



Entropy and Minimal Data Rates for State Estimation and Model Detection*

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ABSTRACT

We investigate the problem of constructing exponentially converging estimates of the state of a continuous-time system from state measurements transmitted via a limited-data-rate communication channel, so that only quantized and sampled measurements of continuous signals are available to the estimator. Following prior work on topological entropy of dynamical systems, we introduce a notion of *estimation entropy* which captures this data rate in terms of the number of system trajectories that approximate all other trajectories with desired accuracy. We also propose a novel alternative definition of estimation entropy which uses approximating functions that are not necessarily trajectories of the system. We show that the two entropy notions are actually equivalent. We establish an upper bound for the estimation entropy in terms of the sum of the system's Lipschitz constant and the desired convergence rate, multiplied by the system dimension. We propose an iterative procedure that uses quantized and sampled state measurements to generate state estimates that converge to the true state at the desired exponential rate. The average bit rate utilized by this procedure matches the derived upper bound on the estimation entropy. We also show that no other estimator (based on iterative quantized measurements) can perform the same estimation task with bit rates lower than the estimation entropy. Finally, we develop an application of the estimation procedure in determining, from the quantized state measurements, which of two competing models of a dynamical system is the true model. We show that under a mild assumption of *exponential separation* of the candidate models, detection is always possible in finite time. Our numerical experiments with randomly generated affine dynamical systems suggest that in practice the algorithm always works.

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1. INTRODUCTION

Entropy is a fundamental notion in the theory of dynamical systems. Roughly speaking, it describes the rate at which the uncertainty about the system's state grows as time evolves. One can think of this alternatively as the exponential growth rate of the number of system trajectories distinguishable with finite precision, or in terms of the growth rate of the size of reachable sets. Different entropy definitions (notably, topological and measure-theoretic ones) and relationships between them are studied in detail in the book [10] and in many other sources, and continue to be a subject of active research in the dynamical systems community. The concept of entropy of course also plays a central role in thermodynamics and in information theory, as discussed, e.g., in [5].

In the context of control theory, if entropy describes the rate at which uncertainty is generated by the system (when no measurements are taken), then it should also correspond to the rate at which information about the system should be collected by the controller in order to induce a desired behavior (such as invariance or stabilization). This link has been recognized in the control community, and suitable entropy definitions for control systems have been proposed and related to minimal data rates necessary for controlling the system over a communication channel. The first such result was obtained by Nair et al. in [15], where topological feedback entropy for discrete-time systems was defined in terms of cardinality of open covers in the state space. An alternative definition was proposed later by Colonius and Kawan in [3], who instead counted the number of "spanning" open-loop control functions. The paper [4] summarized the two notions and established an equivalence between them. Colonius subsequently extended the formulation of [3] from discrete-time to continuous-time dynamics and from invariance to exponential stabilization in [2]. The survey paper [16] provides a broader overview of control under data rate constraints.

In this work we are concerned with the problem of estimating the state of a continuous-time system when state measurements are transmitted via a limited-data-rate communication channel, which means that only quantized and sampled measurements of continuous signals are available to the estimator. We do not address control problems here, although such observation problems and control problems

are known to be closely related (through duality and the fact that state estimates can be used to close a feedback loop). Observability over finite-data-rate channels and its connection to topological entropy has been studied, most notably by Savkin [17]. Our point of departure in this paper is a synergy of ideas from Savkin [17] and Colonius [2]. As in [17], we focus on state estimation rather than control. However, we follow [2] in that we consider continuous-time dynamics and require that state estimates converge at a prescribed exponential rate. As a result, our definition of *estimation entropy* combines some features of the definitions used in [17] and [2]. We also propose a novel alternative definition of entropy which uses approximating functions that are not necessarily trajectories of the system. We show that, somewhat surprisingly, the two entropy notions turn out to be equivalent (Theorem 1). We proceed to establish an upper bound of $(L + \alpha)n/\ln 2$ for the estimation entropy of an n -dimensional nonlinear dynamical system with Lipschitz constant L , when the desired exponential convergence rate of the estimate is α (Proposition 2).

State estimation and monitoring of continuously evolving processes over data networks arise in a variety of engineering applications ranging from power grids to vehicular embedded control systems. Typically these estimation algorithms share a communication bus with many other competing protocols, and therefore, a principled approach to bandwidth allocation is necessary. One of the goals of this work is to develop algorithms for state estimation of continuous system behavior that are optimal with respect to sensing and communication data rates. To this end, we propose an iterative procedure that uses quantized and sampled state measurements to generate state estimates that converge to the true state at the desired exponential rate. The main idea in the algorithm, which borrows some elements from [13] and earlier work cited therein, is to exponentially increase the resolution of the quantizer while keeping the number of bits sent in each round constant. This is achieved by using the quantized state measurement of each round to compute a bounding box for the state of the system for the next round. Then, at the beginning of the next round, this bounding box is partitioned to make a new and more precise quantized measurement of the state. We show that the bounding box is exponentially shrinking in time at a rate α when the average bit rate utilized by this procedure matches the upper bound $(L + \alpha)n/\ln 2$ on the estimation entropy (Theorem 3 and Proposition 4). We also show that no other algorithm that performs state estimation based on iterative quantized measurements can perform the same estimation task with bit rates lower than the estimation entropy (Proposition 5). In other words, the “efficiency gap” of our estimation procedure is at most as large as the gap between the estimation entropy of the dynamical system and the above upper bound on it.

In the last part of the paper, we show an application of the estimation procedure in solving model detection problems. Suppose we are given two competing candidate models of a dynamical system and from the quantized state measurements we would like to determine which one is the true model. For example, the different models may arise from different parameter values or they could model “nominal” and “failure” operating modes of the system. This can be viewed as a variant of the standard system identification or model (in)validation problem (see, e.g., [9, 18]) except,

unlike in classical results which rely on input/output data, here we use quantized state measurements and do not apply a probing input to the system. We show that under a mild assumption of *exponential separation* of the candidate models’ trajectories, a modified version of our estimation procedure can always definitively detect the true model in finite time (Theorem 6). Our experiments with an implementation of this model detection procedure on randomly generated affine dynamical systems suggest that the model detection algorithm always works in practice.

1.1 Notation and terminology

By default, the base of all logarithms is 2 (when we use the natural logarithm we write \ln). We denote by $|\cdot|$ some chosen norm in \mathbb{R}^n . In general definitions and results this norm can be arbitrary, but in specific quantized algorithm implementations we will find it convenient to use the ∞ -norm $\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|$; in those places, the choice of the ∞ -norm will be explicitly declared. For any $x \in \mathbb{R}^n$ and $\delta > 0$, $B(x, \delta) \subseteq \mathbb{R}^n$ is the closed ball of radius δ centered at x , that is, $B(x, \delta) = \{y \in \mathbb{R}^n : |x - y| \leq \delta\}$; for the ∞ -norm this is a hypercube. For a bounded set $S \subseteq \mathbb{R}^n$ and $\delta > 0$, a δ -cover is a finite collection of points¹ $C = \{x_i\}$ such that $\cup_{x_i \in C} B(x_i, \delta) \supseteq S$. For a hyperrectangle $S \subseteq \mathbb{R}^n$ and $\delta > 0$, a δ -grid is a special type of δ -cover of S by hypercubes centered at points along axis-parallel planes that are 2δ apart. The boundaries of the δ -hypercubes centered at adjacent δ -grid points overlap. For a given set S , there are many possible ways of constructing specific δ -grids. We can choose any strategy for constructing them without changing the results in this paper. For example, we can construct a special grid on, say, the unit interval. Then, when working with a general interval I (a cross-section of S in any given dimension), we map I to the unit interval, mark the chosen grid on it, and then map it back to I . We denote the δ -grid on S by $grid(S, \delta)$.

