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A unified approach to controller design for systems with quantization and time scheduling

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Abstract— **We generalize and unify a range of recent results in quantized control systems (QCS) and networked control systems (NCS) literature and provide a unified framework for controller design for control systems with quantization and time scheduling via an emulation-like approach. A crucial step in our approach is finding an appropriate Lyapunov function for the quantization/time-scheduling protocol which verifies its uniform global exponential stability (UGES). We construct Lyapunov functions for several representative protocols that are commonly found in the literature, as well as some new protocols not considered previously.**

I. Introduction

Currently, there are two main approaches to modelling band-limited communication channels in control loops: (i) the channel is digital and due to the finite word length effects only a finite number of bits can be transmitted over the channel at any transmission instant. The main issue in control (stabilization) of systems with such channels is that of *quantization* and we use the term *quantized control systems (QCS)* to denote systems exhibiting this feature; (ii) the channel is a serial bus and only a subset of sensors and/or actuators can transmit their data over the channel at each transmission instant (in this case, the quantization effects are ignored). The main issue in this class of systems is time scheduling of transmissions of various signals in the system and these systems are often referred to in the literature as *networked control systems (NCS)*.

These two modelling approaches evolved separately in the literature with little cross-referencing or cross-fertilization. However, a very similar controller design approach has been proposed for both QCS and NCS and this approach consists of the following steps: (i) ignore the communication constraints of the channels (quantization or time scheduling) and design a controller using the classical techniques; (ii) design/choose a particular quantizer or partition of the sensor/actuator vector, as well as an algorithm (protocol) that governs the quantization or time-scheduling during the operation of the system; (iii) determine a sufficiently small upper bound on the inter-transmission intervals, the so-called *maximal allowable transmission interval (MATI)*, that guarantees the stability of the system. For instance, this controller design approach was proposed in [16] for NCS and it was used in [7] for QCS. We note that this approach is a natural generalization of the *emulation* approach to controller design for sampled-data systems.

The main purpose of this paper is to unify the controller design approach mentioned above for QCS and NCS which naturally leads to the following contributions: (i) We provide a unified framework for the emulation design approach that is flexible, general and amenable to further extensions and modifications; (ii) Our unified controller emulation framework brings two seemingly unrelated areas (QCS and NCS) under one umbrella and facilitates a cross-fertilization between them. For instance, we show that the notion of UGES protocols that was introduced in the NCS literature (see [12]) has a natural interpretation in QCS. We believe that this is the first time in the literature that this connection is made.

We mention that, while there have been no systematic attempts to unify formulations and techniques from the NCS and QCS literature, some specific designs combining quantization and time scheduling have been proposed, for instance, in [2], [11], and [10]. We believe that this fact confirms the need for developing a general framework encompassing a large class of such protocols, which is the goal of this work. In the sequel, we refer to systems that combine time scheduling with quantization as *networked and quantized control systems (NQCS)*.

II. Preliminaries

Given $t \in \mathbb{R}_{\geq 0}$ and a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}^n$, we use the notation $f(t^+) := \lim_{s \searrow t} f(s)$. The Euclidean and infinity norms on $\mathbb{R}^n$ are respectively denoted by $|\cdot|$ and $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$; the corresponding induced matrix norms are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$. Given a piecewise continuous signal $\varphi : [t_0, t] \to \mathbb{R}^n$, we define its $\mathcal{L}_\infty$ norm as follows: $\|\varphi\|_\infty := \sup_{s\in[t_0,t]} |\varphi(s)|$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, zero at zero, strictly increasing and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if for each $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing to zero and for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}$. A function β is said to be of class exp-$\mathcal{KL}$ if there exist $K, c > 0$ such that $\beta(s, t) = K \exp(-ct)s$. To shorten notation, we often use $(x, y) := (x^T \; y^T)^T$. We write diag$\{A_1, \ldots, A_\ell\}$ for the (block-)diagonal matrix with the indicated elements on the diagonal and zeros elsewhere. For the systems we consider in this paper, a monotonically increasing sequence of times $t_i \in \mathbb{R}_{\geq 0}$ is given where $i \in \mathbb{N}$ and $t_0 = 0$. Moreover, we assume that there exist $\varepsilon > 0$ and

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$\tau > \varepsilon$ such that

$$\varepsilon \le t_i - t_{i-1} \ \le \ \tau \qquad \forall\, i \in \mathbb{N}\ . \tag{1}$$

