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Stabilization of Linear Systems Under Coarse Quantization and Time Delays

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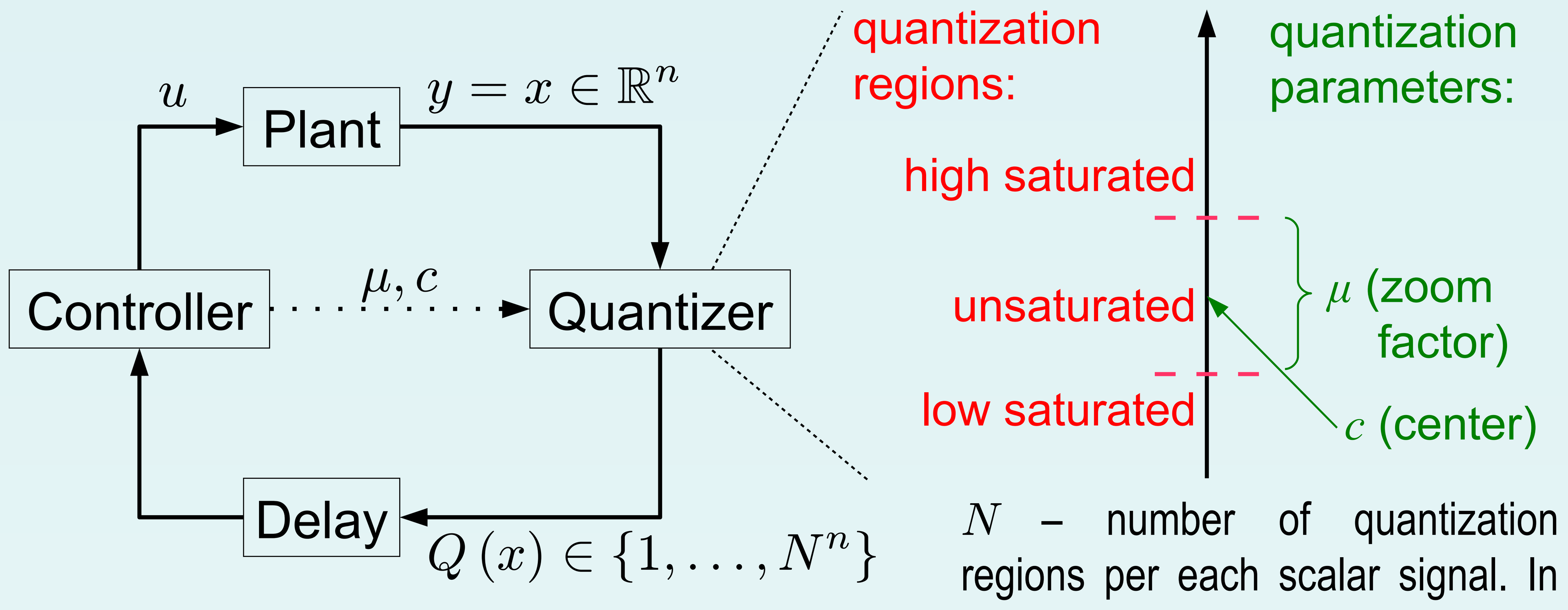
Problem Formulation



N – number of quantization regions per each scalar signal. In the example above $N = 3$

Motivation:

- Networked control systems / cheap sensors / feedback from image
  - Effects: quantization, time scheduling, time delays, packet dropout, interference
- General control systems
  - Unknown disturbance, nonlinear dynamics, output feedback, modeling uncertainties
- Objectives
  - Stability, delay-dependent ultimate bound on the state
  - Minimum data rate / resolution
- Underlined effects addressed in [1]. Here – combination of quantization & time delays