2. ESTIMATION ENTROPY

Consider the (continuous-time) system model

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (1)$$

where f is a Lipschitz continuous function.² Let $\xi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ denote the trajectories or solutions of (1), i.e., for $x \in \mathbb{R}^n$, $\xi(x, \cdot)$ denotes the solution from the initial point x . We assume that these solutions are defined globally in time. Suppose that initial states of the system live in a known compact set $K \subset \mathbb{R}^n$. Let there be given a time horizon $T > 0$ and a desired convergence rate $\alpha \geq 0$.

For each $\varepsilon > 0$, we say that a finite set of functions $\hat{X} = \{\hat{x}_1(\cdot), \dots, \hat{x}_N(\cdot)\}$ from $[0, T]$ to \mathbb{R}^n is $(T, \varepsilon, \alpha, K)$ -approximating if for every initial state $x \in K$ there exists some function $\hat{x}_i(\cdot) \in \hat{X}$ such that

$$|\xi(x, t) - \hat{x}_i(t)| < \varepsilon e^{-\alpha t} \quad \forall t \in [0, T]. \quad (2)$$

¹With a slight abuse of terminology, we take the elements of a cover to be the centers of the balls covering S and not the balls themselves.

²The Lipschitz continuity assumption is quite standard in nonlinear systems theory; in particular, it is needed to ensure the system’s well-posedness (existence of unique solutions) [11].

Let $s_{\text{est}}(T, \varepsilon, \alpha, K)$ denote the minimal cardinality of such a $(T, \varepsilon, \alpha, K)$ -approximating set. We define *estimation entropy* as

$$h_{\text{est}}(\alpha, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log s_{\text{est}}(T, \varepsilon, \alpha, K).$$

It is easy to see that instead of $\lim_{\varepsilon \searrow 0}$ we could equivalently write $\sup_{\varepsilon > 0}$, because $s_{\text{est}}(T, \varepsilon, \alpha, K)$ grows as $\varepsilon \rightarrow 0$ for fixed T, α, K . Intuitively, since s_{est} corresponds to the minimal number of functions needed to approximate the state with desired accuracy, h_{est} is the average number of bits needed to identify these approximating functions. The inner \limsup extracts the base-2 exponential growth rate of s_{est} with time and the outer limit computes the worst case over $\varepsilon > 0$.

As a special case, further considered below, we can define the $\hat{x}_i(\cdot)$'s to be trajectories $\xi(x, \cdot)$ of the system from different initial states. Then, s_{est} corresponds to the number of different quantization points needed to identify the initial states, and h_{est} gives a measure of the long-term bit rate needed for communicating sensor measurements to the estimator. We pursue this connection in more detail in Section 3. We note that the norm in the above definition can be arbitrary.

2.1 Alternative entropy notion

In the above definition, the functions $\hat{x}_i(\cdot)$ are arbitrary functions of time and not necessarily trajectories of the system (1). If we insist on using system trajectories, then we obtain the following alternative definition: a finite set of points $S = \{x_1, \dots, x_N\} \subset K$ is $(T, \varepsilon, \alpha, K)$ -spanning if for every initial state $x \in K$ there exists some point $x_i \in S$ such that the corresponding solutions satisfy

$$|\xi(x, t) - \xi(x_i, t)| < \varepsilon e^{-\alpha t} \quad \forall t \in [0, T]. \quad (3)$$

Letting $s_{\text{est}}^*(T, \varepsilon, \alpha, K)$ denote the minimal cardinality of such a $(T, \varepsilon, \alpha, K)$ -spanning set, we could define estimation entropy differently as

$$h_{\text{est}}^*(\alpha, K) := \lim_{\varepsilon \searrow 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log s_{\text{est}}^*(T, \varepsilon, \alpha, K).$$

Since every $(T, \varepsilon, \alpha, K)$ -spanning set gives rise to a $(T, \varepsilon, \alpha, K)$ -approximating set via $\hat{x}_i(t) := \xi(x_i, t)$, and since entropy is determined by the minimal cardinality of such a set, it is clear that

$$s_{\text{est}}(T, \varepsilon, \alpha, K) \leq s_{\text{est}}^*(T, \varepsilon, \alpha, K) \quad \forall T, \varepsilon, \alpha, K \quad (4)$$

and therefore

$$h_{\text{est}}(\alpha, K) \leq h_{\text{est}}^*(\alpha, K) \quad \forall \alpha, K.$$

We will now show that, interestingly, this last inequality is actually always equality. In other words, there is no advantage—as far as estimation entropy is concerned—in using any approximating functions (even possibly discontinuous ones) other than system trajectories.

Theorem 1 *For every $\alpha \geq 0$ and every compact set K we have $h_{\text{est}}(\alpha, K) = h_{\text{est}}^*(\alpha, K)$.*

To prove this, we bring in the notion of separated sets. The arguments that follow are along the lines of [10, Section 3.1.b], see also Lemma III.1 of [17]. With $T, \varepsilon, \alpha, K$ given as before, let us call a finite set of points $E = \{x_1, \dots, x_N\} \subset K$

a $(T, \varepsilon, \alpha, K)$ -separated set if for every pair of points $x_1, x_2 \in E$ the solutions of (1) with these points as initial states have the property that

$$|\xi(x_1, t) - \xi(x_2, t)| \geq \varepsilon e^{-\alpha t} \quad \text{for some } t \in [0, T]. \quad (5)$$

Let $n_{\text{est}}^*(T, \varepsilon, \alpha, K)$ denote the *maximal* cardinality of such a $(T, \varepsilon, \alpha, K)$ -separated set. The next two lemmas relate n_{est}^* to the previously defined quantities s_{est}^* and s_{est} , respectively.³

Lemma 1 *For all $T, \varepsilon, \alpha, K$ we have*

$$s_{\text{est}}^*(T, \varepsilon, \alpha, K) \leq n_{\text{est}}^*(T, \varepsilon, \alpha, K). \quad (6)$$

PROOF. The inequality (6) follows immediately from the observation that every maximal $(T, \varepsilon, \alpha, K)$ -separated set E is also $(T, \varepsilon, \alpha, K)$ -spanning; indeed, if E is not $(T, \varepsilon, \alpha, K)$ -spanning then there exists an $x \in K$ such that for every $x_i \in E$ the inequality (3) is violated at least for some t , but then we can add this x to E and the separation property will still hold, contradicting maximality. \square

Lemma 2 *For all $T, \varepsilon, \alpha, K$ we have*

$$n_{\text{est}}^*(T, 2\varepsilon, \alpha, K) \leq s_{\text{est}}(T, \varepsilon, \alpha, K).$$

PROOF. Let $\hat{X} = \{\hat{x}_1(\cdot), \dots, \hat{x}_N(\cdot)\}$ be an arbitrary $(T, \varepsilon, \alpha, K)$ -approximating set of functions, and let $E = \{x_1, \dots, x_M\}$ be an arbitrary $(T, 2\varepsilon, \alpha, K)$ -separated set of points in K . We claim that $M \leq N$ which would prove the lemma. By the approximating property of \hat{X} , for every $x \in K$ there exists some $\hat{x}_i(\cdot) \in \hat{X}$ such that (2) holds. Suppose that $M > N$. Then, for at least one function $\hat{x}_i(\cdot) \in \hat{X}$ we can find (at least) two points $x_p, x_q \in E$ such that (2) holds both with $x = x_p$ and with $x = x_q$. By the triangle inequality, this implies $|\xi(x_p, t) - \xi(x_q, t)| < 2\varepsilon e^{-\alpha t}$ for all $t \in [0, T]$. But this contradicts the $(T, 2\varepsilon, \alpha, K)$ -separating property of E , and the claim is established. \square