We set the stage by outlining the emulation approach and recalling a result that follows from [12]. The emulation approach was pursued, for instance in [16], [12] for NCS and in [7] for QCS. First, we design a controller for a given plant ignoring the network (i.e. quantization and/or time scheduling). Namely, given a plant

$$\dot{x} \ = \ \tilde{f}(t,x,u) \tag{2}$$

one first designs a "nominal" controller

$$u \ = \ k(t,x)\ . \tag{3}$$

However, in the presence of time scheduling and/or quantization, the state x is not directly available for control. So, one instead closes the loop with the controller

$$u = k(t,\hat{x}) \tag{4}$$

where $\hat{x}$ is an estimate of x generated using quantized state values transmitted over the network. In Section III we will explain how $\hat{x}$ is to be generated in various cases. To model the systems arising in this way, we consider

$$\dot{x} \ = \ f(t,x,z) \qquad \forall\, t \in [t_{i-1}, t_i] \tag{5}$$

$$\dot{z} \ = \ g(t,x,z) \qquad \forall\, t \in [t_{i-1}, t_i] \tag{6}$$

$$z(t_i^+) \ = \ h(i, x(t_i), z(t_i))\ , \tag{7}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, t_i satisfy (1) and $0 < \varepsilon < \tau$. Here (5) gives the closed-loop plant dynamics, while z includes the network-induced error variables as well as some other auxiliary variables needed to implement the quantization procedure. We adopt terminology from [16] and refer to τ as the *maximum allowable transmission interval* (MATI). We can assume that the system $\dot{x} = f(t,x,0)$ is stable in an appropriate sense. Note that this system is the "nominal" closed-loop system without the quantization and/or time scheduling, obtained from (5) by setting $z \equiv 0$. We use standard notions (see [14]) to characterize robustness of the plant with respect to z:

Definition 1 *The system $\dot{x} = f(t,x,z)$ is* input-to-state stable (ISS) *from z to x if there exist functions $\kappa \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that for all $t_0 \ge 0$, $x_0 \in \mathbb{R}^{n_x}$, $z \in \mathcal{L}_\infty$ and each corresponding solution $x(\cdot)$, we have that*

$$|x(t)| \le \beta(|x_0|, t-t_0) + \kappa(\|z\|_\infty) \qquad \forall\, t \ge t_0 \ge 0\ .$$

If the system is ISS with $\kappa(s) = \gamma \cdot s, \gamma \ge 0$ a linear function and $\beta(\cdot,\cdot)$ an exp-$\mathcal{KL}$ function, then we say that the system is ISS with a linear gain and an exp-$\mathcal{KL}$ function. A system with state x and no input ($z \equiv 0$) is uniformly globally asymptotically stable (UGAS) *if for all $x_0 \in \mathbb{R}^{n_x}$ and all corresponding solutions $x(\cdot)$, we have that*

$$|x(t)| \le \beta(|x_0|, t-t_0) \qquad \forall\, t \ge t_0 \ge 0\ .$$

The system is uniformly globally exponentially stable (UGES) *if it is UGAS and β is of class exp-$\mathcal{KL}$.* □

Motivated by the results in [12], we refer to the jump equation (7) as the network "protocol". Our main results in Section V are presented for a large class of "UGES protocols" that are characterized in Definition 2 stated below. To define this class of protocols, we introduce an auxiliary discrete-time system:

$$z^+ = h(i,x,z) \qquad i \in \mathbb{N}\ , \tag{8}$$

where h comes from (7).

Definition 2 [1] *We say that the protocol (7) is* uniformly globally exponentially stable (UGES) with Lyapunov function W *if there exist $W : \mathbb{N} \times \mathbb{R}^{n_z} \to \mathbb{R}_{\ge 0}$, $\rho \in [0,1)$ and $a_1, a_2 > 0$ such that we have*

$$a_1 |z| \ \le \ W(i,z) \ \le \ a_2 |z| \tag{9}$$

$$W(i+1, h(i,x,z)) \ \le \ \rho W(i,z)\ , \tag{10}$$

for all $i \in \mathbb{N}$, $x \in \mathbb{R}^{n_x}$ and all $z \in \mathbb{R}^{n_z}$. □

Section IV contains many examples of various UGES protocols. The following result is a corollary of main results in [12] and it is central to the emulation approach.

Theorem 1 *Consider the system (5)–(7) with (1). Suppose that the following conditions hold:*
(i) System (5) is ISS from z to x with linear gain $\kappa(s) = \gamma \cdot s, \gamma \ge 0$;
(ii) Inequalities (9), (10) hold, i.e. the protocol (7) is Lyapunov UGES with a Lyapunov function $W(i,z)$;
(iii) For some $L, c \ge 0$, we have that W from item (ii) and g in (6) satisfy $\left\langle \frac{\partial W(i,z)}{\partial z}, g(t,x,z) \right\rangle \le LW(i,z) + c\,|x|$ for almost all $z \in \mathbb{R}^{n_z}$, $x \in \mathbb{R}^{n_x}$ and $t \ge 0$.
(iv) MATI in (1) satisfies $\tau \in (\varepsilon, \tau^*)$ where $\tau^* := \frac{1}{L}\ln\left(\frac{L + c\gamma/a_1}{\rho L + c\gamma/a_1}\right)$, $\varepsilon \in (0,\tau^*)$ is arbitrary, L and c come from the item (iii), γ from item (i), a_1 from (9), and ρ from (10).

