Observer-based quantized output feedback control of nonlinear systems

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Abstract— **This paper addresses the problem of stabilizing a nonlinear system by means of quantized output feedback. A framework is presented in which the control input is generated by an observer-based feedback controller acting on quantized output measurements. A stabilization result is established under the assumption that this observer-based controller possesses robustness with respect to output measurement errors in an input-to-state stability (ISS) sense. Designing such observers and controllers is a largely open problem, some partial results on which are discussed. The main goal of the paper is to encourage further work on this important topic.**

Keywords: **Input-to-state stability, nonlinear system, observer design, output feedback, quantized control.**

I. INTRODUCTION

The problem studied in this paper is that of stabilizing, in an asymptotic or weaker sense, a nonlinear system using quantized measurements of its output. This problem arises, for example, in the presence of a finite-capacity communication channel between the sensor and the actuator. In this setting, the quantizer models the encoding/decoding scheme used to transmit data along such a channel. Transmitting the output (i.e., a partial measurement of the system state) rather than the full state is not only a necessity in limited-sensing scenarios, but may also be a choice of the designer since this relieves the communication burden. The price to pay, however, is that the missing state information needs to be recovered on the decoder side. Any observer that is employed for this purpose must possess sufficient robustness to the errors affecting the output due to quantization.

For linear systems, these issues are addressed in some detail in the previous work by the author [1], [2], [3], [4]. The linear case is relatively straightforward since standard linear observer designs are automatically robust with respect to additive errors at the output, and it can be shown that the resulting overall state estimation error is a product of the output quantization error and a quantity that characterizes observability of the system. For nonlinear systems, the problem is much more difficult and no parallel results have been obtained. Stabilization of nonlinear systems via quantized output feedback has recently been studied in [5], [6]. However, the setting in these papers is different from ours in that the observer

is implemented on the encoder side, i.e., the full state is estimated *before* being encoded and transmitted. This formulation bypasses the issue of observer robustness to errors altogether, and is also arguably less relevant in applications because it requires the sensor to have local access to extensive computational resources.

In Section II of this paper, we present a general framework for quantized output feedback stabilization of nonlinear systems guided by the above considerations. The *input-to-state stability* (ISS) methodology, introduced by Sontag in [7] and by now well established in the nonlinear systems literature, is used to characterize the robustness property required from the feedback controller and the observer. *Assuming* that this ISS robustness property holds, we present a general result which guarantees the existence of two nested invariant regions such that all trajectories of the quantized output feedback system starting in the larger region enter the smaller one. We then show how, combining this property with the idea of dynamic quantization from [1], [2], [4], (global) asymptotic stability of the closed-loop system can be achieved. For linear systems, the ISS assumption is automatically satisfied by any stabilizing output feedback design, and the earlier results from [2], [4] are recovered as a special case.

Rather than seeing it as a goal in itself, we consider the above result primarily as a motivation for pursuing the design of controllers and observers with ISS robustness properties that render this result applicable. In fact, quantization is just one possible reason for corrupted output measurements, and there are other contexts in which these design problems remain relevant (cf., for example, deterministic Kalman filtering discussed in [8, p. 375]). These problems are essentially open, although some constructions that work in certain special cases are available. These partial results are discussed in Section III, and it is hoped that the present paper will stimulate further work on these topics.

II. QUANTIZED OUTPUT FEEDBACK STABILIZATION

We consider a general nonlinear system (plant)

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measured output, and f

and h are sufficiently regular functions with $h(0) = 0$. By an *output quantizer* we mean a piecewise constant function $q : \mathbb{R}^p \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^p$. Following [2], [4], we assume that there exist positive numbers M and Δ (which we call the *range* and the *error bound* of the quantizer, respectively) such that the following condition holds:

$$|y| \leq M \ \Rightarrow \ |q(y) - y| \leq \Delta. \tag{2}$$

As a "nominal" controller (i.e., the controller we would apply in the absence of quantization), we take a dynamic (observer-based) output feedback law of the form