PROOF OF THEOREM 1. Combining Lemmas 1 and 2 and (4), we obtain for all $T, \varepsilon, \alpha, K$

$$\begin{aligned} n_{\text{est}}^*(T, 2\varepsilon, \alpha, K) &\leq s_{\text{est}}(T, \varepsilon, \alpha, K) \\ &\leq s_{\text{est}}^*(T, \varepsilon, \alpha, K) \leq n_{\text{est}}^*(T, \varepsilon, \alpha, K) \end{aligned}$$

This implies that

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{1}{T} \log n_{\text{est}}^*(T, 2\varepsilon, \alpha, K) &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log s_{\text{est}}(T, \varepsilon, \alpha, K) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log s_{\text{est}}^*(T, \varepsilon, \alpha, K) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log n_{\text{est}}^*(T, \varepsilon, \alpha, K) \end{aligned} \quad (7)$$

for all $T, \varepsilon, \alpha, K$. We can now take the limit as $\varepsilon \rightarrow 0$ in (7). This limit always exists (but may be infinite) because all quantities in (7) are monotonically non-decreasing as $\varepsilon \rightarrow 0$ (so taking the limit is actually equivalent to taking the supremum over $\varepsilon > 0$). The difference between 2ε in the first term and ε in the last term disappears as we pass to the limit, hence all inequalities become equalities. This proves that $h_{\text{est}}(\alpha, K) = h_{\text{est}}^*(\alpha, K)$ as claimed in Theorem 1. \square

³We do not define a quantity n_{est} corresponding to separation between arbitrary curves (not necessarily system trajectories) as such a notion does not seem to be useful here.

Remark 1 The above proof shows, in addition, that the two entropy quantities appearing in the statement of Theorem 1 are also equal to

$$\lim_{\varepsilon \searrow 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log n_{\text{est}}^*(T, \varepsilon, \alpha, K).$$

By compactness of K and by the property of continuous dependence of solutions of (1) on initial conditions (see, e.g., [11]), for given ε, α, T there exists a $\delta > 0$ such that (3) holds whenever x and x_i satisfy $|x - x_i| < \delta$. From this it immediately follows that $s_{\text{est}}^*(T, \varepsilon, \alpha, K)$, and hence also $s_{\text{est}}(T, \varepsilon, \alpha, K)$, is finite for every $\varepsilon > 0$. This does not in principle preclude $h_{\text{est}}^*(\alpha, K)$ and $h_{\text{est}}(\alpha, K)$ from being infinite (the supremum over positive ε could still be ∞). However, we will see next that this does not happen because the system’s right-hand side is Lipschitz.

2.2 Entropy bounds

In this section, we establish an upper bound on the estimation entropy of nonlinear systems. This bound is in terms of the *global* Lipschitz constant L of the system’s right-hand side f . In case the system trajectories are confined to a compact invariant set, the result holds for a local Lipschitz constant over that set. We will also see that the entropy bound is independent of the choice of the initial set K ; without significant loss of generality, we assume in the sequel that K is a set of positive measure and “regular” shape, such as a hypercube, large enough to contain all initial conditions of interest.

Proposition 2 *For the system (1), the estimation entropy $h_{\text{est}}(\alpha, K)$ is finite and does not exceed $(L + \alpha)n/\ln 2$ where L is the Lipschitz constant of f .*

PROOF. This proceeds along the lines of the proof of Theorem 3.3 in [2] (see also [1] and the references therein for earlier results along similar lines). We fix the convergence parameters $\varepsilon, \alpha > 0$, the initial set K , and the time horizon $T > 0$, and try to come up with a bound on $s_{\text{est}}(T, \varepsilon, \alpha, K)$. Let us consider an open cover C of K with balls of radii $\varepsilon e^{-(L+\alpha)T}$ centered at points x_1, \dots, x_N ; N is the cardinality of the set C .

Consider any initial state $x \in K$. By the construction of C , we know that there exists an $x_i \in C$ such that $|x - x_i| \leq \varepsilon e^{-(L+\alpha)T}$. For any $t \leq T$,

$$\begin{aligned} |\xi(x, t) - \xi(x_i, t)| &\leq |x - x_i| + \int_0^t |f(\xi(x, s)) - f(\xi(x_i, s))| ds \\ &\leq |x - x_i| + L \int_0^t |\xi(x, s) - \xi(x_i, s)| ds \end{aligned}$$

using the Lipschitz constant of f . By the Bellman-Gronwall inequality (see, e.g., [11]), this implies

$$\begin{aligned} |\xi(x, t) - \xi(x_i, t)| &\leq |x - x_i| e^{Lt} \leq \varepsilon e^{-(L+\alpha)T} e^{Lt} \\ &\leq \varepsilon e^{-(L+\alpha)t} e^{Lt} = \varepsilon e^{-\alpha t}. \end{aligned}$$

It follows that the cover $C = \{x_1, \dots, x_N\}$ defines a $(T, \varepsilon, \alpha, K)$ -approximating set: $\hat{X} = \{\xi(x_1, \cdot), \dots, \xi(x_N, \cdot)\}$. That is, $s_{\text{est}}(T, \varepsilon, \alpha, K)$ is upper bounded by N which is the minimum cardinality of the cover of $K \subseteq \mathbb{R}^n$ with balls of radii $\varepsilon e^{-(L+\alpha)T}$. Let $c(\delta, S)$ denote the minimal cardinality of a cover of a set S with balls of radius δ . Then we can write

that $s_{\text{est}}(T, \varepsilon, \alpha, K) \leq c(\varepsilon e^{-(L+\alpha)T}, K)$. Next we proceed to compute a bound on h_{est} as follows:

$$\begin{aligned} &\limsup_{T \rightarrow \infty} \frac{1}{T} \log s_{\text{est}}(T, \varepsilon, \alpha, K) \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \log c(\varepsilon e^{-(L+\alpha)T}, K) \\ &= (L + \alpha) \limsup_{T \rightarrow \infty} \frac{\log c(\varepsilon e^{-(L+\alpha)T}, K)}{T(L + \alpha)} \\ &= \frac{(L + \alpha)}{\ln 2} \limsup_{T \rightarrow \infty} \frac{\ln c(\varepsilon e^{-(L+\alpha)T}, K)}{\ln(e^{(L+\alpha)T}/\varepsilon) + \ln \varepsilon} \\ &= \frac{(L + \alpha)}{\ln 2} \limsup_{T \rightarrow \infty} \frac{\ln c(\varepsilon e^{-(L+\alpha)T}, K)}{\ln(e^{(L+\alpha)T}/\varepsilon)} \\ &[\text{constant does not affect lim sup}] \\ &= \frac{(L + \alpha)}{\ln 2} \limsup_{\delta \searrow 0} \frac{\ln c(\delta, K)}{\ln(1/\delta)} \\ &[\text{defining } \delta = \varepsilon e^{-(L+\alpha)T}] \\ &\leq \frac{(L + \alpha)n}{\ln 2}. \end{aligned}$$

The last step follows from the fact that for any $K \subseteq \mathbb{R}^n$, the quantity $\limsup_{\delta \searrow 0} \frac{\ln c(\delta, K)}{\ln(1/\delta)}$, also called the upper box dimension of K , is no larger than (and typically equal to) n ; cf. [10, Section 3.2.f]. By taking the limit $\varepsilon \rightarrow 0$, we obtain the result $h_{\text{est}}(\alpha, K) \leq (L + \alpha)n/\ln 2$. \square