Then, the system (5)–(7) is UGAS. Moreover, if the system (5) is ISS from z to x with linear gain and exp-$\mathcal{KL}$ *function β, then the system (5)–(7) is UGES.* □

III. SYSTEM MODELS

In this section we demonstrate that models of various NCS and QCS that were previously considered in the literature, as well as NQCS that were not considered in the literature, can be written in the form (5), (6), (7). This modelling framework allows us to unify, generalize, and compare many results in the NCS and QCS literature, such as [12], [16], [7], [8], [15], [2], [11], [10]. For simplicity, we consider static state feedback controllers and plants without disturbances, and we assume that only the state is sent over the network.

[1] This definition was first used in [12] for a smaller class of protocols of the form $z^+ = h(i,z)$ whose right hand side is independent of x. However, with the definition we use here, all main results in [12] still hold for the system (5)–(7).

A. NCS

For the purpose of comparison and to illustrate the unifying nature of our results, we reproduce the class of models for NCS considered in [12], [16] and show that it is a special case of (5), (6), (7). In this and the following subsections, we always assume that (1) holds. Consider a general nonlinear plant (2) where $x \in \mathbb{R}^n$. As we said, we first design a "nominal" stabilizing controller (3). We then implement it over the network in the following manner. We assume that the vector x is partitioned into ℓ, $1 \leq \ell \leq n$ different subvectors enumerated from 1 to ℓ, i.e., $x = (x_1, x_2, \ldots, x_\ell)$. We refer to the i^{th} subvector as the i^{th} "node". At each transmission time t_i, the protocol gives access to the network to one of the nodes $i \in \{1, 2, \ldots, \ell\}$. We consider general nonlinear NCS of the following form:

$$\begin{array}{rcll} \dot{x} & = & \tilde{f}(t, x, k(t, \hat{x})) & \forall\, t \in [t_{i-1}, t_i] \\ \dot{\hat{x}} & = & 0 & \forall\, t \in [t_{i-1}, t_i] \\ \hat{x}(t_i^+) & = & x(t_i) + h(i, e(t_i)) & \end{array} \qquad (11)$$

where $\hat{x}$ is the vector of most recently transmitted plant state values via the network, $e := \hat{x} - x$ is the network-induced error, and we closed the loop with the controller (4). We assume that $\hat{x}$ is held constant between the transmission instants (i.e. we use a zero-order hold in each node). The function h is typically such that, if the j^{th} node gets access to the network at some transmission time t_i, the corresponding part of the error vector is reset to zero. Rewriting the system in (x, e) coordinates, we obtain the following NCS model:

$$\dot{x} = f(t, x, e) \qquad \forall\, t \in [t_{i-1}, t_i] \qquad (12)$$

$$\dot{e} = g(t, x, e) \qquad \forall\, t \in [t_{i-1}, t_i] \qquad (13)$$

$$e(t_i^+) = h(i, e(t_i)) \;, \qquad (14)$$

The system (12), (13), (14) has exactly the same form as (5), (6), (7) if we think of e in (13), (14) as z in (6), (7). The error vector e models the effects of the network and, since we assumed that NCS has ℓ nodes, it can be partitioned as $e = (e_1, e_2, \ldots, e_\ell)$. We refer to the jump equation (14) as a time-scheduling protocol. In [12], time-scheduling protocols with the maps h of the following form were considered:

$$h(i, e) = (I - \Psi(s))e \;, \qquad (15)$$

where $s = s(i, e) : \mathbb{N} \times \mathbb{R}^n \to \{1, \ldots, \ell\}$ is some *scheduling function*,

$$\Psi(s) = \mathrm{diag}\{\delta_{1s} I_{n_1}, \ldots, \delta_{\ell s} I_{n_\ell}\} \;, \qquad (16)$$

ℓ is the number of nodes, δ_{ij} is the standard Kronecker delta, and I_{n_j} are identity matrices of dimension n_j, with $\sum_{j=1}^{\ell} n_j = n$. This protocol model assumes that if the node j is transmitted at time t_i over the network, then $e_j(t_i^+) = 0$. Examples of this class of protocols are given in Section IV-A.

B. QCS

We now consider the model for QCS described in [7] for linear plants and in [8] for nonlinear plants. Models considered elsewhere (e.g., [15], [11], [2]) are very similar. However, in the QCS literature these models have previously not been written in the way we do it here.