$$\begin{aligned} \dot{z} &= g(z, u, y) \\ u &= k(z) \end{aligned} \tag{3}$$

where $z \in \mathbb{R}^k$ is the controller state. Assuming that only quantized measurements $q(y)$ of the plant output y are available to the controller, we consider the "certainty equivalence" quantized output feedback law based on (3), namely,

$$\begin{aligned} \dot{z} &= g(z, u, q(y)) \\ u &= k(z) \end{aligned} \tag{4}$$

It is convenient to introduce the notation

$$e := q(y) - y \tag{5}$$

for the output quantization error. Then, the closed-loop system that results from interconnecting (1) and (4) can be written as

$$\begin{aligned} \dot{x} &= f(x, k(z)) \\ \dot{z} &= g(z, k(z), h(x) + e) \end{aligned} \tag{6}$$

Note that in this system, e models the output quantization error but may in general also model other measurement disturbances which affect the output y of (1) before it is passed to the observer-based controller (3). Our main assumption states that the controller (3) should guarantee robustness with respect to such disturbances in the input-to-state stability (ISS) sense.

Assumption 1. The system (6) is ISS with respect to e.

According to the results of [7], [9]—to which we also refer the reader for background on ISS and related terminology—Assumption 1 is equivalent to the existence of a $\mathcal{C}^1$ function $V : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ such that for some class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3, \rho$ and for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, $e \in \mathbb{R}^p$ we have

$$\alpha_1(|(x, z)|) \leq V(x, z) \leq \alpha_2(|(x, z)|)$$

and

$$|(x, z)| \geq \rho(|e|)$$
$$\Downarrow \tag{7}$$
$$\frac{\partial V}{\partial x} f(x, k(z)) + \frac{\partial V}{\partial z} g(z, k(z), h(x) + e) \leq -\alpha_3(|(x, z)|)$$

Here and below, (x, z) is a shorthand for the concatenated state vector $(x^T, z^T)^T$.

Take κ to be some class $\mathcal{K}_\infty$ function with the property that

$$\kappa(r) \geq \max_{|x| \leq r} |h(x)| \qquad \forall\, r \geq 0.$$

Then we have

$$|h(x)| \leq \kappa(|x|) \qquad \forall\, x. \tag{8}$$

We are ready to state the following result.

Proposition 1 *Assume that M is large enough compared to Δ so that*

$$\alpha_1 \circ \kappa^{-1}(M) > \alpha_2 \circ \rho(\Delta). \tag{9}$$

Then the sets

$$\mathcal{R}_1 := \{(x, z) : V(x, z) \leq \alpha_1 \circ \kappa^{-1}(M)\}$$

and

$$\mathcal{R}_2 := \{(x, z) : V(x, z) \leq \alpha_2 \circ \rho(\Delta)\}$$

are invariant regions for the system (5), (6). *Moreover, all solutions that start in the set $\mathcal{R}_1$ enter the smaller set $\mathcal{R}_2$ in finite time. An upper bound on this time is*

$$T = \frac{\alpha_1 \circ \kappa^{-1}(M) - \alpha_2 \circ \rho(\Delta)}{\alpha_3 \circ \rho(\Delta)}.$$

Proof: . Whenever $|y| = |h(x)| \leq M$, in view of (2) the quantization error e given by (5) satisfies $|e| \leq \Delta$. Using (7) and (8), we obtain the following formula for the derivative of V along solutions of the system (6):

$$\rho(\Delta) \leq |(x, z)| \leq \kappa^{-1}(M) \ \Rightarrow \ \dot{V} \leq -\alpha_3(|(x, z)|). \tag{10}$$

All the claims easily follow from this (cf. [2], [4] where similar results are derived for the cases of state quantization and input quantization). $\qquad\square$