Remark 2 In the case when (1) is a linear system

$$\dot{x} = Ax \tag{8}$$

the result of Proposition 2 can be sharpened. Namely, in this case one can show that the exact expression (not just an upper bound) for the estimation entropy is

$$\frac{1}{(\ln 2)} \sum_{\text{Re } \lambda_i(A+\alpha I) > 0} \text{Re } \lambda_i(A+\alpha I) = \frac{1}{(\ln 2)} \sum_{\text{Re } \lambda_i(A) > -\alpha} (\text{Re } \lambda_i(A) + \alpha) \tag{9}$$

where $\text{Re } \lambda_i(A)$ are the real parts of the eigenvalues of A . This follows from results that are essentially well known, although not well documented in the literature (especially for continuous-time systems); for discrete-time systems this is shown, e.g., in [17]. Namely, since the flow is $\xi(x, t) = e^{At}x$, the volume of the reachable set at time T from the initial set K is $\det(e^{AT})\text{vol}(K)$ which by Liouville’s trace formula equals $e^{(\text{tr}A)T}\text{vol}(K)$. The decaying factor $e^{-\alpha t}$ on the right-hand side of (2) can be canceled by multiplying by $e^{\alpha t}$ on both sides; the effect of doing this on the left-hand side is that of replacing solutions of $\dot{x} = Ax$ by solutions of $\dot{x} = (A + \alpha I)x$, and suitably modifying the approximating functions. Projecting onto the unstable subspace of $A + \alpha I$, we can refine the trace to be the sum of only unstable eigenvalues of this matrix. The number of approximating functions must be at least proportional to the above volume (since the ε -balls around their endpoints must have enough volume to cover the reachable set), and after taking the logarithm, dividing by T , and letting $T \rightarrow 0$ we obtain (9) as the lower bound. The upper bound is obtained by reducing A to Jordan normal form followed by an argument similar to the proof of Proposition 2 above applied to each Jordan block

(with the corresponding eigenvalue replacing the Lipschitz constant L), and ends up giving the same expression (9).

3. ESTIMATION OVER INFINITE HORIZON

We will first describe a procedure for state estimation of the system (1) over infinite time horizon. Next, we will show that the output from this estimation procedure exponentially converges to the actual state of the system. Finally, we will prove a bound on the bit rate that is sufficient to achieve this convergence. This is a measure of the rate at which information has to be communicated from the sensors of the plant to the estimator.

3.1 Estimation procedure

From this point on in this section, we will discuss a specific estimation procedure based on quantized state measurements. The norm used here will be the infinity norm $\|\cdot\|_\infty$. Accordingly, the $B(x, \delta)$ balls will be the hypercubes and the grids will be sets of hypercubes. We will treat all previous definitions and results related to entropy in terms of the infinity norm.

The estimation procedure computes a function $v : [0, \infty) \rightarrow \mathbb{R}^n$ and an exponentially shrinking envelope around $v(t)$ such that the actual state of the system $\xi(x, t)$ is guaranteed to be within this envelope. It has several inputs: (1) a sampling period $T_p > 0$, (2) a desired exponential convergence rate $\alpha > 0$, (3) an initial set K and an initial partition size $d_0 > 0$, and (4) the Lipschitz constant L of the function f in (1), and (5) a subroutine for computing solutions of the differential equation (1). In this paper we do not distinguish between this subroutine for computing solutions and the actual solutions $\xi(\cdot, \cdot)$. The procedure works in rounds $i = 1, 2, \dots$ and each round lasts T_p time units. In each round, a new state measurement q_i is obtained and the values of three state variables S, δ, C are updated. We denote these updated values in the i^{th} round as q_i, δ_i, S_i , and C_i . Roughly, $S_i \subseteq \mathbb{R}^n$ is a hypercubic over-approximation of the state estimate, δ_i is the radius of the set S_i , and C_i is a grid on S_i which defines the set of possible state measurements q_{i+1} for the next round. We think of the quantized state measurements q_i as being transmitted from the sensors to the estimator via a finite-data-rate communication channel, while the variables δ_i, S_i , and C_i are generated independently and synchronously on both sides of the channel.

The initial values of these state variables are: $\delta_0 = d_0$; S_0 is a hypercube with center, say x_c , and radius $r_c = \frac{\text{diam}(K)}{2}$, such that $K \subseteq B(x_c, r_c)$; and $C_0 = \text{grid}(S_0, \delta_0 e^{-(L+\alpha)T_p})$. Recall the definition of a grid cover from Section 1.1: C_0 is a specific collection of points in \mathbb{R}^n such that $S_0 \subseteq \cup_{x \in C_0} B(x, \delta_0 e^{-(L+\alpha)T_p})$.

At the beginning of the i^{th} round, the algorithm takes as input (from the sensors) a measurement q_i of the current state of the system with respect to the cover C_{i-1} computed in the previous round. The measurement q_i is obtained by choosing a grid point $c \in C_{i-1}$ such that the corresponding $\delta_{i-1} e^{-(L+\alpha)T_p}$ -ball $B(c, \delta_{i-1} e^{-(L+\alpha)T_p})$ contains the current state $\xi(x, iT_p)$ of the system. (If there are multiple grid points satisfying this condition—and this may happen as C_{i-1} is a cover with closed sets having overlapping boundaries—then one is chosen arbitrarily.) Using this measurement, the algorithm computes the following:

(1) $v_i : [0, T_p] \rightarrow \mathbb{R}^n$, which is an approximation function for the state over the interval spanning this round, defined as the solution of the system (1) from q_i , (2) δ_i is updated as $e^{-\alpha T_p} \delta_{i-1}$, (3) $S_i \subseteq \mathbb{R}^n$ is an estimate of the state after T_p time, that is, at the beginning of round $i + 1$, and (4) C_i is a $\delta_i e^{-(L+\alpha)T_p}$ -grid on S_i , where L is the Lipschitz constant of f . Specifically, S_i is computed by first evaluating the solution $v_i(T_p) = \xi(q_i, T_p)$ of the system starting from q_i after time T_p , and then constructing the hypercube $B(v_i(T_p), \delta_i)$. Note that the size of this hypercube decays geometrically at the rate $e^{-\alpha T_p}$ with each successive round. Recall Section 1.1 where we defined grids and provided examples of specific ways of constructing them. For what follows, the specific construction is less important than the fact that each C_i can be computed from q_i by translating and scaling C_{i-1} .

Consider the beginning of the i^{th} round for some $i > 0$. From the algorithm it follows that if the current state x is contained in the estimate S_{i-1} computed in the last iteration, then the measurement q_i is one of the points in the cover C_{i-1} computed in the last iteration, and further, the error in the measurement $|q_i - x|$ is at most the precision of the cover which is $\delta_{i-1} e^{-(L+\alpha)T_p}$. This property will be used in the analysis below.

```

1  input:  $T_p, \alpha, K, d_0, L, \xi(\cdot, \cdot)$ 
2   $i = 0$ ;
3   $\delta_0 \leftarrow d_0$ ;
4   $S_0 \leftarrow B(x_c, r_c)$ ; //  $x_c$  is the center of  $K$ 
5   $C_0 \leftarrow \text{grid}(S_0, \delta_0 e^{-(L+\alpha)T_p})$ ;
6  while (true)
   // at  $i^{\text{th}}$  round,  $i > 0$ 
7    $i++$ ;
8   input  $q_i \in C_{i-1}$ ;
9   // measurement of current state
10   $v_i(\cdot) \leftarrow \xi(q_i, \cdot)|_{[0, T_p]}$ ;
11   $\delta_i \leftarrow e^{-\alpha T_p} \delta_{i-1}$ ;
12   $S_i \leftarrow B(v_i(T_p), \delta_i)$ ;
13   $C_i \leftarrow \text{grid}(S_i, \delta_i e^{-(L+\alpha)T_p})$ ;
14  output  $S_i \subseteq \mathbb{R}^n, C_i, v_i : [0, T_p] \rightarrow \mathbb{R}^n$ ;
15  wait ( $T_p$ );

```

Figure 1: Estimation procedure.