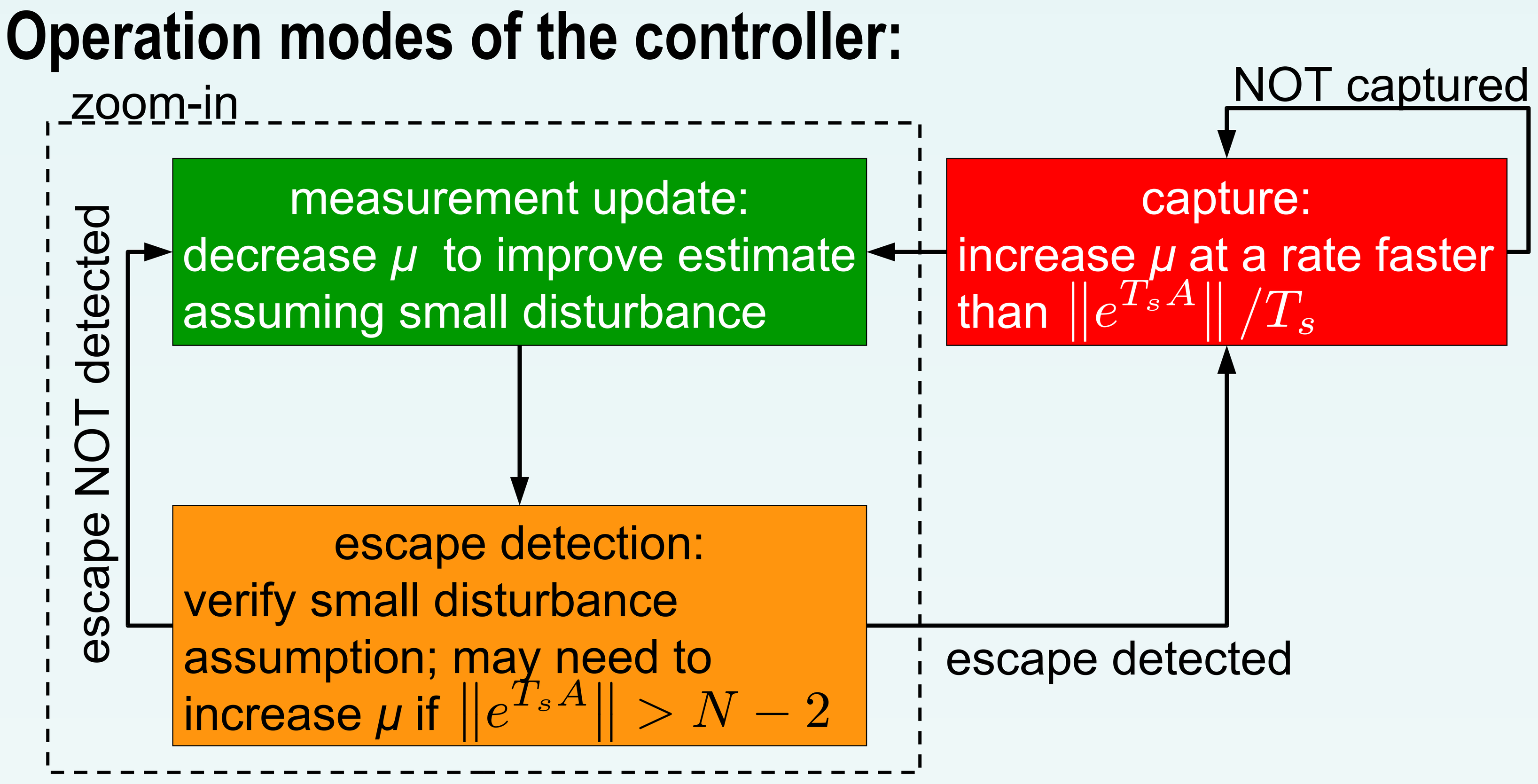
Control Algorithm

Notations:

x	state	u	control input
$\hat{x}$	state estimate	T_s	sampling interval
δ_k	time delay at time kT_s	$\delta_{\max}$	$\max_k \delta_k$

plant model: $\dot{x}(t) = Ax(t) + Bu(t)$

Operation modes of the controller:



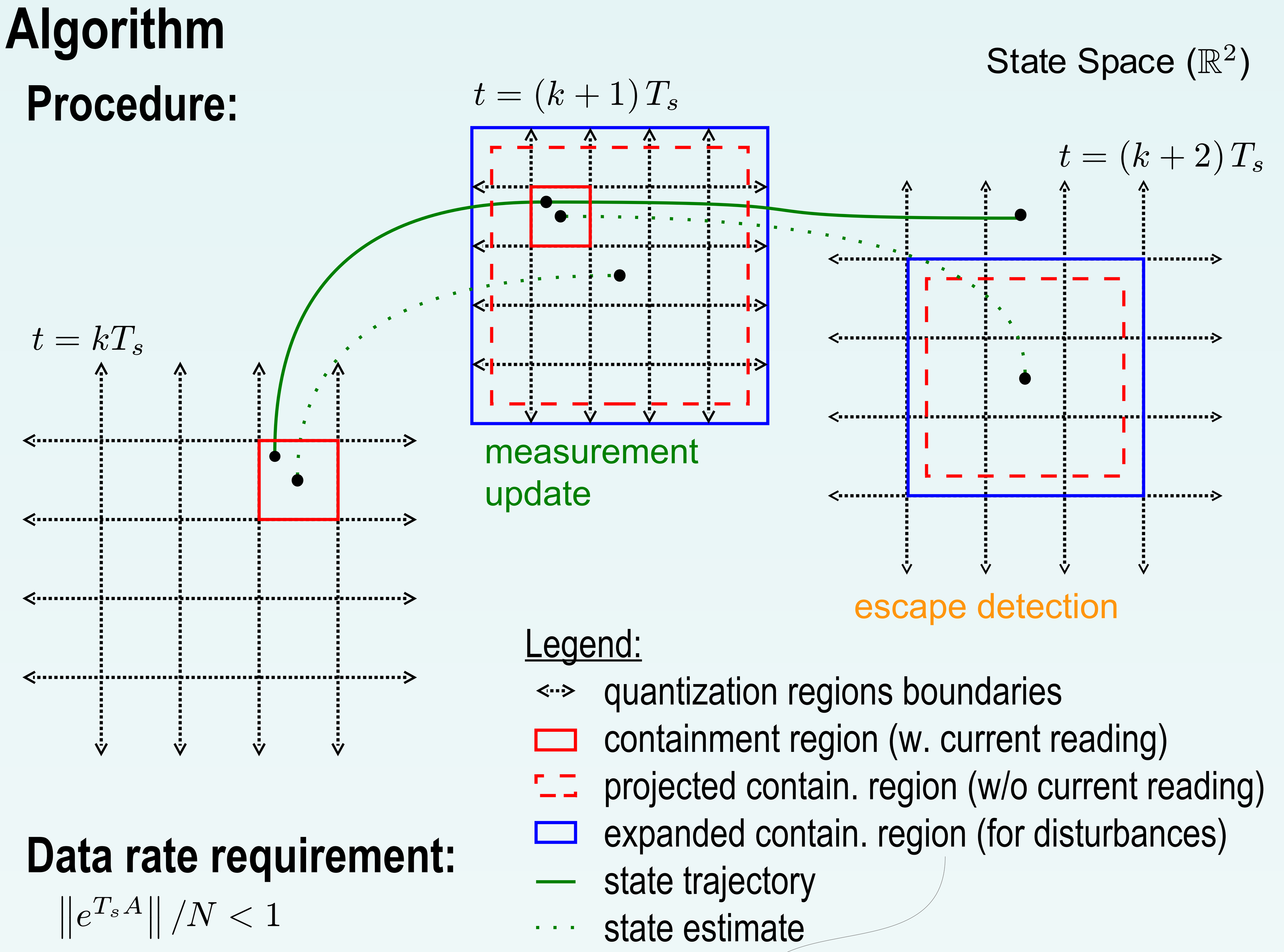
Control Law:

$u(t) = K\hat{x}(t)$, where $A + BK$ is Hurwitz

Disturbances?

Effects of delays are modeled as external disturbances, see (3) below

Procedure:



Data rate requirement:

$\|e^{T_s A}\| / N < 1$

External Disturbances:

measurement update: $\mu_{k+1} = \mu_k \left(\|e^{T_s A}\| + \alpha\right) / N$

α is used to expand containment region proportionally, allowing the controller to converge while external disturbances are small compared to estimation error

Results

Main result:

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(\delta_{\max}) \qquad (1)$$

Input to State Stability (ISS) property:

$y \xrightarrow{ISS} z \;:\; |z(t)| \le \beta(|z(t_0)|, t - t_0) + \gamma\left(\|y\|_{[t_0,t]}\right), \quad \forall t \ge t_0 \ge 0$

where β is a function of class $\mathcal{KL}$ and γ is of class $\mathcal{K}_\infty$

