the time domain from this point on, although we found it more convenient to rely on a Lyapunov argument. These remarks lead at once to the following generalization of Theorem 1.

Theorem 2 *Suppose that Assumption A2 is relaxed to the following:*

A2' *For each $p \in \mathcal{P}$, the vector field f_p takes the block-triangular form*

$$f_p(x) = \begin{pmatrix} f_{p1}(x_1, x_2, \dots, x_k) \\ f_{p2}(x_2, \dots, x_k) \\ \dots \\ f_{pk}(x_k) \end{pmatrix}$$

where $x_i \in \mathbb{R}^{n_i}$, $i = 1, \dots, k$, and $n_1 + \dots + n_k = n$.

Suppose that the switched system $\dot{x}_k = f_{\sigma k}(x_k)$ is uniformly globally asymptotically stable, and that for each $i \in \{1, \dots, k-1\}$ the switched system

$$\dot{x}_i = f_{\sigma i}(x_i, x_{i+1}, \dots, x_k)$$

is uniformly input-to-state stable (uniformly ISS) with respect to the input $u = (x_{i+1}, \dots, x_n)^T$. Then the switched system (2) is uniformly globally asymptotically stable.

As we mentioned earlier, the requirement that the switched system $\dot{x}_k = f_{\sigma k}(x_k)$ be uniformly globally asymptotically stable is automatically satisfied if the last component of the state vector is scalar, i.e., if we have $n_k = 1$.

4 Concluding remarks

As we recalled in the Introduction, a compact family of asymptotically stable linear systems in triangular form gives rise to a uniformly globally asymptotically stable switched linear system. We sketched two possible derivations of this result, one using a direct iterative argument and the other relying on the existence of a common Lyapunov function. Clearly, the conclusion remains valid if the triangular structure is obtained after

applying a linear change of coordinates. It thus follows from *Lie's Theorem* that uniform asymptotic stability is guaranteed if the Lie algebra generated by the individual matrices is *solvable* (see [6] for details, and [1] for further results in this direction).

In this paper we demonstrated that for nonlinear systems the triangular structure does not automatically ensure that a switched system is uniformly globally asymptotically stable (see the counterexample in Section 2). We singled out a set of conditions which can be added to guarantee uniform global asymptotic stability (see Theorem 1 in Section 3). As in the linear case, stability can be established in two different ways: by a direct time-domain analysis or via constructing a common Lyapunov function (we preferred to take the latter approach to prove Theorem 1). We also provided a generalization of this result to block-triangular systems (Theorem 2 in Section 3).

An important and challenging problem for future research is to try to obtain a coordinate-free version of Theorem 1. The first step is to find conditions, formulated in terms of the original data, which would imply that all systems in a given family can be simultaneously triangularized by a suitable change of coordinates. Imposing certain additional assumptions, it is possible to obtain analogues of Lie's Theorem which yield triangular structure in the nonlinear setting (see [3, 4, 9]). However, the methods described in these papers require that the Lie algebra spanned by a given family of vector fields have full rank, and so typically they do not apply to families of systems with common equilibria of the type treated here. Moreover, even if this first task were accomplished, it is not clear under what circumstances the ISS conditions of Section 3 would be satisfied.

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