Remark 3 Line 10 of the estimation procedure uses a subroutine for computing numerical solutions of the differential equation (1) from a given quantized initial state q_i over a fixed time horizon T_p . In this paper, we assume that these computations are precise. Extending the algorithms and results to accommodate numerical imprecisions would proceed along the lines of the techniques used in numerical reachability computations (for example, in [6, 12]). The present case, however, is significantly simpler as the solutions have to be computed from a single initial state and up to a fixed time horizon.

In order to analyze the accuracy of this estimation procedure, we define a piecewise continuous estimation function $v : [0, \infty) \rightarrow \mathbb{R}^n$ by $v(0) := v_1(0)$ and

$$v(t) = v_i(t - (i-1)T_p) \text{ for all } t \in ((i-1)T_p, iT_p], i = 1, 2, \dots \quad (10)$$

The following theorem establishes an exponentially decaying upper bound on the error between the actual state of the system and the approximating function computed by the procedure.

Theorem 3 *For any choice of the parameters $\alpha, d_0, T_p > 0$, the procedure in Figure 1 has the following properties: for $i = 0, 1, 2, \dots$ and for any initial state $x \in K$,*

- (a) for any $t = iT_p, \xi(x, t) \in S_i$, and
- (b) for any $t \in [iT_p, (i+1)T_p), \|\xi(x, t) - v(t)\|_\infty \leq d_0 e^{-\alpha t}$.

PROOF. Part (a): We fix $x \in K$ and proceed to prove the statement by induction on the iteration index i . The base case: $i = 0$, that is, $t = 0$ and $\xi(x, 0) = x$. The required condition follows since $x \in K \subseteq B(x_c, r_c) = S_0$.

For the inductive step, we assume that $\xi(x, iT_p) \in S_i$ and have to show that $\xi(x, (i+1)T_p) \in S_{i+1}$. We proceed by establishing an upper bound on the distance between the actual trajectory of the system at $t = (i+1)T_p$ and the computed approximation $v(t)$:

$$\begin{aligned} & \|\xi(x, (i+1)T) - v((i+1)T_p)\|_\infty \\ &= \|\xi(\xi(x, iT_p), T_p) - v_{i+1}(T_p)\|_\infty \\ & \quad \text{[From Equation (10) defining } v(t)\text{]} \\ &= \|\xi(\xi(x, iT_p), T_p) - \xi(q_{i+1}, T_p)\|_\infty \quad (11) \\ & \quad \text{[From Line 10 } v_{i+1}(T_p) = \xi(q_{i+1}, T_p)\text{]} \\ &\leq e^{LT_p} \|\xi(x, iT_p) - q_{i+1}\|_\infty. \quad (12) \\ & \quad \text{[Bellman-Gronwall inequality]} \end{aligned}$$

The measurement q_{i+1} is the input received at the beginning of round $i+1$ for the actual state $\xi(x, iT_p)$ with respect to the cover C_i of S_i . From the induction hypothesis we know that $\xi(x, iT_p) \in S_i$, and therefore, $q_{i+1} \in C_i$. Since C_i is a $\delta_i e^{-(L+\alpha)T_p}$ -cover of S_i , it follows that

$$\|\xi(x, iT_p) - q_{i+1}\|_\infty \leq \delta_i e^{-(L+\alpha)T_p}. \quad (13)$$

We have $\|\xi(x, (i+1)T_p) - v((i+1)T_p)\|_\infty$

$$\begin{aligned} &\leq \delta_i e^{-(L+\alpha)T_p} e^{LT_p} \\ &= \delta_i e^{-\alpha T_p} \\ &= \delta_{i+1}. \quad \text{[Using definition of } \delta_{i+1}\text{]} \end{aligned}$$

Thus, it follows that $\xi(x, (i+1)T_p) \in B(v((i+1)T_p), \delta_{i+1}) = S_{i+1}$.

Part (b): We fix an iteration index $i \geq 0$ and an initial state $x \in K$. If $t = iT_p$ then the result follows from Part (a) because $\delta_i = d_0 e^{-\alpha iT_p}$. For any $t \in (iT_p, (i+1)T_p)$, we establish an upper bound on the distance between the actual trajectory $\xi(x, t)$ of the system at time t and the computed

approximation $v(t)$:

$$\begin{aligned} \|\xi(x, t) - v(t)\|_\infty &= \|\xi(\xi(x, iT_p), t - iT_p) - v_{i+1}(t - iT_p)\|_\infty \\ & \quad \text{[From equation (10) defining } v(t)\text{]} \\ &= \|\xi(\xi(x, iT_p), t - iT_p) - \xi(q_{i+1}, t - iT_p)\|_\infty \\ & \quad \text{[From } v_{i+1}(t) = \xi(q_{i+1}, t)\text{]} \\ &\leq \|\xi(x, iT_p) - q_{i+1}\|_\infty e^{L(t-iT_p)} \\ & \quad \text{[Bellman-Gronwall inequality]} \\ &\leq \delta_i e^{-(L+\alpha)T_p} e^{L(t-iT_p)} \\ & \quad \text{[From (13)]} \\ &= d_0 e^{-\alpha iT_p} e^{-(L+\alpha)T_p} e^{L(t-iT_p)} \\ & \quad \text{[} \delta_i = d_0 e^{-\alpha iT_p}\text{]} \\ &= d_0 e^{-\alpha(i+1)T_p} e^{L(t-(i+1)T_p)} \\ &\leq d_0 e^{-\alpha t}. \text{ [Since } iT_p \leq t \leq (i+1)T_p\text{]} \end{aligned}$$

□

3.2 Bit rate of estimation scheme and its relation to entropy

Now we estimate the communication bit rate needed by the estimation procedure in Figure 1. As the states S_{i-1} and C_{i-1} are maintained and updated by the algorithm in each round, the only information that is communicated from the system to the estimation procedure in each round is the measurement q_i . The number of bits needed for that is $\log(\#C_i)$, where $\#$ stands for the cardinality of a set. The long-term average bit rate of the algorithm is given by

$$b_r(\alpha, d_0, T_p) := \limsup_{j \rightarrow \infty} \frac{1}{jT_p} \sum_{i=1}^j \log(\#C_{i-1}).$$

We proceed to characterize this quantity from the description of the estimation procedure in Figure 1. We calculate $\#C_0 = \lceil \frac{\text{diam}(K)}{2d_0 e^{-(L+\alpha)T_p}} \rceil^n$. For each successive iteration i , $\#C_i = \lceil \frac{\delta_i}{\delta_i e^{-(L+\alpha)T_p}} \rceil^n = \lceil e^{(L+\alpha)T_p} \rceil^n$. Thus, $b_r(\alpha, d_0, T_p) = \lim_{i \rightarrow \infty} \frac{1}{T_p} \log(\#C_i) = (L + \alpha)n/\ln 2$ is the bit rate utilized by the procedure for any d_0 and T_p . Since it is independent of d_0 and T_p , we write it as $b_r(\alpha)$ from now on. We state our conclusion as follows.