Let the plant be given by (2), where $x \in \mathbb{R}^n$. We assume that a nominal feedback law (3) is given. When the state is quantized, this feedback law is not implementable. Instead, the control law will depend not on the state x but on its estimate, $\hat{x}$, which we generate as follows. In between the times t_i, we let

$$\dot{\hat{x}} = \tilde{f}(t, \hat{x}, u) \qquad (17)$$

which is a "copy" of the plant dynamics. The initial condition can be arbitrary, e.g., $\hat{x}(t_0) = 0$. At each t_i, we reset $\hat{x}$ to a new value obtained from the quantized measurement. The quantizer, q, is defined by three parameters: an integer $N > 1$ (a given constant which defines the number of quantization levels), $\hat{x} \in \mathbb{R}^n$ (the current state estimate), and $\xi \in \mathbb{R}_{\geq 0}$ (an auxiliary variable which defines the size of the quantization regions). Consider a hypercubic box centered at $\hat{x}$ with edges 2ξ and divide it into N^n equal smaller sub-boxes (N in each dimension), numbered from 1 to N^n in some specific way.[2] We let $q(x)$ be the number of the sub-box that contains x (provided that x is indeed contained in one of them). The new value of $\hat{x}$ is then defined to be the center of this box. This can be described by a jump equation of the form $\hat{x}(t_i^+) = g(i, q(x(t_i)), \hat{x}(t_i), \xi(t_i))$. We also update ξ to reflect the fact that the size of the box known to contain x was divided by N as the result of the above procedure: $\xi(t_i^+) = \xi(t_i)/N$. Until the time t_{i+1}, we propagate ξ according to some differential equation $\dot{\xi} = g_\xi(t, \xi)$ and then the procedure is repeated. We can think of $\hat{x}$ and ξ as being implemented synchronously on both ends of the communication channel, i.e., in the encoder and the decoder, starting from some known initial values. Having defined $\hat{x}$, we can now also close the loop using the control law (4).

It is convenient to rewrite the closed-loop system dynamics in terms of the quantization-induced error $e := \hat{x} - x$, suppressing the equations for $\hat{x}$:

$$\dot{x} = f(t, x, e) \qquad \forall t \in [t_{i-1}, t_i] \qquad (18)$$

$$\dot{e} = g_e(t, x, e) \qquad \forall t \in [t_{i-1}, t_i] \qquad (19)$$

$$\dot{\xi} = g_\xi(t, \xi) \qquad \forall t \in [t_{i-1}, t_i] \qquad (20)$$

$$e(t_i^+) = h_e(i, x(t_i), e(t_i), \xi(t_i)) \;, \qquad (21)$$

$$\xi(t_i^+) = h_\xi(\xi(t_i)) \qquad (22)$$

This is similar to (12), (13), (14) and becomes a special case of (5), (6) and (7) if we define $z := (e, \xi)$.

To make sure that the quantizer is well defined, we need to assume that the initialization of ξ and its evolution during continuous flows and jumps fulfills the following.

Assumption 1 *A bound on the initial state $x(0)$ is known and ξ and $\hat{x}$ are such that $\|e(t)\|_\infty \leq \xi(t) \qquad \forall\, t \geq 0$.*

[2] In [7] and [8], the total number of sub-boxes is N, i.e., $N^{1/n}$ in each dimension. While our present choice of notation somewhat simplifies the formulas that follow, it is a trivial matter to adapt the results to the notation used in [7], [8]. Similarly, we could let ξ be an n-vector instead of a scalar and thus allow more general rectilinear boxes, as done, e.g., in [2], [15].

This assumption basically means that the quantizer will never saturate. It is easy to enforce it when the plant is linear [7].

C. NQCS

In this subsection, we combine quantization with time scheduling and show that the closed-loop system can be written in the form (5), (6), (7). As a special case, when there is only one node ($\ell = 1$), we obtain the previous QCS model. We note also that a slightly different class of quantization protocols with "zoom" studied in [1], [6], [9] can also be incorporated into our framework.

We again start with the plant (2) and the nominal control law (3). The next step is to implement this controller over a band-limited channel that will involve quantization and time scheduling. Namely, we apply the control law (4), where $\hat{x}$ is an estimate of x generated as explained next. In between the transmission times, $\hat{x}$ is obtained by running a copy of the plant, given by (17), in the encoder and decoder. As in Section III-A, we partition the state vector as $x = (x_1, x_2, \ldots, x_\ell)$, where ℓ is the number of nodes and each x_j has dimension n_j. The vector $\hat{x}$ is partitioned accordingly as $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_\ell)$, and the error vector $e := \hat{x} - x$ is partitioned as $e = (e_1, e_2, \ldots, e_\ell)$. We assume that for each $j \in \{1, 2, \ldots, \ell\}$, we are given a quantizer q_j with parameters $N \in \mathbb{N}$, $\hat{x}_j \in \mathbb{R}^{n_j}$, $\xi_j \in \mathbb{R}_{\geq 0}$ defined as explained in Section III-B. Here we take N to be the same for all j, but this is not necessary (see also footnote 2).