The above result is especially useful in situations where the quantization can be *dynamic*, in the sense that the parameters of the quantizer can be changed on-line by the control designer (cf. [1], [2], [4]). This is possible in many applications, the only hard constraint typically being the number of quantization levels (i.e., the cardinality of $\mathcal{Q}$). Such a dynamic quantizer can be formalized by introducing the one-parameter family of quantizers

$$q_\mu(x) := \mu q\left(\frac{x}{\mu}\right), \qquad \mu > 0$$

where μ is an adjustable parameter which can be viewed as a "zoom" variable. For each fixed value of μ, the range of the quantizer q_μ is $M\mu$ and the error bound is $\Delta\mu$. Increasing μ corresponds to "zooming out" while decreasing μ corresponds to "zooming in." Substituting q_μ (for a fixed μ) instead of q in the foregoing developments, we obtain a counterpart of Proposition 1 in which M and Δ are multiplied by μ wherever they appear, and accordingly the regions $\mathcal{R}_1$, $\mathcal{R}_2$ and the time T depend on μ. In particular, if initially

$$(x(t_0), z(t_0)) \in \mathcal{R}_1(\mu(t_0)) \tag{11}$$

then at time $t_1 := t_0 + T(\mu(t_0))$ we have $(x(t_1), z(t_1)) \in \mathcal{R}_2(\mu(t_0))$. "Zooming in" on this smaller invariant region by means of setting

$$\mu(t_1) := \frac{1}{M} \kappa \circ \alpha_1^{-1} \circ \alpha_2 \circ \rho(\Delta\mu(t_0))$$

we have $(x(t_1), z(t_1)) \in \mathcal{R}_1(\mu(t_1))$. At the time $t_2 := t_1 + T(\mu(t_1))$, this procedure can be repeated. Of course, we need to be sure that the sequence $\mu(t_0)$, $\mu(t_1)$, $\mu(t_2)$, ... is indeed decreasing to 0. This is guaranteed if we strengthen (9) to

$$\alpha_1 \circ \kappa^{-1}(M\mu) > \alpha_2 \circ \rho(\Delta\mu) \qquad \forall \mu \in (0, \mu(t_0)] \quad (12)$$

Then, proceeding iteratively in this way, we recover asymptotic stability[1] of the closed-loop system. If the inequality in (12) only holds for $\mu \in (\varepsilon, \mu(t_0)]$ for some $\varepsilon > 0$, then convergence to the set $\mathcal{R}_2(\varepsilon)$ is obtained.

To achieve *global* asymptotic stability, we first need to generate an upper bound on the state of (6). Since z is available to the control designer, we only need to have a bound on $|x(\bar{t})|$ for some $\bar{t}$. The basic idea is to "zoom out" to obtain a bound on $|y|$ from the quantized measurements $q_\mu(y)$, and then invoke an appropriate observability assumption to generate a bound on $|x|$. Throughout this "zooming-out" stage, the control is set to 0. Assume that the unforced system

$$\dot{x} = f(x, 0) \quad (13)$$

is forward complete (i.e., has globally defined solutions). Assume also that the quantizer satisfies, in addition to (2), the condition

$$|z| > M \implies |q(z)| > M - \Delta. \quad (14)$$

Suppose that we can increase μ (in a piecewise constant fashion) fast enough to dominate the rate of growth of $|x(t)|$ along (13). (A generic procedure for doing this is described in [2], [10]; in practice, this requires some information about the growth of reachable sets of the system (13).) Then, we will eventually encounter an interval, say $[\bar{t}, \bar{t}+\tau]$, on which we have $|q_\mu(y(t))| \le (M-\Delta)\mu(t)$ and hence, by (14), $|y(t)| \le M\mu(t) \le M\mu(\bar{t}+\tau)$. We can use this to obtain a bound on $|x(\bar{t})|$ if the system (13) is *small-time*[2] *norm-observable* in the sense of [11], namely, if there exists a function $\gamma \in \mathcal{K}_\infty$ such that

$$|x(\bar{t}+\tau)| \le \gamma(\|y\|_{[\bar{t},\bar{t}+\tau]}). \quad (15)$$

Then we have $|x(\bar{t}+\tau)| \le \gamma(M\mu(\bar{t}+\tau))$, and the previous closed-loop "zooming-in" stage can now be commenced at time $\bar{t} + \tau$ after resetting μ to a sufficiently large value so that (11) holds with $\bar{t} + \tau$ in place of t_0. We refer the reader to [11] for equivalent characterizations and Lyapunov-based sufficient conditions for the above

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[1]This argument only shows asymptotic convergence. Stability of the origin in the sense of Lyapunov can be proved as in [2], [10], as long as we assume that $q(y) = 0$ on a neighborhood of 0 in $\mathbb{R}^p$.