Proposition 4 *The average bit rate used by the estimation procedure in Figure 1 is $(L + \alpha)n/\ln 2$.*

By Proposition 2, the bit rate $(L + \alpha)n/\ln 2$ used by the above algorithm is an upper bound on the entropy $h_{\text{est}}(\alpha, K)$. We now establish that no other similar algorithm can perform the same task with a bit rate lower than the entropy $h_{\text{est}}(\alpha, K)$. In other words, the ‘‘efficiency gap’’ of the algorithm is at most as large as the gap between the entropy and its upper bound known from Proposition 2. (Incidentally, combining this result with Proposition 4 we can arrive at an alternative proof of Proposition 2.) The lower bound in terms of entropy is proved below for an algorithm that uses a constant number of bits at each round; since in the above algorithm $\#C_0$ may be higher than $\#C_i$ for $i \geq 1$, we can think of this comparison as being valid once the algorithm has reached ‘‘steady state.’’ Instead of giving a more formalized description of the class of algorithms to which Proposition 5 applies, we refer the reader to [17, Section 2]

and the references therein for these details (which are by now quite standard).

Proposition 5 Consider an algorithm of the above type with an arbitrary choice of the cover C_i but such that at each step i the set C_i has the same number of elements: $\#C_i = N \forall i$ (i.e., the coding alphabet is of fixed size). If this algorithm achieves the properties listed in Theorem 3 for an arbitrary $d_0 > 0$, then its bit rate cannot be smaller than $h_{\text{est}}(\alpha, K)$.

PROOF. This proof follows along the same lines as the proof of Statement 1 of Theorem III.1 in [17]. Here the choice of norm does not matter so we revert to an arbitrary norm $|\cdot|$ on \mathbb{R}^n . Seeking a contradiction, suppose that an algorithm achieves the properties listed in Theorem 3 and has a bit rate smaller than $h_{\text{est}}(\alpha, K)$. Recall (see the proof of Lemma 2 and Remark 1) that

$$\begin{aligned} h_{\text{est}}(\alpha, K) &= \lim_{\varepsilon \searrow 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log n_{\text{est}}^*(T, 2\varepsilon, \alpha, K) \\ &= \sup_{\varepsilon > 0} \limsup_{T \rightarrow \infty} \frac{1}{T} \log n_{\text{est}}^*(T, 2\varepsilon, \alpha, K). \end{aligned}$$

Thus for some $\varepsilon > 0$ small enough we have

$$b_r(\alpha) < \limsup_{T \rightarrow \infty} \frac{1}{T} \log n_{\text{est}}^*(T, 2\varepsilon, \alpha, K).$$

Let d_0 be equal to this ε . Next, for a sufficiently large fixed integer j we must have

$$b_r(\alpha) < \frac{1}{jT_p} \log n_{\text{est}}^*(jT_p, 2\varepsilon, \alpha, K)$$

where T_p is the sampling period in the algorithm. Since the average bit rate is given by

$$b_r(\alpha) = \frac{1}{T_p} \log N$$

we obtain

$$N^j < n_{\text{est}}^*(jT_p, 2\varepsilon, \alpha, K)$$

The left-hand side of the above inequality is the number of possible sequences of codewords $\{q_i\}$ that can be produced by the algorithm over j rounds, while the right-hand side is the cardinality of a maximal $(jT_p, 2\varepsilon, \alpha, K)$ -separated set. This means that there must exist two different initial conditions x_1, x_2 in this $(jT_p, 2\varepsilon, \alpha, K)$ -separated set such that the corresponding solutions $\xi(x_1, t), \xi(x_2, t)$ will produce the same sequence of q_i 's, and hence will be approximated within $\varepsilon e^{-\alpha t}$ by the same approximating function $v(t)$:

$$|\xi(x_i, t) - v(t)| < \varepsilon e^{-\alpha t} \quad \forall t \in [0, jT_p], \quad i = 1, 2. \quad (14)$$

On the other hand, by the definition of a $(jT_p, 2\varepsilon, \alpha, K)$ -separated set it must hold that

$$|\xi(x_1, t) - \xi(x_2, t)| \geq 2\varepsilon e^{-\alpha t} \quad \text{for some } t \in [0, jT_p]$$

which contradicts (14) in view of the triangle inequality. \square

We note that the algorithm described in [17] performs a similar estimation task (with $\alpha = 0$ and in discrete time) and operates at an arbitrary bit rate above the entropy. However, that algorithm is quite abstract, since it relies on the existence of a suitable spanning set and does block coding over a sufficiently large time window using sequences from

this spanning set. By contrast, our algorithm given in Section 3.1 is constructive in that it utilizes a specific quantization procedure and works with an arbitrary fixed sampling period.

Remark 4 For the case of a linear system (8), the algorithm of Section 3.1 can be modified so that its average bit rate equals the entropy of the linear system given by the formula (9). This can be achieved by aligning the grids C_i used in the algorithm with eigenvectors of the matrix A and replacing the Lipschitz constant L with eigenvalues of A (i.e., using a different number of quantization points for each dimension). Constructions of this type for linear systems are well established in the literature; see, e.g., [8, 19].

4. MODEL DETECTION

In this section, we show that the estimation algorithm of Figure 1 can be used to distinguish two system models, provided they are in some sense adequately different.

Consider two continuous-time system models:

$$\dot{x} = f_1(x), \quad x \in \mathbb{R}^n, \quad (15)$$

$$\dot{x} = f_2(x), \quad x \in \mathbb{R}^n \quad (16)$$

where the initial state is in the known compact set $K \subset \mathbb{R}^n$ and f_1 and f_2 are Lipschitz functions with Lipschitz constants L_1 and L_2 . Here we assume that these are *global* Lipschitz constants; in case the system trajectories are confined to a compact invariant set, the result holds for local Lipschitz constants over that set. We denote the trajectories of the systems (15) and (16) by $\xi_1 : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ and $\xi_2 : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, respectively. From runtime data, we are interested in distinguishing whether the true dynamics of the system is f_1 or f_2 . For example, if f_1 and f_2 correspond to models with different sets of parameter values, then solutions to this problem could be used for model parameter identification. As another example application, consider a scenario where f_1 captures the nominal dynamics of the system and f_2 models a known aberration or failure mode. Then, solution to the above detection problem can be used for failure detection. It is straightforward to generalize the solution proposed below to handle multiple competing models.

For $L_s, T_s > 0$ we say that the two models are (L_s, T_s) -*exponentially separated* if there exists a constant $\varepsilon_{\min} > 0$ such that for any $\varepsilon \leq \varepsilon_{\min}$, for any two states $x_1, x_2 \in \mathbb{R}^n$ with $|x_1 - x_2| \leq \varepsilon$,

$$|\xi_1(x_1, T_s) - \xi_2(x_2, T_s)| > \varepsilon e^{L_s T_s}.$$

Remark 5 The exponential separation property can be shown to hold if there exist constants $\alpha_{\min} \in (0, 2\pi)$ and $v_{\min} > 0$ such that the two models satisfy the following two conditions at each $x \in \mathbb{R}^n$ (or at each x reachable from K): (1) the two vector fields have a separation angle of at least α_{\min} (here we are assuming $n \geq 2$), and (2) at least one of them has a velocity of at least v_{\min} (in particular, they have no common equilibria). Under these conditions, trajectories of the two systems with nearby initial conditions diverge from each other at the rate of at least $a := v_{\min} \sin(\alpha_{\min})$. Since for every $L > 0$ and every $T > 0$ we have $aT - \varepsilon > \varepsilon e^{LT}$ if $\varepsilon > 0$ is small enough, the exponential separation property follows (with arbitrary L_s, T_s). If the above transversality

condition (1) fails, we may still be able to establish exponential separation for L_s small enough. We also believe that conditions (1) and (2) are “generic” in the sense that we expect them to hold for almost all pairs of systems; for example, for affine systems this claim can be made precise and is confirmed by the numerical experiments discussed below.