We assume that at any transmission time t_i, a quantized value of only one x_j, $j \in \{1, 2, \ldots, \ell\}$ will be transmitted over the network. To decide which component x_j we are going to transmit, we use a scheduling function $s : \mathbb{N} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^\ell \to \{1, 2, \ldots, \ell\}$:

$$s = s(i, x, e, \xi) \ . \tag{23}$$

Then, we quantize the component $x_{s(t_i)}(t_i)$ using the corresponding quantizer $q_{s(t_i)}$ (which we write simply as q since it is always clear which quantizer is being used) and send the quantized value over the network to the decoder. The decoder takes the transmitted quantized value and resets the value of $\hat{x}_{s(t_i)}$ to be the center of the box corresponding to $q(x_{s(t_i)}(t_i))$, while keeping all other components of $\hat{x}$ unchanged. In other words, at transmission times we have $\hat{x}_j(t_i^+) = g_j(i, q(x_{s(t_i)}(t_i)), \hat{x}_{s(t_i)}(t_i), \xi_{s(t_i)}(t_i))$ if $j = s(t_i)$. This leads to

$$e_j(t_i^+) = \begin{cases} h_j(i) & \text{if } \ j = s(t_i) \\ e_j(t_i) & \text{if } \ j \neq s(t_i) \end{cases} \tag{24}$$

where $h_j(i) = h_j(i, x_{s(t_i)}(t_i), e_{s(t_i)}(t_i), \xi_{s(t_i)}(t_i))$ and $h_j(i, x_s, e_s, \xi_s) := g_j(i, q(x_s), x_s + e_s, \xi_s) - x_s$. We find it convenient to rewrite (24) in the following form:

$$e(t_i^+) = (I - \Psi(s(t_i)))e(t_i) + \Psi(s(t_i))H(i) \ , \tag{25}$$

where $\Psi(s)$ was defined in (16), $H(i) = H(i, x_{s(t_i)}(t_i), e_{s(t_i)}(t_i), \xi_{s(t_i)}(t_i))$, and

$$H := (h_1, \ldots, h_\ell)^T \tag{26}$$

At each transmission time we need to adjust the size of the box (ξ_j) that corresponds to the component x_j that is being transmitted. We consider update laws of the form

$$\xi(t_i^+) = h_\xi(i, s(t_i), \xi(t_i)) \ . \tag{27}$$

In particular, $\xi_{s(t_i)}$ is typically divided by N as before. In between the transmission times, the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_\ell)$ is propagated according to some differential equation of the form (20). To summarize, the closed-loop dynamics are

$$\begin{aligned} \dot{x} &= f(t, x, e) && \forall\, t \in [t_{i-1}, t_i] \\ \dot{e} &= g_e(t, x, e) && \forall\, t \in [t_{i-1}, t_i] \\ \dot{\xi} &= g_\xi(t, \xi) && \forall\, t \in [t_{i-1}, t_i] \\ e(t_i^+) &= h_e(i, x(t_i), e(t_i), \xi(t_i)) \\ \xi(t_i^+) &= h_\xi(i, s(t_i), \xi(t_i)) \ , \end{aligned} \tag{28}$$

where s is given in (23). Note that we are again suppressing the dynamics of $\hat{x}$. Finally, it is clear that this system can be written in the form (5), (6) and (7) by defining $z := (e, \xi)$.

Assumption 2 *A bound on the initial state $x(0)$ is known and ξ and $\hat{x}$ are such that $\|e_i(t)\|_\infty \leq \xi_i(t) \quad \forall i \in \{1, \ldots, \ell\},\ \forall t \geq 0$.*

IV. UGES PROTOCOLS

In this section we consider various examples of protocols that may arise in Subsections III-A, III-B and III-C and show that they are UGES in the sense of Definition 2. Many other protocols, not considered here, can be treated in a similar manner.

A. NCS protocols

In this subsection, we consider protocols of the form

$$e^+ = (I - \Psi(s))e = h(i, e) \qquad s = s(i, e) \ , \tag{29}$$

which arise in NCS considered in Subsection III-A; the function Ψ is defined by (16). We typically assume that if a node j transmits at time t_i, then e_j is reset to zero at time t_i^+, i.e., $e_j(t_i^+) = 0$. In other words, we ignore possible quantization effects in this subsection. However, we emphasize that this assumption is not needed in general and this will become clear in the sequel. We present two examples of protocols from [12], [16] and quote results from [12] that show that these protocols are UGES. Besides serving to illustrate the unifying nature of our results, the RR and TOD protocols that we present next are used in the sequel to generate several genuinely new protocols that combine quantization and time scheduling.