[2]The quantifier *small-time* refers to the fact that a γ for which (15) holds exists for every τ. On the other hand, if (15) can only hold for a specific value of τ (*large-time norm-observability* [11]), then we need to carry out the "zooming-out" stage until we have a bound on $|y(t)|$ over an interval of this length τ.

norm-observability property. (See also the discussion of norm-estimators in [12].)

We remark that the framework just described encompasses, as a special case, the linear results developed in [2, Section 5] and [4, Section 5.3.5]. Suppose that we have an LTI plant which is stabilizable and observable. Then we can use a dynamic output feedback law based on the standard Luenberger observer, and Assumption 1 is satisfied. This is because for linear systems, internal asymptotic stability implies ISS, and a quadratic ISS-Lyapunov function V can be explicitly obtained from the Lyapunov equation. The functions α_1, α_2, and α_3 are then all quadratic, while ρ and κ are linear. Their expressions are easily computed; see [2], [4]. We also have that the time T is independent of μ and that (9) automatically implies (12). The values of μ during the "zooming-in" stage form a decreasing geometric sequence. The "zooming-out" procedure also becomes much more transparent; in particular, the norm-observability property (15) is established constructively by inverting the observability Gramian.

III. DISCUSSION

The problem motivated by the developments of Section II is to characterize classes of nonlinear systems for which one can design an output feedback controller (3) satisfying Assumption 1. One way to achieve the desired ISS property is to first obtain a static state feedback $u = k(x)$ that provides ISS with respect to measurement errors, and then to augment it with a full-order observer which generates an estimate z of the state x with the estimation error satisfying an ISS contraction property with respect to e. Namely, we can ask for the following two properties:

1) The system

$$\dot{x} = f(x, k(z)) = f(x, k(x + z - x))$$

satisfies

$$|x(t)| \le \beta_1(|x(0)|, t) + \gamma_1\left(\|x - z\|_{[0,t]}\right) \quad (16)$$

for some $\beta_1 \in \mathcal{KL}$ and $\gamma_1 \in \mathcal{K}_\infty$. (Here $\|\cdot\|_{[0,t]}$ stands for the (essential) supremum norm of a signal restricted to the interval $[0, t]$.)

2) The system (6) satisfies

$$|x(t) - z(t)| \le \beta_2(|x(0) - z(0)|, t) + \gamma_2\left(\|e\|_{[0,t]}\right) \quad (17)$$

for some $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$.

If Properties 1 and 2 hold, then Assumption 1 is satisfied. Indeed, a cascade argument along the lines of [13] shows that (16) and (17) together imply

$$\left|\begin{pmatrix} x(t) \\ x(t) - z(t) \end{pmatrix}\right| \le \beta\left(\left|\begin{pmatrix} x(0) \\ x(0) - z(0) \end{pmatrix}\right|, t\right) + \gamma\left(\|e\|_{[0,t]}\right)$$

for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, and this is nothing but ISS of the overall closed-loop system (6) modulo the change of coordinates from (x, z) to $(x, x - z)$. This

observation is useful in that it decouples Assumption 1 into two properties which express the requirements on the controller design and observer design, respectively. In what follows, we discuss each of these two properties separately in some more detail.

A. ISS controller design

Property 1 states that the static state feedback $u = k(x)$ should render the x-subsystem ISS with respect to measurement errors, i.e., the system $\dot{x} = f(x, k(x + d))$ should be ISS with respect to d. In our case, d is the difference between the observer state and the plant state, but for the purposes of control design it can be viewed as a general measurement disturbance.