4.1 Distinguishing algorithm

In the above definition of exponential separation the norm can be arbitrary, but in the algorithm below we work with the infinity norm. With some modifications, the procedure in Figure 1 can detect models using observations. In Figure 2, we show the procedure for detecting models. First of all, before taking the measurement in each round (T_p time) it makes an additional check. If the current state is not in the set S_i (line 8) computed from the previous round, then the procedure immediately halts by detecting model 2. If the current state is in S_i , then it proceeds as before and records a measurement q_i of the current state as one of the points in the cover C_i . Secondly, the function v_i (line 13) is now computed as a solution $\xi_1(q_i, \cdot)$ of the system given by (15). Finally, in computing the radius of the elements in the cover C_i (line 16), the Lipschitz constant L_1 of the system (15) is used.

```

1   input :  $T_p, \alpha, K, d_0, L_1, \xi_1(\cdot, \cdot)$ 
2    $i = 0$ ;
3    $\delta_0 \leftarrow d_0$ ;
4    $S_0 \leftarrow B(x_c, r_c)$ ;
5    $C_0 \leftarrow \text{grid}(S_0, \delta_0 e^{-(L_1 + \alpha)T_p})$ ;
6   while (true) // at  $i^{\text{th}}$  round,  $i > 0$ 
7      $i++$ ;
8     if current state  $\notin S_{i-1}$ 
9       output ‘‘second model’’;
10      break;
11    else
12      input  $q_i \in C_{i-1}$ ;
13       $v_i(\cdot) \leftarrow \xi_1(q_i, \cdot)|_{[0, T_p]}$ ;
14       $\delta_i \leftarrow e^{-\alpha T_p} \delta_{i-1}$ ;
15       $S_i \leftarrow B(v_i(T_p), \delta_i)$ ;
16       $C_i \leftarrow \text{grid}(S_i, \delta_i e^{-(L_1 + \alpha)T_p})$ ;
17    wait ( $T_p$ );

```

Figure 2: Procedure for detecting models.

Theorem 6 *Suppose that the true system model is either equation (15) or (16) and that the two models are (L_1, T_p) -exponentially separated. Then the procedure in Figure 2 outputs “second model” if and only if the system model is (16).*

PROOF. For the “if” part, assume that the true model is the second model, that is, given by equation (16). Fixing an initial state of the system x_0 , we have the true trajectory $\xi_2(x_0, \cdot)$. Let us also fix the parameters T_p, d_0, α of the detection algorithm. Since the value of the program variable $\delta_i = d_0 e^{-\alpha i T_p}$ decays geometrically in each iteration, there exists an i^* such that for any iteration $k - 1 \geq i^*$, $\delta_{k-1} e^{-(L_1 + \alpha)T_p} \leq \varepsilon_{\min}$. We consider the execution of the algorithm at one such iteration $k - 1$ and show that the condition in line 8 will be satisfied at the next iteration k .

We denote the actual state of the system at the beginning of the $(k - 1)^{\text{st}}$ iteration as $x_2 = \xi_2(x_0, (k - 1)T_p)$.

Assume that the condition in line 8 is not satisfied, i.e., $x_2 \in S_{k-1}$; otherwise, the algorithm would have already produced the correct “second model” output. The measurement q_k of x_2 obtained in this iteration is an element of C_{k-1} . Thus, $\|x_2 - q_k\|_\infty \leq \delta_{k-1} e^{-(L_1 + \alpha)T_p} \leq \varepsilon_{\min}$. By the (L_1, T_p) -separation with the infinity norm, it follows that

$$\begin{aligned} \|\xi_2(x_2, T_p) - \xi_1(q_k, T_p)\|_\infty &> \delta_{k-1} e^{-(L_1 + \alpha)T_p} e^{L_1 T_p} \quad (17) \\ &= \delta_{k-1} e^{-\alpha T_p} = \delta_k. \end{aligned}$$

As $v_k(\cdot) = \xi_1(q_k, \cdot)$, from the above strict inequality it follows that $\xi_2(x_0, kT_p) = \xi_2(x_2, T_p) \notin B(v_k(T_p), \delta_k) = S_k$. Thus, at the beginning of the k^{th} iteration, the condition in line 8 will hold.

For the “only-if” part, assume that the true model is not the second (equation (16)). Let us fix an initial state of the system x_0 . From the hypothesis we know that the true model is the first model and the true trajectory of the system is $\xi_1(x_0, t)$. From Theorem 3, it follows that at every iteration i , the state of the system at that round $\xi_1(x_0, iT_p) \in S_i$. Thus the **if**-condition in Line 8 is not satisfied at any iteration and consequently the algorithm never outputs “second model.” \square

Remark 6 The definition of exponential separation does not imply that the value of the upper bound ε_{\min} is known, and short of that we cannot conclude for sure that the true model is the first model even if the state measurements conform with the constructed bound S_i in every round. However, if we know such an upper bound ε_{\min} for which the models are (L_1, T_p) -exponentially separated, then with one extra conditional, the above algorithm can be made to decisively halt with the output “first model.” For this, the conditional statement

```

else if  $\delta_i e^{-L_1 T_p} < \varepsilon_{\min}$ 
  output ‘‘first model’’; break;

```

is to be inserted after line 10. This branch is executed by the algorithm at the i^{th} round only if we had $\delta_{i-1} e^{-(L_1 + \alpha)T_p} \leq \varepsilon_{\min}$ at the $(i - 1)^{\text{st}}$ round and the measured state was in S_j for each of the preceding rounds $j < i$. At this point the algorithm can soundly infer “first model” because, according to the above proof of Theorem 6, the second model would have already triggered line 8 in the current round or one of the earlier rounds.

Remark 7 It is possible to run two versions of the detection algorithm, one with each of the candidate models, in parallel. While this may speed up detection in practice, in the worst case the two versions would take the same amount of time to reach a decision. This would also double the data rate without guaranteeing faster model detection. We thus opted for an approach which, while “asymmetric,” works with the minimal needed data rate.

4.2 Experimental evaluation of detection algorithm

We have implemented the detection algorithm of Figure 2 in Python⁴. In this section, we discuss certain details about this implementation and numerical simulation-based results.

All sets in \mathbb{R}^n in the implementation, including the initial set K and the S_i ’s, are n -dimensional hyperrectangles

⁴Available at <https://bitbucket.org/mitras/detection>.

and they are represented either by two corner points or by a center point and a radius. The choice of this representation has implications on the efficiency of the algorithms. It enables the implementation of all the necessary operations such as testing membership in S , computing a grid on S , and quantizing a point with respect to a grid, in time that is linear in the number of dimensions n . Specifically, the $grid(S, \delta)$ function computes n lists of points in \mathbb{R} where the i^{th} list is generated by uniformly partitioning the i^{th} dimension of S into intervals of length 2δ . This list representation of $grid(S, \delta)$ is adequate for quantizing a state with respect to it. The detection algorithm has to compute solutions $\xi_1(\cdot, \cdot)$ of the system (15) over $[0, T_p]$. Moreover, in order to simulate the algorithm we have to compute the actual trajectories $\xi_2(\cdot, \cdot)$ of the system (16). For affine models $\dot{x} = Ax + b$, considered below, the analytic solution is given by

$$\xi(x, t) = e^{At}(x + A^{-1}b) - A^{-1}b$$

(provided A is invertible). Our implementation can handle more general models using the Python ODE solvers.