1) Round robin (RR) protocol: The simplest time-scheduling protocol is round robin in which the node j is transmitted periodically with period ℓ, where ℓ is the total number of nodes (see, for instance, [3], [4], [5]). Note that the scheduling function becomes in this case:

$$s = s(i) = j \text{ if } i = j + k\ell \text{ for some } k = 0, 1, 2, \ldots \ . \tag{30}$$

Proposition 2 ([12]) *The RR protocol (29), (30) is UGES with the Lyapunov function* $W(i,e) := \sqrt{\sum_{k=i}^{\infty} |\phi(k,i,e)|^2}$*, where* ϕ *denotes the solution of (29), (30) at time* k *starting at time* i *and initial condition* e*. In particular, we can take* $a_1 = 1$*,* $a_2 = \sqrt{\ell}$*, and* $\rho = \sqrt{\frac{\ell-1}{\ell}}$*.* □

2) Try-once-discard (TOD) protocol: The Try-Once-Discard (TOD) time-scheduling protocol proposed by Walsh et al in [16] and its stability was analysed in [12]. In TOD protocol, the scheduling function takes the following form:

$$s = s(e) = \min\{\arg\max_i |e_i|\} \ . \tag{31}$$

Proposition 3 ([12]) *The TOD protocol (29), (31) is UGES with the Lyapunov function* $W(e) := |e|$*. In particular, we can take* $a_1 = a_2 = 1$ *and* $\rho = \sqrt{\frac{\ell-1}{\ell}}$*.* □

B. QCS protocols

We now prove that the quantization "box" protocol from Section III-B is UGES. As far as we are aware, this is the first analysis of stability properties for this protocol, taken separately from the continuous-time dynamics. However, protocols of this kind have been widely used in the literature as encoder/decoder models of communication channels between the plant and the controller; see, e.g., [7], [8], [15], [11], [13], [2]. The protocol is given by

$$e^+ = h_e(i,x,e,\xi); \qquad \xi^+ = \frac{\xi}{N} \ . \tag{32}$$

From Assumption 1 and other constructions in Section III-B, it is easy to show that there exists a $d_1 \geq 0$ such that h_e in (32) satisfies

$$|h_e(i,x,e,\xi)| \leq d_1 \xi \qquad \forall\, i,x,e,\xi \ , \tag{33}$$

where we can take $d_1 = \frac{\sqrt{n}}{N}$.

Proposition 4 *Suppose that* (33) *holds. Then, the box quantization protocol (32) is UGES with the Lyapunov function* $W(e,\xi) := \varepsilon|e| + \xi$*, where* $\varepsilon \in (0, \widetilde{\rho})$ *and* $\widetilde{\rho} = \frac{N-1}{d_1 N}$*. In particular, we can take* $a_1 = \min\{1, \varepsilon\}$*,* $a_2 = 1 + \varepsilon$*, and* $\rho = \varepsilon d_1 + \frac{1}{N}$*.* □

C. NQCS protocols

The focus of this subsection are protocols of the form

$$\begin{aligned} e^+ &= (I - \Psi(s))e + \Psi(s)H(i,x_s,e_s,\xi_s) \\ \xi^+ &= h_\xi(i,s,\xi) \\ s &= s(i,x,e,\xi) \end{aligned} \tag{34}$$

that arise in Subsection III-C. The functions H and h_ξ depend on the quantization procedure, whereas the scheduling function s depends on the time-scheduling procedure. Hence, the protocol (34) combines time scheduling and quantization. Using (26) and Assumption 2, there exists a $d \geq 0$ such that

$$|H(i,q(x),e,\xi)| \leq d|\xi| \qquad \forall\, i,x,e,\xi \ . \tag{35}$$

In what follows, this is all we need to know about the function H. Various NQCS protocols are possible and we consider two such protocols where we combine the RR and TOD time-scheduling protocols with the "box" quantization protocol. We are not aware of this class of protocols having been systematically studied in the literature.

1) RR protocol with quantization: In this section we combine the RR protocol considered in Subsection IV-A with the "box" quantization protocol considered in Subsection IV-B. A somewhat similar (but more complicated) protocol was considered in [11]. The function s in (34) is given by (30) and the function Ψ is defined in (16). We know that H satisfies (35). We let h_ξ in (34) be given by

$$h_\xi(i,s,\xi) = (I - \widetilde{\Psi}(s(i)))\xi + \frac{1}{N}\widetilde{\Psi}(s(i))\xi \ , \tag{36}$$

where N^{n_s} is the number of quantization levels for the corresponding quantizer (q_s),

$$\widetilde{\Psi}(s) := \mathrm{diag}\{\delta_{1i}, \ldots, \delta_{\ell i}\} \tag{37}$$