The existence of feedback laws providing ISS with respect to measurement errors is studied in several references. It was demonstrated by way of counterexamples in [14] and later in [15] that not every stabilizable nonlinear system, even affine in controls, is input-to-state stabilizable with respect to measurement errors by means of time-invariant feedback. In [16] and [17, Chapter 6], time-invariant feedback laws guaranteeing ISS with respect to measurement errors were designed for the class of single-input plants in strict feedback form, via backstepping and "flattened" Lyapunov functions. In that work, the function $g(x)$ multiplying the control was assumed to be sign-definite and known. For the case when the sign of $g(x)$ is unknown, a time-varying feedback solution was developed for one-dimensional systems and then extended to feedback passive systems of any dimension in [18]. In [15], time-varying feedback was designed to handle affine systems for which $g(x)$ is allowed to have zero crossings, but only in one dimension. In [19], small-gain techniques were applied to a class of systems with unknown parameters and unmodeled dynamics. In the recent paper [20], a hybrid control solution was developed for systems possessing an output function whose dynamics take the form considered in [18] and with respect to which the system is minimum phase (in a suitable sense); this class covers the counterexample from [14] but not the one from [15].

It can be seen from the above discussion that, despite significant efforts, results on designing controllers to achieve ISS with respect to measurement errors are available only for fairly restricted classes of systems. In quantized control of nonlinear systems, this ISS property appears to be fundamental, and thus further progress in this area hinges upon extending the ISS controller design to broader classes of systems. In fact, ISS with respect to measurement errors is a standing assumption in the results on quantized *state* feedback developed in [2], [10], [4]. So-called "relaxations" of this assumption, discussed in [21] and [22], replace ISS by just global asymptotic stability under the zero error (0-GAS) and compensate for this by working on a bounded region and arranging for the quantization error to be sufficiently small. However, this amounts to not really relaxing ISS but just (implicitly) using the fact that for suitably small errors on a given bounded region, ISS automatically follows from 0-GAS. (This can be shown via a small-gain argument as in [21] or via Lyapunov analysis as in [22].) In other words, ISS still plays a crucial role. On the other hand, it is not a problem to allow the state feedback law to be time-varying and/or hybrid, as in some of the design results mentioned earlier.

B. ISS observer design

We now turn to Property 2. The ISS-like contraction condition (17) means that the z-subsystem is a full-order state observer for the x-subsystem in the sense of [12, Definition 20]; in particular, if $e \equiv 0$, then the estimate z asymptotically converges to the true state x. (The control is set to $u = k(z)$ throughout this discussion.)

The design of observers with this ISS property has apparently not been pursued in the literature. We need to investigate for what classes of systems such an observer can be constructed. One (quite restricted) class of systems for which this is possible is given by

$$\begin{aligned} \dot{x} &= Ax + g(u, y) \\ y &= Cx \end{aligned} \tag{18}$$

where (A, C) is a detectable pair and g is a globally Lipschitz function. It is not hard to show that the observer

$$\dot{z} = (A + LC)z + g(u, y + e) - L(y + e)$$

guarantees (17) provided that $A + LC$ is a Hurwitz matrix. Intersecting the class of systems (18) with the class of single-input systems in strict feedback form, we arrive at a nonempty but severely restricted class of nonlinear systems for which Assumption 1 can be satisfied. Note that in the context of Proposition 1, which works with a bounded invariant region, the global Lipschitzness of g can be relaxed to local Lipschitzness.

Another possibility, suggested to us by Murat Arcak, is to relax the requirement (17) by allowing an additional ISS gain from the state x to the observer error $x - z$:

$$\begin{aligned} |x(t) - z(t)| &\leq \beta_2(|x(0) - z(0)|, t) \\ &\quad + \gamma_2\left(\|e\|_{[0,t]}\right) + \gamma_3\left(\|x\|_{[0,t]}\right) \end{aligned} \tag{19}$$

where $\gamma_3 \in \mathcal{K}_\infty$. While the admissible class of observers is potentially broadened by passing to (19), the class of control laws $u = k(z)$ must be further restricted to preserve closed-loop ISS. Namely, the ISS gain γ_1 from the observer error $x - z$ to x must be small enough to compensate the ISS gain γ_3 of the observer, so that the two gains satisfy the small-gain condition. Then, Assumption 1 can be verified using the ISS small-gain theorem of [23] (this includes the cascade argument based on (17) as a special case). Such an approach was taken in [24], but in a context different from ours (robustness was sought with respect to unmodeled dynamics rather than output measurement errors). Further work is needed to see whether other observer design methodologies (e.g., high-gain observers [25]) might be helpful for achieving the ISS property.

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