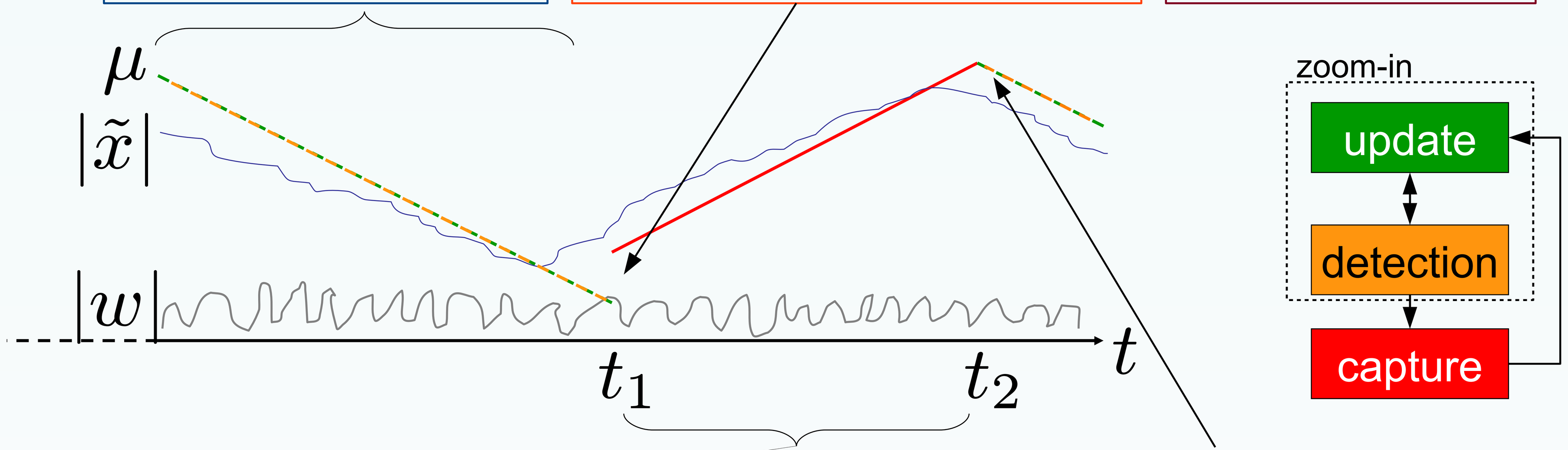
Main steps in proof of (1):

- New signal definitions: $\tilde{x}(t) \doteq \hat{x}(t) - x(t - \delta_{k(t)})$,
  $\theta_x(t) \doteq x(t - \delta_{k(t)}) - x(t)$, $\quad \theta_e(t) \doteq \tilde{x}(t - \delta_{k(t)}) - \tilde{x}(t)$
- Using Small Gain Theorem between $x(t)$ & $\theta_x(t)$: $\tilde{x} \xrightarrow{ISS} x$
- Intermediate result from [1]:
  For $\dot{x}(t) = Ax(t) + Bu(t) + w(t)$ and with no delays, $w \xrightarrow{ISS} \tilde{x}$ (2)
- Estimation error dynamics
  $$\dot{\tilde{x}}(t) = A\tilde{x}(t) - \underbrace{BK(\theta_e(t) + \theta_x(t - \delta))}_{w(t)} \qquad (3)$$
- Using Small Gain Theorem between $\tilde{x}(t)$ & $\int_{t-T_s}^{t} \theta_e(\tau)\, d\tau$: $x \xrightarrow{ISS} \tilde{x}$
- Small Gain Theorem between x & $\tilde{x}(t)$ (if $\gamma_x(\gamma_e(\delta_{\max} r)) \le r$)

First step in proof of (2):

showing that bound on steady state value depends only on $\|w\|$, independently of $|x(t_0)|$ (sketch):

1. While in zoom-in: $|\tilde{x}(t)| \le \mu(t)$
2. Switch to capture only if: $\mu(t_1 - 1) \le \eta \|w\|_{[t_1 - P, t_1]}$
3. $|\tilde{x}(t_1)| \le \zeta \|w\|$ (1+2)



4. Time to capture: $t_2 - t_1 \le f\left(|\tilde{x}(t_1)|, \|w\|_{[t_1,t_2]}\right)$
5. Size of μ when state is captured: $\mu(t_2) = g(\mu(t_1), t_2 - t_1)$

$\Rightarrow \|\tilde{x}\| \le \mu(t_2) \le \delta(\|w\|)$ (1; 5+4+3)

Remaining steps for proving (2), and derivation of $\tilde{\beta}_{\tilde{x}}, \tilde{\gamma}_{\tilde{x}}$, are detailed in [1]

Acknowledgment:

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References:

[1] Sharon, Y. & Liberzon, D., *IEEE Trans. Automat. Control.* To appear