We generate pairs of random affine dynamical systems $sys1: \dot{x} = A_1x + b_1$, $sys2: \dot{x} = A_2x + b_2$, and then $sys1$ is used as the input model for the algorithm while $sys2$ is used as the true model of the system. With this set-up we performed many experiments to arrive at the following empirical conclusions. First of all, the detection algorithm always works (unless we deliberately choose $A_2 = A_1$ and $b_2 = b_1$). The detection time depends on several factors. As is expected from the algorithm, it increases with smaller values of α and T . If A_2 and b_2 are generated by perturbing A_1 and b_1 (not independently at random) then the detection time increases with smaller perturbations. Finally, on the average, the detection time increases with smaller-dimensional systems. This is possibly because with increasing n , there is a higher probability of having a larger separation in at least one of the eigenvalues of the models, and therefore, a faster detection.

5. CONCLUSIONS AND FUTURE DIRECTIONS

This paper proposed a framework for studying state estimation algorithms that have guaranteed efficiency with respect to sensing and communication data rates. We introduced two different notions of *estimation entropy* and established their equivalence. We derived an upper bound of $(L + \alpha)n/\ln 2$ for the estimation entropy of an n -dimensional nonlinear dynamical system with Lipschitz constant L , when the desired exponential convergence rate of the estimate is α . We developed an iterative procedure whose average bit rate matches this upper bound on the entropy. We showed that no other iterative estimation algorithm can work with bit rates lower than the entropy. Finally, we presented an application of the estimation procedure in picking out one from a pair of candidate models using measurement data. We showed that under a mild assumption of *exponential separation*—which holds almost surely for randomly chosen model pairs—the algorithm can always detect the true model in finite time.

This work suggests several avenues for future investigations. First of all, the bounds given in this paper using Lipschitz constants could be refined to bounds using suitable matrix norms of the Jacobian matrix, following the results

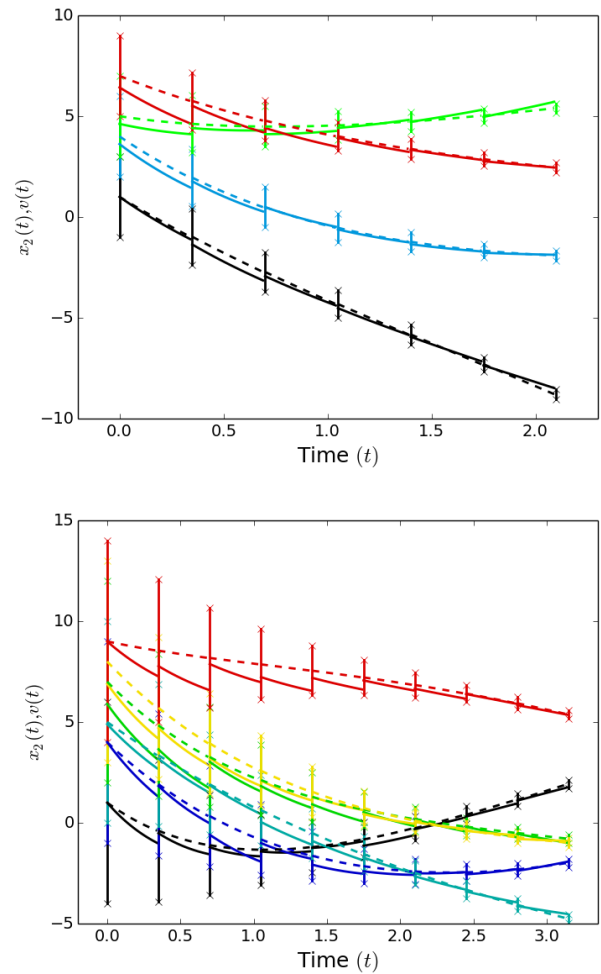


Figure 3: Two sample executions of the detection algorithm for four- and six-dimensional systems ($n = 4, 6$) from initial state to the detection of second model. The dashed lines (---) show the trajectories of the four (six) state variables of the actual system, $sys2$, and the solid lines show the estimates computed by the detection algorithm, $v(t)$. The vertical segments show the exponentially decaying envelope defined by $diam(S_i)$ in each round. Any one of the state components falling outside of S_i triggers detection.

in [1, 7, 14]. Second, it would be desirable to have more rigorous and readily checkable versions of the sufficient conditions for exponential separation, building on what is described in Remark 5. Third, it would be interesting to establish a lower bound on the estimation entropy for the general nonlinear case; this result would parallel Theorem 3.2 of [2] which gives a lower bound for the control version of entropy. Finally, while here the digital communication channel was assumed to be error-free, it would be of interest to incorporate packet losses, delays, noise, etc.

6. REFERENCES

- [1] V. A. Boichenko and G. A. Leonov. Lyapunov's direct method in estimates of topological entropy. *J. Math. Sci.*, 91:3370–3379, 1998.
- [2] F. Colonius. Minimal bit rates and entropy for exponential stabilization. *SIAM J. Control Optim.*, 50:2988–3010, 2012.
- [3] F. Colonius and C. Kawan. Invariance entropy for control systems. *SIAM J. Control Optim.*, 48:1701–1721, 2009.
- [4] F. Colonius, C. Kawan, and G. Nair. A note on topological feedback entropy and invariance entropy. *Systems Control Lett.*, 62:377–381, 2013.
- [5] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley, New York, 1991.
- [6] P. S. Duggirala, S. Mitra, and M. Viswanathan. Verification of annotated models from executions. In *Proceedings of the International Conference on Embedded Software, EMSOFT 2013, Montreal, QC, Canada, September 29 - Oct. 4, 2013*, pages 1–10. IEEE, 2013.
- [7] C. Fan and S. Mitra. Bounded verification with on-the-fly discrepancy computation. In *Automated Technology for Verification and Analysis - 13th International Symposium, ATVA 2015, Shanghai, China*, volume 9364 of *LNCS*, pages 446–463. Springer, 2015.
- [8] J. P. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. In *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*, 2002.
- [9] P. A. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice Hall, New Jersey, 1996.
- [10] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
- [11] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, New Jersey, 3rd edition, 2002.
- [12] K.-D. Kim, S. Mitra, and P. R. Kumar. Computing bounded epsilon-reach set with finite precision computations for a class of linear hybrid automata. In *Proceedings of Hybrid Systems: Computation and Control (HSCC 2011)*, 2011.
- [13] D. Liberzon and J. P. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Trans. Automat. Control*, 50:910–915, 2005.
- [14] J. Maidens and M. Arcak. Reachability analysis of nonlinear systems using matrix measures. *IEEE Trans. Automat. Control*, 60:265–270, 2015.
- [15] G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran. Topological feedback entropy and nonlinear stabilization. *IEEE Trans. Automat. Control*, 49:1585–1597, 2004.
- [16] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans. Feedback control under data rate constraints: An overview. *Proc. IEEE*, 95:108–137, 2007.
- [17] A. V. Savkin. Analysis and synthesis of networked control systems: Topological entropy, observability, robustness and optimal control. *Automatica*, 42:51–62, 2006.
- [18] R. S. Smith and J. C. Doyle. Model validation: a connection between robust control and identification. *IEEE Trans. Automat. Control*, 37:942–952, 1992.
- [19] S. Tatikonda and S. K. Mitter. Control under communication constraints. *IEEE Trans. Automat. Control*, 49:1056–1068, 2004.