(note that Ψ and $\widetilde{\Psi}$ have different dimensions); δ_{ij} is the Kronecker symbol; ℓ denotes the number of nodes used in the time scheduling. This means that $\xi_{s(i)}$ is divided by N while other components of ξ remain unchanged. To summarize, the discrete-time system induced by the protocol is

$$\begin{aligned} e^+ &= (I - \Psi(s(i)))e + \Psi(s(i))H(i,x_{s(i)},e_{s(i)},\xi_{s(i)}) \\ \xi^+ &= (I - \widetilde{\Psi}(s(i)))\xi + \frac{1}{N}\widetilde{\Psi}(s(i))\xi \end{aligned} \tag{38}$$

Proposition 5 *Suppose that* (35) *holds. Then, the protocol (38) is Lyapunov UGES. In particular, there exists*[3] W *such that (9) and (10) hold with* $a_1 = \min\{1, \varepsilon\}$*,* $a_2 = \varepsilon\sqrt{\ell} + \sqrt{\frac{N^2\ell}{N^2-1}}$*, and* $\rho = \max\left\{\sqrt{\frac{\ell-1}{\ell}}, \varepsilon d\sqrt{\ell} + \widetilde{\rho}\right\}$*, where* $\varepsilon \in \left(0, \frac{1-\widetilde{\rho}}{d\sqrt{\ell}}\right)$ *and* $\widetilde{\rho} = \sqrt{\frac{N^2\ell - N^2 + 1}{N^2\ell}}$*.* □

2) TOD protocol with quantization: In this subsection we consider a combination of TOD protocol and quantization. We believe that this protocol has not been considered previously in the literature. The protocol is given by:

$$\begin{aligned} e^+ &= (I - \Psi(s))e + \Psi(s)H(i,x_s,e_s,\xi_s) \\ \xi^+ &= (I - \widetilde{\Psi}(s))M(s(e),\xi) + \frac{1}{N}\widetilde{\Psi}(s)\xi \end{aligned} \tag{39}$$

where Ψ is defined in (16), $\widetilde{\Psi}(s)$ is defined in (37), s is defined in (31), H satisfies (35), and $M(s,\xi) = (m_1(s,\xi), \ldots, m_\ell(s,\xi))$, where $m_j(s,\xi) := \min\{\xi_s, \xi_j\}$. In other words, the updated value of ξ_j for any $j \in \{1, \ldots, \ell\}$ satisfies the following:

$$\xi_j^+ = \begin{cases} \frac{\xi_j}{N} & \text{if } j = s \\ \xi_s & \text{if } \ \xi_j \geq \xi_s \text{ and } j \neq s \\ \xi_j & \text{if } \ \xi_j \leq \xi_s \text{ and } j \neq s \ . \end{cases} \tag{40}$$

The above protocol first compares the errors e_i in individual nodes and then transmits a quantized version of the measurement in the node with the largest error. Note that the

[3]We omit the explicit formula for W due to space reasons. It will be reported in the journal version of this paper.

ξ_j corresponding to this node is divided by N after the transmission. Moreover, since we already know which node has the largest error, any of the $\xi_i's$ corresponding to other nodes that are larger than ξ_j are reset to be equal to ξ_j (recall Assumption 2). While not immediately obvious, this protocol is quite a natural way to combine TOD with a box quantization protocol. We note that a naive update for ξ as follows: $\xi^+ = (I - \widetilde{\Psi}(s))\xi + \frac{1}{N}\widetilde{\Psi}(s)\xi$, where s is defined in (31), may not work and, in particular, we were unable to prove that this modified protocol is Lyapunov UGES.

Proposition 6 *Suppose that* (35) *holds. Then, the protocol (39) is Lyapunov UGES. In particular, (9) and (10) hold with* $a_1 = \min\{1, \varepsilon\}$, $a_2 = 1 + \varepsilon$, *and* $\rho = \max\left\{\sqrt{\frac{\ell-1}{\ell}}, \varepsilon d + \widetilde{\rho}\right\}$, *where* $\varepsilon \in \left(0, \frac{1-\widetilde{\rho}}{d}\right)$, $\widetilde{\rho} = \max\left\{\sqrt{\frac{\ell+\alpha^2-1}{\ell}}, \sqrt{\frac{N^2\ell-\alpha^2N^2+\alpha^2}{N^2\ell}}\right\}$, *and* $\alpha \in (0,1)$ *is arbitrary.* □

V. Main results

In this section we demonstrate the utility and generality of our unifying approach to the controller emulation design for systems considered in Section III. For space reasons, we only state two corollaries that provide MATI bounds for *linear* NQCS[4]. As special cases, we recover linear versions of results for NCS from [12] as well as results for QCS from [7] and elsewhere (when $\ell = 1$). We thus consider the plant $\dot{x} = Ax + Bu$. We pick a feedback gain K such that the matrix $A + BK$ is Hurwitz. In the presence of the network, we apply $u = K\hat{x}$, where $\dot{\hat{x}} = A\hat{x} + Bu$. Then the error e evolves according to

$$\dot{e} = Ae \qquad \forall\, t \in [t_{i-1}, t_i] \tag{41}$$

Using e we rewrite the closed-loop system as follows:

$$\dot{x} = (A + BK)x + BKe\ . \tag{42}$$

We will consider the NQCS for the two protocols given in Subsection IV-C. To define the quantization protocol, we need to generate the auxiliary variable $\xi \in \mathbb{R}^{\ell}_{\geq 0}$:

$$\dot{\xi} = A_\xi \xi \qquad \forall t \in [t_{i-1}, t_i] \tag{43}$$

and A_ξ is chosen appropriately so that Assumption 2 holds. For example, we can define (43) via

$$\dot{\xi}_i = \|A_{ii}\|_\infty \xi_i + \sum_{j\neq i} \|A_{ij}\|_\infty \xi_j, \qquad i = 1, \ldots, \ell\ , \tag{44}$$

where the initial data satisfies $\|e_i(0)\|_\infty \leq \xi_i(0)\ \forall i$, the indices $i = 1, 2, \ldots, \ell$ are consistent with the decomposition of the vector $e = (e_1, e_2, \ldots, e_\ell)$ used in the time-scheduling protocol, and the matrices A_{ij} form the corresponding partition of the matrix A. If the matrix A has some structure, such as block-diagonal, then (44) simplifies (cf. [2], [15]).

The following are direct consequences of Theorem 1 and Propositions 5 and 6.

[4]Similar corollaries can be stated for NCS and QCS and they will be reported in the journal version of this paper.

Corollary 7 *Consider the NQCS (42), (41), (43) with (1) and the protocol (38) that has* ℓ *nodes. Suppose that* K *is designed so that* $A + BK$ *is Hurwitz. Then, the system is UGES if MATI satisfies* $\tau \in (0, \tau^*)$, *where* $\tau^* := \frac{1}{\max\{\sqrt{\ell}\|A\|, \widetilde{a}\|A_\xi\|\}} \ln\left(\frac{1}{\rho}\right)$, ρ *comes from Proposition 5, and* $\widetilde{a} = \sqrt{\frac{N^2\ell}{N^2-1}}$. □

Corollary 8 *Consider the NQCS (42), (41), (43) with (1) and the protocol (39). Suppose that* K *is designed so that* $A + BK$ *is Hurwitz. Then, the system is UGES if MATI satisfies* $\tau \in (0, \tau^*)$, *where* $\tau^* := \frac{1}{\max\{\|A\|, \|A_\xi\|\}} \ln\left(\frac{1}{\rho}\right)$ *and* ρ *comes from Proposition 6.* □

VI. Conclusions

In this paper, we unified results on emulation for QCS and NCS by generalizing recently reported results for NCS in [12]. A central issue in our approach is proving stability properties of the corresponding quantization/time scheduling protocols. We illustrated how this can be done for several representative protocols. Our approach is amenable to various generalizations, modifications and extensions as will be discussed in the forthcoming journal version.

References

[1] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Control*, 45:1279–1289, 2000.

[2] J. P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. In *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*, 2002.

[3] D. Hristu and K. Morgansen. Limited communication control. *Systems Control Lett.*, 37:193–205, 1999.

[4] H. Ishii and B. Francis. Stabilization with control networks. *Automatica*, 38:1745–1751, 2002.

[5] S. Lall and G. Dullerud. An LMI solution to the robust synthesis problem for multi-rate sampled-data systems. *Automatica*, 37:1909–1922, 2001.

[6] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39:1543–1554, 2003.

[7] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Trans. Automat. Control*, 48:304–307, 2003.

[8] D. Liberzon and J. P. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Trans. Automat. Control*, 50:910–915, 2005.

[9] D. Liberzon and D. Nešić. Input-to-state stabilization of linear systems with quantized state measurements. *IEEE Trans. Automat. Control*, 52:767–781, 2007.

[10] Q. Ling and M. D. Lemmon. Stability of quantized control systems under dynamic bit assignment. *IEEE Trans. Automat. Control*, 50:734–740, 2005.

[11] G. N. Nair and R. J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM J. Control Optim.*, 43:413–436, 2004.

[12] D. Nešić and A. R. Teel. Input-output stability properties of networked control systems. *IEEE Trans. Automat. Control*, 49:1650–1667, 2004.

[13] I. R. Petersen and A. V. Savkin. Multi-rate stabilization of multivariable discrete-time linear systems via a limited capacity communication channel. In *Proc. 40th IEEE Conf. on Decision and Control*, pages 304–309, 2001.

[14] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34:435–443, 1989.

[15] S. Tatikonda and S. K. Mitter. Control under communication constraints. *IEEE Trans. Automat. Control*, 49:1056–1068, 2004.

[16] G. C. Walsh, O. Beldiman, and L. G. Bushnell. Asymptotic behavior of nonlinear networked control systems. *IEEE Trans. Automat. Control*, 46:1093–1097, 2001.